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Relabeling Symmetries in Hydrodynamics and Magnetohydrodynamics

NIKHIL PADHYE and P.J. MORRISON
Department of Physics and Institute for Fusion Studies
The University of Texas at Austin
Austin, Texas 78712 USA

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Nikhil Padhye and P. J. Morrison
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Abstract

Lagrangian symmetries and concomitant generalized Bianchi identities associated with the relabeling of fluid elements are found for hydrodynamics and magnetohydrodynamics (MHD). In hydrodynamics relabeling results in Ertel's theorem of conservation of potential vorticity, while in MHD it yields the conservation of cross helicity. The symmetries of the reduction from Lagrangian (material) to Eulerian variables are used to construct the Casimir invariants of the Hamiltonian formalism.

1 Introduction

It is well known that one can find continuous symmetries of the Lagrangian for a physical system that lead to conservation laws according to Noether's (first) theorem. Many familiar physical systems have actions that are invariant under infinitesimal space-time translations, space rotations, and Galilean boosts, elements of the ten-parameter Lie group called the Galilei group. These symmetries lead to the conservation laws of energy, linear and angular momenta, and uniform motion of the center-of-mass. One also has the possibility, especially in field theories, that the action is invariant under infinitesimal transformations of an *infinite* continuous group parametrized by *arbitrary functions*. For such symmetries one has generalized Bianchi identities (Noether's second theorem) in addition to the usual statement of Noether's first theorem. In this letter we explore the consequences of both Noether's theorems for the ideal compressible fluid.

After presenting Noether's first and second theorems in the next section, we find a symmetry of an infinite continuous group for the ideal, compressible fluid Lagrangian in Sec. 3 and for MHD in Sec. 5. These symmetries give rise to Ertel's theorem [1] for the fluid case and the conservation of cross helicity for MHD. The first discussion of such symmetries seems to have been made in [2], where they were called exchange symmetries. Since both, [2] and [3] connect Lagrangian symmetries to Kelvin's circulation theorem, we point out that the circulation theorem can be derived from Ertel's theorem. More recently, Ertel's theorem has been connected to fluid element relabeling [4, 5], however our treatment is more general than [4] and differs from [5].

The symmetries here involve only a continuous transformation of the fluid element labels, hence we follow [5] in naming them "relabeling symmetries." Conservation of cross helicity in MHD has previously been linked to Lagrangian symmetries [6], but not to fluid element relabeling. In Sec. 3 it is also shown that the potential energy functional obtained by expanding about a stationary equilibrium possesses a Bianchi identity and relates to spontaneous symmetry breaking, which gives rise to null eigenfunctions.

In Sec. 4 we are concerned with the Hamiltonian framework and show that the map from Lagrangian variables (which are synonymously called material variables) to Eulerian variables for a fluid has the same relabeling symmetry. This symmetry is then used to directly construct the Casimir invariants for the noncanonical Poisson bracket [7, 8] for the fluid in Eulerian form. This rounds out the usual picture of reduction from Lagrangian to Eulerian variables (see e.g. [8, 9]). Later we do the same for MHD in Sec. 5, which results in the familiar cross helicity invariant for barotropic flows. Other symmetries of the reduction from material to Eulerian variables give rise to Casimir invariants too, including a family of *invariants* which incorporates magnetic helicity as a special case.

2 Noether's theorems

Here we briefly outline the derivation of Noether's first and second theorems [10, 11].

The action for a classical field theory may be written as

$$S[q] := \int_{\mathcal{D}} \mathcal{L}(q, \partial q, x)\, d^n x\,, \tag{1}$$

where $\mathcal{L}$ is the Lagrangian density and $q(x) = (q^1, q^2, \cdots, q^m)$ are the fields which depend on the variables $x = (x^0, x^1, \cdots, x^n)$ — x^0 may be regarded as the time variable — and ∂q denotes the derivatives of the fields with respect to the variables.

