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ELLIPTIC INTEGRALS: SYMMETRY AND SYMBOLIC INTEGRATION

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Abstract. Computation of elliptic integrals, whether numerical or symbolic, has been aided by the contributions of Italian mathematicians. Tricomi had a strong interest in iterative algorithms for computing elliptic integrals and other special functions, and his writings on elliptic functions and elliptic integrals have taught these subjects to many modern readers (including the author). The theory of elliptic integrals began with Fagnano's duplication theorem, a generalization of which is now used iteratively for numerical computation in major software libraries. One of Lauricella's multivariate hypergeometric functions has been found to contain all elliptic integrals as special cases and has led to the introduction of symmetric canonical forms. These forms provide major economies in new integral tables and offer a significant advantage also for symbolic integration of elliptic integrals. Although partly expository the present paper includes some new proofs and proposes a new procedure for symbolic integration.

Key words. elliptic integral, symbolic integration, hypergeometric R-function, computer algebra, integral table
AMS(MOS) subject classifications. primary 33E05, 65D20, 41-04; secondary 33C75, 33C65

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1 Introduction

Regrettably I never had an opportunity to meet Professor Tricomi, but his chapter on elliptic functions and integrals in the Bateman Series [12, Chap. XIII] was my introduction to the subject. That chapter is a condensed version of his book [23], of which I read the German edition in 1962-63. Before vacationing in Italy during the Christmas holidays of 1971, I tried to arrange a meeting in Turin, but he was on vacation near Genoa, and his cordial reply (still in my possession) reached me too late. On several occasions he sent me reprints because of our common interest in iterative algorithms and particularly in generalizing the Schwab-Borchardt algorithm for computing an inverse circular or inverse hyperbolic function [24][25, pp. 23-36][26], an algorithm generalized also by Gatteschi [14][15] and Allasia [1][2]. One generalization, stimulated when John Todd told me about Tricomi's work, is now used for computing elliptic integrals of the first kind and consists in iterating the duplication theorem of a symmetric integral [6, § 5]. The Schwab-Borchardt algorithm iterates an unsymmetric special case of this theorem, and another special case was the starting point of the theory of elliptic integrals. Two Italian mathematicians, Fubini and Fagnano, are respectively associated with these two cases; we shall take up Fagnano's theorem first. Later in the paper, after explaining how a multivariate hypergeometric function defined by Lauricella is connected with symmetric elliptic integrals, we shall show their advantages for integral tables and symbolic integration.

2 Duplication theorem and iterative algorithms

Let s^2 be a polynomial of degree three or four in t with simple zeros. If $R(t,s)$ is a rational function of t and s containing at least one odd power of s, then

$$\int R(t, s(t))\, dt \tag{2.1}$$

is called an elliptic integral. In the early history of the calculus many familiar plane curves like the ellipse, the hyperbola, and the lemniscate were found to have arclengths represented by elliptic integrals.

Bernoulli's lemniscate, described in plane polar coordinates by

$$r^2 = \cos 2\theta\,, \tag{2.2}$$

has the shape of a figure 8 lying on its side (∞). The arc of this curve from the origin to a point in the first quadrant with radial coordinate r has length

$$\ell = \int_0^r \frac{dt}{\sqrt{1-t^4}}\,, \qquad 0 \le r \le 1\,. \tag{2.3}$$

Let this arclength be double the arclength from the origin to a point in the first quadrant with radial coordinate ρ:

$$\int_0^r \frac{dt}{\sqrt{1-t^4}} = 2\int_0^\rho \frac{dt}{\sqrt{1-t^4}}\,, \qquad 0 < \rho < r \le 1. \tag{2.4}$$

In 1718 Giulio Carlo di Fagnano (1682-1766) found r^2 as a rational function of ρ^2,

$$r^2 = \frac{4\rho^2(1-\rho^4)}{(1+\rho^4)^2}. \tag{2.5}$$

Discussions of this result and speculations about how Fagnano might have discovered it are given in [22, pp. 1-7] and [27]. Fagnano's duplication theorem was the first major discovery in the theory of elliptic integrals and was destined, by a stroke of good fortune 33 years later, to be extremely influential: it stimulated Euler to find the addition theorem for lemniscatic arcs, which generalizes Fagnano's theorem, and subsequently the addition theorem for general elliptic integrals.

To generalize Fagnano's theorem without invoking the addition theorem, we map the interval $(0, r)$ onto the positive real line by substituting $t = 1/\sqrt{u + r^{-2}}$, obtaining

$$\int_0^r \frac{dt}{\sqrt{1-t^4}} = \frac{1}{2}\int_0^\infty \frac{du}{\sqrt{(u+r^{-2}-1)(u+r^{-2})(u+r^{-2}+1)}}. \tag{2.6}$$

We shall prove a duplication theorem for a more general integral,

$$\int_0^\infty \frac{du}{\sqrt{(u+z_1)(u+z_2)(u+z_3)}}. \tag{2.7}$$

An earlier proof [6, § 5] used the properties of Jacobian elliptic functions, and an earlier version [8] of the proof given here used a change of integration variable based on prior knowledge of the desired result. We begin with an elementary lemma that is symmetric, like (2.7), in the subscripts $1, 2, 3$.

LEMMA 2.1 *Define*

$$A_i = A_i(u) = \sqrt{u + z_i}, \qquad i = 1, 2, 3, \tag{2.8}$$

$$f_i = f_i(u) = (A_i + A_j)(A_i + A_k), \qquad \{i, j, k\} = \{1, 2, 3\}, \tag{2.9}$$

where z_i is a constant and $A_i \neq 0$. Then $f_i - z_i$ is independent of i, and

$$\frac{df_i}{du} = \frac{\sqrt{f_1 f_2 f_3}}{2A_1A_2A_3}, \qquad i = 1, 2, 3. \tag{2.10}$$

PROOF. We see that

$$f_i - z_i = A_i^2 - z_i + A_iA_j + A_iA_k + A_jA_k = u + A_iA_j + A_iA_k + A_jA_k. \tag{2.11}$$

The last member is symmetric in i, j, k, and hence the first member is independent of i. Because $dA_i/du = 1/2A_i$, differentiation of (2.9) gives

$$2\frac{df_i}{du} = (A_i + A_j)\left(\frac{1}{A_i} + \frac{1}{A_k}\right) + \left(\frac{1}{A_i} + \frac{1}{A_j}\right)(A_i + A_k)$$

$$= (A_i + A_j)(A_i + A_k)\left(\frac{1}{A_iA_k} + \frac{1}{A_iA_j}\right)$$
$$= (A_i + A_j)(A_i + A_k)(A_j + A_k)\frac{1}{A_iA_jA_k}$$
$$= \frac{\sqrt{f_if_jf_k}}{A_iA_jA_k}. \quad \square$$

