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OF U(N)

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Submitted to: Proceedings of the Third Symposium of
Symmetry and Structural Properties of
Condensed Matter, World Scientific,
Singapore, September, 1996

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COMBINATORIAL ASPECTS OF REPRESENTATIONS OF U(n)

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(Dedicated to the memory of L. C. Biedenharn)

Abstract. The boson operator theory of the representations of the unitary group, its Wigner-Clebsch-Gordan , and Racah coeffficients is reformulated in terms of the ring of polynomials in any number of indeterminates with the goal of bringing the theory, as nearly as possible, under the purview of combinatorial oriented concepts. Four of the basic relations in unitary group theory are interpreted from this viewpoint.

1. INTRODUCTION

It is convenient to formulate the theory of the family of unitary groups $U(n), n = 1, 2, \ldots$, in terms of the ring of all polynomials in any number of indeterminates

$$z = (z_1, z_2, \ldots, z_i, \ldots), \tag{1.1}$$

where we restrict our discusssion to the case in which the scalars are real or complex numbers. The inner product of two such polynomials P and P' is defined to be

$$(P, P') = \overline{P}(\partial / \partial z) P'(z)\big|_{z=0}, \tag{1.2}$$

where $\overline{P}$ denotes the complex conjugate polynomial to P and

$$\partial / \partial z = (\partial / \partial z_1, \partial / \partial z_2, \ldots, \partial / \partial z_i, \ldots). \tag{1.3}$$

For a comprehensive theory dealing with the unitary irreducible representations of $U(n)$ and its Wigner-Clebsch-Gordan (WCG) coefficients, there are two basic polynomials of interest together with the relationships that they satisfy. For the description of these results, it is convenient to introduce the following notations at the outset in which the variables x_i and z_{ij} are commuting indeterminates:

$$a = (a_1, a_2, \ldots, a_n), \quad a! = \prod_i a_i!, \ a_i = nonnegative \ integer;$$

$$x = (x_1, x_2, \ldots, x_n), \quad x^a = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n};$$

$$A = (a_{ij})_{1 \le i, j \le n}, \quad A! = \prod_{ij} a_{ij}!, \ a_{ij} = nonnegative \ integer; \tag{1.4}$$

$$Z = (z_{ij})_{1 \le i, j \le n}, \quad Z^A = \prod_{ij} (z_{ij})^{a_{ij}}.$$

We also introduce an array T of n^2 operators $t_{i,\tau}$ with $i, \tau = 1, 2, \cdots, n,$, where operators having the same column index τ are commuting, but those of different column indices are noncommuting, in general:

$$T = (t_{i,\tau})_{1 \le i, \tau \le n}, \quad T^A = \prod_i (t_{i,1})^{a_{i1}} \prod_i (t_{i,2})^{a_{i2}} \cdots \prod_i (t_{i,n})^{a_{in}}. \tag{1.5}$$

DISCLAIMER

The first polynomial is denoted $P_{m\ m'}^{\ \mu}(Z)$ and is a polynomial over the commuting indeterminates $Z = (z_{i,j})$. The second polynomial is denoted $W_{m\ m'}^{\ \mu}(T)$ and is a polynomial over the noncommuting operators $T = (t_{i,\tau})$. Symbols such as

$$_{m\ m'}^{\ \mu} = \begin{pmatrix} m' \\ \mu \\ m \end{pmatrix} \ and \ _{m\ \gamma}^{\ \mu} = \begin{pmatrix} \gamma \\ \mu \\ m \end{pmatrix} \tag{1.6}$$

denote double Gel'fand-Zetlin patterns, as explained below. It is the purpose of this paper to give precise meaning to each of the following relationships and give their interpretation in the context of the unitary group $U(n)$:

$$P_{m\ m'}^{\ \mu}(Z) = \sum_{(\alpha:A:\alpha')} C_{m\ m'}^{\ \mu}(A)Z^A/A!,, \tag{I}$$

$$P_{m\ m'}^{\ \mu}(Z)P_{q\ q'}^{\ \nu}(Z) = \sum_{\left(_{\ell\ \ell'}^{\ \lambda}\right)} L\left[\left(_{\ell\ \ell'}^{\ \lambda}\right)\left(_{m\ m'}^{\ \mu}\right)\left(_{q\ q'}^{\ \nu}\right)\right] P_{\ell\ \ell'}^{\ \lambda}(Z), \tag{II}$$

$$W_{m\ \gamma}^{\ \mu}(T) = \sum_{(\alpha:A:\Delta(\gamma))} I_{m\ \gamma}^{\ \mu}(A)T^A/A!, \tag{III}$$

$$W_{m\ \gamma}^{\ \mu}(T)W_{q\ \rho}^{\ \nu}(T) = \sum_{\left(_{\ell\ \kappa}^{\ \lambda}\right)} I\left[\left(_{\ell\ \kappa}^{\ \lambda}\right)\left(_{m\ \gamma}^{\ \mu}\right)\left(_{q\ \rho}^{\ \nu}\right)\right] W_{\ell\ \kappa}^{\ \lambda}(T). \tag{IV}$$

To explain these relations, we must first define the labeling symbols.

2. COMBINATORIAL ENTITIES ENTERING INTO THE LABELING SCHEME

Partition:

$\mu = (\mu_1,\mu_2,...,\mu_n)$, $\mu_1 \geq \mu_2 \geq ...\geq \mu_n \geq 0$, μ_i *an integer.* (2.1)

Young frame of shape $\mu = (\mu_1,\mu_2,...,\mu_n)$: An array of n rows of "boxes" of equal size with μ_1 boxes in the first (top) row, μ_2 in the second row, ...,μ_n boxes in the $n-th$ (bottom) row, where the boxes are justified to the left, and a 0 means no box.

Standard Young-Weyl tableau: This is a Young frame with the boxes "filled in" with the integers 1,2,...,n with repetitions such that the following rules are obeyed:
The sequence of integers appearing in a given column is strictly increasing;
The sequence of integers appearing in a given row is nondecreasing.

Gel'fand-Zetlin pattern: This is a triangular array of nonnegative integers that contains the information for filling in every standard Young-Weyl tableau. It is written

$$\begin{matrix} m_{1,n} & m_{2,n} & \cdots & & & \cdots & m_{n,n} \\ & \ddots & & & & \cdots & \\ & & m_{1,3} & & m_{2,3} & m_{3,3} & \\ & & & m_{1,2} & & m_{2,2} & \\ & & & & m_{1,1} & & \end{matrix} \tag{2.2}$$

The Gel'fand pattern encodes the information on how to fill in the Young frame of shape $\mu = (m_{1,n}, m_{2,n}, ..., m_{n,n})$ to obtain a standard tableau:

$$row\ 1:\quad 1^{m_{1,1}}\quad 2^{m_{1,2}-m_{1,1}}\quad 3^{m_{1,3}-m_{1,2}} ...\quad n^{m_{1,n}-m_{1,n-1}}$$

$$row\ 2:\quad 2^{m_{2,2}} 3^{m_{2,3}-m_{2,2}} 4^{m_{2,4}-m_{2,3}} ...(n-1)^{m_{2,n}-m_{2,n-1}}$$

$$\bullet$$
$$\bullet \qquad\qquad\qquad\qquad\qquad\qquad (2.3)$$
$$\bullet$$

$$row\ n-1:\ (n-1)^{m_{n-1,n-1}}\, n^{m_{n-1,n}-m_{n-1,n-1}}$$

$$row\ n:\quad\ n^{m_{n,n}}$$

We have the one-to-one relation expressed by

$$\{betweenness\ conditions\}\ \leftrightarrow\ \{standard\ tableau\ conditions\}$$

Content or weight of the standard tableau (Gel'fand pattern): This is the $n-$ tuple of nonnegative integers $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ given by

$\alpha_j = number\ of\ j\ in\ tableau$

$\quad = (sum\ of\ entries\ in\ row\ j) - (sum\ of\ entries\ in\ row\ j-1)$

$$= \sum_{i=1}^{j} m_{i,j} - \sum_{i=1}^{j-1} m_{i,j-1}, \left(\alpha_1 = m_{1,1}\right). \qquad (2.4)$$

Kostka number $K(\mu, \alpha)$: This is the number of times weight α appears in all the standard tableaux of shape μ.