Under point transformations,

$$\hat{x}^i = \hat{x}^i(x)\,, \quad \hat{q}^i = \hat{q}^i(q, x)\,, \tag{2}$$

the action transforms to

$$\hat{S}[\hat{q}] := \int_{\hat{\mathcal{D}}} \hat{\mathcal{L}}(\hat{q}, \hat{\partial}\hat{q}, \hat{x})\, d^n\hat{x} = S[q]\,, \tag{3}$$

where the second equality expresses covariance of the action and implies that the Lagrangian density must transform as

$$\hat{\mathcal{L}}(\hat{q}, \hat{\partial}\hat{q}, \hat{x}) = \frac{\partial(x)}{\partial(\hat{x})}\mathcal{L}(q, \partial q, x)\,, \tag{4}$$

where $\partial(x)/\partial(\hat{x})$ stands for the Jacobian of the transformation. Furthermore we seek transformations that leave the form of the Euler-Lagrange equations invariant, i.e. we seek (a subset of) symmetry transformations. Evidently, for such transformations $\hat{S}[\hat{q}] = S[\hat{q}]$, which implies

$$\hat{\mathcal{L}}(\hat{q}, \hat{\partial}\hat{q}, \hat{x}) - \mathcal{L}(\hat{q}, \hat{\partial}\hat{q}, \hat{x}) = \partial_i \Lambda^i \,, \tag{5}$$

where Λ is a vector with zero flux across the boundary of $\hat{\mathcal{D}}$. (Repeated indices are summed throughout this paper.) Such transformations, for which the Lagrangian density differs at most by a divergence, are called invariant transformations. In particular, if $\partial_0 \Lambda^0 = 0$, i.e. the divergence is only spatial, the Lagrangian is invariant and if the divergence term is altogether absent, the Lagrangian *density* is invariant.

We now consider invariant point transformations that have the following infinitesimal form:

$$\hat{x}^i = x^i + \delta x^i(x) \,, \qquad \hat{q}^j(\hat{x}) = q^j(x) + \Delta q^j(q, x) \,. \tag{6}$$

Derivatives of the fields change accordingly:

$$\Delta(\partial_j q^i) = \partial_j(\Delta q^i) - (\partial_j \, \delta x^k)(\partial_k q^i) \,, \tag{7}$$

where $\Delta(\partial_j q^i)$ is defined to be the first order piece of $\hat{\partial}_j \hat{q}^i - \partial_j q^i$. Finite transformations can be constructed by iteration of such infinitesimal ones. Up to first order, δx^j and Δq^i may be considered functions of either the new or the old variables and the Jacobian may be written as

$$\partial(\hat{x})/\partial(x) = 1 + \partial_i \, \delta x^i \,. \tag{8}$$

The differential form of Eq. (5) is thus

$$\bar{\delta}\mathcal{L} + \mathcal{L} \, \partial_i \, \delta x^i + \partial_i \, \delta \Lambda^i = 0 \,, \tag{9}$$

where $\bar{\delta}\mathcal{L}$ is defined to be the first order piece of $\mathcal{L}(\hat{q}, \hat{\partial}\hat{q}, \hat{x}) - \mathcal{L}(q, \partial q, x)$ and Λ^i is written as $\delta\Lambda^i$ to indicate that it is also of first order. For convenience we define, to first order,

$$\delta q^i(x) := \hat{q}^i(x) - q^i(x) = \Delta q^i - (\partial_j q^i)\delta x^j \,. \tag{10}$$

Thus while δq^i is the change in the field at a fixed point, Δq^i is the change relative to a transformed point. Equation (9) may now be written as

$$\partial_i \, \delta J^i + S_i \, \delta q^i = 0 \,, \tag{11}$$

where S_i's denote functional derivatives of the action with respect to q^i's, that is,

$$S_i := \frac{\delta S}{\delta q^i} = \frac{\partial \mathcal{L}}{\partial q^i} - \partial_j \frac{\partial \mathcal{L}}{\partial(\partial_j q^i)} \tag{12}$$

and the current,

$$\delta J^i := \mathcal{L} \, \delta x^i + \frac{\partial \mathcal{L}}{\partial(\partial_i q^j)} \, \delta q^j + \delta \Lambda^i \,. \tag{13}$$

We now note that when the equations of motion are satisfied, i.e. $S_i \equiv 0$, we are left with

$$\partial_i \, \delta J^i = 0 \, . \tag{14}$$

The conservation law expressed by Eq. (14) may be recognized as the usual expression of Noether's (first) theorem.

