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Title: SIMULATION AND SEQUENTIAL DYNAMCIAL SYSTEMS

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Abstract

Computer simulations have a generic structure. Motivated by this we present a new class of discrete dynamical systems that captures this structure in a matematically precise way. This class of systems consists of (i) a loopfree graph Y with vertex set $\{1, 2, \ldots, n\}$ where each vertex has a binary state, (ii) a vertex labeled set of functions $(F_{i,Y} : \mathbb{F}_2^n \to \mathbb{F}_2^n)_i$ and (iii) a permutation $\pi \in S_n$. The function $F_{i,Y}$ updates the state of vertex i as a function of the states of vertex i and its Y-neighbors and leaves the states of all other vertices fixed. The permutation π represents the update ordering, i.e., the order in which the functions $F_{i,Y}$ are applied. By composing the functions $F_{i,Y}$ in the order given by π one obtains the dynamical system

$$[F_Y, \pi] = \prod_{i=1}^{n} F_{\pi(i),Y} : \mathbb{F}_2^n \longrightarrow \mathbb{F}_2^n ,$$

which we refer to as a sequential dynamical system, or SDS for short. We will present bounds for the number of functionally different systems and for the number of nonisomorphic digraphs $\Gamma[F_Y, \pi]$ (having vertex set $\mathbb{F}_2^n$ and edge sets $\{(x, [F_Y, \pi](x)) \mid x \in \mathbb{F}_2^n\}$) that can be obtained by varying the update order and applications of these to specific graphs and graph classes.

1. Introduction

Computer simulations have become an important technique in both science and business. Simulations share a generic structure. They typically consist of (i) entities or agents with some state, where (ii) each agent or entity has communication capabilities and (iii) a schedule that gives the order in which the agents or entities (sequentially) update their states. The agents (entities) are usually only able to establish communication with agents (entities) located 'close' to themselves, i.e., they have a local region of perception. One possible way to translate this into mathematical terms is to

- interprete agents or entities as symmetric Boolean functions,
- interprete the endpoints of the communication links as the (state) variables over which the Boolean function is defined, and
- interprete the schedule that decides the order in which the agents update their states as a permutation.

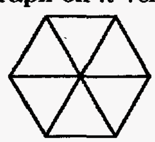
This brings us to the concept of SDS. An SDS consists of

- a set of symmetric functions F associated to the agents,

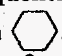
- a graph Y where vertices are the agents and where edges represent the communication links, and
- a permutation π (the update schedule).

Thus an SDS can be written written as a triple $(F = (f_i), Y, \pi)$. If all the functions f_i are the same we write (f, Y, π).

Example 1. Consider the following example of a market where there is only one stock and where each agent (labeled 0 to n) in turn is allowed to buy or sell one share. Agent 0 knows the most recent action of all other agents. Agent i ($2 \leq i \leq n-1$) only knows the most recent action of agent 0, agent $i-1$ and agent $i+1$. Agent 1 sees agent 0, 2 and n and agent n sees agent 0, 1 and $n-1$. Assume that every agent sells one share if the majority of its neighbors (and itself included) just sold one share and buys one share otherwise. Let **Maj** be the symmetric function that maps $(x_1, \ldots, x_k)$ to 0 if 0 occurs more often than 1 and to 1 otherwise. Assign a binary state to each agent that is 0 if the agent just sold a share and 1 if it just bought a share. The communication among the agents are given by the graph $W_n = \mathrm{Circ}_n \otimes 0$, i.e., the vertex join of the circle graph on n vertices with the vertex 0. Thus, e.g., the graph W_6 is



Finally, let π be the order in which the agents are acting. Thus our example is accurately described by the SDS with **Maj** as the local functions, W_n as the dependency graph and π as the update order, i.e., $(\mathbf{Maj}, W_n, \pi)$. Of course, this is a very simple example. One can easily derive more realistic/complex models. Nevertheless, it shows how the sequential nature of markets can be modeled by SDS in a very natural way.

