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Title: CONTINUED FRACTIONS: YET ANOTHER TOOL TO OVERCOME THE CURSE OF DIMENSIONALITY

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Author(s): Andrew Zardecki

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CONTINUED FRACTIONS: YET ANOTHER TOOL TO OVERCOME THE CURSE OF DIMENSIONALITY

ANDREW ZARDECKI
Los Alamos National Laboratory, MS E541, Los Alamos, NM 87545, USA
E-Mail: azz@lanl.gov

We provide a rapid prediction method, in which a larger number of antecedents than currently considered is accounted for. To this end, we encode the successive (possibly rescaled) values of a time series, as the partial quotients of a continued fraction, resulting in a number from the unit interval. The accuracy of a ruled-based system utilizing this coding is investigated to some extent. Qualitative criteria for the applicability of the algorithm are formulated.

1 Introduction

Fuzzy logic control is an effective approach to utilizing linguistic rules, whereas neural control is suited for using numerical data pairs. Fuzzy basis functions, which are algebraic superpositions of fuzzy membership functions, can combine both numerical data and linguistic information. In parallel with neural net numerical techniques, an increasing effort has been devoted to rule-based forecasting by employing fuzzy logic controllers. Wang and Mendel[1] developed a general method to generate fuzzy rules from numerical data and used their method for time series prediction. Subsequently, Mendel and coworkers also represented fuzzy systems as series expansions of fuzzy basis functions (FBF), obtained as algebraic superpositions of fuzzy membership functions.[2] The FBF method avoids the combinatorial explosion problem associated with fuzzy logic systems having a large number of antecedents in the rule base.[3] The prize one needs to pay, though, are long running times needed for the algorithm to converge. An alternative fuzzy rule configuration that avoids the combinatorial explosion has recently been advanced by Combs and Andrews.[4]

The objective of this paper is to provide a rapid prediction method, in which a larger number of antecedents than currently considered is accounted for. To this end, we encode the successive (possibly rescaled) values of a time series, as the partial quotients of a continued fraction. (We recall that a set of natural numbers, called partial quotients, determines a simple continued fraction whose value belongs to the unit interval; conversely, given the value of a continued fraction, its partial quotients are readily recovered.) Within the 64 bit representation of the double precision numbers, we can thus encode up to 40 antecedents in the continued fraction form. When this representation is processed by a standard fuzzy logic controller, one obtains the rule base including the historical data for each term of the time series. One can speak about the dressed rules, in which not only the values, but also their history, is encapsulated. We study to some extent the accuracy of the continued fraction representation. We also investigate the different decoding schemes that lead to different forecasting accuracy. In the simplest case, when the continued fraction is expressed in terms of its first partial quotient, a considerable increase in the forecasting power, as compared to the standard fuzzy controller-based technique, is achieved. This method can be viewed as a data compression technique for the time series. In the context of nuclear safeguards, we consider it a refinement of the anomaly detection algorithm proposed earlier.[5]

2 Continued Fractions

Let x be a real number from the interval (0,1). An expression of the form

$$x = \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \dots}}} \tag{1}$$

is called a (simple) continued fraction[6] representation of x; for reasons of technical convenience, Eq. 1 is often written as

$$x = [q_1, q_2, q_3, \dots] \ . \tag{2}$$

The numbers q_1, q_2, q_3, ... are called the partial quotients, whereas the successive approximations of x, obtained by retaining an increasing number of partial quotients are referred to as complete quotients or convergents.

For our purpose, the most important results of the theory of continued fractions can be expressed as two theorems.[7]

Theorem 1. To every $x \in (0, 1)$, there corresponds a unique continued fraction with value equal to x. This fraction is finite if x is rational and infinite if x is irrational.

Theorem 2. Let us agree to call a rational fraction a/b ($b > 0$) a best approximation of a real number x if every other rational fraction with the same or smaller denominator differs from x by a greater amount. Then every best approximation of x is a convergent or an intermediate fraction of the continued fraction representing that number.

The one-to-one correspondence between x and the partial quotients of the continued fraction representing X is readily obtained. In fact, given a number x in the unit interval, we can write $x = (1/(q_1 + x'))$, so that q_1 is the integer part $[1/x]$ of $1/x$, and x' is its fractional part $\{1/x\}$. If we define the functions

$$T(x) = \{1/x\} \ , \tag{3}$$

and

$$q(x) = [1/x] \ , \tag{4}$$

then $q_n(x) = q(T^{n-1}(x))$, are just the partial quotients of the continued fraction expansion of x. Conversely, given the partial quotients, q_1, q_2, q_3, ..., the number x represented by them can be recovered through the successive approximations $x = A_n/B_n$, where, for $n \geq 2$, the A_n and B_n are given recursively as

$$\begin{aligned} A_n &= q_n A_{n-1} + A_{n-2} \ , \\ B_n &= q_n B_{n-1} + B_{n-2} \ . \end{aligned} \tag{5}$$

The initial values, for $n = 0$, are $A_0 = 1$, $B_0 = 0$; for $n = 1$, we have $A_1 = 0$, $B_1 = 1$.