Another possibility is to integrate Eq. (11) to get

$$\int_{\mathcal{D}} S_i \, \delta q^i \, d^n x = 0 \, . \tag{15}$$

Consider, for example, transformations which have the form [12]

$$\delta x^i = \epsilon(x) \, \chi^i(x) \, , \quad \Delta q^i = \epsilon(x) \, \phi^i(x) + \partial_j \epsilon(x) \, \psi^{ij}(q, x) \, , \tag{16}$$

where $\epsilon(x)$ is an infinitesimal, arbitrary function of x. (In general there can exist a set of independent symmetries, in which case one may wish to add a subscript to the ϵ's.) For such transformations

$$\int_{\mathcal{D}} S_i \, \delta q^i \, d^n x = \int_{\mathcal{D}} \epsilon(x) \left[S_i \phi^i - S_i (\partial_j q^i) \chi^j - \partial_j (S_i \psi^{ij}) \right] \, d^n x = 0 \, , \tag{17}$$

where we have used Eq. (10) to express δq^i and integrated by parts to get rid of the derivative on ϵ. The arbitrariness of ϵ allows us to choose it so that the boundary terms disappear. And since the integral in Eq. (17) vanishes for arbitrary $\epsilon(x)$, the Dubois-Reymond lemma then implies

$$S_i \left[\phi^i - (\partial_j q^i) \chi^j \right] - \partial_j (S_i \psi^{ij}) = 0 \, . \tag{18}$$

Note that when the equations of motion are satisfied the terms $S_i \left[\phi^i - (\partial_j q^i) \chi^j \right]$ and $\partial_j (S_i \psi^{ij})$ vanish separately (and trivially); this is replaced by the weaker condition, Eq. (18), when the equations of motion are not necessarily satisfied. Equation (18), which depends crucially on $\epsilon(x)$ being an arbitrary function of x rather than a constant parameter, is an example of the identity of Noether's second theorem, also referred to as a generalized Bianchi identity. It is particularly interesting since it is satisfied independently of the equations of motion and its existence indicates that not all Euler-Lagrange equations of motion are independent. For this reason it is also called a *strong* conservation law as opposed to the *weak* conservation law expressed by Eq. (14) which requires the equations of motion.

It is also noteworthy that for such transformations, with an arbitrary $\epsilon(x)$ as in Eq. (16), the weak conservation law itself splits into more than one statement. This follows from $\epsilon(x)$ and its derivatives being independent, hence terms multiplying them must vanish independently.

3 Relabeling symmetry in hydrodynamics

We now apply the discussion of the previous section to the case of an ideal fluid Lagrangian. The variable x^0 of the previous section is replaced explicitly by time, t, and three other components of x are to be interpreted as the labels, a, of the Lagrangian fluid elements; e.g. these could be the initial positions of the fluid elements, $q(t=0)$. The variables $q(a,t)$ keep track of the position of the fluid element labeled a. At any time the mapping between q and a is an invertible mapping of a domain, D, and to simplify matters, D is assumed time independent although the fluid is compressible.

The fluid Lagrangian density, $\mathcal{L}$, may be written as [14, 13, 15]

$$\mathcal{L} = \rho_0 \left[\frac{1}{2}\dot{q}^2 - U(\rho, s) - \Phi(q)\right] , \tag{19}$$

where $\rho_0 = \rho_0(a)$ is the initial density distribution and $\dot{q}$ denotes the time derivative of q keeping the label fixed. The internal or potential energy per unit mass is denoted by U and is assumed to be a function of two thermodynamic quantities, viz. the density, ρ and the entropy, s. Additional forces on the fluid can be accounted for by including a potential, $\Phi(q)$.