Example 2. A sequential cellular automaton (SCA). An SCA consists of the circle graph on n vertices (Circ_n), e.g., Circ_6 is the graph . Each vertex i has associated a binary state x_i. The state of each vertex is updated with the same function $f : \mathbb{F}_2^3 \to \mathbb{F}_2$ in the order specified by a permutation π. Thus an SCA is a triple $(f, \mathrm{Circ}_n, \pi)$. It is interesting to note that every (parallel) cellular automaton (PCA) on n vertices (or cells) can be realized as an SDS over a graph on $2n$ vertices. The theory developed in [1, 2, 5, 4] applies in particular to the study of SCA.

In this paper we continue the work on the class of discrete dynamical systems referred to as sequential dynamical systems. The work on SDS was initiated in [1, 2, 5, 4]. In these papers combinatorial, probabilistic and dynamical system aspects are analyzed. This paper focuses on the combinatorial aspects. The emphasis is on ideas and applications of the theory and for this reason some proofs have been left out. In these cases we refer to [5, 4] for full proofs. We start by reviewing the concepts and the setting needed for the definition of a sequential dynamical system (SDS).

Let Y be a finite, loopfree, undirected graph with vertex set $\mathrm{v}[Y] = \{1, \ldots, n\}$ and edge set $\mathrm{e}[Y]$. Let $B_0(i)$ be the set of Y-vertices adjacent to vertex i and let $\delta_i = |B_0(i)|$. We denote the increasing sequence of elements of the set $B_0(i) \cup \{i\}$ by

$$B_1(i) = (j_1, \ldots, i, \ldots, j_{\delta_i}) \ , \tag{1.1}$$

and set $d = \max_{1 \leq i \leq n} \delta_i$. To each vertex i there is associated a state $x_i \in \mathbb{F}_2$, and for each $k = 1, \ldots, d+1$ let $f_k : \mathbb{F}_2^k \to \mathbb{F}_2$ be a given symmetric function. For each vertex $i \in \mathbb{N}_n = \{1, 2, \ldots, n\}$ we introduce the map

$$\mathrm{proj}_Y[i] : \mathbb{F}_2^n \to \mathbb{F}_2^{\delta_i + 1}, \quad (x_1, \ldots, x_n) \mapsto (x_{j_1}, \ldots, x_i, \ldots, x_{j_{\delta_i}}) \ .$$

Further, let S_k with $k \in \mathbb{N}$ denote the symmetric group on k letters. Let $(f_k)_{1 \leq k \leq d(Y)+1}$ be a multiset of symmetric functions $f_k : \mathbb{F}_2^k \to \mathbb{F}_2$. Set $x = (x_1, x_2, \ldots, x_n)$. For each $i \in \mathbb{N}_n$ there is a Y-local map $F_{i,Y}$ given by

$$\begin{aligned} y_i &= f_{\delta_i+1} \circ \mathrm{proj}_Y[i], \\ F_i(x) &= (x_1, \ldots, x_{i-1}, y_i(x), x_{i+1}, \ldots, x_n). \end{aligned}$$

