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Title: ON THEORETICAL ISSUES OF COMPUTER SIMULATIONS SEQUENTIAL DYNAMICAL SYSTEMS

CONF-980772--

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Submitted to: Information Systems Analysis & Synthesis
4th International Conference (ISAS'98)
July 12-16,1998
Orlando, Florida

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ON THEORETICAL ISSUES OF COMPUTER SIMULATIONS SEQUENTIAL DYNAMICAL SYSTEMS

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ABSTRACT

This paper is a short version of [3]. We study a class of discrete dynamical systems that is motivated by the generic structure of simulations. The systems consist of the following data: (a) a finite graph Y with vertex set $\{1,\dots,n\}$ where each vertex has a binary state, (b) functions $F_i : \mathbb{F}_2^n \to \mathbb{F}_2^n$ and (c) an update ordering π. The functions F_i update the binary state of vertex i as a function of the state of vertex i and its Y-neighbors and leave the states of all other vertices fixed. The update ordering is a permutation of the Y-vertices. By composing the functions F_i in the order given by π one obtains the dynamical system $(\mathfrak{F}_Y,\pi) = \prod_{i=1}^n F_{\pi(i)} : \mathbb{F}_2^n \longrightarrow \mathbb{F}_2^n$, i.e., a representative for the equivalence class of sequential dynamical systems $[\mathfrak{F}_Y,\pi] = \{(\mathfrak{F}_Y,\pi') \mid (\mathfrak{F}_Y,\pi') = (\mathfrak{F}_Y,\pi)\}$ which we refer to as SDS. We derive a decomposition result, characterize invertible SDS and study fixed points. In particular we analyse how many different SDS that can be obtained by reordering a given multiset of update functions and give a criterion for when one can derive concentration results on this number. Finally, some specific SDS are investigated.

Keywords. Sequential dynamical systems, fixed points, structure, orderings.

1. INTRODUCTION

We build on the ideas presented in the paper [2] and introduce *Sequential Dynamical Systems*, (SDS), a new class of dynamical systems implied by the formalization of simulation as composed local maps.

An SDS basically consists of (i) a graph Y, (ii) local maps, i.e., Boolean functions indexed by the vertices and defined on the states of the vertex itself and its corresponding nearest neighbors and (iii) a permutation of the vertices. The full update for the states of the entities gives a class of discrete dynamical systems which we will refer to as sequential dynamical systems or simply SDS [2].

Note that the mathematical constituents of SDS correspond to the essential elements of a computer simulation. Simulations typically are comprised of entities having state values and local rules governing state transitions, a spatial environment in which the entities act or interact, and some method with which to trigger an update of the state of each entity. Schedules for updates can be time stepped, event driven, scripted, etc., and result in the dynamical properties in state space that we call a "simulated system".

As is seen above and in [2], the general form of the support structure for SDS is discrete. It is not that this theory is being constructed to apply only to simulations that represent space discretely. Rather, what is captured is that the idea of entity *adjacency* in the support structure is defined by the *causal dependency* among local maps. That is, entities are adjacent in the support structure if and only if they can interact. This spatial representation (support structure), perhaps called "interaction space" or "cause space", is an inherently discrete (graph) structure having maps associated to vertices and dependency denoted by edges. The support structure is a transformation of the "natural" space that particular entities could be defined with respect to and is, in that important sense, general and context free. This is obviously an essential issue for a truly general simulation theory.

Locality, a property of the maps, is defined in terms of adjacency, a property of the of the support structure. The resulting interplay between the topological and algebraic properties of SDS is very interesting and seems to open new areas of purely mathematical investigation.