THEOREM 2.2 (DUPLICATION THEOREM) *Let z_1, z_2, z_3 lie in the complex plane cut along the negative real axis, at most one of them being 0, and take all square roots in the right half-plane. Define*

$$\lambda = \sqrt{z_1}\sqrt{z_2} + \sqrt{z_1}\sqrt{z_3} + \sqrt{z_2}\sqrt{z_3}. \tag{2.12}$$

Then

$$\int_0^\infty \frac{dt}{\prod_{i=1}^3 \sqrt{t+z_i}} = 2\int_0^\infty \frac{dt}{\prod_{i=1}^3 \sqrt{t+z_i+\lambda}}. \tag{2.13}$$

PROOF. Because Lemma 2.1 states that $f_i - z_i$ is independent of i, we can define $v = f_i - z_i$. By (2.11) we have

$$v(u) = u + \sqrt{u+z_1}\sqrt{u+z_2} + \sqrt{u+z_1}\sqrt{u+z_3} + \sqrt{u+z_2}\sqrt{u+z_3}, \tag{2.14}$$
$$v(0) = \lambda, \quad \frac{dv}{du} = \frac{df_i}{du}. \tag{2.15}$$

Then (2.10) implies

$$\frac{dv}{du} = \frac{1}{2}\prod_{i=1}^3 \frac{\sqrt{v+z_i}}{\sqrt{u+z_i}}, \tag{2.16}$$

and integration gives

$$\int_0^\infty \prod_{i=1}^3 (u+z_i)^{-1/2}\, du = 2\int_\lambda^\infty \prod_{i=1}^3 (v+z_i)^{-1/2}\, dv, \tag{2.17}$$

which becomes (2.13) on putting $v = t + \lambda$. Since $z_i + \lambda = (\sqrt{z_i} + \sqrt{z_j})(\sqrt{z_i} + \sqrt{z_k})$, where both factors lie in the open right half-plane, $z_i + \lambda$ lies in the plane cut along the nonpositive real axis. Thus both integrals in (2.13) are well defined. $\square$

Because of (2.6) this duplication theorem reduces to Fagnano's theorem if we put

$$(z_1, z_2, z_3) = (r^{-2} - 1, r^{-2}, r^{-2} + 1), \tag{2.18}$$
$$(z_1 + \lambda, z_2 + \lambda, z_3 + \lambda) = (\rho^{-2} - 1, \rho^{-2}, \rho^{-2} + 1). \tag{2.19}$$

The equation

$$z_2 + \lambda = (\sqrt{z_2} + \sqrt{z_1})(\sqrt{z_2} + \sqrt{z_3}) \tag{2.20}$$

becomes

$$\begin{aligned}\rho^{-2} &= (\sqrt{r^{-2}}+\sqrt{r^{-2}-1})(\sqrt{r^{-2}}+\sqrt{r^{-2}+1}) \\ &= \frac{\sqrt{r^{-2}}+\sqrt{r^{-2}+1}}{\sqrt{r^{-2}}-\sqrt{r^{-2}-1}} \\ &= \frac{1+\sqrt{1+r^2}}{1-\sqrt{1-r^2}}\end{aligned}$$

or

$$\rho^2 = \frac{1-\sqrt{1-r^2}}{1+\sqrt{1+r^2}}. \tag{2.21}$$

This inverse of Fagnano's relation (2.5) can be checked by noting that (2.5) implies

$$\sqrt{1\pm r^2} = \frac{1\pm 2\rho^2-\rho^4}{1+\rho^4}. \tag{2.22}$$

If two of z_1, z_2, z_3 are equal, the elliptic integral (2.7) loses its symmetry and degenerates to an inverse circular or inverse hyperbolic function,

$$\int_0^\infty \frac{du}{\sqrt{u+z_1}\,(u+z_2)} = \frac{2}{\sqrt{z_2-z_1}}\arccos\sqrt{\frac{z_1}{z_2}}, \qquad 0\le z_1<z_2, \tag{2.23}$$

$$\int_0^\infty \frac{du}{\sqrt{u+z_1}\,(u+z_2)} = \frac{2}{\sqrt{z_1-z_2}}\operatorname{arccosh}\sqrt{\frac{z_1}{z_2}}, \qquad 0<z_2<z_1. \tag{2.24}$$

(Substitute $u+z_2=(z_2-z_1)/(1-t^2)$ to prove either equation.) Since $z_1-z_2 = (z_1+\lambda)-(z_2+\lambda)$, Theorem 2.2 becomes the duplication formula for the arccos and arccosh functions,

$$\arccos\sqrt{\frac{z_1}{z_2}} = 2\arccos\sqrt{\frac{z_1+\lambda}{z_2+\lambda}}, \qquad \lambda = 2\sqrt{z_1}\sqrt{z_2}+z_2, \tag{2.25}$$

and the same equation with arccos replaced by arccosh.

The duplication theorem (2.13) can be rewritten as an invariance,

$$\int_0^\infty \frac{dt}{\prod_{i=1}^3\sqrt{t+z_i}} = \int_0^\infty \frac{dt}{\prod_{i=1}^3\sqrt{t+w_i}}, \qquad w_i = \frac{z_i+\lambda}{4}, \tag{2.26}$$

if t is replaced by $4t$ on the right side of (2.13). In this form it is useful for iterative computation because

$$w_i - w_j = \frac{z_i+\lambda}{4} - \frac{z_j+\lambda}{4} = \frac{1}{4}(z_i-z_j). \tag{2.27}$$

Each iteration of the invariance reduces the separation of the variables by a factor 4, and they converge to a common limit L. Although the rate of convergence is linear, it can be accelerated to give a simple and stable method of numerical computation with an error of order 4^{-6n} after n iterations [10]. This method is now used in major software libraries. Since the proof of convergence is tedious in the complex case, we shall assume here that the variables are nonnegative.

COROLLARY 2.3 *Let x, y, z be real and nonnegative, at most one of them being 0. Let $x_0 = x$, $y_0 = y$, $z_0 = z$, and*

$$x_{n+1} = \frac{x_n + \lambda_n}{4}, \quad y_{n+1} = \frac{y_n + \lambda_n}{4}, \quad z_{n+1} = \frac{z_n + \lambda_n}{4}, \quad n = 0, 1, 2, \dots, \tag{2.28}$$

$$\lambda_n = \sqrt{x_n y_n} + \sqrt{x_n z_n} + \sqrt{y_n z_n}. \tag{2.29}$$

Then x_n, y_n, z_n have a common limit $L = L(x, y, z)$ as $n \to \infty$, and

$$\int_0^\infty \frac{dt}{\sqrt{(t+x)(t+y)(t+z)}} = \frac{2}{\sqrt{L}}. \tag{2.30}$$