Double-standard tableau / double Gel'fand pattern: The single triangular array (Gel'fand pattern) (2.2) containing n rows is denoted by

$$\binom{\mu}{m}, \mu = (m_{1,n}, m_{2,n}, \cdots, m_{n,n}), \qquad (2.5)$$

in which the single symbol m denotes a Gel'fand pattern of n-1 rows that "fits" below partition μ. A double Gel'fand pattern is two patterns that share the same partition μ:

$$\binom{\mu}{m} and \binom{\mu}{m'};\quad written: \begin{pmatrix} m' \\ \mu \\ m \end{pmatrix} with \binom{m'}{\mu} inverted, or \quad \begin{matrix} & \mu & \\ m & & m' \end{matrix}. \qquad (2.6)$$

It is these double Gel'fand patterns corresponding to partitions μ, v, and λ having n parts that appear in relations I-IV.

Let us consider now in greater detail Relation I:

$$P_{m\ m'}^{\mu}(Z) = \sum_{(\alpha:A:\alpha')} C_{m\ m'}^{\mu}(A) Z^A / A!. \qquad (2.7)$$

The double Gel'fand pattern also labels the C-coefficients on the right, which is also labeled by the matrix arrary A of nonnegative integers:

$$C_{m\ m'}^{\ \mu}(A);\ A = (a_{ij}),\ each\ a_{ij}\ a\ nonnegative\ integer. \qquad (2.8)$$

This coefficient is defined to be zero unless the array A has row sums α and column

sums α' as determined by the weights of the Gel'fand patterns $\begin{pmatrix} \mu \\ m \end{pmatrix}$ and $\begin{pmatrix} \mu \\ m' \end{pmatrix}$; that is,

$$row\ i: \sum_j a_{ij} = \alpha_i\ ;\quad column\ j: \sum_i a_{ij} = \alpha'_j\ . \tag{2.7}$$

The summation

$$\sum_{(\alpha:A:\alpha')} \tag{2.9}$$

is to be carried out over all such arrays A.

The Knuth bijective algorithm ([1], [2]): This important result proves that the number of double Gel'fand patterns (double-standard tableaux) with given weights α and α' corresponding to all partitions $\mu = (\mu_1, \mu_2, ..., \mu_n)$ of a positive integer N equals the number of arrays A having fixed row sums α and fixed column sums α':

$$M_n(\alpha, \alpha') = \sum_{\mu \mapsto N} K(\mu, \alpha) K(\mu, \alpha'). \tag{2.10}$$

It is this combinatorial identity that assures that the discretized functions

$$C^{\mu}_{m\ m'}(A) \tag{2.11}$$

can be arranged into a square matrix of dimension $M_n(\alpha, \alpha')$, where row enumeration is given by the double patterns ${}_m\mu_{m'}$ as μ runs over all partitions of N, and for each such μ, the patterns m and m' over all arrays having given weights α and α', respectively; and column enumeration is given by the A running over all arrays having the row and columns sums corresponding to the given weights α and α'.

These results complete the description of the meaning of the **labels** occurring in Relation I. We still must give the definition of the polynomials by defining the coefficients (2.8). We refer to these coefficients as discretized functions, since they may be viewed as follows:

The $C^{\mu}_{m\ m'}$ are functions defined over the nonnegative integers (a_{ij})

satisfying the square array conditions $(\alpha:A:\alpha')$;

We sometime refer to the $C^{\mu}_{m\ m'}(A)$ as the discrete $P^{\mu}_{m\ m'}(Z)$. Both of these objects carry labels that ate purely combinatoric in structure.

3. COMPLETE DEFINITION OF THE POLYNOMIALS $P^{\mu}_{m\ m'}$

We have described the meaning of the labels occurring in Relation I, but have still to define the objects themselves. We will not be able to do this explicitly, but only implicitly through a recursive definition of the discretized functions. But let us note that these polynomials are well-studied objects: They are the so-called boson polynomials discussed at great length in [2]-[9], where we have here renamed the boson operators a^j_i in those references to be the indeterminates z_{ij}. The natural boson inner product then gives exactly the same number as the inner product defined by (1.2), since the operator corresponding to the Hermitian conjugate boson $(a^j_i)^* = \bar{a}^j_i$ is $\partial / \partial z_{ij}$. The results we state here are transcriptions of the many of the known properties of the so-called boson polynomials, interpreted now in a new context.

Since we will be unable to give a fully explicit formula for the discretized functions, we state some of the many properties of the polynomials $P_m{}^\mu{}_{m'}$ and the $C_m{}^\mu{}_{m'}$, giving finally a recursive definition of the latter that defines them uniquely, hence, also the functions $P_m{}^\mu{}_{m'}$.

3.1. Properties of the $P_m{}^\mu{}_{m'}(Z)$

• **maximal polynomial:**

$$P_{\max}{}^\mu{}_{\max}(Z) = \prod_{k=1}^{n} (\det Z_k)^{\mu_k - \mu_{k+1}}, \ \mu_{n+1} = 0, \tag{3.1}$$

where max denotes the Gel'fand pattern of weight μ, and

$$\det Z_k = k \times k \ \textit{principal minor of } Z. \tag{3.2}$$

• **homogeneity:**

of degree α_i in $(z_{i1}, z_{i2}, ..., z_{in})$,

of degree α'_j in $(z_{1j}, z_{2j}, ..., z_{nj})$,

of degree $\mu_1 + \mu_2 + ... + \mu_n$ in all n^2 variables.

• **transposition and multiplication:**

$$P^\mu(Z^T) = (P^\mu(Z))^T, \ T = \textit{matrix transposition.}$$

$$P^\mu(X)P^\mu(Y) = P^\mu(XY); \ X \text{ and } Y \text{ arbitrary.} \tag{3.3}$$

• **orthogonality in the inner product (1.2):**

$$(P_\ell{}^\lambda{}_{\ell'}, P_m{}^\mu{}_{m'}) = \delta_{\lambda,\mu} \delta_{\ell,m} \delta_{\ell',m'} M(\lambda), \tag{3.4}$$

where the normalization factor $M(\lambda)$ is given by

$$M(\lambda) = \prod_i (\lambda_i + n - i)! \Big/ \prod_{i<j} (\lambda_i - \lambda_j + j - i)!. \tag{3.5}$$

This factor has a tableau interpretation in terms of hooks ([3], Vol. 8. p. 236; Macdonald [10], p.9).

• **generating functions:**

$$\prod_{k=1}^{n} [\det(X^T Z Y)_k]^{\mu_k - \mu_{k+1}} = \sum_{m,m'} P_m{}^\mu{}_{\max}(X) P_m{}^\mu{}_{m'}(Z) P_{m'}{}^\mu{}_{\max}(Y),$$

$$\prod_{k=1}^{n} [\det(X^T Z)_k]^{\mu_k - \mu_{k+1}} = \sum_m P_m{}^\mu{}_{\max}(X) P_m{}^\mu{}_{\max}(Z). \tag{3.6}$$

• **reduction property:**

$$P_m{}^\mu{}_{m'}\begin{pmatrix} Z_{n-1} & 0 \\ 0 & z_{nn} \end{pmatrix} = \delta_{v,v'}(z_{nn})^{\alpha_n} P_q{}^v{}_{q'}(Z_{n-1}), \tag{3.7}$$

where

$$v = (m_{1,n-1}, m_{2,n-1}, ..., m_{n-1,n-1}), \ v' = (m'_{1,n-1}, m'_{2,n-1}, ..., m'_{n-1,n-1}),$$

$$q = (m)_{n-2}, \ q' = (m')_{n-2}. \tag{3.8}$$

This reduction property is to apply for each n=2,3, ... where, for n = 2, we have

$$P_\phi{}^{v_1}{}_\phi(z_{11}) = (z_{11})^{v_1}, \ \phi = \textit{empty set} = (m)_0 = (m')_0. \tag{3.9}$$