2. Sequential Dynamical Systems

2.1. Definitions. We set $\mathbb{N}_n = \{1, 2, \ldots, n\}$. Let the set of Y-vertices adjacent to vertex i be denoted by $\Delta_1(i)$ and set $\delta_i = |\Delta_1(i)|$. We denote the increasing sequence of elements of the set $B_1(i) = \Delta_1(i) \cup \{i\}$ by

$$\hat{B}_1(i) = (j_1, \ldots, i, \ldots, j_{\delta_i}) , \qquad (2.1)$$

and set $d = \max_{1 \le i \le n} \delta_i$. Each vertex i has associated a binary state x_i. Also, let $(f_k)_k$ with $f_k : \mathbb{F}_2^k \to \mathbb{F}_2$ where $1 \le k \le d+1$ be some given multiset of symmetric functions. For each vertex $i \in \mathbb{N}_n$ we define the map

$$\mathrm{proj}_Y[i] : \mathbb{F}_2^n \to \mathbb{F}_2^{\delta_i+1},$$
$$(x_1, \ldots, x_n) \mapsto (x_{j_1}, \ldots, x_i, \ldots, x_{j_{\delta_i}})$$

Finally, let S_k with $k \in \mathbb{N}$ denote the permutation group on k letters.

Definition 1 (Y-local maps). Let $(f_k)_{1 \le k \le d(Y)+1}$ be a multiset of S_k-symmetric functions $f_k : \mathbb{F}_2^k \to \mathbb{F}_2$. For each $i \in \mathbb{N}_n$ there is a Y-local map $F_{i,Y}$ given by

$$\begin{aligned} y_i &= f_{\delta_i+1} \circ \mathrm{proj}_Y[i] \\ F_{i,Y}((x_j)_j) &= (x_1, \ldots, x_{i-1}, y_i(x), x_{i+1}, \ldots, x_n) . \end{aligned}$$

$F_{i,Y}$ is a map $F_{i,Y} : \mathbb{F}_2^n \to \mathbb{F}_2^n$ that updates the state of vertex i as a function of the states contained in $B_1(i)$ and leaves all other vertex states fixed. We refer to the multiset $(F_{i,Y})_i$ as $\mathfrak{F}_Y$.

In particular, let $(f_k)_{1 \le k \le n}$ be a fixed multiset of S_k-symmetric functions as defined above. Then for each $Y < K_n$ the multiset $(f_k)_{1 \le k \le n}$ induces a multiset $\mathfrak{F}_Y$, i.e., we have a map $\{Y < K_n\} \to \{\mathfrak{F}_Y\}$. Let $\pi \in S_n$. The introduction of the maps $F_{i,Y}$ allows us to consider products of the form

$$(\mathfrak{F}_Y, \pi) = \prod_{i=1}^{n} F_{\pi(i),Y} : \mathbb{F}_2^n \longrightarrow \mathbb{F}_2^n . \qquad (2.2)$$

Definition 2 (sequential dynamical system). A sequential dynamical system (SDS) over a graph Y w.r.t. π is an equivalence class

$$[\mathfrak{F}_Y, \pi] = \{(\mathfrak{F}_Y, \pi') \mid (\mathfrak{F}_Y, \pi') = (\mathfrak{F}_Y, \pi)\} . \qquad (2.3)$$

In this paper we will be particularly interested in computing the number of different SDS, i.e.,

$$a_{(f_k)_k}(Y) = |\{[\mathfrak{F}_Y, \pi] \mid \pi \in S_n\}| \qquad (2.4)$$

for a given multiset $(f_k)_k$ and for a given graph Y. That is, how many different dynamical systems can be obtained by rescheduling.

Sometimes the multiset $(f_k)_k$ is induced by a single Boolean function $\mathbf{B} : \mathbb{F}_2^n \to \mathbb{F}_2$. In this case we will say that the corresponding SDS is induced by **B**. The Boolean functions listed below have this property and will be studied later in some detail. Here $x = (x_1, \ldots, x_k)$. Also let $A_t(x) = |\{x_j \mid x_j = t\}|$.