We refer to the multiset $(F_{i,Y})_i$ as F_Y. It is clear that for each $Y < K_n$ the multiset $(f_k)_{1\le k\le n}$ induces a multiset F_Y, i.e., we have a map $\{Y < K_n\} \to \{F_Y\}$. Let $\pi \in S_n$. Now define the map $\phi_F : S_n \to \mathbb{F}_2^{n\,\mathbb{F}_2^n}$ by $\phi_F(\pi) = \prod_{i=1}^n F_{\pi(i),Y}$. The sequential dynamical system (SDS) over Y with respect to the ordering π is $\phi_F(\pi)$ which we also denote as $[F_Y, \pi]$. Introduce the equivalence relation $\sim_{(Y,F)}$ on $S_n \times S_n$ by $\pi \sim_{(Y,F)} \sigma$ iff $\phi(\pi) = \phi(\sigma)$ and let $\mathbb{S}_{(f_k)_k}(Y) = \phi_F(S_n)$. The digraph $\Gamma[F_Y, \pi]$ is the directed graph having vertex set $\mathbb{F}_2^n$ and edge set $\{(x, [F_Y, \pi](x)) \mid x \in \mathbb{F}_2^n\}$. For $\pi = (i_1, \dots, i_n)$ write $i <_\pi j$ if $i = i_k$, $j = i_l$ and $k < l$. For each graph $Y < K_n$ we define the update graph $U(Y)$ as the graph having vertex set S_n and in which two different vertices $(i_1, \dots, i_n)$ and $(h_1, \dots, h_n)$ are adjacent iff (a) $i_\ell = h_\ell$, $\ell \neq k, k+1$ and (b) $\{i_k, i_{k+1}\} \notin \mathrm{e}[Y]$. Let $\sim_Y$ be the transitive closure of the adjacency relation in $U(Y)$ and set $[\pi]_Y = \{\pi' \mid \pi' \sim_Y \pi\}$. Clearly, $\pi' \in [\pi]_Y$ implies $[F_Y, \pi] = [F_Y, \pi']$.

One symmetric function that will be referred to in the following is **Nor** defined by $(x_1, \dots, x_k) \mapsto \overline{x_1 \vee \cdots \vee x_k}$. The function **Par** defined by $(x_1, \dots, x_k) \mapsto \sum_{i=1}^k x_i$ has the important property that the induced SDS is invertible independent of the underlying graph Y.

2. Combinatorial results

In [2] the emphasis is put on counting the number of different SDS, that is, determining the number of equivalence classes of $\sim_{(Y,F)}$. In [3] it is shown that there is a bijection

$$f(Y,) : [S_n / \sim_Y] \longrightarrow \mathrm{Acyc}(Y), \tag{2.1}$$

where $\mathrm{Acyc}(Y)$ denotes the set of all acyclic orientations of Y. Thus $a(Y) = |\mathrm{Acyc}(Y)| \geq |\phi_F(S_n)|$. Moreover, in [5] it is shown that for any graph there is always a set of local symmetric functions for which this upper bound is achieved. It is interesting to note that this result is not limited to SDS, but applies just as well to non-symmetric local functions. Thus the question of determining the best possible bound for the number of functionally different systems that can be obtained by varying the update order is completely answered.

The bijection in (2.1) gives rise to a map

$$^{\#} : \mathrm{Acyc}(Y) \to S_n, \quad \mathfrak{O} \mapsto \mathfrak{O}^{\#}, \tag{2.2}$$

that assigns to an acyclic orientation its canonical permutation, see [3].

Example 3. Example: Circ_4. There are 14 different acyclic orientations of this graph and consequently the maximum number of SDS that can be obtained by varying the update order is 14. Note that there are $4! = 24$ possible update orderings in this case.

Example 4. Consider again the graph W_n defined in example 1. Let $a(Y)$ denote the number of acyclic orientations of the graph Y. For the computation of $a(W_n)$ we make use of the recursion relation $a(Y) = a(Y') + a(Y'')$ where Y' is the graph obtained from Y by deleting the edge e, say, and Y'' is the graph obtained from Y by contracting the edge e. Let W_n' be the graph obtained from W_n by deleting the edge $\{1, n\}$. Applying the recursion relation then gives

$$a(W_n) = a(W_n') + a(W_n).$$

It is easily seen that $a(W_n') = 2 \cdot 3^{n-1}$ and by solving the recursion relation we obtain $a(W_n) = 3^n - 3$. Thus for the graph W_n with fixed local maps one can at most obtain $3^n - 3$ different systems by changing the update ordering. This number should be compared to number of different orderings which is $n!$.

For applications the concept of dynamically equivalent SDS is of importance. In, e.g., a modeling setting one wants to prescribe systems with a given number of orbits and orbit sizes and possibly also a given transient behaviour.