3 Time Series Coding

We encode a mapping X: $Z \rightarrow \Re$ from integers to real numbers representing a time series by viewing each time series element X_i as a partial fraction in the continued fraction expansion.[a] For reasons of accuracy, we employ a moving window covering about 40 elements of the time series. If X_i are restricted to interval (0,1), we scale them by a factor s ranging from 50 to 1000. Thus for $i = k_1$,..., k_{max}, we make the assignments $q_i = sX_i$, leading to the continued fraction expansion given by Eq. 1 through the correspondence

$$[X_{k_1}, X_{k_2}, \ldots, X_{k_{max}}] \rightarrow x \tag{6}$$

between an aggregate of the time series elements and its continued fraction representation. The aggregation of the time series elements can be viewed as a mechanism of data compression. An interesting application of continued fractions to cryptography is the described by Jan and Kowng.[8]

The accuracy of this assignment depends on the relative values of the partial quotients. For example, the fractional value of π has the following continued fraction expansion

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[a]. To distinguish between the elements X_i of a time series, and their continued fraction transforms, we use the lower case letters to denote the continued fractions, and the upper case letters to denote the elements of a time series.

$$\{\pi\} = [7, 15, 1, 292, 1, 1, 1, 2, 2, 3, \ldots] \; . \tag{7}$$

When $\{\pi\}$ is replaced by its decimal approximation 0.141592654, only the first four partial quotients are correctly recovered when the numerical algorithm of Sec. 2 is used. Because of the large value of q_4, the first three partial quotients lead already to a good approximation of $\{\pi\}$. On the other hand, the partial quotients of the number $(1+\sqrt{5})/2-1$ are all equal to 1; in the double precision arithmetic, their numeric assignment is correct up to q_{37}.

In traditional rule-based systems, a library of rules with n antecedents X_1, ..., X_n and output Y is constructed from input-output data pairs of the form

$$(X_1^{(m)}, X_2^{(m)}, \ldots, X_n^{(m)}; Y^{(m)}) \; , \tag{8}$$

where the index m labels the rules. Under the continued fraction encoding scheme, the rules can, similarly, have more than one antecedent. For example, with $max = 5$, we transform the first 5 elements of X_i, $i = 1, \ldots, 5$, into x_1; the elements X_i, i = 2, ..., 6, are transformed into x_2; the elements X_i, i = 3, ..., 7, are transformed into x_3, etc., as shown schematically in Fig. 1.

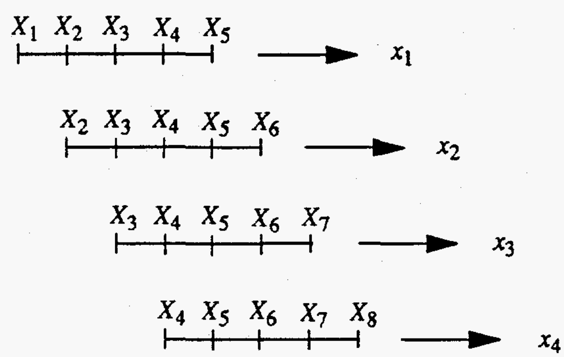

Figure 1: Aggregating the elements of a time series into continued fractions.

Once the time series has been encoded, fuzzy rules are generated from examples according the scheme of Wang and Mendel.[1] The five steps of their algorithm are well known and will not be reproduced here. In the last step, which determines a mapping from the combined fuzzy rule base, the output is generated by adopting a centroid or center of gravity defuzzification schemes. This results in the numeric output for the continued fraction transform, from which the actual, non-transformed value needs to be decoded. The two possible encoding-decoding schemes, correspond to the order in which the elements X_i are aggregated into a continued fraction. For the natural order, as exemplified by Eq. 8, the most ancient element is the dominant partial fraction, q_1; when the reverse order is used, the most recent element, $X_{k_{max}}$, is dominant.

4 Numerical Results

The Lorenz model[9] provides a well-known example of the chaotic motion; the solution to the system of differential equations

$$\begin{aligned} \dot{X} &= -a(X-Y)\,, \\ \dot{Y} &= -XZ + bX + Y\,, \\ \dot{Z} &= XY - cZ\,, \end{aligned} \tag{9}$$

in which a, b, and c are parameters, exhibits a strange attractor.