$$\mathbf{NOR}_k : \mathbb{F}_2^k \to \mathbb{F}_2$$
$$x \mapsto \overline{x_1 \vee \cdots \vee x_k} \tag{2.5}$$
$$\mathbf{NAND}_k : \mathbb{F}_2^k \to \mathbb{F}_2$$
$$x \mapsto \begin{cases} 0 & \text{iff} \quad x = (1, 1, \ldots, 1) \\ 1 & \text{else} \end{cases} \tag{2.6}$$
$$\mathbf{PAR}_k : \mathbb{F}_2^k \to \mathbb{F}_2$$
$$x \mapsto \sum_{i=1}^{k} x_i \tag{2.7}$$
$$\mathbf{MIN}_k : \mathbb{F}_2^k \to \mathbb{F}_2$$
$$x \mapsto \begin{cases} 1 & A_1(x) < A_0(x) \\ 0 & \text{else} \end{cases} \tag{2.8}$$
$$\mathbf{DF}_{1k} : \mathbb{F}_2^k \to \mathbb{F}_2$$
$$x \mapsto \begin{cases} 1 & \text{iff} \quad A_1(x) = 1 \\ 0 & \text{else} \end{cases} \tag{2.9}$$

Although a slight abuse of terminology, we will simply write, e.g., $a_{\mathbf{PAR}}$, for these functions instead of using the full multiset $(f_k)_k$ as index.

2.2. **Combinatorial analysis.** The function $a_{(f_k)_k}(Y)$ is closely related to a combinatorial invariant of Y itself, namely the number of acyclic orientations of Y denoted by $a(Y)$. An acyclic orientation is a map that assigns a direction to each Y-edge such that the resulting directed graph is a forest. Some comments on this relation are in order. We will write a permutation π as an n-tuple $(i_1, \ldots, i_n)$ and when nothing else is stated the natural ordering $(1, \ldots, n)$ is assumed. Now, SDS can be analysed from a purely combinatorial perspective [2]. This approach is based on the simple observation that if $\pi = (i_1, \ldots, i_n)$ and $\pi' = (i'_1, \ldots, i'_n)$ are two permutations differing by a transposition of consecutive coordinates (i_k, i_{k+1}) where $\{i_k, i_{k+1}\} \notin \mathrm{e}[Y]$, then independently of the choice of the maps $F_{i,Y}$ we have $[\mathfrak{F}_Y, \pi] = [\mathfrak{F}_Y, \pi']$. This leads to an analysis which is independent of the structure of the local maps, that is, it only considers formal dependencies and is thus determined by the underlying graph Y alone. It motivates the introduction of the update graph $U(Y)$:

Definition 3. Let $U(Y)$ be the graph having vertex set S_n and in which two different vertices $(i_1, \ldots, i_n)$, $(h_1, \ldots, h_n)$ are adjacent iff (a) $i_\ell = h_\ell$, $\ell \neq k, k+1$ and (b) $\{i_k, i_{k+1}\} \notin \mathrm{e}[Y]$.

Write $\pi \sim_Y \pi'$ iff π and π' occur in a $U(Y)$-path and set $[\pi] = \{\pi' \mid \pi' \sim_Y \pi\}$. Then for $\pi, \pi' \in [\sigma]_Y$ we have $[\mathfrak{F}_Y, \pi] = [\mathfrak{F}_Y, \pi']$. That is, $U(Y)$-components do independently of the maps $F_{i,Y}$ represent equivalence classes of SDS. As shown in [7] the combinatorial analysis allows us to interprete an equivalence class $[\pi]_Y$ as an acyclic orientation of Y. That is, there is a bijection

$$f(Y, \ ) : [S_n / \sim_Y] \longrightarrow \mathrm{Acyc}(Y), \tag{2.10}$$

where $\mathrm{Acyc}(Y)$ is the set of all acyclic orientations[1] of Y. We set $a(Y) = |\mathrm{Acyc}(Y)|$. The bijection given in (2.10) shows that each $U(Y)$-component corresponds uniquely to an acyclic

[1] The number of acyclic orientations are of independent interest in theoretical computer science, since they provide lower bounds on the computational complexity of various decision and sorting problems [9].

orientation of Y, i.e., the map

$$h_{(f_k)_k} : [S_n / \sim_Y] \longrightarrow \{[\mathfrak{F}_Y, \pi] \mid \pi \in S_n\},$$
$$h_{(f_k)_k}([\pi]_Y) = [\mathfrak{F}_Y, \pi] \qquad (2.11)$$

is surjective and we have $a_{(f_k)_k}(Y) \leq a(Y)$. Another way of stating this is that some components in the update graph $U(Y)$ may merge as a result of the specific structure of the Y-local maps, see Lemma 1.