3.2. Properties of the discretized functions $C_{m\,m'}^{\mu}(A)$

The first important property of these discretized functions is that in their normalized version

$$N\left(_{m}{}^{\mu}{}_{m'}\right)(A) = [M(\mu)]^{-1/2}(A!)^{-1/2}C\left(_{m}{}^{\mu}{}_{m'}\right)(A), \qquad (3.10)$$

they are the elements of a real orthogonal matrix and satisfy the following orthogonality relaltions:

$$\sum_{(\alpha:A:\alpha')} N_{\ell}{}^{\lambda}{}_{\ell'}(A)N_{m}{}^{\mu}{}_{m'}(A) = \delta_{\lambda,\mu}\delta_{\ell,m}\delta_{\ell',m'}, \qquad (3.11)$$

$$\sum_{\lambda\mapsto N}\sum_{m,m'} N_{m}{}^{\lambda}{}_{m'}(A)N_{m}{}^{\lambda}{}_{m'}(A') = \delta_{A,A'}, N = \alpha_1+...+\alpha_n = \alpha_1'+...+\alpha_n'. \quad (3.12)$$

These relations are a consequence of the known orthogonality relations for the boson polynomials. The fact that we are, indeed, dealing with a *square,* matrix is assured by the Knuth algorithm.

The second important property of these discretized functions is that they may be generated recursively from the coefficients at level $n-1$. We require some notations to given this recursive definition. Let $\mu = (\mu_1,\mu_2,...,\mu_n)$ and $v = (v_1, v_2,..., v_n)$ denote partitions in which each part of μ is at least as great as the corresponding part of v, that is, $\mu_i \geq v_i$. We define the abbreviated symbol on the left below in terms of the notation for a totally symmetric WCG-coefficient used in 2] and also in Section 5:

$$\left\langle \begin{matrix} \mu \\ m \end{matrix} \middle\| \begin{matrix} v \\ q \end{matrix} \right\rangle = \left\langle \begin{matrix} \mu \\ m \end{matrix} \middle| \left\langle \begin{matrix} \gamma \\ k\ 0^{n-1} \\ \alpha \end{matrix} \right\rangle \middle| \begin{matrix} v \\ q \end{matrix} \right\rangle = \left[\begin{pmatrix} \mu \\ m \end{pmatrix} \begin{pmatrix} \gamma \\ k\ 0^{n-1} \\ \alpha \end{pmatrix} \begin{pmatrix} v \\ q \end{pmatrix} \right], \qquad (3.13)$$

where $k = \sum_{i=1}^{n}(\mu_i - v_i)$. This WCG-coefficient is known explicitly (see [11]), although it

is quite complicated. The notation on the left is complete in the sense that the pattern γ

and the pattern α are fully determined by the patterns $\begin{pmatrix} \mu \\ m \end{pmatrix}$ and $\begin{pmatrix} v \\ q \end{pmatrix}$:

$$\alpha_{1j} = \sum_{i=1}^{j}(m_{ij} - q_{ij}), \quad j = 1,2,...,n;\ and\ \alpha_{ij} = 0, i = 2,3,...,j; j = 2,3,\cdots,n,$$

$$\gamma_{1j} = \sum_{i=1}^{j}(\mu_i - v_i),\ j = 1,2,...,n\ and\ \gamma_{ij} = 0,\ i = 2,3,...,j; j = 2,3,\cdots,n. \qquad (3.14)$$

We also introduce an abbreviated notation for a totally symmetric reduced $U(n):U(n-1)$ WCG-coefficient. Let μ and v denote partitions as in (3.13), and let μ' and v' denote partitions satisfying $\mu' \prec \mu$ and $v' \prec v$, where the symbol $\prec$ denotes that the primed pátrition lies between the unprimed one in the sense of row $n-1$ and row n of a Gel'fand pattern. Then, we define the abbreviated symbol on the left below in terms of a standard notation ([6], [7]) for a reduced $U(n):U(n-1)$ coefficient by

$$\left[\begin{matrix} \mu \\ \mu' \end{matrix} \middle\| \begin{matrix} v \\ v' \end{matrix} \right] = \left\langle \begin{matrix} \mu \\ \mu' \end{matrix} \middle\| \begin{matrix} \gamma \\ 1\ 0^{n-1} \\ \gamma' \end{matrix} \middle\| \begin{matrix} v \\ v' \end{matrix} \right\rangle , \quad k = \sum_{i=1}^{n}(\mu_i - v_i). \qquad (3.15)$$

The operator patterns γ and γ' are uniquely determined by the partitions $\mu' \prec \mu$ and $v' \prec v$ to be

$$\gamma_{1j} = \sum_{i=1}^{j}(\mu_i - \nu_i), j = 1,2,\cdots,n; \ \gamma_{ij} = 0, i = 2,3,...,j, \ j = 2,3,...,n;$$

$$\gamma'_{1j} = \sum_{i=1}^{j}(\mu'_i - \nu'_i), j = 1,2,\cdots,n; \ \gamma'_{ij} = 0, i = 2,3,...,j; j = 2,3,...,n-1. \tag{3.16}$$

In terms of these notations, we have the following recurrence formula for the normalized n-coefficients defined by (3.10). This recurrence relation defines completely the general N – coefficient:

$$N\left(\begin{pmatrix}\nu\\q\end{pmatrix}^{\mu}\begin{pmatrix}\nu'\\q'\end{pmatrix}\right)(A_n) = \sum_{\begin{pmatrix}\lambda\\\ell\,\ell'\end{pmatrix}} \begin{bmatrix}\mu\\\nu\end{bmatrix}\begin{vmatrix}\nu'\,0\\\lambda\end{vmatrix}\left\langle\begin{matrix}\nu\\q\end{matrix}\middle\|\begin{matrix}\lambda\\\ell\end{matrix}\right\rangle\left\langle\begin{matrix}\nu'\\q'\end{matrix}\middle\|\begin{matrix}\lambda\\\ell'\end{matrix}\right\rangle N^{\ \ \lambda}_{\ell\ \ell'}(A_{n-1}). \tag{3.17}$$

For n = 2, this result reduces to (using now the first (inverted) notation from (2.6)):

$$N\left(\mu_1 \begin{matrix}\nu'_1\\ \\ \nu_1\end{matrix} \mu_2\right)\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix} = \sum_{\lambda_1}\begin{bmatrix}\mu_1&\mu_2\\ &\nu_1\end{bmatrix}\begin{vmatrix}\nu'_1&0\\ &\lambda_1\end{vmatrix}\langle\nu_1\|\lambda_1\rangle\langle\nu'_1\|\lambda_1\rangle N(\lambda_1)(a_{11}),$$

in which the $U(1)$ WCG-coefficients $\langle\nu_1\|\lambda_1\rangle$ and $\langle\nu'_1\|\lambda_1\rangle$ are both equal to 1, and the discrete coefficient for the representations of $U(1)$ is $N(\lambda_1)(a_{11}) = \delta_{\lambda_1,a_{11}}$. Thus, we obtain

$$N\left(\mu_1 \begin{matrix}\nu'_1\\ \\ \nu_1\end{matrix} \mu_2\right)\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix} = \begin{bmatrix}\mu_1&\mu_2\\ &\nu_1\end{bmatrix}\begin{vmatrix}\nu'_1&0\\ &a_{11}\end{vmatrix} = C^{\ j\ \ J\ \ j'}_{\ m\ M\ m'}, \tag{3.18}$$

where the C-coefficient to the right is an $SU(2)$ WCG-coefficient (see [3] for this notation) with

$$j = \tfrac{1}{2}\nu'_1, \ J = \tfrac{1}{2}(\mu_1 + \mu_2) - \nu'_1, \ j' = \tfrac{1}{2}(\mu_1 - \mu_2),$$

$$m = a_{11} - \tfrac{1}{2}\nu'_1, \ M = m' - m, \ m' = \nu_1 - \tfrac{1}{2}(\mu_1 - \mu_2) \ . \tag{3.19}$$

