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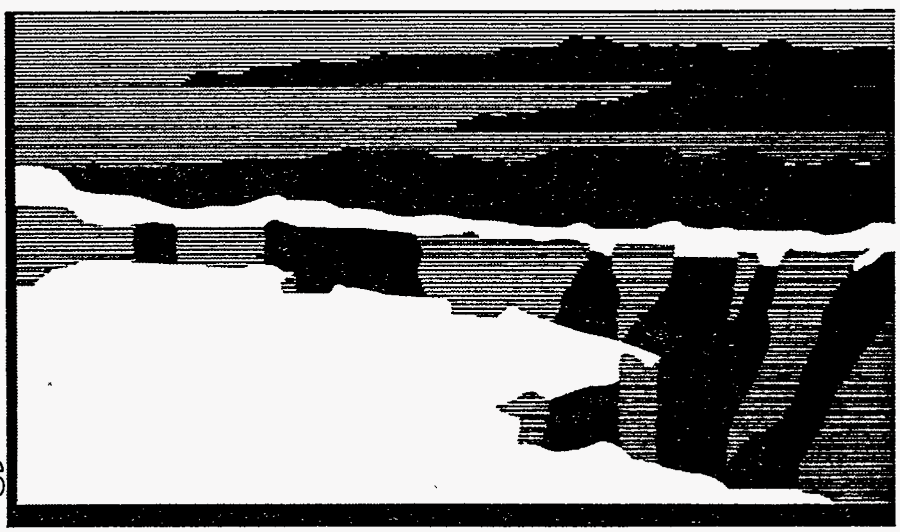

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Highest-Weight Representations of Borcherds Algebras

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Abstract

General features of highest-weight representations of Borcherd's algebras are described. To show their typical features, several representations of Borcherds extensions of finite-dimensional algebras are analyzed. Then the example of the extension of affine-su(2) to a Borcherds algebra is examined. These algebras provide a natural way to extend a Kac-Moody algebra to include the hamiltonian and number-changing operators in a generalized symmetry structure.

1 Introduction

This talk examines the highest-weight representation theory of Borcherd's algebras with the goal of giving some rather primitive notions on how these algebras might be used in quantum field theory. Borcherd's algebras were proposed in a 1988 paper by Richard Borcherds, Ref. [1]. They are extensions of Kac-Moody algebras, a topic to which J. Patera has contributed significantly, e.g., Ref. [2]. The work I am reporting today is derived from results that Patera and I have explored together.

The physics setting of this work is the energy operator of a quantum field theory, which is associated with the time-translation invariance of the Lagrangian. Its role in the overall symmetry structure of the Lagrangian varies: in 4-dimensional theories it belongs to the Lie algebra of the Poincaré group, which is a contraction of noncompact $SO(5)$. In two-dimensional conformal field theories the hamiltonian is a member of the infinite-dimensional Virasoro algebra. If the symmetry is further extended by a finite-dimensional Lie algebra, the conformal theory is a representation of a Kac-Moody algebra (the vertex construction), and is a solution to the corresponding two-dimensional current algebra.

The usual construction of a Lie algebra and its representation theory requires a nondegenerate bilinear form that is derived from the Cartan matrix. In the affine Kac-Moody case, this nondegenerate form can be obtained if the Cartan subalgebra is extended by an independent diagonalizable operator, which is the hamiltonian $-L_0$ in simple two-dimensional conformal field theories. However, it does not have the full status of the other members of the Cartan subalgebra of a Kac-Moody algebra, since from one point of view its role is defined through this extension, but from another point of view, it is contained in the Virasoro algebra, and the full algebra is a semi-direct product of the Virasoro algebra with the affine Kac-Moody algebra.

It is the purpose of this talk to show how the number operator and the hamiltonian are promoted to full fledged members of the Cartan subalgebra of a Borcherds algebra, and to study a few simple representations of these Lie algebras, Ref. [3]. There is an interesting twist in this construction when applied to affine Kac-Moody algebras: the extended representations contain states of all numbers of particles. The original hamiltonian becomes a member of the Cartan subalgebra, as does the operator that measures the level (or number of particles). By explicit construction we show these diagonal operators are in the Cartan subalgebra. The highest-weight representations of the extended affine Kac-Moody algebra contain affine representations of all levels. A possible shortcoming of this construction is that states of all statistics are in the representations. There is is no restriction to Bose or Fermi statistics only.

The important problem not solved here is to construct a field theory in some number of space-time dimensions where this extended symmetry structure emerges. Instead, we use quantum mechanical and conformal field theory examples to motivate the interpretation of representations of these algebras. It may be useful to have particle-number changing operators as part of the symmetry structure.

One goal of this talk is to explore several representations of Borcherds algebras. (In practical terms, this means that the branching rules of various highest-weight representations are "sliced" or organized as a function of the eigenvalue of the number operator.) Section 2 is a brief summary of results concerning Borcherds algebras, including a summary of the theory of its highest-weight representations. These algebras are extensions of Kac-Moody algebras, and most of the representation theory of Kac-Moody algebras directly applies to Borcherds algebras, Ref. [1]. The review of Sec. 2 is completed by computing the roots of several simple Borcherds algebras and the weights and multiplicities of several highest-weight representations.