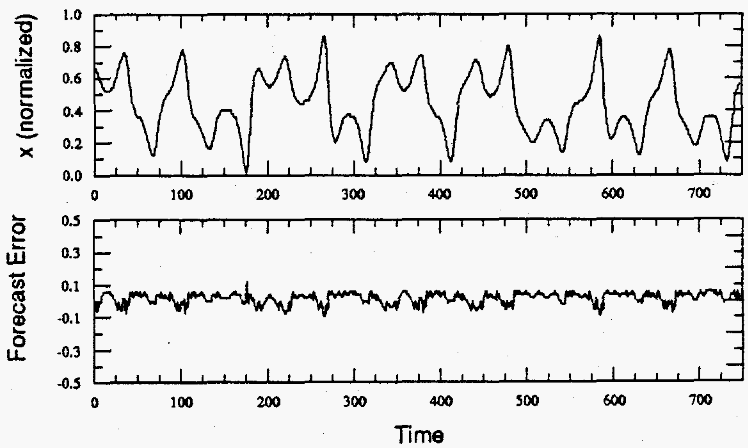

Figure 2: The normalized x component of the Lorenz attractor, together with the forecast error.

Setting $a = 10$, $b = 28$, $c = 8/3$, the computed values of the x-component of the Lorenz system is displayed in the upper part of Fig. 2, whereas the forecast error is shown in the lower part of the figure. We used the continued fraction encoding with three antecedents.

When the transformed data are used, the forecast error becomes much smaller. In Fig. 3 we show the situation after the times series is written in terms of the continuous fractions, using the algorithm of the preceding section.

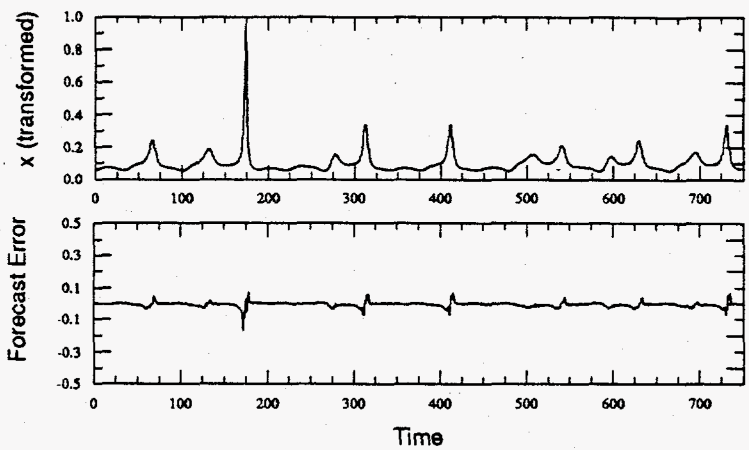

Figure 3: The continued fraction transformation of the x component of the Lorenz system, displayed in Fig. 2, and its forecast error.

As a quantitative measure of the forecasting efficiency, we can use the square root deviation per time step; with the continued fraction encoding, we gain about 10% in the efficiency as compared to the standard fuzzy controller.

For highly chaotic time series, exhibiting large oscillations between neighboring values, the rule system fails to capture the time evolution to a satisfactory degree of accuracy. For example, using the water flow data of Kasabov,[10] we observe sizable forecast errors when the water flow rate changes abruptly, Fig. 4. In the continued fractions interpretation, this is due to differences in the dominant partial fraction of the time series encoding.

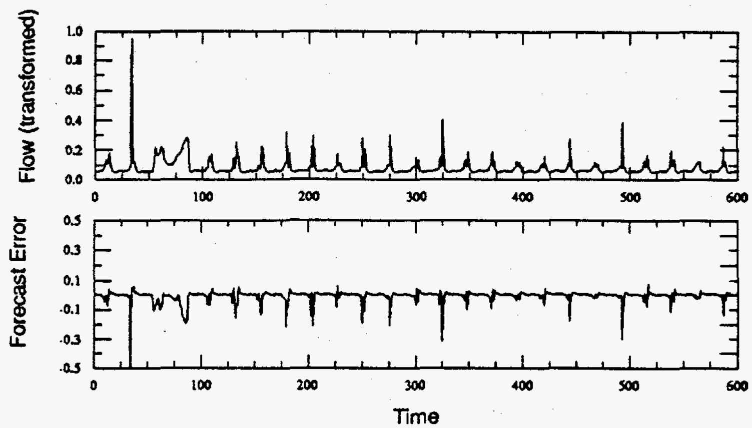

Figure 4: The transformed time series and its forecast error for the water flow data of Ref. 10.

5 Conclusions

To account for the past information contained in a time series and, simultaneously, to avoid the combinatorial explosion in the rule library, we have encoded the overlapping measurement windows into continued fractions through a one-to-one transformation. The algorithm potentially captures the history of a large number of past events, leading to quantitatively better prediction results than the simple fuzzy controller. When the time series exhibits rapid oscillations, the algorithm is less successful. Future research will explore the applicability of this encoding scheme to radial basis functions approach;[11] we will also study the optimization of parameters through genetic algorithms.

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