For n = 3, we have:

$$N\left(\mu_1\begin{matrix}q1\\\nu'_1\\ \\ \nu_1\\q1\end{matrix}\mu_2\begin{matrix}\nu_2\\ \\ \nu_2\end{matrix}\mu_3\right)\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{32}&a_{32}&a_{33}\end{pmatrix} = \sum_{(\lambda_1,\lambda_2)}\begin{bmatrix}\mu_1&\mu_2&\mu_3\\ &\nu_1&\nu_2\end{bmatrix}\begin{vmatrix}\nu'_1&\nu'_2&0\\ &\lambda_1&\lambda_2\end{vmatrix}$$

$$\times \sum_{\ell_1}\sum_{\ell'_1}\left\langle\begin{matrix}\nu_1&\nu_2\\ &q1\end{matrix}\middle\|\begin{matrix}\lambda_1&\lambda_2\\ &\ell_1\end{matrix}\right\rangle\left\langle\begin{matrix}\nu'_1&\nu'_2\\ &q'_1\end{matrix}\middle\|\begin{matrix}\lambda_1&\lambda_2\\ &\ell'_1\end{matrix}\right\rangle N\left(\lambda_1\begin{matrix}\ell'_1\\ \\ \ell_1\end{matrix}\lambda_2\right)\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix},$$

$$\tag{3.20}$$

where, except for the square-bracket reduced U(3):U(2) WCG-coefficient, all other symbols on the right are SU(2) WCG-coefficients as identified explicitly by

$$\left\langle\begin{matrix}\nu_1&\nu_2\\ &q1\end{matrix}\middle\|\begin{matrix}\lambda_1&\lambda_2\\ &\ell_1\end{matrix}\right\rangle = C^{\ \tfrac{1}{2}(\lambda_1-\lambda_2)\quad\quad \tfrac{1}{2}(\nu_1+\nu_2-\lambda_1-\lambda_2)\quad\quad \tfrac{1}{2}(\nu_1-\nu_2)}_{\ \ell_1-\tfrac{1}{2}(\lambda_1+\lambda_2)\ \ q_1-\ell_1-\tfrac{1}{2}(\nu_1+\nu_2-\lambda_1-\lambda_2)\ \ q_1-\tfrac{1}{2}(\nu_1+\nu_2)},$$

$$\left\langle\begin{matrix}\nu'_1&\nu'_2\\ &q'_1\end{matrix}\middle\|\begin{matrix}\lambda_1&\lambda_2\\ &\ell_1\end{matrix}\right\rangle = C^{\ \tfrac{1}{2}(\lambda_1-\lambda_2)\quad\quad \tfrac{1}{2}(\nu'_1+\nu'_2-\lambda_1-\lambda_2)\quad\quad \tfrac{1}{2}(\nu'_1-\nu'_2)}_{\ \ell'_1-\tfrac{1}{2}(\lambda_1+\lambda_2)\ \ q'_1-\ell'_1-\tfrac{1}{2}(\nu'_1+\nu'_2-\lambda_1-\lambda_2)\ \ q'_1-\tfrac{1}{2}(\nu'_1+\nu'_2)},$$

$$N\begin{pmatrix} & \ell_1' & \\ \lambda_1 & & \lambda_2 \\ & \ell_1 & \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = C\begin{matrix} \frac{1}{2}\ell_1' & \frac{1}{2}(\lambda_1+\lambda_2-\ell_1') & \frac{1}{2}(\lambda_1-\lambda_2) \\ a_{11}-\frac{1}{2}\ell_1' & \ell_1-a_{11}-\frac{1}{2}(\lambda_1+\lambda_2-\ell_1') & \ell_1-\frac{1}{2}(\lambda_1-\lambda_2) \end{matrix}.$$

All C—quantities in these relations are SU(2) WCG-coefficients of the form in the right-hand side of (3.18). It is quite remarkable that the reduced U(3):U(2) WCG-coefficient in (3.20) is itself essentially an SU(2) quantity , at least for $\mu_3 = 0$, where it may be shown to be a $6 - j$ symbol. *This result may be regarded as the fundamental reason that SU(2) quanaities play an unexpected role in expressions for the SU(3) WCG-coefficients* (see [12] and references therein).

The genesis of the important recurrence relation (3.17), from which one builds all irreducibke unitary representations of $U(n)$, indeed, the general polynomial (2.7), is an expression given in [2] ,eq. 2.28). It is only necessary in that result to identify the boson polynomial matrix with the matrix Z of indeterminates. This relation gives a unique construction of all unitasy irreps of $U(n)$ from those of $U(n-1)$. *As pointed out in [12], this result is of fundamental importance.*

3.3. The unitary irreducible representations of U(n)

As already indicated at the end of the last section, the significance of the polynomials $P_{m\ m'}^{\mu}(Z)$, which are defined for all indeterminates $Z = (z_{ij})$, is that when one chooses $Z = U \in U(n)$, one obtains the unitary irreducible representations of U(n) given by

$$D_m{}^{\mu}{}_{m'}(U) = P_m{}^{\mu}{}_{m'}(U), U \in U(n); D^{\mu}(U)D^{\mu}(U') = D^{\mu}(UU'),$$

$$D^{\mu}(I_n) = I_{\dim \lambda}; \dim \lambda = [\prod_{i<j}(\lambda_i - \lambda_j + j - i)]/1!2!...(n-1)! .$$

$$(3.21)$$

All unitary irreps are then obtained from these by multiplying by the appropriate power of $\det U$. The structure of the unitary irreps (3.21) reflect the information encoded in the Gel'fand-Zetlin patterns that upon restricting $U \in U(n)$ to $V \in U(n-1)$ by setting

$$U = \begin{pmatrix} V & 0 \\ 0 & 1 \end{pmatrix},$$

$$(3.22)$$

one obtains already in reduced form the unitary irreps of $U(n-1)$ given by

$$D^{\nu}(V), \text{ each } \nu \prec \mu.$$

$$(3.23)$$

Since this paper is about connections with combinatorics, let us point out that there is a very nice relation between the totally symmetric unitary irreps of $U(n)$ and MacMahon's master theorem. This is developed in detail in [13].

4. REDUCTION OF THE KRONECKER PRODUCT

Relation II given by

$$P_m{}^{\mu}{}_{m'}(Z)P_q{}^{\nu}{}_{q'}(Z) = \sum_{\begin{pmatrix} \lambda \\ \ell\ \ell' \end{pmatrix}} L\left[\begin{pmatrix} \lambda \\ \ell\ \ell' \end{pmatrix}\begin{pmatrix} \mu \\ m\ m' \end{pmatrix}\begin{pmatrix} \nu \\ q\ q' \end{pmatrix}\right]P_\ell{}^{\lambda}{}_{\ell'}(Z)$$

$$(4.1)$$

has its origin in the reduction of the Kronecker product of two unitary irreps of $U(n)$ into unitary irreps. The summation in this relation extends over all partitions λ that occur in the abstract Clebsch-Gordan series

$$\mu \times \nu = \sum_{\lambda} g_{\mu\nu\lambda} \lambda,$$

$$(4.2)$$

and expressed explicitly by the matrix relation ($\times$ denotes matrix direct product and $\oplus$ matrix direct sum):

$$C^T(D^\mu(U) \times D^v(U))C = \sum_\lambda \oplus g_{\mu v \lambda} D^\lambda(U). \qquad (4.3)$$

The quantities $g_{\mu v \lambda}$ are the Littlewood-Richardson numbers giving the number of occurrences of λ in the Kronecker product $\mu \times v$. The direct sum matrix on the right consists of the matrix $D^\lambda(U)$ repeated $g_{\mu v \lambda}$ times along the diagonal for each λ that occurs in the direct product $\mu \times v$, which we denote by $\lambda \in \mu \times v$. The matrix C is of dimension $\dim \mu \dim v$ and is real orthogonal. The balance of dimensions on each side of relation (4.3) requires the following identity between dimensions:

$$\dim \mu \dim v = \sum_\lambda g_{\mu v \lambda} \dim \lambda . \qquad (4.4)$$

In relation (4.3), we can also move the orthogonal matrix C to the right-hand side:

$$D^\mu(U) \times D^v(U) = C\left(\sum_\lambda \oplus g_{\mu v \lambda} D^\lambda(U)\right)C^T. \qquad (4.5)$$

Setting $Z = U \in U(n)$ in this relation and writing it in matrix element form, we obtain (4.1). That this gives a valid identity for arbitrary Z is a consequence of its known validity for bosons ([2], [6]).