The solution to the problem of constructing the extended Cartan subalgebra that includes the hamiltonian in an affine Kac-Moody algebra, Ref. [2] is summarized in Sec. 3. In this construction the natural choice of the linear functional of the hamiltonian with itself is zero; if this bilinear form were used as a Cartan matrix for some new algebra, it would violate the rule that the diagonal elements of the Cartan matrix all be 2. However, this construction is interesting from the point of view of this talk because the nondegenerate bilinear form defines a Borcherds Lie algebra. It may be useful to analyze the representations in terms of this new algebra. The Borcherds algebras extend and generalize Kac-Moody algebras by adding an imaginary root to the set of simple roots, so a zero-valued diagonal element of the symmetrized Cartan matrix is possible.

Section 4 explores how a two-dimensional current algebra is extended to include number changing operators. Adding an imaginary simple root to an affine algebra leads to two infinite directions, one hamiltonian-like and the other counts the number of particles in the state. Both operators are constructed as sums of the diagonal operators that correspond to the simple roots of the extended algebra. Results and speculations are briefly restated in Sec. 5.

This introduction concludes with a brief review of those aspects of Kac-Moody algebras that are of greatest importance to the Borcherds generalization. Speaking rather roughly, the approach to Lie algebra theory that

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has been the most fruitful for generalization starts from an analysis of the space $\mathcal{P}$ spanned by the eigenvalues of a set of independent simultaneously diagonalizable operators in the Cartan subalgebra. In the finite-dimensional case $\mathcal{P}$ is a Euclidean space with a positive definite metric and of dimension equal to the rank ℓ of the Lie algebra. The vectors in this space are the roots of the Lie algebra and the weights of its representations.

The basis of the Cartan subalgebra is a set of ℓ linearly independent diagonalizable operators h_i, $i = 1, \ldots, \ell$, conveniently chosen. A root α is an ℓ-component vector in $\mathcal{P}$ whose components are the eigenvalues of h_i in the sense of the Lie bracket:

$$[h_i, e_\alpha] = \alpha(h_i)e_\alpha, \tag{1}$$

where e_α is one of the ladder operators that increases a Hilbert space representation vector with weight λ to one with weight $\lambda+\alpha$ (if is $\lambda+\alpha$ in the representation; the functional $\alpha(h_i)$ is the i-th component of the ℓ-dimensional root α.

For a vector $|\mathbf{r}, \lambda\rangle$ in the Hilbert space of a representation $\mathbf{r}$, the ℓ-dimensional vector λ in $\mathcal{P}$ is called a weight. The components of the weight λ are defined by

$$h_i|\mathbf{r}, \lambda\rangle = \lambda(h_i)|\mathbf{r}, \lambda\rangle. \tag{2}$$

Additional labels of the vector $|\mathbf{r}, \lambda\rangle$ when the multiplicity of λ is greater than unity have been suppressed.

The Cartan subalgebra is dual to the space of roots, so $\alpha(h_i)$ and $\lambda(h_i)$ are linear functionals. The weight or root component $\lambda(h_i)$ is defined by

$$\lambda(h_i) = (\lambda|\alpha_i), \tag{3}$$

where this definition of $\lambda(h_i)$ differs from the usual one by a normalization factor $\frac{2}{(\alpha_i|\alpha_i)}$, see Refs. [2, 4]. The simple root α_i in $\mathcal{P}$ corresponds to the diagonal operator h_i in the Cartan subalgebra, and the component of α_i associated with h_j is $\alpha_i(h_j) = (\alpha_i|\alpha_j)$, where $(\alpha_i|\alpha_j)$ is the natural scalar product in $\mathcal{P}$ of the simple roots.

The commutation relations that define the Lie algebra are defined in terms of a set of "generators," that generate the full Lie algebra through multiple commutators. For a rank ℓ algebra there are 3ℓ generators, e_i, f_i, and h_i, $i = 1, \ldots, \ell$. The "presentation" of the Lie algebra is then

$$
\begin{aligned}
[h_i, e_j] &= (\alpha_j|\alpha_i)e_j, \\
[h_i, f_j] &= -(\alpha_j|\alpha_i)f_j, \\
[e_i, f_j] &= \frac{2\delta_{ij}}{(\alpha_i|\alpha_i)}h_i.
\end{aligned}
\tag{4}
$$

The values of the scalar products $(\alpha_j|\alpha_i)$ are specified by the Cartan matrix that defines the Kac-Moody algebra. The presentation Eq. (4) normalizes the Cartan subalgebra somewhat differently from the usual conventions, but is convenient for Borcherds algebras.