Clearly, two systems can differ as functions but even so they may have the same dynamical properties. To make this more precise let $[F_Y, \pi]$ and $[F_Y, \pi']$ be two SDS. If there exists a bijection $\varphi : \mathbb{F}_2^n \to \mathbb{F}_2^n$ such that $\Gamma[F_Y, \pi'] = \varphi \circ [F_Y, \pi'] \circ \varphi^{-1}$ we will say that $[F_Y, \pi]$ and $[F_Y, \pi']$ are dynamically equivalent SDS. Note that φ can also be regarded as an isomorphism of the corresponding digraphs. For classical dynamical systems the definition of dynamically equivalent systems coincides with the concept of topologically conjugate systems when φ is a homeomorphism.

An object important in some of the questions to be raised later is the set

$$\Sigma_Y[F_Y, \pi] = \{\mathfrak{O} \in \mathrm{Acyc}(Y) \mid \Gamma[F_Y, \mathfrak{O}^{\#}] \cong \Gamma[F_Y, \pi]\} . \tag{2.3}$$

In the following we will simply write $\mathbb{S}(Y)$ for $\mathbb{S}_{(f_k)_k}(Y)$.

Proposition 1. *Let $Y < K_n$ and define the S_n-action on $\mathbb{F}_2^n$ by $\rho(x) = (x_{\rho(1)}, \ldots, x_{\rho(n)})$.*

1. *The map $\mathrm{Aut}(Y) \times S_n/\sim_Y \longrightarrow S_n/\sim_Y$ defined by $(\gamma, [\pi]_Y) \mapsto [\gamma \circ \pi]_Y$ is an $\mathrm{Aut}(Y)$-action on $S_n/\sim_Y$. This action induces an $\mathrm{Aut}(Y)$-action on $\mathrm{Acyc}(Y)$ given by $\{\gamma\, \mathfrak{O}\}(\{i, k\}) = \mathfrak{O}(\{\gamma^{-1}(i), \gamma^{-1}(k)\})$.*
2. *For $\pi \in S_n$ and $\gamma \in \mathrm{Aut}(Y)$ we have $[F_Y, \gamma\pi] = \gamma \circ [F_Y, \pi] \circ \gamma^{-1}$.*
3. *The map $\mathrm{Aut}(Y) \times \mathbb{S}(Y) \longrightarrow \mathbb{S}(Y)$ given by $(\gamma, [F_Y, \pi]) \mapsto [F_Y, \gamma \circ \pi]$ is an $\mathrm{Aut}(Y)$-action on $\mathbb{S}(Y)$ with the property $[F_Y, \gamma \circ \pi] = \gamma \circ [F_Y, \pi] \circ \gamma^{-1}$. In particular $\Sigma_Y[F_Y, \pi]$ is an $\mathrm{Aut}(Y)$-set.*

For the proof of this proposition we refer to [5].

As a consequence of this proposition we derive the following bound for the number of nonequivalent SDS.

Corollary 1. *Let $Y < K_n$. We have*

$$|\{\Gamma[F_Y, \pi] \mid \pi \in S_n\}| \leq \frac{1}{|\mathrm{Aut}(Y)|} \sum_{\gamma \in \mathrm{Aut}(Y)} |\mathrm{Fix}(\gamma)| , \tag{2.4}$$

where $\mathrm{Fix}(\gamma) = \{\mathfrak{O} \in \mathrm{Acyc}(Y) \mid \gamma \circ \mathfrak{O} = \mathfrak{O}\}$.

The corollary is essentially a consequence of Burnside's theorem and the details can be found in [5]. In the following we will set $\Delta(Y) = \sum_{\gamma \in \mathrm{Aut}(Y)} |\mathrm{Fix}(\gamma)| / |\mathrm{Aut}(Y)|$. In order to efficiently calculate the bound $\Delta(Y)$ we may proceed as follows. Let G be a group and let Y be an undirected graph. We will denote Y-automorphisms by γ. The group G is said to act on Y if there exists a group homomorphism $u : G \longrightarrow \mathrm{Aut}(Y)$.