3. Fixed Points, Bijectivity and a concentration Result

In the following we will write $x = (x_1, \dots, x_n)$. We begin by showing that an SDS over the graph Y is a direct product of SDS over the Y-components.

Proposition 1. *Let Y be a graph, $[\mathfrak{F}_Y, \pi]$ an* SDS*, C a Y-component, $n_C = |C|$ and $(\mathfrak{F}_C, \pi_C) = \prod_{i_1^C < i_2^C < \cdots < i_{n_C}^C} F_{\pi_C(i_j^C), Y}$. Then we have*

$$[\mathfrak{F}_Y, \pi] = \prod_{C<Y} [\mathfrak{F}_C, \pi_C] \, , \qquad (3.1)$$

where π_C denotes the restriction of the bijective map π to the elements $j \in \mathrm{v}[C]$.

Proposition 2. *Let Y be a graph and $[\mathfrak{F}_Y, \pi]$ an* SDS *over Y. Denote by $\mathrm{Fix}([\mathfrak{F}_Y, \pi])$ the set $\{x \mid [\mathfrak{F}_Y, \pi](x) = x\}$. Then we have*

$$\forall\, \sigma \in S_n : \qquad \mathrm{Fix}([\mathfrak{F}_Y, \sigma]) = \mathrm{Fix}([\mathfrak{F}_Y, \pi]). \qquad (3.2)$$

We will now give a characterization of bijective SDS.

Proposition 3. [5] *Let $Y < K_n$, and (f_k) a multiset $f_k : \mathbb{F}_2^k \to \mathbb{F}_2$. An* SDS *$[\mathfrak{F}_Y, \pi]$ is bijective if and only if*

$$f_{\delta_{i,Y}+1} \circ \mathrm{proj}_Y[i](x) = x_i \quad or$$
$$f_{\delta_{i,Y}+1} \circ \mathrm{proj}_Y[i](x) = \overline{x_i} \, .$$

Furthermore let $\pi = (i_1, \dots, i_{n-1}, i_n) \in S_n$, $\pi^* = (i_n, i_{n-1}, \dots, i_1)$ and $[\mathfrak{F}_Y, \pi]$ be a bijective SDS*. Then we have*

$$[\mathfrak{F}_Y, \pi]^{-1} = [\mathfrak{F}_Y, \pi^*] \, .$$

Remark 1. Obviously, the bijectivity of one particular SDS $[\mathfrak{F}_Y, \pi]$ implies that any SDS $[\mathfrak{F}_Y, \sigma]$ is bijective.

In particular we have

Corollary 1. *Let $(\mathbf{PAR}_k)_{1 \leq k \leq n}$ be the multiset of maps defined in (2.7). Then for arbitrary $Y < K_n$ all* SDS *induced by $(\mathbf{PAR}_k)_{1 \leq k \leq n}$ are invertible.*

Proof. Obviously, if $\sum_{j \in \Delta_1(i)} x_j = 0$, then $x_i \mapsto x_i$ and if $\sum_{j \in \Delta_1(i)} x_j = 1$, we derive $x_i \mapsto \overline{x_i}$. The corollary now follows from Proposition 3. □

Proposition 3 immediately allows one to determine all bijective sCA[2] [2].

Corollary 2. *There are, independent of n, exactly $2^{(2^2)} = 16$ different bijective* sCA.

Proof. An sCA is an SDS over the base graph $Y = \mathrm{Circ}_n$, i.e., the cycle graph on n vertices.