Equation (4) does not completely define the Lie algebra. The "Serre relations" impose the requirement that certain multiple commutators vanish:

$$
\begin{aligned}
(ad\ e_i)^{1-\frac{2(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)}} e_j &= 0, \\
(ad\ f_i)^{1-\frac{2(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)}} f_j &= 0,
\end{aligned}
\tag{5}
$$

where, for example, $(ad\ e_i)^2 e_j$ means $[e_i, [e_i, e_j]]$. The presentation of the Lie algebra, Eqs. (4) and (5), agrees with the usual one up to the normalization of the Cartan subalgebra. Equation (5) makes sense if $(\alpha_i|\alpha_i) > 0$, but this is always the case for the simple roots of Kac-Moody algebras.

Any straight line in $\mathcal{P}$ that crosses more than one root defines an $su(2)$ subalgebra, so the Cartan matrix indicates all the nontrivial ways that ℓ "independent" $su(2)$s can be connected and yield a Lie algebra. This relation is characterized by the angles and relative lengths of the simple roots in $\mathcal{P}$.

Kac-Moody algebras are defined by the geometrical relations (scalar products) among the eigenvalues (roots) of the diagonalizable operators. The first steps of the theory are dedicated to finding the best strategy for defining these relations from the axioms for Lie algebras. The method then requires proving that the construction of the eigenvalues may be lifted to a definition of the Lie algebra through its commutation relations. The geometry is defined by a matrix $A_{ij} = (\alpha_i|\alpha_j)$, where the ℓ-component euclidean vectors α_i are called the simple roots, and the matrix A_{ij} is called the symmetrized Cartan matrix, where the Cartan matrix itself for symmetrizable Kac-Moody algebras is defined by

$$
C_{ij} = \frac{2(\alpha_i|\alpha_j)}{(\alpha_j|\alpha_j)}. \tag{6}
$$

Kac-Moody algebras and the Borcherds generalization are completely defined by this matrix of scalar products of simple roots. The list of rules for C_{ij} that characterizes a Kac-Moody algebra is quite simple. The Cartan matrix is an integer matrix with diagonal elements $C_{ii} = 2$ and non-positive integers for the off diagonal elements with zeros matching pairwise. If C has positive determinant, then the algebra is finite-dimensional. If C has zero determinant, then the algebra is an infinite-dimensional affine algebra; this is the case of interest in a two-dimensional current algebra. Finally, if the determinant is negative, the resulting algebra is one of the hyperbolic or other Kac-Moody algebras, which are not easily listed but are rather easily studied at the level of description of this paper.

As the notation in Eqs. (3) and (6) suggests, only "symmetrizable" Cartan matrices are considered here. The symmetrized Cartan matrix $A = CD$ is

$$\begin{aligned} A_{ij} &= (CD)_{ij} = (\alpha_i|\alpha_j), \\ D_{ij} &= \tfrac{1}{2}\delta_{ij}(\alpha_j|\alpha_j). \end{aligned} \tag{7}$$

A generalization of Kac-Moody algebras by modifying the definition of the Cartan matrix was discovered by Borcherds: for symmetrizable Cartan matrices the requirement that $C_{ii} = 2$ may be dropped by introducing a simple root α_1 of zero length, $A_{11} = (\alpha_1|\alpha_1) = 0$. The next section summarizes an example of Borcherds' results [1].

2 Borcherds Algebras

The extension from Kac-Moody algebras to Borcherds algebras is accomplished by relaxing the rules for forming the Cartan matrix. The simplest statement of the extension is: the set of simple roots α_i, $i = 1, \ldots, \ell$, may include imaginary roots. Our discussion is restricted to the case of just one imaginary simple root, selected to be α_1 and to have zero length. (The choice of zero length turns out to be a convention so long as α_1 is imaginary.)

The new rules for the Cartan matrix must be supplemented with new rules for the presentation of the Lie algebra: the presentation given in Eq. (4) and (5) generalizes to Borcherds algebras except where $(\alpha_1|\alpha_1) = 0$ causes nonsense. This includes the last relation of Eq. (4), where the right hand side of the equation is zero for $i = 1$, and the Serre relations Eq. (5), where the relation is simply ignored for $i = 1$. Note that this is a very strong assumption since $(ad\ e_1)^n e_j \neq 0$ for $j \neq 1$ and any positive integer n. Thus, the Lie algebra places no constraints on the "1" direction. For all other simple roots, the presentation is unchanged, and the Cartan matrix C_{ij}, $j \neq 1$ satisfies the usual rules to be a Cartan matrix, which continues to be a symmetrizable integer matrix with nonpositive off-diagonal elements and $C_{ii} = 2$ for $i \neq 1$. Ref. [1].

The Borcherds paper summarizes the algebraic structure and representation theory of these extended algebras [1]. The focus in this section is the root and weight multiplicity formulas for Borcherds algebras and their representations. The representation theory of highest-weight representations is almost identical to the Kac-Moody case: the Weyl-Kac character formula is valid for Borcherds algebras. Ref. [1]. Thus, the root and weight multiplicities can be computed in a manner identical to those of Kac-Moody algebras. In particular, the Peterson formula [4] is valid and provides a computational method for determining root and weight multiplicities.