PROOF. Let T_n be the smallest interval containing x_n, y_n, z_n (assumed to be not all equal). It is easy to see that $\lambda_n/3$ is an interior point of T_n, and so are $x_{n+1} = \frac{1}{4} x_n + \frac{3}{4}(\lambda_n/3)$, y_{n+1}, and z_{n+1}. Hence T_{n+1} lies in the interior of T_n and is shorter by a factor 4 because of relations like $x_{n+1} - y_{n+1} = (x_n - y_n)/4$. This implies that the sequence $\{T_n\}$ of nested intervals converges to a point L. If the integral in (2.30) is denoted by $I(x, y, z)$, then (2.26) shows that $I(x_n, y_n, z_n) = I(x_{n+1}, y_{n+1}, z_{n+1})$. It follows from the continuity of $I(x, y, z)$ that

$$I(x, y, z) = I(x_0, y_0, z_0) = \lim_{n\to\infty} I(x_n, y_n, z_n) = I(L, L, L) = \frac{2}{\sqrt{L}}. \quad \square \tag{2.31}$$

If $y = z$ Corollary 2.3 becomes the Schwab-Borchardt algorithm for iterative computation of inverse circular or inverse hyperbolic functions. For convenience of notation we return to (2.26) and put $z_1 = x^2$ and $z_2 = z_3 = y^2$, whence $\lambda = 2xy + y^2$, to obtain

$$\int_0^\infty \frac{dt}{\sqrt{t + x^2}\,(t + y^2)} = \int_0^\infty \frac{dt}{\sqrt{t + a^2}\,(t + ay)}, \qquad a = \frac{x + y}{2}. \tag{2.32}$$

where x and y lie in the open right half-plane, except that x may be 0. For an easy proof of convergence we assume these variables are real and nonnegative.

COROLLARY 2.4 (SCHWAB-BORCHARDT ALGORITHM) *Let $x_0 = x \geq 0$, $y_0 = y > 0$, and*

$$x_{n+1} = \frac{x_n + y_n}{2}, \qquad y_{n+1} = \sqrt{x_{n+1}\, y_n}, \qquad n = 0, 1, 2, \dots. \tag{2.33}$$

Then x_n and y_n approach a common limit $\psi = \psi(x, y)$ as $n \to \infty$, and

$$\int_0^\infty \frac{dt}{\sqrt{t + x^2}\,(t + y^2)} = \frac{2}{\psi}. \tag{2.34}$$

PROOF. It is easy to verify that the sequences $\{x_n\}$ and $\{y_n\}$ are monotonic and that $x_{n+1}^2 - y_{n+1}^2 = (x_n^2 - y_n^2)/4$; therefore the two sequences have a common limit, say ψ, as $n \to \infty$. If the integral in (2.34) is denoted by $I(x,y)$, then (2.32) shows that $I(x_n, y_n) = I(x_{n+1}, y_{n+1})$. It follows from the continuity of $I(x,y)$ that

$$I(x,y) = I(x_0, y_0) = \lim_{n\to\infty} I(x_n, y_n) = I(\psi,\psi) = \frac{2}{\psi}. \quad \square \tag{2.35}$$

The early history of this algorithm involves Gauss, Pfaff, Schwab (whose geometrical version [21, pp. 103-107] was the first to be published), and Borchardt [3]. See [7]and [20, Chap. 12] for more details and for an elementary proof using the duplication formula of the cos and cosh functions. The circular and hyperbolic cases are usually stated separately but are unified here by using the integral. In 1897 Guido Fubini (1879-1943), who was apparently unaware of [3], discussed this algorithm carefully while still a student in his first published paper [13]. He emphasized that $\psi(x,y)$ could be used to represent, and the algorithm to compute numerically, not only the inverse circular functions but also the inverse hyperbolic functions and the logarithm. I learned of Fubini's paper from Luigi Gatteschi, who kindly sent me a copy.

The Schwab-Borchardt algorithm has a lemniscatic twin [7, § 4], but the similarity between the recurrence relations is deceptive. The purpose of the proof given here is to show that the algorithm can be deduced from Theorem 2.2. The easier proof given in [7] uses an integral representation of a Gauss hypergeometric function, obtained by substituting $t = r(1+u)^{-1/4}$ in (2.3).

COROLLARY 2.5 (LEMNISCATE ALGORITHM) *Let* $x_0 = r^{-2} \geq 1$, $y_0 = \sqrt{r^{-4} - 1}$, *and*

$$x_{n+1} = \frac{x_n + y_n}{2}, \qquad y_{n+1} = \sqrt{x_{n+1}\, x_n}, \qquad n = 0, 1, 2, \ldots . \tag{2.36}$$

Then x_n *and* y_n *approach a common limit* $G = G(r)$ *as* $n \to \infty$, *and*

$$\int_0^r \frac{dt}{\sqrt{1-t^4}} = \frac{1}{\sqrt{G}}. \tag{2.37}$$

PROOF. In contrast with the Schwab-Borchardt case, $x_n - y_n$ alternates in sign as n increases, and two iterations are needed for invariance of an integral. Because x_{n+1} and y_{n+1} lie in the open interval with endpoints x_n and y_n, and because $x_{n+1}^2 - y_{n+1}^2 = \frac{1}{4}(y_n^2 - x_n^2)$, x_n and y_n have a common limit G as $n \to \infty$. For $n = 0, 1, 2, \ldots$, let

$$X_n = x_n, \quad Y_n = x_n + \sqrt{x_n^2 - y_n^2}, \quad Z_n = x_n - \sqrt{x_n^2 - y_n^2}, \tag{2.38}$$

These three quantities also have the common limit G as $n \to \infty$. From

$$x_{n+2}^2 - y_{n+2}^2 = \frac{y_{n+1}^2 - x_{n+1}^2}{4} = \frac{x_n^2 - y_n^2}{16}, \tag{2.39}$$

we see that

$$Y_{n+2} - X_{n+2} = \frac{Y_n - X_n}{4}, \qquad X_{n+2} - Z_{n+2} = \frac{X_n - Z_n}{4}. \tag{2.40}$$

Because

$$x_n = X_n = \frac{Y_n + Z_n}{2}, \qquad y_n = \sqrt{Y_n Z_n}, \tag{2.41}$$

we find

$$x_{n+1} = \frac{x_n + y_n}{2} = \left(\frac{\sqrt{Y_n} + \sqrt{Z_n}}{2}\right)^2, \tag{2.42}$$

$$y_{n+1} = \sqrt{x_{n+1} x_n} = \frac{\sqrt{X_n Y_n} + \sqrt{X_n Z_n}}{2}. \tag{2.43}$$

Thus we get

$$\begin{aligned} 4x_{n+2} &= 2x_{n+1} + 2y_{n+1} \\ &= x_n + y_n + \sqrt{X_n Y_n} + \sqrt{X_n Z_n} \\ &= X_n + \sqrt{Y_n Z_n} + \sqrt{X_n Y_n} + \sqrt{X_n Z_n}, \end{aligned} \tag{2.44}$$

which is the same as

$$X_{n+2} = \frac{X_n + \lambda_n}{4}, \quad \lambda_n = \sqrt{X_n Y_n} + \sqrt{X_n Z_n} + \sqrt{Y_n Z_n}. \tag{2.45}$$