In the mathematics literature, relation (4.1) is often called the linearization of a product of polynomials. The fact that the extension is valid for general variables z_{ij} is quite significant. For example, one has the inner product relation

$$\left(P_\ell{}^\lambda{}_{\ell'}, P_m{}^\mu{}_{m'} P_q{}^v{}_{q'}\right) = M(\lambda)L\left[\binom{\lambda}{\ell \ \ell'}\binom{\mu}{m \ m'}\binom{v}{q \ q'}\right], \qquad (4.6)$$

which, in consequence of Relation I, allows one to express the left-hand side of this relation explicitly in terms of the discretized coefficients leading to

$$M(\lambda)L\left[\binom{\lambda}{\ell \ \ell'}\binom{\mu}{m \ m'}\binom{v}{q \ q'}\right] =$$

$$= \sum_{(\alpha:A:\alpha')} \sum_{(\beta:B:\beta')} \sum_{(\gamma:C:\gamma')} \frac{(B+C)!}{B!C!} \delta(A, B+C) C_\ell{}^\lambda{}_{\ell'}(A) C_m{}^\mu{}_{m'}(B) C_q{}^v{}_{q'}(C). \qquad (4.7)$$

Moreover, it is a consequence of relation (4.5) that the $L-$coefficients must be expressible as a sum over products of the elements (WCG-coefficients) of the matrix C:

$$L\left[\binom{\lambda}{\ell \ \ell'}\binom{\mu}{m \ m'}\binom{v}{q \ q'}\right] = \sum_{\kappa=1}^{g_{\mu v \lambda}} C\left[\binom{\lambda}{\ell}_\kappa \binom{\mu}{m}\binom{v}{q}\right]C\left[\binom{\lambda}{\ell'}_\kappa \binom{\mu}{m'}\binom{v}{q'}\right], \quad (4.8)$$

each $\lambda \in \mu \times v$, where the elements of the matrix C are denoted by

$$C\left[\binom{\lambda}{\ell}_\kappa \binom{\mu}{m}\binom{v}{q}\right]. \qquad (4.9)$$

The columns and rows of C are enumerated as follows:

columns: all $\mu \atop m$ and $v \atop q$ patterns, giving $\dim \mu \dim v$ labels;

rows: all $\lambda \in \mu \times v$, and for each such λ, by an index κ

which assumes values $1, 2, \cdots, g_{\mu v \lambda}$, giving $\sum_\lambda g_{\mu v \lambda} \dim \lambda$ labels.

The left-hand side of relation (4.8) is known (at least in principle from (4.7), and the most important problem in unitary group theory for physics and chemistry is a procedure for determining the WCG-coefficients under the summation on the right. For the group $U(2)$, there is no multiplicity in the Kronecker product, and no summation in relation (4.8). In this case, relations (4.8) may be used to determine the WCG-coefficients uniquely to within phase conventions as was done by Wigner (see, for example, [3]).

Larry Biedenharn had the remarkable insight to realize that there is a *universal label* for the multiplicity for general $U(n)$. This is achieved by introducing into the notation for a WCG-coefficient still another Gel'fand pattern γ; it is associated with the partition μ, and is in addition to the pattern m. For convenience of display, it is inverted over the partition μ in the manner of the symbol (2.6):

$$\begin{pmatrix} \gamma \\ \mu \\ m \end{pmatrix}. \tag{4.10}$$

There is, of course, a weight associated with the (inverted) pattern $\begin{pmatrix} \gamma \\ \mu \end{pmatrix}$ in the standard way, given by (2.4). But, because the role of the pattern γ is conceptually different from that of a Gel'fand pattern, which has the Weyl group-subgroup interpretation, we call it an *operator pattern*, denote its weight by $\Delta(\gamma)$, and call this weight a *shift-weight.*:

$$\Delta(\gamma) = \text{weight of } \begin{pmatrix} \gamma \\ \mu \end{pmatrix} = (\Delta_1(\gamma), \Delta_2(\gamma), \cdots, \Delta_n(\gamma)). \tag{4.11}$$

One can prove that for every $\lambda \in \mu \times \nu$ there exists a unique shift-weight Δ of μ such that $\lambda = \nu + \Delta$. Accordingly, we denote the WCG-coefficient (4.9) by

$$C\left[\begin{pmatrix} \nu+\Delta \\ \ell \end{pmatrix}_\kappa \begin{vmatrix} \mu \\ m \end{pmatrix} \begin{pmatrix} \nu \\ q \end{pmatrix} \right] = \left[\begin{pmatrix} \nu+\Delta \\ \ell \end{pmatrix} \begin{pmatrix} \gamma \\ \mu \\ m \end{pmatrix} \begin{pmatrix} \nu \\ q \end{pmatrix} \right], \tag{4.12}$$

where the relation between the multiplicity index κ and the operator pattern γ is yet to be explained. It appears, at first, that this notation is flawed, since, for a given shift-weight Δ of μ, the number of distinct operator patterns γ, which numerically runs over the set of Gel'fand patterns, is given by the Kostka number $K(\mu, \Delta)$, while the index κ assumes $g_{\mu, \nu, \nu+\Delta}$ distinct values. Indeed, it is a well-known fact that the Littlewood-Richardson numbers assume values in the set given by

$$g_{\mu, \nu, \nu+\Delta} \in \Lambda(\mu, \Delta) = \{0, 1, ..., K(\mu, \Delta)\}. \tag{4.13}$$

Thus, in general, there are $K(\mu, \Delta)$ symbols on the right of (4.12), corresponding to all operator patterns γ having shift-weight Δ, and $g_{\mu, \nu, \nu+\Delta}$ symbols on the left corresponding to the values of κ. It was Biedenharn's idea that there should exist a canonical definition of the coefficients on the right such that these coefficients would *automatically* assume the value 0 for exactly

$$K(\mu, \Delta) - g_{\mu, \nu, \nu+\Delta} \tag{4.14}$$

of the operator patterns, thus achieving the correct number of families of WCG-coefficients.

It is imperative to realize here that there are denumerably many partitions ν where the maximum value $g_{\mu, \nu, \nu+\Delta} = K(\mu, \Delta)$ of the Littlewood-Richardson is actually attained. In such cases, the operator pattern notation works exactly; in all other cases, we utilize some as yet undetermined *subset* of these operator patterns, but nonetheless it is always the same patterns entering into the enumeration of the multiplity, and for this reason these

patterns are universal labels. Going beyond this, one would like to find an intrinsic meaning of an operator pattern comparable in significance to the Weyl group-subgroup property of a Gel'fand pattern; this is achieved fully for $U(2)$ and $U(3)$ through the notion of the null space of a unit tensor operator, which is defined in terms of the WCG-coefficients, but it is known from the work of Baclawski [14] that this only partially distinguishes the WCG-coefficients having the same shift-weight for $U(n), n > 3$. This is discussed further in Section 5. Independently of whether or not a natural or canonical significance exists for an operator pattern, it is still an elegant device for labeling the multipliclty, and we use this notation henceforth. Rewritten in this form, relation (4.8) reads

$$L\left[\begin{pmatrix}\lambda \\ \ell\ \ell'\end{pmatrix}\begin{pmatrix}\mu \\ m\ m'\end{pmatrix}\begin{pmatrix}\nu \\ q\ q'\end{pmatrix}\right] = \sum_\gamma \left[\begin{pmatrix}\lambda \\ \ell\end{pmatrix}\begin{pmatrix}\gamma \\ \mu \\ m\end{pmatrix}\begin{pmatrix}\nu \\ q\end{pmatrix}\right]\left[\begin{pmatrix}\lambda \\ \ell'\end{pmatrix}\begin{pmatrix}\gamma \\ \mu \\ m'\end{pmatrix}\begin{pmatrix}\nu \\ q'\end{pmatrix}\right], \qquad (4.15)$$

where for each $\lambda \in \mu \times \nu$, the summation is over all γ with shift-weight $\Delta = \lambda - \nu$ of μ.