The derivation of the Peterson formula for Kac-Moody algebras is worked out in detail in Sec. 22 of Ref. [2]. It is an iterative formula for the multiplicities of positive roots $\beta = \sum_i n_i\alpha_i$, in terms of the multiplicities of lower positive roots. These are nonzero roots with components $0 \leq m_i \leq n_i$:

$$(\beta|\beta - 2\rho)c_\beta = \sum_{\beta',\beta''} (\beta'|\beta'')c_{\beta'}c_{\beta''}. \tag{8}$$

where the sum is over all positive roots β' and β'' with $\beta' + \beta'' = \beta$, and ρ is defined by $(\rho|\alpha_i) = 1$, $i = 1, \ldots, \ell$. The quantity c_β is defined by

$$c_\beta = \sum_{n>0} \frac{1}{n}\mathrm{mult}(\frac{\beta}{n}). \tag{9}$$

where "mult" is the multiplicity of the root, which is nonzero only when β/n is a root. This sum necessarily terminates at some finite n for any root β.

The iteration formula Eq. (8) requires several "boundary conditions" to be defined. The multiplicity of a simple root is always unity. If β is a real simple root, Eq. (8) reads $0 = 0$, so we set c_β and $\mathrm{mult}(\beta)$ to unity. If $(\beta|\beta - 2\rho) = 0$, then $\mathrm{mult}(\beta) = 0$. Equations (8) and (9) can be used to calculate the root multiplicities of any Kac-Moody or Borcherds algebra.

Table 1 gives the root multiplicities for the Borcherds algebra defined by the symmetrized Cartan matrix.

$$A(su(2)_B) = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}. \tag{10}$$

The simple root α_1 is imaginary and α_2 is real. The scalar product of roots $\alpha = n_1\alpha_1 + n_2\alpha_2$ and $\beta = m_1\alpha_1 + m_2\alpha_2$ is

$$(\alpha|\beta) = -n_1 m_2 - n_2 m_1 + 2n_2 m_2. \tag{11}$$

The root system is characterized by a set of $su(2)$ representations for each value of n_1, and n_1 behaves like a number-operator eigenvalue. The name "$su(2)_B$" refers to the Lie algebra with symmetrized Cartan matrix Eq. (10).

Table 1: Positive roots of Borcherds algebra with symmetrized Cartan matrix given in Eq. (10) for $su(2)_B$. The imaginary simple root α_1 has zero norm. The positive root system is listed here, except at $n_1 = 0$, where one negative and two zero roots are included. (Recall that if α is a root, so is $-\alpha$.) The $su(2)$ representation is indicated by its highest weight $\Lambda(h_2) = 2j$; for example (3) is the spin or isospin 3/2 representation of dimension 4.

n_1	$su(2)$ content
0	(2) + (0)
1	(1)
2	(0)
3	(1)
4	(2)
5	(3) + (1)
6	(4) + (2) + (0)
7	(5) + 2(3) + 2(1)
8	(6) + 2(4) + 4(2) + (0)
9	(7) + 3(5) + 5(3) + 5(1)

The adjoint representation is not a highest-weight representation, since if α is a root, so is $-\alpha$, and there is an infinite set of roots. Nevertheless, the weight multiplicities of highest-weight representations can also be constructed from the Peterson formula for an enlarged algebra. In particular, the Freudenthal formula for Kac-Moody algebras is derived from the Peterson formula for an extended algebra with simple root α_0 appended to the simple roots, $\alpha_1, \ldots, \alpha_\ell$; the additional scalar products needed to define the extended algebra are $(\alpha_0|\alpha_0) = 2$ and $(\alpha_0|\alpha_i) = -(\Lambda|\alpha_i)$, $i = 1, \ldots, \ell$, where Λ is the highest weight of the representation.

The results are illustrated by calculations of the (1,0) representation of the algebra with symmetrized Cartan matrix Eq. (10). In order to calculate the weight multiplicities for the (1,0) representation, one uses the Peterson formula Eqs.(8) and (9), but in conjunction with the extended Cartan matrix,

$$A^{(1,0)}(su(2)_B) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 2 \end{pmatrix}. \tag{12}$$

The weight multiplicities of the highest weight representation (1,0) are the root multiplicities computed from $A^{(1,0)}(su(2)_B)$ of the form,

$$\alpha = \alpha_0 + n_1\alpha_1 + n_2\alpha_2. \tag{13}$$

The Freudenthal formula follows from this extension, as derived in Refs. [2, 4, 5]. The weight multiplicities of the (1,0) and (0,1) (using a different extended Cartan matrix, $A^{(0,1)}(su(2)_B)$ in this latter case) representations are listed for the first 10 values of n_1 in Table 2. Note the infinity of weights starting from the highest weight extending in the α_1 direction: h_1 is an operator with a semi-infinite spectrum.