Definition 1. Assume G acts on $Y < K_n$. Then $G \setminus Y$ is the graph with

$$\mathrm{v}[G \setminus Y] = \{G(i) \mid i \in \mathrm{v}[Y]\}, \quad \mathrm{e}[[G \setminus Y] = \{G(y) \mid y \in \mathrm{e}[Y]\}$$

and π_G is the surjective graph morphism given by

$$\pi_G : Y \longrightarrow G \setminus Y, \quad i \mapsto G(i) .$$

Proposition 2. *Let $Y < K_n$ be an undirected graph . Then we have*

$$\Delta(Y) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} |a(\langle \gamma \rangle \setminus Y)| . \tag{2.5}$$

Let $(0 \otimes Y)$ denote the vertex join of 0 and Y. The map π_G has the property

$$\pi_G(0 \otimes Y) = 0 \otimes \pi_G(Y). \tag{2.6}$$

The proof can be found in [4]

Some words of caution are in order. To begin, take $\gamma \in \mathrm{Aut}(Y)$ and write it as a product of disjoint cycles where cycles of length 1 are also included, say $\gamma = c_1 \cdot c_2 \cdots c_k$. The vertices in $G \setminus Y$ are (in a 1-1 correspondence with) the cycles $c_1, c_2, \dots, c_k$. However, $G \setminus Y$ is in general not a simple graph as it may contain loops.

There are two main factors making the computations of $a(\langle\gamma\rangle \setminus Y)$ relatively simple. The graph $\langle\gamma\rangle \setminus Y$ is typically of a nature well suited for computing its number of acyclic orientations. The procedure is also simplified the fact that if $\langle\gamma\rangle \setminus Y$ has loops every orientation is necessarily cyclic. The graph $\langle\gamma\rangle \setminus Y$ also has fewer vertices than Y when $\gamma \neq 1$.

Example 5. To illustrate the concepts and theory above let us consider the cube $\mathfrak{Q}_2^3$ as shown in figure 1. Note that cycle notation is used throughout the entire example. We have $\mathrm{Aut}(\mathfrak{Q}_2^3) \cong \mathbb{Z}_2^3 \rtimes S_3$. Here we will simply use the fact

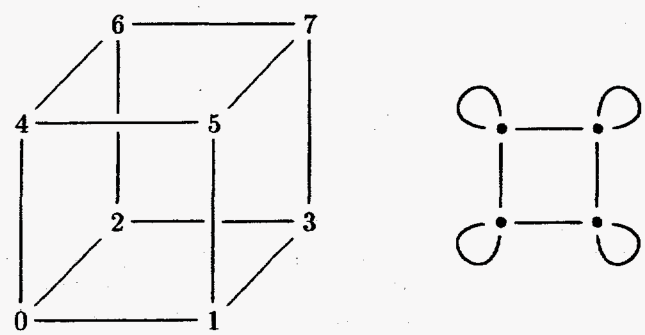


FIGURE 1. The graphs $\mathfrak{Q}_2^3$ and $\langle(04)(15)(26)(37)\rangle \setminus \mathfrak{Q}_2^3$.

that the automorphism group is generated by, e.g., $\rho = (0132)(4576)$, $\mu = (0)(124)(365)(7)$ and $\tau = (04)(15)(26)(37)$. The graph $\langle\tau\rangle \setminus \mathfrak{Q}_2^3$ is shown to the right in figure 1. The only automorphisms contributing to $\Delta(Y)$ are given in figure 2 along with their reduced graphs. Since we have $a(\mathfrak{Q}_2^3) = 1862$ we get $\Delta(\mathfrak{Q}_2^3) = 54$. Thus there are at most