Combining this with (2.40), we have also

$$Y_{n+2} = \frac{Y_n + \lambda_n}{4}, \qquad Z_{n+2} = \frac{Z_n + \lambda_n}{4}. \tag{2.46}$$

It follows from the invariance (2.26) that the integral

$$J_n = \int_0^\infty \frac{dt}{\sqrt{(t + X_n)(t + Y_n)(t + Z_n)}} \tag{2.47}$$

satisfies $J_n = J_{n+2}$. Because $X_0 = r^{-2}$, $Y_0 = r^{-2} + 1$, $Z_0 = r^{-2} - 1$ by (2.38), we see finally from (2.6) that

$$2\int_0^r \frac{dt}{\sqrt{1 - t^4}} = J_0 = \lim_{m \to \infty} J_{2m} = \int_0^\infty \frac{dt}{(t + G)^{3/2}} = \frac{2}{\sqrt{G}}. \quad \square \tag{2.48}$$

3 Symmetric reduction from quartic to cubic

An elementary lemma similar to Lemma 2.1 will be used to replace a quartic polynomial by a cubic polynomial in an important integrand without losing symmetry in the zeros of the quartic.

LEMMA 3.1 *Define*

$$A_i = A_i(u) = \sqrt{u + z_i}\,, \qquad i = 1,2,3,4\,, \tag{3.1}$$

$$f_j = f_j(u) = (A_1 A_j + A_k A_\ell)^2, \qquad \{j,k,\ell\} = \{2,3,4\}\,, \tag{3.2}$$

where z_i is a constant and $A_i \neq 0$. Then $f_j(u) - f_j(0)$ is independent of j, and

$$\frac{df_j}{du} = \frac{\sqrt{f_2 f_3 f_4}}{A_1 A_2 A_3 A_4}\,, \qquad j = 2,3,4\,. \tag{3.3}$$

PROOF. We see that

$$\begin{aligned} f_j(u) - f_j(0) &= (u + z_1)(u + z_j) + (u + z_k)(u + z_l) + 2A_1 A_j A_k A_\ell \\ &\quad - z_1 z_j - z_k z_\ell - 2\sqrt{z_1 z_j z_k z_\ell} \\ &= 2u^2 + u(z_1 + z_j + z_k + z_\ell) + 2A_1 A_j A_k A_\ell - 2\sqrt{z_1 z_j z_k z_\ell}\,. \end{aligned}$$

The last member is symmetric in j,k,ℓ and hence the first member is independent of j. Because $dA_i/du = 1/2A_i$, differentiation of (3.2) gives

$$\begin{aligned} \frac{df_j}{du} &= 2(A_1 A_j + A_k A_\ell)\frac{d}{du}(A_1 A_j + A_k A_\ell) \\ &= \sqrt{f_j}\left(\frac{A_1}{A_j} + \frac{A_j}{A_1} + \frac{A_k}{A_\ell} + \frac{A_\ell}{A_k}\right) \\ &= \sqrt{f_j}\,(A_1^2 A_k A_\ell + A_j^2 A_k A_\ell + A_k^2 A_1 A_j + A_\ell^2 A_1 A_j)/A_1 A_j A_k A_\ell \\ &= \sqrt{f_j}\,(A_1 A_k + A_j A_\ell)(A_1 A_\ell + A_j A_k)/A_1 A_j A_k A_\ell \\ &= \sqrt{f_j f_k f_\ell}\,/A_1 A_j A_k A_\ell\,. \quad \square \end{aligned}$$

THEOREM 3.2 (REDUCTION THEOREM) *Let z_1, z_2, z_3, z_4 lie in the complex plane cut along the negative real axis, at most one of them being 0, and take all square roots in the right half-plane. Define*

$$w_j = \sqrt{z_1}\sqrt{z_j} + \sqrt{z_k}\sqrt{z_\ell}\,, \qquad \{j,k,\ell\} = \{2,3,4\}, \tag{3.4}$$

and assume w_2, w_3, w_4 all lie in the open right half-plane, except that at most one of them may be 0. (A sufficient but not necessary condition is that all z_i lie in the open right half-plane.) Then

$$w_j^2 - w_k^2 = (z_1 - z_\ell)(z_j - z_k) \tag{3.5}$$

and

$$\int_0^\infty \frac{du}{\prod_{i=1}^4 \sqrt{u + z_i}} = \int_0^\infty \frac{dv}{\prod_{j=2}^4 \sqrt{v + w_j^2}}\,. \tag{3.6}$$

PROOF. Writing $w_j^2 = z_1 z_j + z_k z_\ell + 2\sqrt{z_1 z_j z_k z_\ell}$ and subtracting the same equation with j and k interchanged proves (3.5). We define $v(u) = f_j(u) - f_j(0)$, which is independent of j by Lemma 3.1. Since $f_j(0) = w_j^2$ we have $f_j = v + w_j^2$. Then (3.3) implies

$$\frac{dv}{du} = \frac{\prod_{j=2}^4 \sqrt{v + w_j^2}}{\prod_{j=1}^4 \sqrt{u + z_i}}. \tag{3.7}$$

Since $v(0) = 0$ and $v(\infty) = \infty$, integration gives (3.6). □

The cubic polynomial in the right side of (3.6) is symmetric in $z_1, \ldots, z_4$. We shall apply this theorem to an integral with any interval of integration, provided of course that the open interval does not contain a branch point of the integrand. We first map the interval of integration onto the positive real line.

LEMMA 3.3 Let x and y be real, and for $1 \leq i \leq 4$ assume the line segment with endpoints $a_i + b_i x$ and $a_i + b_i y$ lies in the complex plane cut along the nonpositive real axis. Define

$$s(t) = \prod_{i=1}^4 \sqrt{a_i + b_i t}, \qquad z_i = \frac{a_i + b_i y}{a_i + b_i x}, \tag{3.8}$$

and take all square roots in the right half-plane. Then

$$\int_y^x \frac{dt}{s(t)} = \frac{x-y}{s(x)} \int_0^\infty \frac{du}{\prod_{i=1}^4 \sqrt{u + z_i}}. \tag{3.9}$$

PROOF. Since the assumptions imply that $|\text{ph}(z_i)| < \pi$, both integrals are well defined. Substitute $t = (xu + y)/(u + 1)$, whence

$$u = \frac{t-y}{x-t}, \quad \frac{dt}{du} = \frac{x-y}{(u+1)^2}, \quad a_1 + b_i t = \frac{(a_i + b_i x)(u + z_i)}{u+1}. \quad \square \tag{3.10}$$

The following theorem is important for integral tables and symbolic integration because it reduces a general elliptic integral of the first kind to the same form no matter where the interval of integration is located.