It is possible to unravel the Borcherds representation in terms of Fock space operators, just as can be done for the affine Kac-Moody highest weight representations. The structure of the (1,0) representation of $su(2)_B$ is extremely simple: it is possible to build this representation with an $su(2)$ doublet of creation operators $a^{(1)\dagger}_{1/2,m}$ that carries $\Delta n_1 = 1$. All products of $a^{(1)\dagger}_{1/2,m}$ acting on the ground state are linearly independent in this construction; the lack of statistics is a problem for physical particles and is likely due to the assumption of no Serre relation, Eq. (5) for $i = 1$. The $su(2)$ structure at slice $n_1 = n$ is the tensor product of the (1) representation of $su(2)$ with itself n times.

From a direct analysis of the (1,0) representation, it is simple to transform from the (h_1, h_2) basis of simple roots to the $(N, 2I_3)$ basis of the Cartan

Table 2: Branching Rules of the (1.0) and (0.1) representations of the Borcherds algebra $su(2)_B$ of Table 1. Again, we slice the representation with α_1. The $su(2)$ representation is given by its highest weight, $\Lambda(h_2) = 2$. The results for the (1.0) are derived from the Peterson formula with Eq. (12) for the bilinear form.

n_1	$su(2)$ content of (1.0)	$su(2)$ content of (0.1)
0	(0)	(1)
1	(1)	(0)
2	(2) + (0)	(1)
3	(3) + 2(1)	(2) + (0)
4	(4) + 3(2) + 2(0)	(3) + 2(1)
5	(5) + 4(3) + 5(1)	(4) + 3(2) + 2(0)
6	(6) + 5(4) + 9(2) + 5(0)	(5) + 4(3) + 5(1)
7	(7) + 6(5) + 14(3) + 14(1)	(6) + 5(4) + 9(2) + 5(0)
8	(8) + 7(6) + 20(4) + 28(2) + 14(0)	(7) + 6(5) + 14(3) + 14(1)
9	(9) + 8(7) + 27(5) + 48(3) + 48(3) + 42(1)	(8) + 7(6) + 20(4) + 28(2) + 28(2) + 14(0)

subalgebra (I_3 is the third component of isospin). If the weights are written in terms of simple roots,

$$\Lambda = n_1\alpha_1 + n_2\alpha_2, \tag{14}$$

then a glance at the (1.0) representation in Table 2 reveals the following definitions:

$$\begin{aligned} \Lambda(N) &= n_1, \\ \Lambda(2I_3) &= -n_1 + 2n_2, \end{aligned} \tag{15}$$

where N is the number operator and I_3 is the diagonalized operator of $su(2)$. It follows from Eq. (10) and (15) that

$$\begin{aligned} N &= -2h_1 - h_2, \\ 2I_3 &= h_2. \end{aligned} \tag{16}$$

Thus, the number operator is in the Cartan subalgebra; it corresponds to the root,

$$N \longrightarrow \alpha_N = -2\alpha_1 - \alpha_2. \tag{17}$$

with $\alpha_N(N) = (\alpha_N|\alpha_N) = -2$.

The next example is a Borcherds extension of $su(3)$, done in a manner similar to the extension of $su(2)$, where we show by example that the features of the (1.0) representation of Table 2 generalizes to a Borcherds extended $su(3)$ defined by the Cartan matrix.

$$A(su(3)_B) = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \qquad (18)$$

Table 3: Positive roots and weights of the (1.0.0) representation of Borcherds algebra with Cartan matrix Eq. (18). The imaginary simple root is α_1. Except for $n_1 = 0$, only positive roots are listed.

n_1	$su(3)$ content of roots	$su(3)$ content of (1.0.0)
0	(1.1) + (0.0)	(0.0)
1	(0.1)	(0.1)
2	(1.0)	(0.2) + (1.0)
3	(1.1)	(0.3) + 2(1.1) + (0.0)
4	(1.2) + (0.1)	(0.4) + 3(1.2) + 2(2.0) + 3(0,1)
5	(1.3) + (2.1) + (0.2) + (1.0)	(0.5) + 4(1.3) + 5(2.1) + 6(0,2) + 5(1.0)
6	(1.1) + (2.2) + 2(0.3) + (3.0) + 3(1.1)	(0.6) + 5(1.4) + 9(2.2) +10(0,3) + 5(3.0) + 16(1.1) + 5(0.0)
7	(1.5) + 2(2.3) + 2(3.1) + 2(0.1) + 5(1.2) + 3(2.0) + 3(0.1)	(0.7) + 6(1.5) + 11(2.3) + 15(0.4) 11(3.1) + 35(1.2) + 21(2.0) + 35(0,1)

In the case of Eq. (18) only the number operator is added to the Cartan subalgebra. The roots can be written in the form, $\alpha = n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3$. Table 3 gives the root system of this algebra in terms of the $su(3)$ representations for each value of n_1. Only positive roots (except at $n_1 = 0$) are listed and are given in terms of $su(3)$ representations. The $su(3)$ weights are derived by converting the (n_2, n_3) of the root basis to weight basis, so (1.0) is the **3**, (0.1) is the $\overline{\mathbf{3}}$, (1.1) is the **8**, and so on.