We rewrite relation (4.8) in terms of this new notation, combining it with relation (4.7):

$$M(\nu+\Delta) \sum_{\substack{all\ \gamma\ with \\ \Delta(\gamma)=\Delta}}\left[\begin{pmatrix}\nu+\Delta \\ \ell\end{pmatrix}\begin{pmatrix}\gamma \\ \mu \\ m\end{pmatrix}\begin{pmatrix}\nu \\ q\end{pmatrix}\right]\left[\begin{pmatrix}\nu+\Delta \\ \ell'\end{pmatrix}\begin{vmatrix}\gamma \\ \mu \\ m'\end{vmatrix}\begin{pmatrix}\nu \\ q'\end{pmatrix}\right] =$$

$$= \sum_{(\alpha:A:\alpha')}\sum_{(\beta:B:\beta')}\sum_{(\gamma:C:\gamma')}\frac{(B+C)!}{B!C!}\delta(A,B+C)C_{\ell\ \ell'}^{\nu+\Delta}(A)C_{m\ m'}^{\mu}(B)C_{q\ q'}^{\nu}(C).$$
$$(4.16)$$

In this relation, Δ is any selected shift-weight of μ. The solution of this equation for the WCG-coefficients for general $U(n)$ would be a major achievement for physics, not only because one would achieve the explicit reduction of the Kronecker product, but also because the WCG-coefficients are the essential ingredients needed in physical theory for the construction of composite systems having $U(n)$ symmetry from more elementary systems possessing this symmetry. Indeed, this is their principal role.

From the point of view of combinatorics, what one needs for addressing the structure of the complex relationship (4.15) is more detailed information on the properties of the Littlewood-Richardson numbers. For this, it is convenient to reformulate briefly their properties in a manner that focuses on the partitons corresponding to a fixed value in the set $\Lambda(\mu,\Delta)$ given by (4.13).

For each partition μ and each of its weights Δ (interpreted as a shift-weight in the context of WCG-coefficients), we associate a function, denoted $I_{\mu,\Delta}$, whose domain of definition is the set H^n of $n-$tuples of integers (all nonnegative except possibly h_n):

$$H^n = \{(h_1, h_2, ..., h_n) \mid h_1 > h_2 > ... > h_n \geq 0\}, \qquad (4.17)$$

The range of $I_{\mu,\Delta}$ is the set of integers $\Lambda(\mu,\Delta)$, so that

$$I_{\mu,\Delta}: H^n \to \Lambda(\mu,\Delta) \qquad (4.18)$$

with values:

$$I_{\mu,\Delta}(h) = g_{\mu,\nu,\nu+\Delta} \ , for\ h = (\nu_1+n-1, \nu_2+n-2, ..., \nu_{n-1}+1, \nu_n). \quad (4.19)$$

We call the subset of H^n whose image is $g_{\mu,\nu,\nu+\Delta}$ a level subspace. The set of partitions $\{\nu\}$ corresponding to such a level subspace then gives those partitions for which

$K(\mu,\Delta) - g_{\mu,\nu,\nu+\Delta}$ of the WCG-coefficients (4.12) are to be zero.

The study of the functions $I_{\mu,\Delta}$ and their associated level-subspaces is essential for the WCG problem. This study leads naturally to the use of barycentric coordinates associated with the regular simplex, but only partial results have been obtained (see [15] and the important work by Baclawski [14]). This is the purest of problems for combinatorics.

5. UNIT TENSOR OPERATORS (ABSTRACT THEORY)

The concept of a unit tensor operator was intoduced by Racah [15] for the group $SU(2)$, and generalized to all $U(n)$ by Biedenharn [4]. The definition is based on the WCG-coefficients. For the purpose of stating the definition, we assume that we have determined all the $U(n)$ WCG-coefficients (4.12). We also assume that we have a set of finite-dimensional Hilbert spaces H_ν, one for each partition $\nu = (\nu_1, \nu_2, \cdots, \nu_n)$ with n parts, including zeros as parts, and that each such space has an orthornormal basis B_ν of ket vectors given by

$$B_\nu = \left\{ \left| \begin{matrix} \nu \\ q \end{matrix} \right\rangle \,\middle|\, q \text{ runs over all Gel' fand patterns} \right\}. \tag{5.1}$$

It is further assumed that there is an action T_U, each $U \in U(n)$, of the unitary group defined on the space H_ν and that this action on the basis B_ν is expressed by

$$T_U \left| \begin{matrix} \nu \\ q \end{matrix} \right\rangle = \sum_{q'} D^\nu_{q' \, q}(U) \left| \begin{matrix} \nu \\ q' \end{matrix} \right\rangle. \tag{5.2}$$

We next introduce the model Hilbert space H to be the direct sum of the spaces H_ν, each partition taken exactly once:

$$H = \sum_{n \geq 0} \sum_{\nu \mapsto n} \oplus H_\nu, \tag{5.3}$$

and extend, by linearity, the action T_U, each $U \in U(n)$, to the space H. It is this space H in which an abstract unit tensor operator acts.

Corresponding to each partition $\mu = (\mu_1, \mu_2, \cdots, \mu_n)$, including zero as a part, and to each operator pattern γ, each of which satisfies the betweenness conditions for a Gel'fand pattern with partition μ, there exists a unit tensor operator: It is a collection of component operators, $\dim H_\mu = \dim \mu$ in number as given by (3.21), and denoted by

$$\left\langle \begin{matrix} \gamma \\ \mu \\ \bullet \end{matrix} \right\rangle \text{ with components } \left\langle \begin{matrix} \gamma \\ \mu \\ m \end{matrix} \right\rangle \text{ enumerated by the Gel' fand patterns } m. \tag{5.4}$$

Thus, altogether there are $\dim \mu$ unit tensor operators, each having $\dim \mu$ components. Associated with each such tensor operator is a shift-weight $\Delta(\gamma)$.

The component operators are defined by their action on the basis vectors in the model space by specifying their actions on an arbitrary basis vector, using the WCG-coefficients:

$$\left\langle \begin{matrix} \gamma \\ \mu \\ m \end{matrix} \right\rangle \left| \begin{matrix} \nu \\ q \end{matrix} \right\rangle = \sum_{q'} \left[\left(\begin{matrix} \nu + \Delta(\gamma) \\ q' \end{matrix} \right) \left(\begin{matrix} \gamma \\ \mu \\ m \end{matrix} \right) \left(\begin{matrix} \nu \\ q \end{matrix} \right) \right] \left| \begin{matrix} \nu + \Delta(\gamma) \\ q' \end{matrix} \right\rangle. \tag{5.5}$$

Observe that this is a mapping from the space H_ν to the space $H_{\nu+\Delta(\gamma)}$, or to the 0

vector should $\nu + \Delta(\gamma)$ not be a partition. All properties of a unit tensor operator flow from this definition ([6],[7]), One of the most important is the transformation property of the components of each unit tensor operator under the action of T_U, each $U \in U(n)$:

$$T_U \left\langle \begin{matrix} \gamma \\ \mu \\ m \end{matrix} \right\rangle T_{U^{-1}} = \sum_{m'} D_{m' m}^{\mu}(U) \left\langle \begin{matrix} \gamma \\ \mu \\ m' \end{matrix} \right\rangle . \tag{5.6}$$

Unit tensor operators are also referred to as *Wigner operators*, since there definition is based on the WCG-coefficients.