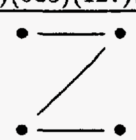
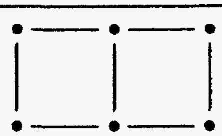
Rotations	Reflection	Rotation+ Reflection	Identity, reflection, implosion
(0)(124)(365)(7)			(0)(1)(2)(3)(4)(5)(6)(7)
(0)(142)(356)(7)	(0)(2)(5)(7)(14)(36)	(0365)(1274)	
(1)(053)(247)(6)	(1)(3)(4)(6)(05)(27)	(0356)(1743)	(03)(12)(47)(56)
(1)(035)(274)(6)	(0)(1)(6)(7)(24)(35)	(0653)(1247)	(06)(17)(24)(35)
(2)(063)(147)(5)	(2)(3)(4)(5)(06)(17)	(0563)(1472)	(05)(14)(27)(36)
(2)(036)(174)(5)	(0)(3)(4)(7)(12)(56)	(0635)(1427)	
(3)(056)(172)(4)	(1)(2)(5)(6)(03)(47)	(0536)(1724)	(07)(16)(25)(34)
(3)(065)(127)(4)			
			$\mathfrak{Q}_2^3$, Circ_4, K_4

FIGURE 2. The elements of $\mathrm{Aut}(\mathfrak{Q}_2^3)$ with their reduced graphs.

54 nonequivalent SDS on the cube. Explicit computations for the function **Nor** shows that this bound is sharp.

In the following we show how one we can take full advantage of the results above when we apply it to families of graphs and vertex joins of such. As usual denote by Circ_n the graph with vertex set $\{1,2,\dots,n\}$ and edge set $\{\{i,i+1\} \mid 1 \le i \le n-1\} \cup \{\{1,n\}\}$. Set $W_n = \mathrm{Circ}_n \otimes 0$ and $P_n = W_n \otimes (n+1)$. Define Q_n by $\mathrm{v}[Q_n] = \mathrm{v}[P_n]$ and $\mathrm{e}[Q_n] = \mathrm{e}[P_n] \setminus \{\{0, n+1\}\}$.

Proposition 3. *Let $n > 2$. Then we have*

$$\Delta(\mathrm{Circ}_n) = \frac{1}{2n}\sum_{d|n}\phi(d)\big[(2^{n/d}-2\big] + (1+(-1)^n)\,\frac{2^{n/2}}{8}, \tag{2.7}$$

$$\Delta(W_n) = \frac{1}{2n}\sum_{d|n}\phi(d)\big[(3^{n/d}-3)\big] + (1+(-1)^n)\,\frac{2\cdot 3^{n/2}}{8}, \quad n \neq 3 \tag{2.8}$$

$$\Delta(P_n) = \Delta(Q_n) = \frac{1}{4n}\sum_{d|n}\phi(d)\big[2\cdot(4^{n/d}-4)\big] + (1+(-1)^n)\,\frac{3\cdot 4^{n/2}}{8}, \quad n \neq 4. \tag{2.9}$$

Here ϕ is the Euler ϕ-function.

Proof. Let Circ_n^i be the graph obtained from Circ_n by deleting the edge $\{i,i+1\}$. Similarly define W_n^i, P_n^i and Q_n^i as the graphs obtained from W_n, P_n and Q_n respectively by deleting the edge $\{i,i+1\}$. Straightforward calculations show that

$$\begin{array}{llll} a(\mathrm{Circ}_n^i) = 2^{n-1}, & a(W_n^i) = 2\cdot 3^{n-1}, & a(P_n^i) = 6\cdot 4^{n-1}, & a(Q_n^i) = a(P_n^i) - a(W_n^i), \\ a(\mathrm{Circ}_n) = 2^n - 2, & a(W_n) = 3^n - 3, & a(P_n) = 2\cdot(4^n - 4), & a(Q_n) = a(P_n) - a(W_n). \end{array}$$

Consider the graph Circ_n. Clearly $\mathrm{Aut}(\mathrm{Circ}_n) \cong D_n$, i.e., the dihedral group on $2n$ elements. Now $D_n = \langle\tau\rangle \rtimes \langle\sigma\rangle$ where, using cycle notation, $\sigma = (1,2,\dots,n)$ and $\tau = \prod_{i=2}^{\lceil n/2\rceil}(i, n-i+2)$.