The Peterson formula gives weight multiplicities, which are then converted into $su(3)$ irreducible representations using the Tables of Ref. [6]. The weight system of the (1,0,0) representation is also given in Table 3. As in the $su(2)_B$ case, the fundamental representation (1,0,0) can be constructed from a triplet of creation operators with $\Delta n_1 = 1$ acting on an $su(3)$ singlet ground state. The trick of adding the imaginary root to a finite dimensional Lie algebra gives a representation that provides the entire Fock space of a set of operators transforming as one of the representations of the finite-dimensional Lie algebra.

Thus, the (1,0,0) of $su(3)_B$ of Eq. (18) has a structure similar to the (1,0) representation of $su(2)_B$. The representation at $n_1 = n$ is the tensor product of the $su(3)$ representation (0,1) with itself n times, with no symmetry or antisymmetry constraints. Once again the representation of a Borcherds extended algebra adjoins to the finite dimensional representation its entire Fock space, including the singlet ground state. The representation is shown in Table 3.

3 Cartan Subalgebra of an Affine Kac-Moody Algebra

The group theoretical role of the hamiltonian in conformal field theory and two-dimensional current algebra arises from the need to define a nondegenerate bilinear form for the algebra. Finite-dimensional Lie algebras are characterized by positive-definite Cartan matrices, and so the definition of the Lie algebra by its presentation Eqs. (1) and (5) has no ambiguity. However, for affine algebras the determinant of A is zero, so the Cartan matrix cannot be selected naively to be the bilinear form that defines the Lie algebra. A nondegenerate bilinear form is constructed by extending the Cartan subalgebra, and consequently extending the space of roots by adding a linearly independent vector corresponding to this new operator. The extension sketched here is worked out in more detail in Sec. 5 of Ref. [2].

The problem of the presentation given in Eq. (1) and (5) for an affine Kac-Moody algebra is that the functional $\alpha_i(h_j) = (\alpha_i, \alpha_j)$ is degenerate, so that for each root α, there is an infinite number of operators e_α and f_α with no immediate way to distinguish among them. The problem of labeling the roots is trivially solved for those affine algebras constructed as central extensions of loop algebras, as is reviewed in Sec. 3 and Sec. 16 of Ref. [2].

The general solution focuses on the geometry of $\mathcal{P}$, which is now outlined.

The solution to the labeling problem is to extend the $\ell \times \ell$ Cartan matrix to an $(\ell + 1) \times (\ell + 1)$ nonsingular bilinear form by introducing the operator L_0:

$$\begin{aligned} [L_0, e_\alpha] &= \alpha(L_0)e_\alpha, \\ [L_0, f_\alpha] &= -\alpha(L_0)f_\alpha, \\ [L_0, h_i] &= 0. \end{aligned} \tag{19}$$

L_0 closes with the remaining operators of the algebra and can be added to the Cartan subalgebra. An example of Eq. (19) is provided by the vertex construction. Our interest in Borcherds algebras was aroused by the result that the resulting nondegenerate bilinear form is the Cartan matrix of a Borcherds algebra.

It is required that $\alpha_i(h_j) = (\alpha_i|\alpha_j)$ not be changed by the extension, which adds $\alpha_i(L_0)$, $L_0(h_i)$ and $\Lambda_0(L_0)$, where Λ_0 is the vector in $\mathcal{P}^{ext}$ that corresponds to L_0. The extension of $\mathcal{P}$ to $\mathcal{P}^{ext}$ is $(\ell + 1)$-dimensional. The symmetry structure imposes several conditions. First the bilinear form must be symmetric:

$$\alpha_i(L_0) = (\alpha_i|\Lambda_0) = (\Lambda_0|\alpha_i) = L_0(h_i). \tag{20}$$

The linear dependence in the affine Cartan matrix must be treated in a consistent fashion. This linear dependence is expressed in terms of a root δ defined by

$$\delta = \sum_i c_i\alpha_i, \tag{21}$$

where the integer coefficients c_i (called "marks") depend only on the algebra. Before the extension, δ is literally zero and Eq. (21) simply expresses the linear dependence among the rows of the Cartan matrix:

$$\delta(h_i) = (\delta|\alpha_i) = 0. \tag{22}$$

The whole point of extending the Cartan subalgebra by L_0 is to be able to require $\delta(L_0) \neq 0$ in $\mathcal{P}^{ext}$, and avoid the degeneracy in $\mathcal{P}$ implied by Eq. (22). The critical definition is

$$\delta(L_0) = (\delta|\Lambda_0) = (\sum_{i=1}^{\ell} c_i\alpha_i|\Lambda_0) = -1. \tag{23}$$

This provides an extended bilinear form that completely labels the operators in the affine Lie algebra. In cases where the Cartan matrix is symmetrical

($c_l = 1$). the simplest solution to Eq. (20) and (23) is to add a zeroth row (and column) of the form $(0, -1, 0, \ldots, 0)$ to the Cartan matrix.