THEOREM 3.4 *For $1 \leq i \leq 4$ let a_i and b_i be real numbers, define $d_{ij} = a_i b_j - a_j b_i$, and assume $d_{ij} \neq 0$ if $i \neq j$. Let x and y be real numbers with $x > y$, define*

$$X_i = \sqrt{a_i + b_i x}, \qquad Y_i = \sqrt{a_i + b_i y}, \qquad 1 \leq i \leq 4, \tag{3.11}$$

and assume that all the X_i and Y_i are real and nonnegative. Define

$$U_{1j} = \frac{X_1 X_j Y_k Y_\ell + Y_1 Y_j X_k X_\ell}{x - y}, \qquad \{j, k, \ell\} = \{2, 3, 4\}. \tag{3.12}$$

Then

$$U_{1j}^2 - U_{1k}^2 = d_{1\ell}\, d_{jk} \tag{3.13}$$

and

$$\int_y^x \frac{dt}{\prod_{i=1}^4 \sqrt{a_i + b_i t}} = \int_0^\infty \frac{dt}{\prod_{j=2}^4 \sqrt{t + U_{1j}^2}}. \tag{3.14}$$

Proof. Because (3.13) ensures that at most one of the U_{1j} is 0, we may assume that all the X_i and Y_i are strictly positive, for otherwise (3.14) remains valid by continuity of the integrals. Let

$$z_i = \frac{Y_i^2}{X_i^2}, \qquad w_j = \sqrt{z_1}\,\sqrt{z_j} + \sqrt{z_k}\,\sqrt{z_\ell} = \frac{Y_1 Y_j}{X_1 X_j} + \frac{Y_k Y_\ell}{X_k X_\ell}. \tag{3.15}$$

Then

$$z_j - z_k = \frac{Y_j^2 X_k^2 - X_j^2 Y_k^2}{X_j^2 X_k^2} = \frac{(x-y) d_{jk}}{X_j^2 X_k^2}, \tag{3.16}$$

and (3.5) becomes

$$w_j^2 - w_k^2 = g^2 d_{i\ell}\, d_{jk}, \qquad g = \frac{x-y}{\prod_{i=1}^4 X_i}. \tag{3.17}$$

Since $U_{1j} = w_j/g$, (3.13) is proved. By Lemma 3.3 and Theorem 3.2 we see that

$$\begin{aligned} \int_y^x \frac{dt}{\prod_{i=1}^4 \sqrt{a_i + b_i t}} &= g \int_0^\infty \frac{du}{\prod_{i=1}^4 \sqrt{u + z_i}} \\ &= g \int_0^\infty \frac{dv}{\prod_{j=2}^4 \sqrt{v + w_j^2}}, \end{aligned} \tag{3.18}$$

which becomes (3.14) on substituting $v = g^2 t$. □

If exactly one of x and y is infinite, the theorem holds by continuity if the right side of (3.12) is replaced by its limit. The assumptions of the theorem can be relaxed by using the analyticity of the integrals and the permanence of functional relations. It suffices that, for all i and j, the open line segment connecting $a_i + b_i x$ and $a_i + b_i y$ lies in the complex plane cut along the nonpositive real axis and that U_{1j} lies in the plane cut along the negative real axis.

By choosing $a_4 = 1$, $b_4 = 0$, and $\ell = 4$, whence $X_\ell = Y_\ell = 1$, we include the case in which the left side of (3.14) contains the square root of a cubic polynomial.

4 Symmetric canonical forms

In 1893 Giuseppe Lauricella (1867-1913) defined four hypergeometric functions of m variables [18], including a function called F_D with integral representation

$$F_D(a;\, b_1,\ldots,b_m;\, c;\, x_1,\ldots,x_m) = \frac{1}{B(a,c-a)} \int_0^1 u^{a-1}(1-u)^{c-a-1} \prod_{i=1}^{m} (1-ux_i)^{-b_i} du\,, \tag{4.1}$$

where $\operatorname{Re} c > \operatorname{Re} a > 0$ and B is the beta function. We shall see that the obvious symmetry of F_D in the subscripts $1,\ldots,m$ is part of a hidden symmetry in $m+1$ subscripts [5].

If $m = 1$, F_D is the Gauss hypergeometric function. If all three exponents in this case are half-odd integers, the right side contains the square root of a cubic polynomial and, as Gauss observed, is a complete elliptic integral. If $m = 2$, F_D is the double hypergeometric function F_1 that Appell had previously defined in 1880. During the nineteenth century attention had shifted from elliptic integrals to elliptic functions, and until 1961 no one (not even Appell, as his later coauthor Kampé de Fériet told me) noticed that F_1 could represent an incomplete elliptic integral.

An illuminating example is the integral (2.7) that has been the main subject of Sections 2 and 3. We find

$$\begin{aligned} \int_0^\infty \frac{dt}{\prod_{i=1}^{3}\sqrt{t+z_i}} &= z_3^{-1/2} \int_0^1 u^{-1/2} \prod_{i=1}^{2} (1-u+uz_i/z_3)^{-1/2} du \\ &= z_3^{-1/2} B\left(\frac{1}{2},1\right) F_1\left(\frac{1}{2};\frac{1}{2},\frac{1}{2};\frac{3}{2};1-\frac{z_1}{z_3},1-\frac{z_2}{z_3}\right), \end{aligned} \tag{4.2}$$

where we have substituted

$$t = z_3\frac{1-u}{u}, \qquad u = \frac{z_3}{t+z_3}, \qquad -\frac{dt}{du} = \frac{z_3}{u^2} = \frac{(t+z_3)^2}{z_3}. \tag{4.3}$$

This shows not only that F_1 can represent an incomplete elliptic integral but also that the F_1 notation hides the symmetry in the subscripts $1,2,3$ that is obvious on the left side. Permutations of the subscripts induce transformations of F_1 that Appell discovered by another method.

To extend these conclusions to F_D, which is needed for incomplete elliptic integrals of the third kind, we apply the substitution (4.3) in reverse (u in terms of t with 3 replaced by n) to find

$$\begin{aligned} z_n^{-a} F_D\left(a;\, b_1,\ldots,b_{n-1};\, c;\, 1-\frac{z_1}{z_n},\ldots,1-\frac{z_{n-1}}{z_n}\right) \\ = \frac{1}{B(a,c-a)} \int_0^\infty t^{c-a-1} \prod_{i=1}^{n} (t+z_i)^{-b_i}\, dt \\ = R_{-a}(b_1,\ldots,b_n;\, z_1,\ldots,z_n)\,, \end{aligned} \tag{4.4}$$

where the parameter c in F_D has been replaced in R_{-a} by $b_n = c - \sum_{i=1}^{n-1} b_i$, whence

$$c = \sum_{i=1}^{n} b_i\,. \tag{4.5}$$

The function defined by the second equality in (4.4) is called the hypergeometric R-function and is symmetric in the subscripts $1, \ldots, n$, as we see from the integral. Whenever n is exchanged with a different subscript, a transformation of F_D is induced because of its hidden symmetry. The first member of (4.4) shows that R_{-a} is homogeneous of degree $-a$ in $z_1, \ldots, z_n$ and that $R_{-a} = 1$ if $z_1 = \ldots = z_n = 1$.