In what follows the following determinant identities (see e.g. [16]) will be found of use. The cofactor, $A_i{}^j$, of the transformation matrix element, $\partial_j q^i$, can be written as $2A_i{}^j = \epsilon_{ik\ell}\, \epsilon^{jmn}\, \partial_m q^k\, \partial_n q^\ell$ and satisfies

$$A_i{}^j\, \partial_j q^k = \partial_j (A_i{}^j q^k) = \mathcal{J}\, \delta_i{}^k\, , \tag{20}$$

where $\mathcal{J}$ is the determinant of the transformation matrix, i.e. the Jacobian, $\partial(q)/\partial(a)$, of the time dependent map $q \to a$. The identity

$$\epsilon^{ijk}\, \partial_i q^\ell\, \partial_j q^m\, \partial_k q^n = \mathcal{J}\, \epsilon^{\ell mn} \tag{21}$$

is of particular use in converting from Lagrangian to Eulerian variables. The volume and surface elements transform as

$$d^3 q = \mathcal{J}\, d^3 a \quad \text{and} \quad d\sigma_i = A_i{}^j\, d\sigma_{0j}\, , \tag{22}$$

where $d\sigma_{0j}$ denotes the area element in the jth direction in label space while $d\sigma_i$ denotes the area element in the ith direction in configuration space.

We also assume adiabaticity, that is $s = s_0(a)$ only. Conservation of mass implies $\rho\, d^3q = \rho_0\, d^3a$ and hence from the first of Eqs. (22) we have

$$\rho(a,t) = \frac{\rho_0(a)}{\mathcal{J}}\, . \tag{23}$$

We now seek an infinitesimal *relabeling* transformation $\hat{a} = a + \delta a(a,t)$, $\Delta q := \hat{q}(\hat{a},t) - q(a,t) \equiv 0$ which leaves the Lagrangian density invariant. Evidently, relabeling means that each component of q transforms as a scalar. The transformed Lagrangian density can be expressed using Eq. (4) and, up to first order, leads to

$$\hat{\mathcal{L}}(\hat{q}, \hat{\partial}\hat{q}, \hat{a}) - \mathcal{L}(\hat{q}, \hat{\partial}\hat{q}, \hat{a}) = -\left[\frac{1}{2}\dot{q}^2 - U - \Phi - \frac{\rho_0}{\mathcal{J}}\frac{\partial U}{\partial \rho}\right] \nabla\cdot(\rho_0 \delta a)$$

$$+ \rho_0 \frac{\partial U}{\partial s} \delta a \cdot \nabla s_0 + \rho_0\, \dot{q}_i\, \dot{\delta a} \cdot \nabla q^i\, , \tag{24}$$

which is invariant if

$$\nabla \cdot (\rho_0 \, \delta a) = 0\,, \quad \delta a \cdot \nabla s_0 = 0 \quad \text{and} \quad \delta \dot{a} = 0\,. \tag{25}$$

These requirements assure that the relabeling does not alter the mass, lies within isentropic surfaces, and does not change the velocity field. They are met by

$$\delta a = \frac{\nabla s_0 \times \nabla \epsilon_0}{\rho_0}\,, \tag{26}$$

where $\epsilon_0 = \epsilon_0(a)$ is an infinitesimal, arbitrary function of the label alone and hence is advected.

For this symmetry, Noether's first theorem, Eq. (14), gives us

$$\frac{\partial}{\partial t}\left[\dot{q}_i \, \nabla \cdot \left(\epsilon_0 \, \nabla q^i \times \nabla s_0\right)\right] + \nabla \cdot \left[\epsilon_0 \, \nabla s_0 \times \nabla \left(\frac{\dot{q}^2}{2} - U - \Phi - \frac{p}{\rho}\right)\right] = 0\,, \tag{27}$$

where $p(\rho, s_0)$, the pressure, is defined by $\rho^2 \partial U/\partial \rho$. Since the conserved current in the above equation is not unique, we integrate Eq. (27) over the label space. The divergence term then vanishes and integration by parts allows us to isolate $\epsilon_0(a)$, giving

$$\frac{d}{dt}\int_D \epsilon_0(a) \, \nabla \dot{q}_i \cdot \nabla q^i \times \nabla s_0 \, d^3a = 0\,. \tag{28}$$