Finally, the value of $\Lambda_0(L_0)$ is not very important so long as it is not 2. The natural choice is to set $(\Lambda_0|\Lambda_0) = 0$.

The extended Cartan subalgebra is selected to include L_0 along with h_i, $i = 1, \ldots, \ell$. The basis vectors of the extended root space corresponding to these operators are then Λ_0, and α_i, $i = 1, \ldots, \ell$. Then the bilinear form is defined as $A_{ij} = (\alpha_i|\alpha_j)$ for $i, j = 1, \ldots, \ell$ plus a zeroth row and column. The zeroth row is constrained by $(\delta|\Lambda_0) = -1$; for the algebras analyzed here, we take $(\alpha_1|\Lambda_0) = -1$, $(\alpha_i|\Lambda_0) = 0$, $i = 2, \ldots, \ell$ and $(\Lambda_0|\Lambda_0) = 0$. In making the extension of the Cartan matrix to a Borcherds algebra, it is necessary to identify the operators corresponding to the simple roots.

4 Adding Energy and Number Operators to the Cartan Subalgebra

In Sec. 2 it was suggested from a simple quantum-mechanical example that a number operator appears in the Cartan subalgebra of a Borcherds extended finite-dimensional Lie algebra. In this section the hamiltonian and number operators are explicitly constructed from the simple roots of the Borcherds algebra by constructing the Borcherds extension of affine-$su(2)$.

As noted in Sec. 3, the addition of an imaginary simple root to an affine algebra according to the constraints Eqs. (20) - (23) gives a symmetrized Cartan matrix of a Borcherds algebra. Thus, we obtain the Borcherds algebra, affine-$su(2)_B$. (In affine-$su(2)$, α_1 and α_2 subtend 180° in $\mathcal{P}$ and have equal lengths.) The Cartan matrix is

$$A(\text{affine-}su(2)_B) = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}. \tag{24}$$

The root system is calculated from the Peterson formula Eq. (8) with the scalar product in $\mathcal{P}$ defined by Eq. (21). Roots are of the form

$$\alpha = n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3. \tag{25}$$

The root system can be broken up into representations of affine $su(2)$ by computing the multiplicities of roots of the form Eq. (25). The roots at $n_1 = 0$ correspond precisely to the root system of affine $su(2)$, which is a series of $su(2)$ triplets at each integer multiple of $\delta = \alpha_2 + \alpha_3$. The positive roots for $n_1 = 1$ are those of the (1.0) representation of affine-$su(2)$. The $n_1 = 2$ slice is a reducible sequence of affine-$su(2)$ representations, each starting at a specific multiple of δ, n_δ:

$$\begin{gathered}(2.0)_{n_\delta=0} + (2.0)_{n_\delta=1} + (2.0)_{n_\delta=2} + (2.0)_{n_\delta=3} + \\ 2(2.0)_{n_\delta=4} + 2(2.0)_{n_\delta=5} + 3(2.0)_{n_\delta=6} + 4(2.0)_{n_\delta=7} \\ +5(2.0)_{n_\delta=8} + 6(2.0)_{n_\delta=9} + \dots \end{gathered} \tag{26}$$

where the multiplicity of the affine-$su(2)$ representation eventually grows exponentially like a typical partition function found in the theory of Kac-Moody representations. In particular, these multiplicities are the coefficients in the expansion of the partition function of the $c = \frac{1}{2}$ (c is the central charge) representation of the Virasoro algebra with highest weight $\Lambda(L_0) = \frac{1}{2}$. These numbers correspond to the dimensions of the Hilbert subspaces gotten from applying an odd number of the Neveu-Schwarz (half-odd integer moded, anticommuting) operators to the ground state.

The weight-system multiplicities of the $(1.0.0)$ representation of affine-$su(2)$ is given by the Peterson formula with the bilinear form

$$A^{(1.0.0)}(\text{affine-}su(2)_H) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -2 & 2 \end{pmatrix}. \tag{27}$$

The weights of the $(1.0.0)$ representation are of the form

$$\alpha = \alpha_0 + n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3. \tag{28}$$

A detailed discussion of affine $su(2)$ representations is contained in Ref. [2], and the tables given there are used to unravel the weight multiplicities computed from Eq. (27) into affine $su(2)$ representations.

For the $(1.0.0)$ representation, the $n_1 = 0$ weight is a singlet at $n_2 = n_3 = 0$ and corresponds to the vacuum. The $n_1 = 1$ weights are in the (1.0) representation of affine-$su(2)$. As with the roots for this algebra, the $n_1 = 2$ weights are reducible under affine-$su(2)$, and can be decomposed into affine-$su(2)$ irreducible representations as

$$
\begin{aligned}
&[(2,0)+(0,2)]_{n_k=0}+(0,2)_{n_k=1}+[(2,0)+(0,2)]_{n_k=2}\\
&+[(2,0)+(0,2)]_{n_k=3}+2[(2,0)+(0,2)]_{n_k=4}\\
&+2[(2,0)+(0,2)]_{n_k=5}+\ldots.
\end{aligned}
\tag{29}
$$