If the b's are all equal, R_{-a} is symmetric in the z's, as in

$$R_{1-n/2}\left(\frac{1}{2}, \ldots, \frac{1}{2}; z_1, \ldots, z_n\right) = \frac{n-2}{2} \int_0^\infty \frac{dt}{\prod_{i=1}^n \sqrt{t+z_i}}, \qquad n \geq 3. \tag{4.6}$$

The case with $n = 3$ will be denoted for brevity by

$$R_F(z_1, z_2, z_3) = \frac{1}{2} \int_0^\infty \frac{dt}{\prod_{i=1}^3 \sqrt{t+z_i}}, \tag{4.7}$$

an elliptic integral of the first kind. Theorem 2.2 is its duplication theorem, and it can be computed numerically by Corollary 2.3. If $z_3 = z_2$ it degenerates to an elementary integral denoted by

$$R_C(z_1, z_2) = \frac{1}{2} \int_0^\infty \frac{dt}{\sqrt{t+z_1}\,(t+z_2)} \tag{4.8}$$

and computable by Corollary 2.3 or 2.4. Any inverse circular function or inverse hyperbolic function or logarithm can be written in terms of $R_C(x^2, y^2)$, which is the reciprocal of the function that Fubini [13] denoted by $\psi(x, y)$.

The case of (4.6) with $n = 4$ reduces to R_F by Theorem 3.2.

The case with $n = 5$ is a hyperelliptic integral, but it degenerates if $z_5 = z_4$ to an elliptic integral of the third kind,

$$R_J(z_1, z_2, z_3, z_4) = \frac{3}{2} \int_0^\infty \frac{dt}{\prod_{i=1}^3 \sqrt{t+z_i}\,(t+z_4)}. \tag{4.9}$$

Further degeneration with $z_4 = z_3$ gives an elliptic integral of the second kind that is symmetric only in z_1 and z_2,

$$R_D(z_1, z_2, z_3) = \frac{3}{2} \int_0^\infty \frac{dt}{\prod_{i=1}^2 \sqrt{t+z_i}\,(t+z_3)^{3/2}}. \tag{4.10}$$

The duplication theorem for R_J is similar to that for R_F but contains also a term in R_C that reduces to an algebraic term when R_J reduces to R_D. Iterative computational algorithms are given in [10].

Any elliptic integral can be expressed in terms of Legendre's three canonical forms,

$$F(\phi, k) = \int_0^\phi \frac{d\theta}{\Delta}, \quad E(\phi, k) = \int_0^\phi \Delta\, d\theta, \quad \Pi(\phi, k, \nu) = \int_0^\phi \frac{d\theta}{\Delta\,(1 + \nu \sin^2\theta)}, \tag{4.11}$$

where $\Delta = \sqrt{1 - k^2 \sin^2\theta}$. Each has five transformations induced by permuting three unseen variables. Substituting $\sin^2\theta = (z-x)/(t+z)$, we find

$$\frac{1}{\sqrt{z-x}} F\left(\arcsin\sqrt{\frac{z-x}{z}}, \sqrt{\frac{z-y}{z-x}}\right) = R_F(x, y, z), \tag{4.12}$$

which is symmetric in x, y, z. Interchange of x and y, for example, induces the reciprocal-modulus transformation,

$$F(\phi, k) = \frac{1}{k} F(\psi, 1/k), \quad \sin\psi = k \sin\phi. \tag{4.13}$$

Interchange of real x and z leads to imaginary ϕ and k. We can dispense with these transformations by using R_F. Legendre's E and Π have corresponding transformations that involve an extra term in F.

Conversion from Legendre's integrals to symmetric integrals is accomplished by the relations

$$F(\phi, k) = R_F(c-1,\, c-k^2,\, c), \qquad c = \csc^2\phi, \tag{4.14}$$

$$F(\phi, k) - E(\phi, k) = \frac{k^2}{3} R_D(c-1,\, c-k^2,\, c), \tag{4.15}$$

$$F(\phi, k) - \Pi(\phi, k, \nu) = \frac{\nu}{3} R_J(c-1,\, c-k^2,\, c,\, c+\nu). \tag{4.16}$$

The integrals are called complete if $\phi = \pi/2$; then $c = 1$ and one variable of each R-function is 0.

In integral tables or symbolic integration one cannot use (4.14) to express the right side of (3.14) in terms of $F(\phi, k)$ (with ϕ and k in the customary ranges $0 \le \phi \le \pi/2$ and $0 \le k \le 1$) without first specifying enough information to determine the relative sizes of U_{12}, U_{13}, U_{14}. With a similar situation for the other integrals, this accounts in part (along with ignorance of (3.14)) for the burdensome number of cases listed in integral tables and creates a serious drawback to the use of Legendre's integrals in symbolic integration.

5 Symbolic integration: the first stage

In 1976 Ng and Polanjar [19] discussed the difficulties in four approaches to symbolic integration of elliptic integrals; one approach used R-functions but had troubles with multiparameter recurrence relations. Labahn and Mutrie [17], in a paper not yet published, improve the classical method of reduction but retain Legendre's canonical forms.

We consider elliptic integrals of the form

$$I(m) = \int_y^x \prod_{i=1}^{h} (a_i + b_i t)^{-1/2} \cdot \prod_{j=1}^{n} (a_j + b_j t)^{m_j}\, dt, \tag{5.1}$$

where $h = 3$ or 4, x and y are real with $x > y$ (at most one of them may be infinite), the a's and b's are real or complex, the b's are nonzero, and $m = (m_1, \ldots, m_n)$ is an n-tuple of integers. We assume that the integral is well defined (in particular that the open interval of integration contains no finite branch point of the integrand) and that no two linear factors are proportional (that is, $d_{ij} = a_i b_j - a_j b_i \neq 0$ if $i \neq j$). We do not assume any other information about x, y, the a's, or the b's, not even qualitative information in the form of inequalities. In contrast, conventional integral tables such as [4] and [16] require inequalities relating to the branch points of the integrand and the interval of integration in order to express the integral in terms of Legendre's integrals with $0 < \phi \leq \pi/2$ and $0 \leq k \leq 1$.

However, we do assume that the integers m_j are known. For nearly all integrals listed in tables, the m_j are obvious, including those with $1 \leq j \leq h$, but in general one needs to find out whether any two polynomials in an integrand have one or more zeros in common, and numerical information may be needed to determine this. Except for finding the m_j, symbolic factorization of polynomials suffices.