[2]here we will assume closed boundary conditions and nearest neighbor rules.

Obviously the corresponding multiset $(f_k)_k$ consists of the single map $f_3 : \mathbb{F}_2^3 \to \mathbb{F}_2$ and Proposition 3 implies that either

$$\begin{aligned} f_3(x_{i-1}, x_i, x_{i+1}) &= x_i \quad \text{or} \\ f_3(x_{i-1}, x_i, x_{i+1}) &= \overline{x_i} \end{aligned}$$

where x_{i-1} and x_{i+1} are arbitray and $i-1, i, i+1 \in \mathbb{Z}/n\mathbb{Z}$, proving the Corollary. □

In contrast to this characterization, bijectivity of parallely updated CA (pCA) does in fact depend on the number of cells. For example, CA-rule 150 is not bijective for $n = 6$ and bijective for $n = 7$ [1, 4].

We next consider SDS over the random graph $G_{n,p}$, i.e., the probability space consisting of all K_n-subgraphs where each edge is selected with independent probability p. We will study $a_{(f_k)_k}$ as a random variable w.r.t. the probability space $G_{n,p}$ and prove a concentration result for $\log_2 a_{\mathbf{NOR}}(G_{n,p})$. The existence of a concentration result for $\log_2 a_{(f_k)_k}(G_{n,p})$ can be interpreted as follows: the number of different SDS depend only on *the number of edges* of Y and not on the particular choice of Y itself. Insofar it can be viewed as a generic property. To begin we will define a key property of real valued $G_{n,p}$ random variables.

Definition 4. Let $\eta_{n,p} : G_{n,p} \longrightarrow \mathbb{R}$ be a random variable (r.v.) Then $\eta_{n,p}$ is called Lipschitz if and only if for any two graphs $Y, Y' < K_n$ that differ by the alteration of exactly one edge one has

$$| \eta_{n,p}(Y) - \eta_{n,p}(Y') | \leq 1 \, . \qquad (3.3)$$

In particular we will be interested in multisets $(f_k)_k$ for which the r.v. $\log_2 a_{(f_k)_k}$ is Lipschitz, i.e.,

$$| \log_2 a_{(f_k)_k}(Y) - \log_2 a_{(f_k)_k}(Y') | \leq 1 \, . \quad (3.4)$$

Lemma 1. *Let $Y < K_n$ be an arbitrary graph. Then the following assertions hold*

$$\begin{array}{lll} (i) & \log_2 a_{\mathbf{NOR}} & : \ \{Y < K_n\} \to \mathbb{N} \\ (ii) & \log_2 a_{\mathbf{NAND}} & : \ \{Y < K_n\} \to \mathbb{N} \\ (iii) & \log_2 a_{\mathbf{DF}_1} & : \ \{Y < K_n\} \to \mathbb{N} \\ & \textit{are Lipschitz} & \\ (iv) & \log_2 a_{\mathbf{PAR}} & : \ \{Y < K_n\} \to \mathbb{N} \\ & \textit{is not Lipschitz} & \end{array}$$

Proof. A detailed proof of (i)–(iii) can be found in [5] and can be sketched as follows: first one proves that

$$\begin{aligned} h_{(f_k)_k} : S_n/\sim_Y &\longrightarrow \{[\mathfrak{F}_Y, \pi] \mid \pi \in S_n\}, \quad (3.5) \\ [\pi]_Y &\mapsto [\mathfrak{F}_Y, \pi] \end{aligned}$$

is injective for $(\mathbf{NOR}_k)_k$. Second one considers the bijection in (2.10) and uses the fact that $\log_2 a(Y)$ is Lipschitz. □

Remark 2. The above Proposition implies that $h_{\mathbf{PAR}}$ is not bijective. A simple example demonstrating this is provided by the graph $Y = \mathrm{Circ}_4$. For instance, the two permutations $\pi_1 = (2134)$ and $\pi_2 = (4132)$ satisfy $\pi_1 \not\sim_Y \pi_2$, that is, they are contained in different components of $U(\mathrm{Circ}_4)$. However, because of the structure of the **PAR**-function these two components give the same sequential dynamical system. The number of different SDS is 11 whereas the number of acyclic orientations is 14.