i) If $o(\sigma^k) = n$ then $\langle\sigma^k\rangle \setminus \mathrm{Circ}_n$ contains loops and consequently $|\,\mathrm{Fix}(\sigma^k)| = 0$. Here $o(\ )$ denotes order.
ii) If $o(\sigma^k) = n/2$ then $\langle\sigma^k\rangle \setminus \mathrm{Circ}_n$ is a graph with two vertices connected by an edge and we obtain $|\,\mathrm{Fix}(\sigma^k)| = 2 = 2^{\frac{n}{n/2}} - 2$.
iii) In the case where σ^k has order $\frac{n}{d}$, $d > 2$ we have $\langle\sigma^k\rangle \setminus \mathrm{Circ}_n \cong \mathrm{Circ}_{n/d}$ and thus $|\,\mathrm{Fix}(\sigma^k)| = 2^{n/d} - 2$.
iv) Finally it is seen that the only case in which $\langle\tau\sigma^k\rangle \setminus \mathrm{Circ}_n$ does not contain loops is when $n, k \equiv 0 \mod 2$, and in this case $\langle\tau\sigma^k\rangle \setminus \mathrm{Circ}_n \cong \mathrm{Circ}_{n/2+1}^i$ and $|\,\mathrm{Fix}(\tau\sigma^k)| = 2^{n/2}$ for all k.
Thus

$$\begin{aligned} \Delta(Y) &= \frac{1}{2n}\Big(\sum_k |\,\mathrm{Fix}(\sigma^k)| + \sum_k |\,\mathrm{Fix}(\tau\sigma^k)|\Big) \\ &= \frac{1}{2n}\Big(\sum_{d|n}\phi(d)\big[2^{n/d}-2\big] + \frac{1}{2}(1+(-1)^n)\cdot\frac{n}{2}\cdot 2^{n/2}\Big). \end{aligned}$$

Now consider W_n. Clearly we also have $\mathrm{Aut}(W_n) \cong D_n$. The calculation of $\Delta(W_n)$ now follows effortlessly from what we did above. To be specific:
i) By the same argument we have $|\,\mathrm{Fix}(\sigma^k)| = 0$ whenever σ^k has order n.
ii) Since $\pi_G(0\otimes Y) = 0\otimes G(Y)$ we have $\langle\sigma^k\rangle \setminus W_n \cong \mathrm{Circ}_3$ when σ^k has order $n/2$ and thus $|\,\mathrm{Fix}(\sigma^k)| = 6 = 3^{\frac{n}{n/2}} - 3$.
iii) When $o(\sigma^k) = n/d$, $d > 2$ we have obtain $\langle\sigma^k\rangle \setminus W_n \cong W_{n/d}$ and we get $|\,\mathrm{Fix}(\sigma^k)| = 3^{n/d} - 3$.

iv) Using the property of π_G again we obtain $\langle\tau\sigma^k\rangle \setminus W_n \cong W^i_{n/2+1}$ when $n, k \equiv 0 \mod 2$ and consequently $|\operatorname{Fix}(\tau\sigma^k)| = 2 \cdot 3^{n/2}$. Adding up produces the given formula.

The graphs P_n and Q_n can be dealt with similarly. The only difference is that now we have $\operatorname{Aut}(P_n) = \operatorname{Aut}(Q_n) = \langle\delta\rangle \rtimes (\langle\tau\rangle \rtimes \langle\sigma\rangle)$ where $\delta = (0, n+1)$. $\square$

An important question is the following: Are there graphs $Y < K_n$ such that no matter what the choice of the local symmetric functions are it turns out that the $\Delta(Y)$ is not the best possible bound? A closely related issue is that of determining if $\Sigma_Y[\mathbf{Nor}, \pi] = \{\mathfrak{O} \in \operatorname{Acyc}(Y) \mid \mathfrak{O} \in \operatorname{Aut}(Y)(f(Y,\pi))\}$. A positive answer to the latter question will of course imply an affirmative answer to the former. Although this is hard to determine we are able to show it for some graph classes, or more precisely:

Let Star_n, be the graph with vertex set $\{1, 2, 3, \ldots, n\}$ and edge set $\{\{1, k\} \mid 2 \le k \le n\}$.