The rational part of the integrand of (5.1) can be decomposed explicitly [11] into partial fractions:

$$\begin{aligned}\prod_{j=1}^{n}(a_j+b_j\,t)^{m_j} &= B\,b_i^{-M}\sum_{q=0}^{M}C_{M-q}(i)\,(a_i+b_i\,t)^q \\ &+ \sum_{j=1}^{n}D_j\,b_j^{m_j-M}\sum_{q=1}^{-m_j}C_{m_j+q}(j)\,(a_j+b_j\,t)^{-q}\,, \end{aligned} \tag{5.2}$$

where the polynomial part is independent of the choice of i, and each sum over q is empty if the upper limit is less than the lower limit. Various quantities are defined by

$$M=\sum_{j=1}^{n}m_j\,, \qquad B=\prod_{j=1}^{n}b_j^{m_j}, \qquad d_{jk}=a_jb_k-a_kb_j\,, \qquad D_j=\prod_{\substack{k=1\\k\neq j}}^{n}d_{kj}^{m_k}\,, \tag{5.3}$$

$$\mu_{\pm s}(j)=\frac{-1}{s}\sum_{\substack{k=1\\k\neq j}}^{n}m_k\left(\frac{d_{jk}}{b_k}\right)^{\pm s}, \qquad s=1,2,3,\ldots, \tag{5.4}$$

$$C_0(j)=1, \qquad C_{\pm s}(j)=\sum\frac{\mu_{\pm1}^{\alpha_1}(j)\cdots\mu_{\pm s}^{\alpha_s}(j)}{\alpha_1!\cdots\alpha_s!}\,, \qquad s=1,2,3,\ldots, \tag{5.5}$$

where upper (lower) signs go together and the last sum extends over all nonnegative integers $\alpha_1,\ldots,\alpha_s$ such that $\alpha_1+2\alpha_2+\ldots+s\alpha_s=s$. In particular we have

$$\begin{aligned} C_{\pm1}(j) &= \mu_{\pm1}(j)\,, \qquad C_{\pm2}(j)=\frac{1}{2}\,\mu_{\pm1}^2(j)+\mu_{\pm2}(j)\,, \\ C_{\pm3}(j) &= \frac{1}{6}\,\mu_{\pm1}^3(j)+\mu_{\pm1}(j)\,\mu_{\pm2}(j)+\mu_{\pm3}(j)\,. \\ C_{\pm4}(j) &= \frac{1}{24}\,\mu_{\pm1}^4(j)+\frac{1}{2}\,\mu_{\pm1}^2(j)\mu_{\pm2}(j)+\mu_{\pm1}(j)\mu_{\pm3}(j)+\frac{1}{2}\,\mu_{\pm2}^2(j)+\mu_{\pm4}(j)\,. \end{aligned} \tag{5.6}$$

Substitution of (5.2) in (5.1) reduces $I(m)$ to a set of simpler integrals in which at most one component of m is nonzero. Defining n-tuples

$$\begin{aligned} e_0 &= (0,0,\ldots,0), \\ e_1 &= (1,0,\ldots,0), \\ &\vdots \\ e_n &= (0,\ldots,0,1), \end{aligned} \tag{5.7}$$

we see that

$$I(m) = B\, b_i^{-M} \sum_{q=0}^{M} C_{M-q}(i)\, I(qe_i) + \sum_{j=1}^{n} D_j\, b_j^{m_j - M} \sum_{q=1}^{-m_j} C_{m_j+q}(j)\, I(-qe_j)\,. \tag{5.8}$$

The first term on the right side is independent of the choice of i, and each sum over q is empty if the upper limit is less than the lower limit.

Recurrence relations make it possible to express the integrals $I(\pm qe_j)$, $1 \le j \le n$, in terms of a set of basic integrals. If $h = 3$ in (5.1) these are $I(-e_j)$, $j = 0, 1, \ldots, n$, but if $h = 4$ the basic integrals include also $I(+e_j)$, $j = 1, \ldots, 4$. The recurrence relations involve algebraic terms representing the difference between the values at $t = x$ and $t = y$ of an integrand like that in (5.1):

$$A(m) = J_m(x) - J_m(y)\,, \tag{5.9}$$

$$J_m(t) = \prod_{i=1}^{h} (a_i + b_i t)^{-1/2} \prod_{j=1}^{n} (a_j + b_j t)^{m_j}\,. \tag{5.10}$$

The principal recurrence relations, proved in [11], have one less term if $1 \le j \le h$ than if $h + 1 \le j \le n$.

THEOREM 5.1 *For $1 \le j \le n$ let $E_s(j)$ be the elementary symmetric function of degree s in*

$$\frac{d_{1j}}{b_1}, \ldots, \frac{d_{hj}}{b_h}$$

and define

$$\sigma_0(j) = \sigma_0 = b_1 \cdots b_h\,, \qquad \sigma_s(j) = \sigma_0\, E_s(j)\,, \qquad 1 \le j \le n\,. \tag{5.11}$$

If $1 \le i \le h$ then

$$\sum_{r=0}^{h-1} (q + \frac{r+1}{2})\, \sigma_{h-1-r}(i)\, I((q+r)e_i) = b_i^{h-1} A(qe_i + \sum_{j=1}^{h} e_j)\,. \tag{5.12}$$

If $h + 1 \le \alpha \le n$ then

$$\sum_{r=0}^{h} (q + \frac{r}{2} + 1)\, \sigma_{h-r}(\alpha)\, I((q+r)e_\alpha) = b_\alpha^{h-1} A((q+1)e_\alpha + \sum_{j=1}^{h} e_j)\,. \tag{5.13}$$

If we choose $i \le h$ in the first term on the right side of (5.8), then $I(qe_\alpha)$ is never needed for $q > 2$, but (5.13) with $q = -2$ involves $I(e_\alpha)$ if $h = 3$ and both $I(e_\alpha)$ and $I(2e_\alpha)$ if $h = 4$. These two integrals can be reduced by specializing (5.8) to get

$$I(qe_a) = \left(\frac{b_\alpha}{b_i}\right)^q \sum_{r=0}^{q} C_{q-r}(i)\, I(re_i), \quad q > 0, \quad i \le h\,. \tag{5.14}$$

The following table shows how 136 integrals from [16] with $h = 4$ are represented in terms of basic integrals by five formulas.

TABLE 5.2 Each entry starts with a list of those integrals in [16] that have the form $I(m) = I(\sum_{j=1}^{5} m_j e_j)$ displayed in the entry and defined in (5.1). Also displayed is the expression for $I(m)$ in terms of the basic integrals $I(e_0)$, $I(-e_5)$, and $I(\pm e_i)$, $i = 1, \ldots, 4$. Because $\{i, j, k, \ell\} = \{1, 2, 3, 4\}$, no two of i, j, k, ℓ can be equal.