Theorem 1. *Let $\eta_{n,p}$ be Lipschitz. Then for $G_{n,p}$ and arbitrary probability p one has*

$$\mu_{n,p}(\{ \, |\eta_{n,p}(G_{n,p}) - \mathbb{E}[\eta_{n,p}(G_{n,p})]| > \lambda\sqrt{n(n-1)/2} \, \}) < 2e^{-\lambda^2/2}. \quad (3.6)$$

In particular, if for some Boolean function $\mathbf{B}$ *the map* $h_{\mathbf{B}}$ *(see 3.5) is bijective, we have*

$$n[\log_2(n) - \log_2 e - \log_2 p - o(1)] \leq \mathbb{E}[\log_2 a_{\mathbf{B}}(G_{n,p})]. \quad (3.7)$$

The first assertion of Theorem 1 is a consequence of a general result of Milman and Schechtman [8]. It is proved by (a) constructing a finite martingale $(X_i)_i$ that converges to the random variable $\eta_{n,p}(G_{n,p})$, (b) showing that $\eta_{n,p}$ being Lipschitz implies $|X_{i+1} - X_i| \leq 1$ and (c) by applying Azuma's inequality [6]. The second assertion of Theorem 1 follows from Theorem 2 of [7].

In particular we have

Corollary 3. *For the random graph $G_{n,p}$, $\mathbf{B} \in \{\mathbf{NOR}, \mathbf{NAND}, \mathbf{DF}_1\}$ and arbitrary probability p one has*

$$\mu_{n,p}(\{ \, |\log_2 a_{\mathbf{B}}(G_{n,p}) - \mathbb{E}[\log_2 a_{\mathbf{B}}(G_{n,p})]| > \lambda\sqrt{n(n-1)/2} \, \}) < 2e^{-\lambda^2/2} , \quad (3.8)$$

and we have $n[\log_2(n) - \log_2 e - \log_2 p - o(1)] \leq \mathbb{E}[\log_2 a_{\mathbf{B}}(G_{n,p})]$.

4. Analysis of some special systems

In this section we will present some analysis of SDS induced by the Boolean functions **NOR**, **PAR MIN** as listed in (2.5) – (2.8).

As will be shown below the dynamics for the complete graph and the empty graph is well understood. To convey information on what one can expect for a graph $Y < K_n$ we make use of random graph theory. For an SDS $[\mathfrak{F}_Y, \pi]$ we denote by $\nu(\mathfrak{F}_Y, \pi)$ and $\gamma(\mathfrak{F}_Y, \pi)$ the number of different periodic orbits and the size of a largest periodic orbit respectively. In the following we will study the random variables

$$\begin{aligned}
\mathrm{Fix}_{(f_k)_k}(Y) &= |\,\mathrm{Fix}(\mathfrak{F}_Y)|, && (4.1)\\
\mathrm{N}_{(f_k)_k}(Y) &= \max_{\pi \in S_n}\{\nu(\mathfrak{F}_Y, \pi)\}, && (4.2)\\
\Gamma_{(f_k)_k}(Y) &= \max_{\pi \in S_n}\{\gamma(\mathfrak{F}_Y, \pi)\} \quad \text{and} && (4.3)\\
a_{(f_k)_k}(Y) &= |\{[\mathfrak{F}_Y, \pi] \mid \pi \in S_n\}| && (4.4)
\end{aligned}$$

for the functions in (2.5)–(2.8).