Proposition 4.

$$\Sigma_{\mathrm{Star}_n}[\mathbf{Nor}, \pi] = \{\mathfrak{O} \in \operatorname{Acyc}(\mathrm{Star}_n) \mid \mathfrak{O} \in \operatorname{Aut}(\mathrm{Star}_n)(f(\mathrm{Star}_n, \pi))\} = \{[\pi]\}.$$

Proof. It is shown in [5] that $h_{\mathbf{Nor}} : S_n/\sim_{\mathrm{Star}_n} \to \mathbb{S}(Y)$ is a bijective $\operatorname{Aut}(\mathrm{Star}_n)$-map and thus it suffices to consider the $\operatorname{Aut}(\mathrm{Star}_n)$-action defined in Proposition 1. For $j = 1, \ldots, n$ we select $\pi_j = (i_1, \ldots, i_n) \in S_n$ such that $i_j = 1$. It follows immediately from $\operatorname{Aut}(\mathrm{Star}_n) \cong S_{n-1}$ that

$$S_n/\sim_{\mathrm{Star}_n} = \bigcup_{j=1}^{n} \operatorname{Aut}(\mathrm{Star}_n)([\pi_j]_{\mathrm{Star}_n}) \,.$$

It remains to prove that the SDS $[\mathbf{Nor}, \pi_i]$, $i = 1, \ldots, n$ exhibit pairwise non-isomorphic digraphs $\Gamma[\mathbf{Nor}, \pi_i]$. Let $\pi \in S_n$ be a permutation with $\pi(i) = 1$. Set $x = (x_{\pi(1)}, \ldots, x_{\pi(i-1)})$ and $y = (x_{\pi(i+1)}, \ldots, x_{\pi(n+1)})$. If $i \neq 1, n$ one obtains the following orbits in phase space: (underline denotes vectors.)

(2.10) $$(x1y) \longmapsto (\underline{0}0\bar{y}) \rightleftarrows (\underline{1}0y), \qquad y \neq \underline{0},$$

(2.11) $$(x1\underline{0}) \longmapsto (\underline{0}0\underline{1}) \longmapsto (\underline{1}0\underline{0}) \longmapsto (\underline{0}1\underline{0}) \longmapsto (\underline{0}0\underline{1}), \qquad x \neq \underline{0}$$

(2.12) $$(x0y) \rightleftarrows (\bar{x}0\bar{y}) \qquad x \neq \underline{0}, \underline{1},$$

In the case $i = 1$ one obtains

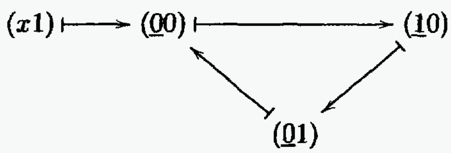
(2.13) $$(1y) \longmapsto (0\bar{y}) \rightleftarrows (0y) \longleftarrow (1\bar{y}), \qquad y \neq \underline{0}, \underline{1}.$$

(2.14) $$(1\underline{1}) \longmapsto (0\underline{0}) \longmapsto (1\underline{0}) \longmapsto (0\underline{1}) \longmapsto (0\underline{0}),$$

In the case $i = n$ one has

(2.15) $$(x0) \rightleftarrows (\bar{x}0), \qquad x \neq \underline{0}, \underline{1}$$

(2.16)



$x \neq \underline{0}$.

It is clear from the above diagrams that for any i the associated digraph has a unique component containing a 3-cycle and on this cycle there is a unique element v_i with indegree$(v_i) > 1$. In the first case indegree$(v_i) = 2^{i-1}$, in the second case indegree$(v_i) = 2$ and in the third case indegree$(v_i) = 2^{n-1}$. The only case in which these numbers are not all different is for $i = 2$. But in this case one can use, e.g., the structure in (2.13) to distinguish the corresponding digraphs. It follows that if $i \neq j$ the corresponding digraphs are not isomorphic. □

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