[§ 3.147, 1-8]:

$$I(e_0)\,. \tag{5.15}$$

[§ 3.148, 1-8]:

$$I(e_5) = \frac{1}{b_i}\,[b_5\, I(e_i) - d_{i5}\, I(e_0)]\,. \tag{5.16}$$

[§ 3.149, 1-8; § 3.151, 1-8]:

$$I(-e_5)\,. \tag{5.17}$$

[§ 3.167, 1-32]:

$$I(e_i)\,. \tag{5.18}$$

[§ 3.168, 1-72]:

$$I(e_k - e_i) = \frac{1}{b_i}\,[d_{ki}\, I(-e_i) + b_k\, I(e_0)]\,. \tag{5.19}$$

If the basic integrals are to be expressed in terms of Legendre's canonical forms, each of 136 formulas has to be accompanied by inequalities relating the branch points of the integrand and the interval of integration. In the next Section we shall use R-functions to avoid most of this complexity.

6 Symbolic integration: the second stage

In the quartic case ($h = 4$) the simplest of the basic integrals is

$$I(e_0) = \int_y^x \frac{dt}{\prod_{i=1}^{4} \sqrt{a_i + b_i t}} = \int_0^\infty \frac{dt}{\prod_{j=2}^{4} \sqrt{t + U_{1j}^2}}\,, \tag{6.1}$$

where the second equality is (3.14). In terms of the function R_F defined by (4.7), we have

$$I(e_0) = 2\, R_F(U_{12}^2\,, U_{13}^2\,, U_{14}^2)\,, \tag{6.2}$$

where the U's are defined by (3.12).

It is convenient to replace 1 by i and rewrite (3.12) and (3.13) as

$$U_{ij} = \frac{X_i X_j Y_k Y_\ell + Y_i Y_j X_k X_\ell}{x-y}, \qquad \{i,j,k,\ell\} = \{1,2,3,4\}. \tag{6.3}$$

$$U_{ij}^2 - U_{ik}^2 = d_{i\ell}\, d_{jk}\,, \qquad d_{ij} = a_i b_j - a_j b_i\,. \tag{6.4}$$

Because $U_{ij} = U_{ji} = U_{k\ell} = U_{\ell k}$ there are only three distinct U's, and we can avoid the subscript 4 (with an eye to the case $h = 3$) by writing (6.2) as

$$I(e_0) = 2\, R_F(U_{12}^2\,, U_{13}^2\,, U_{23}^2)\,, \tag{6.5}$$

The other basic integrals are integrals of the second and third kinds. Their reduction to R_D and R_J depends on an analogue of Theorem 3.2 that reduces

$$\int_0^\infty \frac{u+z_1}{u+z_\alpha} \frac{du}{\prod_{i=1}^4 \sqrt{u+z_i}}, \qquad 2 \le \alpha \le n, \tag{6.6}$$

to a sum of two integrals expressible in terms of R_J and R_C respectively. The proof [9], too long to give here, splits the integral into two parts that are recombined by the addition theorem for R_J, which contains a term in R_C. A similar proof can be given for Theorem 3.2, but Lemma 3.1 does not seem to be directly applicable to the more complicated integral.

With $\{i;j,k,\ell\} = \{1,2,3,4\}$ and $\alpha \ge h+1 = 5$, the results for the other basic integrals are

$$d_{ji}\, I(-e_i) = \frac{2}{3} b_i d_{jk}\, d_{j\ell}\, R_D(U_{ik}^2\,, U_{jk}^2\,, U_{ij}^2) + \frac{2b_i X_j Y_j}{X_i Y_i U_{ij}} - b_j\, I(e_0)\,, \tag{6.7}$$

$$d_{i\alpha}\, I(-e_\alpha) = \frac{2b_\alpha}{3d_{i\alpha}} d_{ij}\, d_{ik}\, d_{i\ell}\, R_J(U_{12}^2\,, U_{13}^2\,, U_{23}^2\,, U_{i\alpha}^2) + 2b_\alpha\, R_C(S_{i\alpha}^2\,, Q_{i\alpha}^2) - b_i\, I(e_0)\,, \tag{6.8}$$

$$b_i\, I(e_i) = \frac{2}{3} d_{ij}\, d_{ik}\, d_{\ell i}\, R_J(U_{12}^2\,, U_{13}^2\,, U_{23}^2\,, U_{i0}^2) + 2b_i\, R_C(S_{i0}^2\,, Q_{i0}^2)\,, \tag{6.9}$$

where $5 \le \alpha \le n$ and

$$U_{i\alpha}^2 = U_{ij}^2 - \frac{d_{ik}\, d_{i\ell}\, d_{j\alpha}}{d_{i\alpha}}, \qquad Q_{i\alpha} = \frac{X_\alpha Y_\alpha}{X_i Y_i} U_{i\alpha}\,, \qquad S_{i\alpha}^2 - Q_{i\alpha}^2 = \frac{d_{j\alpha}\, d_{k\alpha}\, d_{\ell\alpha}}{d_{i\alpha}}, \tag{6.10}$$

$$S_{i\alpha} = (X_i^{-1} X_j X_k X_\ell Y_\alpha^2 + Y_i^{-1} Y_j Y_k Y_\ell X_\alpha^2)/(x-y)\,, \tag{6.11}$$

The corresponding formulas for U_{i0}, S_{i0}, Q_{i0} are obtained from (6.10) and (6.11) by putting $a_\alpha = 1$ and $b_\alpha = 0$, whence $X_\alpha = Y_\alpha = 1$ and $d_{i\alpha} = -b_i$.

If $U_{ij} = 0$ the first two terms on the right side of (6.7) are infinite, but (6.4) ensures that at most one of the three U's can be 0. Given i we can choose j so that $U_{ij} \neq 0$.

If $U_{i\alpha}^2$ is real and negative in (6.8), then R_J is a Cauchy principal value. This happens for at most two of the four choices for i and so can be avoided. The corresponding problem in (6.9) can be avoided by using the identity

$$b_j\, I(e_i) = b_i\, I(e_j) + d_{ij}\, I(e_0)\,. \tag{6.12}$$

The preceding equations remain valid in the cubic case if we put $\{i, j, k\} = \{1, 2, 3\}$ and $m_4 = 0$, retaining $5 \le \alpha \le n$. Also, we put $\ell = 4$, $a_4 = 1$, and $b_4 = 0$, whence $X_\ell = Y_\ell = 1$ and $d_{i\ell} = -b_i$. We recall that $I(e_i)$ is not needed in the cubic case.

The basic integrals have now been expressed in terms of R-functions, showing that symbolic integration of (5.1) is feasible without imposing inequalities on the parameters in the integrand. The number of R-functions in the result is independent of whether a branch point of the integrand is an endpoint of the interval of integration. If numerical computation is to be performed, there are efficient algorithms [10] for the R-functions even if their variables are complex, and Cauchy principal values of R_J can be avoided.

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