Obviously, $\mu_{n,p}$ converges for $n \to \infty$ to the uniform measure on graphs with $p\binom{n}{2}$ edges. However, for small n the deviations between the uniform measure and $\mu_{n,p}$ are significant. Accordingly, we will use an adapted version of the measure $\mu_{n,p}$ for the following computer experiments as follows: for fixed $n \in \mathbb{N}$ and a given set of graphs, $\mathrm{Exp} = \{Y_1, \ldots, Y_M\}$, $M \in \mathbb{N}$ we obtain the multiset of probabilities $\mu(Y_i) = p_i$, $1 \leq i \leq M$. Now we take $\beta_E \in \mathbb{R}$ such that $\beta_E \sum_{i=1}^{M} p_i = 1$ and define $\mu_E : \mathrm{Exp} \to \mathbb{R}$ by $\mu_E = \beta_E \mu_{n,p}$. We will denote expectation value and variance w.r.t. the measure μ_E by $\mathbb{E}_E[\,]$ and $\mathbb{V}_E[\,]$. Figures 1–3 show expectations and variances for basic properites of SDS induced by the Boolean functions mentioned above.

Let $[\mathfrak{F}_Y, \pi]$ be an SDS. The digraph $\Gamma[\mathfrak{F}_Y, \pi]$ has vertex set is $\mathbb{F}_2^n$ and its directed edges are all pairs of the form $(x, [\mathfrak{F}_Y, \pi](x))$. Clearly,

$\Gamma[\mathfrak{F}_Y, \pi]$-cycles correspond to periodic orbits of the SDS $[\mathfrak{F}_Y, \pi]$.

In the following we present some results on SDS induced by the functions **NOR**, **PAR** and **MIN**. An SDS induced by **PAR** over an empty graph only has fixed points, or equivalently, the corresponding digraph has an empty edge set. For SDS induced by **NOR** and **MIN** all points are contained in a period 2 cycle. Accordingly there are 2^n fixed points in the former case and 2^{n-1} period 2 orbits in the latter case. Let now e_k be the kth unit vector and $\langle x, y \rangle$ be the standard inner product of x and y.

Proposition 4 (NOR). *Let* $[\mathfrak{F}_{K_n}, \mathrm{id}]$ *be the* SDS *induced by* **NOR**. *The states x for which $\langle x, e_n \rangle = 1$ are mapped to zero. If $\langle x, e_n \rangle \neq 1$ then x is mapped to e_k where $k = 1 + \max_i\{x_i = 1\}$. The set $L = \{0, e_1, e_2, \dots, e_n\}$ is the unique limit cycle of* $[\mathfrak{F}_{K_n}, \mathrm{id}]$. *Moreover, for arbitrary dependency graph Y the* SDS *induced by* **NOR** *has no fixed points.*

Proposition 5 (PAR). *Let* $[\mathfrak{F}_{K_n}, \mathrm{id}]$ *be the system induced by* **PAR**. *Then all points are contained in a periodic orbit $\mathfrak{O}$ and we have* $n+1 \equiv 0 \pmod{|\mathfrak{O}|}$.

Proposition 6 (MIN). *Let* $[\mathfrak{F}_{K_n}, \mathrm{id}]$ *be the* SDS *induced by* MIN. *For any periodic orbit $\mathfrak{O}$ one has* $n+1 \equiv 0 \pmod{|\mathfrak{O}|}$.

Acknowledgments. Special thanks and gratitude to D. Morgeson, for his continuous support. This research is supported by Laboratory Directed Research and Development under DOE contract W-7405-ENG-36 to the University of California for the operation of the Los Alamos National Laboratory.

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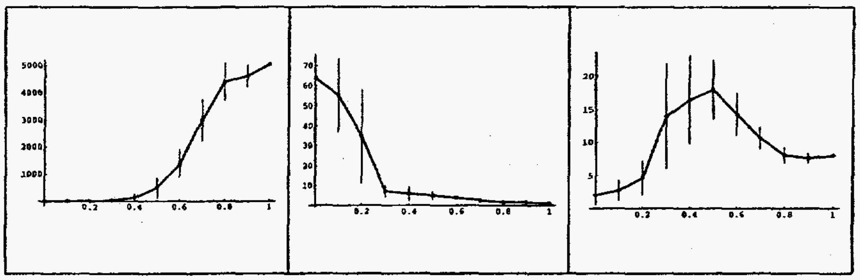