This sequence is the set of representations in the tensor product $(1,0)\times(1,0)$ of affine-$su(2)$, which is computed in Sec. 6 of Ref. [2]. The $n_1 = 3$ slice has the representations of $(1,0)\times(1,0)\times(1,0)$. It appears obvious that this structure generalizes to all larger n_1. Thus, a generalization of a two-dimensional current algebra that includes multiparticle states, where the single-particle states are in the $(1,0)$ representation of affine-$su(2)$, is the $(1,0,0)$ representation of affine-$su(2)_B$. It includes a vacuum at $n_1 = 0$, single particle states at $n_1 = 1$, two-particle states at $n_1 = 2$, three-particle states at $n_1 = 3$, and so on. Thus, the full multiparticle space of states is included in this single representation of extended affine-$su(2)$, and the algebra contains operators that change number of particles.

The final task is to work out the relation of the operators h_1, h_2 and h_3, which correspond to the simple roots α_1, α_2 and α_3, to the operators N, L_0 and I_3. The calculation follows the same path that was followed in Sec. 3. From a direct analysis of the $(1,0,0)$ representation, it is a simple matter to make the transformation from the (h_1, h_2, h_3) basis, corresponding to the simple roots, to the $(N, L_0, 2I_3)$ basis of the Cartan subalgebra. In terms of

$$\lambda = n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3, \tag{30}$$

the following definitions are easily identified:

$$
\begin{aligned}
\lambda(N) &= n_1,\\
\lambda(L_0) &= n_1 + n_3,\\
\lambda(2I_3) &= -n_1 + 2n_2 - 2n_3,
\end{aligned}
\tag{31}
$$

where N is the number operator and I_3 is the diagonal operator in $su(2)$. The energy operator L_0 is defined to be 0 for the highest weight state, and is normalized to unity for the lowest doublet state at $n_1 = 1$. From the definition of the Cartan matrix and Eq. (30) it follows that

$$
\begin{aligned}
N &= -h_2 - h_3,\\
L_0 &= -h_1 - h_2 - \tfrac{1}{2}h_3,\\
2I_3 &= h_2.
\end{aligned}
\tag{32}
$$

Since either α_2 or α_3 can be chosen as the root defining $2I_3$, it is also possible to set $2I_3 = h_3$, which correspondingly rearranges the rest of Eqs.(30) and (32). Thus, both number and hamiltonian operators are in the Cartan subalgebra, although neither corresponds to a simple root. The correspondence is

$$\begin{aligned} N &\longrightarrow \alpha_N = -\alpha_2 - \alpha_3 = -\delta, \\ L_0 &\longrightarrow \alpha_H = -\alpha_1 - \alpha_2 - \tfrac{1}{2}\alpha_3, \\ 2I_3 &\longrightarrow \alpha_2, \end{aligned} \tag{33}$$

with $N(\alpha_N) = (\alpha_N|\alpha_N) = 0$ and $(\alpha_H|\alpha_H) = -3/2$. Although α_N is a null root, it is not the simple null root.

5 Conclusions

It can be interesting to survey new mathematical structures for applications in physics. In this paper we have proposed the use of Borcherds algebras and their representations to describe multiparticle states. It is an algebraic structure that extends quantum mechanics, as in the example of Sec. 3 and the first example in Sec. 4, or simple field theories, as the second example in Sec. 4, from a single particle description to a structure that unifies all numbers of particles. Thus, there is an interesting Fock space structure of the simplest representations of the simplest algebra $su(2)_B$, where the representation space $(1,0)$ is the Fock space of an $su(2)$ doublet of quantum mechanical operators. Similarly the analogous extension of affine $su(2)$ as a function of n_1 is a vacuum for $n_1 = 0$, single particle for $n_1 = 1$, two particle for $n_1 = 2$, and so on. We have not discussed how to construct such a theory in detail, but it seems physically clear that multiparticle states are natural in the representation theory of Borcherds algebras. One might speculate that such a structure is useful for second quantization of a single particle theory. More particularly, we have shown that the representation theory of these algebras is computationally tractable, and have examined several examples.

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References

[1] R. E. Borcherds, Generalized Kac-Moody Algebras, J. of Algebra **115**, 501 (1988).

[2] S. Kass, R. V. Moody, J. Patera, R. Slansky, Affine Lie Algebras, Weight Multiplicities and Branching Rules (University of California Press, Los Angeles, 1990).

[3] R. Slansky, An Algebraic Role for Energy and Number Operators for Multiparticle States, Nucl. Phys. **B389**, 349 (1993).

[4] V. G. Kac, Infinite Dimensional Lie Algebras (Cambridge University Press, Cambridge, 1985).

[5] A review of this notation can be found in R. Slansky, Group Theory for Unified Model Building, Phys. Reports **79**, 1 (1981).

[6] M. R. Bremner, R. V. Moody, J. Patera, Tables of Dominant Weight Multiplicities for Representations of Simple Lie Algebras (Marcel Dekker, Inc., New York, 1985).