The simplest of all tensor operators are the *fundamental* tensor operators, n in number, each with n components, transforming according to the fundamental representation $D^{(1,0^{n-1})}(U) = U$ of $U(n)$. Each such tensor operator is denoted

$$\left\langle \begin{matrix} \gamma \\ 1 \ 0^{n-1} \\ \bullet \end{matrix} \right\rangle \text{ with components } \left\langle \begin{matrix} \gamma \\ 1 \ 0^{n-1} \\ m \end{matrix} \right\rangle \text{ enumerated by m.} \tag{5.7}$$

In this case, the lower (Gel'fand) pattern and the upper (operator) pattern have weight α and shift-weight Δ of the forms given by: $\alpha(m) =$ unit row vector e_k of length n with 1 in position k; $\Delta(\gamma) =$ unit row vector e_τ with $k, \tau \in \{1, 2, \cdots, n\}$. Since in this case, the weights uniquely determine the patterns, it is convenient to introduce the simplified notation $t_{k,\tau}$ for these operators:

$$t_{k,\tau} = \left\langle \begin{matrix} \gamma \\ 1 \ 0^{n-1} \\ m \end{matrix} \right\rangle, \quad \begin{matrix} \text{for } \Delta(\gamma) = e_\tau, \\ \text{for } \alpha(m) = e_k, \end{matrix} \quad k, \tau = 1, 2, \cdots, n. \tag{5.8}$$

Each of the n fundamental tensor operators $t_{\bullet, \tau}$ with components $t_{k,\tau}$, $k = 1, 2, \cdots, n$, transforms under the unitary group action according to

$$T_U t_{k,\tau} T_{U^{-1}} = \sum_{j=1}^{n} u_{jk} t_{j,\tau}, \text{ each } \tau = 1, 2, \cdots, n. \tag{5.9}$$

Using a technique called the *pattern calculus* [9], one can evaluate completely the nonzero matrix elements of each of the fundamental tensor operastors $t_{\bullet, \tau}$ on the model Hilbert space H. Despite its intricatcy, we give this result fully because of its importance:

$$\left\langle \begin{matrix} [m]_n + e_n(\tau_n) \\ [m]_{n-1} + e_{n-1}(\tau_{n-1}) \\ \vdots \\ [m]_k + e_k(\tau_k) \\ (m)_{k-1} \end{matrix} \middle| t_{k,\tau_n} \middle| \begin{matrix} [m]_n \\ [m]_{n-1} \\ \vdots \\ [m]_k \\ (m)_{k-1} \end{matrix} \right\rangle = \left\langle \begin{matrix} [m]_n + e_n(\tau_n) \\ [m]_{n-1} + e_{n-1}(\tau_{n-1}) \\ \vdots \\ [m]_k + e_k(\tau_k) \\ (m)_{k-1} \end{matrix} \middle| \begin{matrix} \tau_n \\ [1 \quad \dot{0}]_n \\ [1 \quad \dot{0}]_{n-1} \\ \vdots \\ [1 \quad \dot{0}]_k \\ (0) \end{matrix} \middle| \begin{matrix} [m]_n \\ [m]_{n-1} \\ \vdots \\ [m]_k \\ (m)_{k-1} \end{matrix} \right\rangle$$

$$= \prod_{j=k+1}^{n} S(\tau_{j-1} - \tau_j) \left[\left| \frac{\displaystyle\prod_{\substack{i=1 \\ i \neq \tau_j}}^{j-1} (p_{\tau_j, j} - p_{i, j-1}) \prod_{\substack{i=1 \\ i \neq \tau_{j-1}}}^{j} (p_{\tau_{j-1}, j-1} - p_{i, j} + 1)}{\displaystyle\prod_{\substack{i=1 \\ i \neq \tau_j}}^{j} (p_{\tau_j, j} - p_{i, j}) \prod_{\substack{i=1 \\ i \neq \tau_{j-1}}}^{j-1} (p_{\tau_{j-1}, j-1} - p_{i, j-1} + 1)} \right| \right]^{1/2}$$

$$\tag{5.10}$$

where

$e_i(\tau) = $ *unit row vector of length i with 1 in position τ and 0 elsewhere,*

$$\tau_j = 1, 2, \ldots, j; \ p_{i,j} = m_{i,j} + j - i, \tag{5.11}$$

$S(j-i) = $ *sign of $j-i$ (1 for $j=i$).*

One can show [2] that the components $t_{k,\tau}, k = 1, 2, \cdots, n,$ of a given fundamental tensor $t_{\bullet,\tau}$ operator commute, but those corresponding to distinct τ do not, in general [ ]. Thus, the transformation (5.9) is among commuting components of the tensor operator $t_{\bullet,\tau}$.

We are now in position to explain Relation III:

$$W_m{}^\mu{}_\gamma(T) = \sum_{(\alpha:A:\Delta(\gamma))} I_m{}^\mu{}_\gamma(A) T^A / A!. \tag{5.12}$$

The elements of the matrix T are the fundamental unit tensor operators:

$$T = (t_{k,\tau})_{1 \le k \le n, 1 \le \tau \le n}. \tag{5.13}$$

The elements in column τ are the components $t_{k,\tau}, k = 1, 2, \cdots, n,$ of the unit tensor operator $t_{\bullet,\tau}$, and these components mutually commute, while elements from distinct columns do not. The order of the unit tensor operator components in each of the products

$$(t_\tau)^{a_\tau} = \prod_{k=1}^{n} (t_{k,\tau})^{a_{k\tau}}, \ t_\tau = (t_{1,\tau}, t_{2,\tau}, \cdots, t_{n,\tau}), a_\tau = (a_{1\tau}, a_{2\tau}, \cdots, a_{n\tau}) \tag{5.14}$$

is immaterial . According to (1.5), we have taken T^A in relation (5.12) to be

$$T^A = t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n}. \tag{5.15}$$

But let us immediatley note that the order of the factors is important, and we can write the factors in this product in $n!$ ways corresponding to the permutations

$\pi(T) = \pi(t_1, t_2, \cdots, t_n) = (t_{\pi_1}, t_{\pi_2}, \cdots, t_{\pi_n})$ of the columns t_τ of the matrix T. We write

$$(\pi(T))^A = t_{\pi_1}^{a_1} t_{\pi_2}^{a_2} \cdots t_{\pi_n}^{a_n}, \ \pi \in S_n (symmetric \ group), \tag{5.16}$$

and, correspondingly,

$$W_m{}^\mu{}_\gamma(\pi(T)) = \sum_{(\alpha:A:\Delta(\gamma))} I_\pi(m{}^\mu{}_\gamma)(A)(\pi(T))^A / A!, \tag{5.17}$$

where, for π the identity permutation, this expression coincides with Relation III. The I-quantities in relations (5.17) are, as yet, undetermined invariant operators, that is, operators that commute with the action T_U. Finally, the left-hand side of expression (5.17) is a general unit tensor operator

$$W_m{}^\mu{}_\gamma(\pi(T)) = \left\langle \begin{matrix} \gamma \\ \mu \\ m \end{matrix} \right\rangle \times (invariant \ factor).. \tag{5.18}$$

The idea of the form (5.12) is quite simple: In analogy to the representation functions $D_{m\,m'}^\mu(U)$ being real homogeneous polynomial forms over the elements u_{ij} of U, the general tensor operator is a homogeneous polynomial form over the fundamental tensor operators, where the scalars are invariant operators, but one must deal with the noncommutivity of the fundamental tensor operators. This can be done for the unitary group $U(2)$ (see [3], Vol. 9 for complete details). It may be shown [2] that the forms discussed above must exist for general $U(n)$, but the theory is far from

complete in that one doen't know the properties of the different orderings, nor the invariant scalaars. One might hope to find a natural way in the above constructin for defining the general unit tensor operator thus fulfilling Biedenahran's intuition.