FIGURE 1. The number of different SDS, the number of orbits and the size of a largest orbit for the **NOR**-function. From the left: $\mathbb{E}_E[a_{\mathbf{NOR}}]$, $\mathbb{E}_E[\mathrm{N}_{\mathbf{NOR}}]$ and $\mathbb{E}_E[\Gamma_{\mathbf{NOR}}]$ with error bars showing the standard deviation with respect to the measure μ_E. Here $n = 7$ with sample size 50.

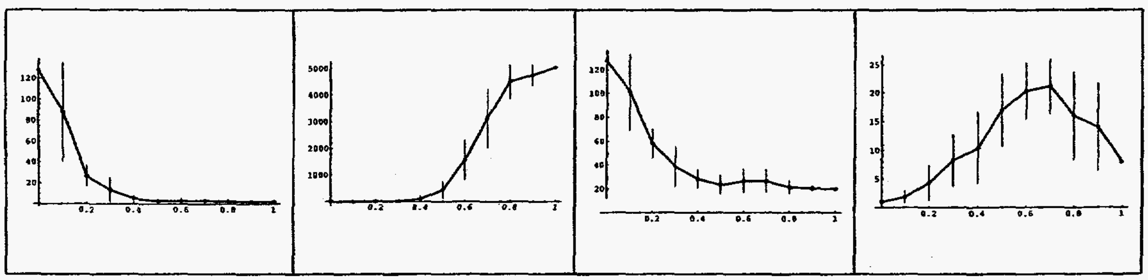

FIGURE 2. The number of fixed points, the number of different SDS, the number of orbits and the size of a largest orbit for the **PAR**-function. From the left: $\mathbb{E}_E[\mathrm{Fix}_{\mathbf{PAR}}]$, $\mathbb{E}_E[a_{\mathbf{PAR}}]$, $\mathbb{E}_E[\mathrm{N}_{\mathbf{PAR}}]$ and $\mathbb{E}_E[\Gamma_{\mathbf{PAR}}]$ with error bars showing the standard deviation with respect to the measure μ_E. Here $n = 7$ with sample size 50.

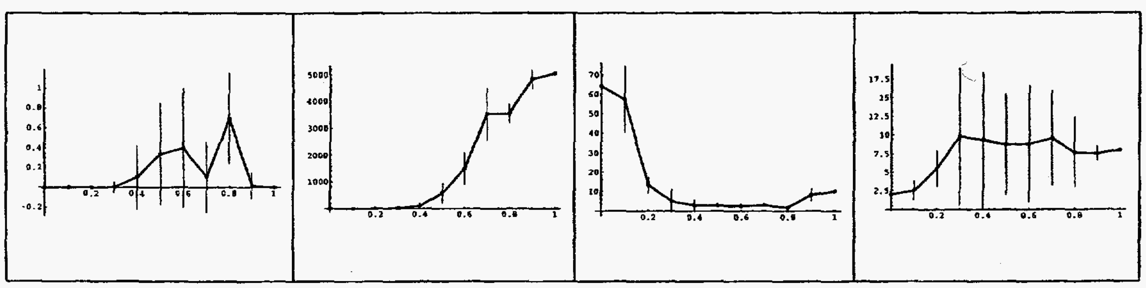

FIGURE 3. The number of fixed points, the number of different SDS, the number of orbits and the size of a largest orbit for the **MIN**-function. From the left: $\mathbb{E}_E[\mathrm{Fix}_{\mathbf{MIN}}]$, $\mathbb{E}_E[a_{\mathbf{MIN}}]$, $\mathbb{E}_E[\mathrm{N}_{\mathbf{MIN}}]$ and $\mathbb{E}_E[\Gamma_{\mathbf{MIN}}]$ with error bars showing the standard deviation with respect to the measure μ_E. Here $n = 7$ with sample size 50.