We can now also give the interpretation of Relation IV: It is the product law for two unit tensor operators, which expresses this product as a linear combination of unit tensor operators, with coefficients that are invariant. In Section 1, we have written this relation in a form that emphasizes its formal analogy with the product law, Relation II, for representation functions. We now rewrite it in a form more often given ([6], [7]), and based directly on the definition (5.5), but which must be compatible with the form (5.17):

$$
\left\langle\begin{matrix}\gamma\\\mu\\m\end{matrix}\right\rangle\left\langle\begin{matrix}\rho\\\nu\\q\end{matrix}\right\rangle = \sum_{\left(\begin{smallmatrix}\lambda\\\ell\ \kappa\end{smallmatrix}\right)} I\left[\left(\begin{smallmatrix}&\lambda\\\ell&\kappa\end{smallmatrix}\right)\left(\begin{smallmatrix}&\nu\\q&\rho\end{smallmatrix}\right)\left(\begin{smallmatrix}&\mu\\m&\gamma\end{smallmatrix}\right)\right]\left\langle\begin{matrix}\kappa\\\lambda\\\ell\end{matrix}\right\rangle, \tag{5.19}
$$

where the I-invariant is a sum over factors , as in relation ( ), but now into a WCG-coefficient and an invariant operator called a Racah invariant, since its eigenvalues are the Racah coefficients of $U(n)$ (see references [] for details):

$$
I\left[\left(\begin{smallmatrix}&\lambda\\\ell&\kappa\end{smallmatrix}\right)\left(\begin{smallmatrix}&\nu\\q&\rho\end{smallmatrix}\right)\left(\begin{smallmatrix}&\mu\\m&\gamma\end{smallmatrix}\right)\right] = \sum_{\tau}\left[\left(\begin{smallmatrix}\lambda\\\ell\end{smallmatrix}\right)\left(\begin{smallmatrix}\tau\\\nu\\q\end{smallmatrix}\right)\left(\begin{smallmatrix}\mu\\q\end{smallmatrix}\right)\right]\left\{\left(\begin{smallmatrix}\lambda\\\kappa\end{smallmatrix}\right)\left(\begin{smallmatrix}\tau\\\nu\\\rho\end{smallmatrix}\right)\left(\begin{smallmatrix}\mu\\\gamma\end{smallmatrix}\right)\right\}, \tag{5.20}
$$

where the curly bracket object is a Racah invariant operator, and the summation is over all operator patterns τ with shift-weight $\Delta = \lambda - \mu$ of partition ν.

6. OPERATOR ACTIONS IN THE RING OF POLYNOMIALS

There would appear to little relationship between the abstract theory outlined in Section 5 and the spaces of polynomials discussed in the earlier sections. But this is not the case, since Relation II allows us to interpret the first polynomial as an operator acting on the second polynomial. Indeed, the structure of that relation is that of a *restricted tensor product space as shown by the correspondences:*

$$
[M(\nu)]^{-1/2}P_{q\ q'}^{\ \nu} \to \left|\begin{matrix}\nu\\q\end{matrix}\right\rangle\otimes\left|\begin{matrix}\nu\\q'\end{matrix}\right\rangle, \quad [M(\lambda)]^{-1/2}P_{\ell\ \ell'}^{\ \lambda} \to \left|\begin{matrix}\lambda\\\ell\end{matrix}\right\rangle\otimes\left|\begin{matrix}\lambda\\\ell'\end{matrix}\right\rangle, \tag{6.1}
$$

$$
P_{m\ m'}^{\ \mu} \to M^{1/2}\sum_{\gamma}\left\langle\begin{matrix}\gamma\\\mu\\m\end{matrix}\right\rangle\otimes\left\langle\begin{matrix}\gamma\\\mu\\m'\end{matrix}\right\rangle M^{-1/2}. \tag{6.2}
$$

Here M is an invariant operator with eigenvalue $M(\nu)$ on the space $H_\nu \otimes H_\nu$. Using these correspondences in Relation II, we obtain exactly relation (5.5) defining a unit tensor operator.

The self-consistency of the above maps means that we can obtain a *reallzation* of a unit tensor operator in the tensor product form, $\left\langle\begin{smallmatrix}\gamma\\\mu\\\bullet\end{smallmatrix}\right\rangle\otimes\left\langle\begin{smallmatrix}\gamma\\\mu\\\bullet\end{smallmatrix}\right\rangle$, on the space of polynomials by the definition

$$
M^{1/2}\left\langle\begin{matrix}\gamma\\\mu\\m\end{matrix}\right\rangle\otimes\left\langle\begin{matrix}\gamma\\\mu\\m'\end{matrix}\right\rangle M^{-1/2}P_{q\ q'}^{\ \nu}(Z) = \sum_{\left(\begin{smallmatrix}\lambda\\\ell\ \ell'\end{smallmatrix}\right)}\left[\left(\begin{smallmatrix}\lambda\\\ell\end{smallmatrix}\right)\left(\begin{smallmatrix}\gamma\\\mu\\m\end{smallmatrix}\right)\left(\begin{smallmatrix}\nu\\q\end{smallmatrix}\right)\right]\left[\left(\begin{smallmatrix}\lambda\\\ell'\end{smallmatrix}\right)\left(\begin{smallmatrix}\gamma\\\mu\\m'\end{smallmatrix}\right)\left(\begin{smallmatrix}\nu\\q'\end{smallmatrix}\right)\right]P_{\ell\ \ell'}^{\ \lambda}(Z), \tag{6.3}
$$

where we must have

$$
P_{m\ m'}^{\ \mu}(Z) = M^{\frac{1}{2}}\left(\sum_{\gamma}\left\langle\begin{matrix}\gamma\\\mu\\m\end{matrix}\right\rangle\otimes\left\langle\begin{matrix}\gamma\\\mu\\m'\end{matrix}\right\rangle\right)M^{-\frac{1}{2}}. \tag{6.4}
$$

The Hermitian conjugate (the * operation) of (6.4) gives

$$P_{m\ m'}^{\ \mu}(\partial \,/\, \partial Z) = M^{\frac{1}{2}} \left(\sum_{\gamma} \left\langle \begin{matrix} \gamma \\ \mu \\ m \end{matrix} \right\rangle^{*} \otimes \left\langle \begin{matrix} \gamma \\ \mu \\ m' \end{matrix} \right\rangle^{*} \right) M^{-\frac{1}{2}}. \tag{6.5}$$

As examples of relations (6.4 and (6.5), we have the following identities expressing the indeterminates z_{ij} and their derivatives in terms of the fundamental unit tensor operators (shift operators):

$$z_{ij} = M^{\frac{1}{2}} \left(\sum_{\tau=1}^{n} t_{i,\tau} \otimes t_{j,\tau} \right) M^{-\frac{1}{2}}, \tag{6.6}$$

$$\partial \,/\, \partial z_{ij} = M^{-\frac{1}{2}} \left(\sum_{\tau=1}^{n} t_{i,\tau}^{*} \otimes t_{j,\tau}^{*} \right) M^{\frac{1}{2}}. \tag{6.7}$$

Let us conclude by noting one more nice relationship, which offers the possiblity of sorting out the various forms discussed above for Relation III. Using (6.4) and (6.6) in Relation I gives:

$$\sum_{\gamma} \left\langle \begin{matrix} \gamma \\ \mu \\ m \end{matrix} \right\rangle \otimes \left\langle \begin{matrix} \gamma \\ \mu \\ m' \end{matrix} \right\rangle = \sum_{(\alpha:A:\alpha')} C_{m\ m'}^{\ \mu}(A) \prod_{i,j=1}^{n} \left(\sum_{\tau} t_{i,\tau} \otimes t_{j,\tau} \right)^{a_{ij}} / \, a_{ij}! . \tag{6.8}$$

We have now come full circle, beginning with an abstract definition of a unit tensor operator, giving it a realization in its action in polynomial space, and arriving at relation (6.8), which must hold abstractly, in the restricted tensor product (denoted $\otimes_d$, d *for diagonal*) model space defined by

$$H \otimes_d H = \sum_{\mu} (H_{\mu} \otimes H_{\mu}). \tag{6.9}$$

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