The arbitrariness of $\epsilon_0(a)$ then leads us to the material conservation law,

$$\frac{\partial}{\partial t}\left(\nabla \dot{q}_i \cdot \nabla q^i \times \nabla s_0\right) = 0\,. \tag{29}$$

Using the chain rule to convert a derivatives to q derivatives and using Eq. (21) yields the corresponding Eulerian expression,

$$\frac{d}{dt}\left(\frac{1}{\rho}\tilde{\nabla} s \cdot \tilde{\nabla} \times v\right) = 0\,. \tag{30}$$

In obtaining the above equation we have also made use of Eq. (23) and noted that $\rho_0(a)$ has no time dependence. Here the gradient operator in q space is denoted by $\tilde{\nabla}$, the velocity is $v(q,t) := \dot{q}(a(q,t),t)$, the entropy $s(q,t) := s_0(a(q,t))$, the density $\rho(q,t) := \rho(a(q,t),t)$, and d/dt denotes the Lagrangian or material derivative,

$$\frac{d}{dt} := \left.\frac{\partial}{\partial t}\right|_a = \left.\frac{\partial}{\partial t}\right|_q + v \cdot \tilde{\nabla}\,.$$

The lack of arbitrariness in time of $\epsilon_0(a)$, which arose due to the last condition of Eqs. (25) and can be traced to the kinetic energy term, prevents us from using Eq. (15) directly. (In essence this is because Hamilton's principle for particles is not parameterization invariant.)

Instead we integrate the equivalent of Eq. (11) over the label space (not time) to get the generalized Bianchi identity

$$\frac{\partial}{\partial t}\left(\nabla \dot{q}_i \cdot \nabla q^i \times \nabla s_0\right) + \nabla\left(\frac{S_i}{\rho_0}\right) \cdot \nabla q^i \times \nabla s_0 = 0\,, \tag{31}$$

It can be verified that the above equation is satisfied for *any* $q(a,t)$ by using the explicit form for S_i,

$$S_i = -\rho_0\,\ddot{q}_i - A_i{}^j\,\partial_j p - \rho_0\,\partial\Phi/\partial q^i\,. \tag{32}$$

When the equations of motion are satisfied $S_i \equiv 0$ and Eq. (31) reduces to Eq. (29), as might be expected.

Note also that Eq. (27) can be expressed in Eulerian form by substituting $\epsilon_0(a) := \epsilon\tau_0(a)$, where $\tau_0(a)$ is an arbitrary function of the label. Then we have the expression

$$\frac{d}{dt}\left(\frac{1}{\rho}\tilde{\nabla}\tau \cdot v \times \tilde{\nabla}s\right) = \frac{1}{\rho}\tilde{\nabla}\tau \cdot \tilde{\nabla}\left(\frac{v^2}{2} - \frac{\partial}{\partial\rho}(\rho U) - \Phi\right) \times \tilde{\nabla}s\,, \tag{33}$$

where $\tau(q,t) := \tau_0(a(q,t))$ is an arbitrary advected quantity. Clearly, even if such an observable, advected quantity that does not affect the potential energy exists (dye, perhaps), the above equation is not elegant as Eq. (30). In Eq. (30), the quantity, Q_s, defined by

$$Q_s := \frac{1}{\rho}\,\tilde{\nabla}s \cdot \tilde{\nabla} \times v \tag{34}$$

is called the potential vorticity associated with the advected quantity, s, and Eq. (30), which expresses the advection of Q_s, is called Ertel's theorem of conservation of potential vorticity.

The conservation of potential vorticity was derived from a (different) Lagrangian symmetry in [4] for incompressible stratified flows. In [5] conservation of potential vorticity is derived from a constrained variational principle. The transformation used has a time dependence in contrast to the symmetry used here, which must be time independent to qualify as a symmetry. In [2] and [3] relabeling symmetry is related to Kelvin's circulation theorem. The treatment in [3] expresses the symmetry in terms of q rather than a. We can easily make correspondence by using the one-to-one mapping between q and a, which implies

$$\delta q = -\delta a \cdot \nabla q = \frac{\tilde{\nabla}\epsilon \times \tilde{\nabla}s}{\rho}\,. \tag{35}$$

The use of relabeling symmetry seems to have been made first in [2] where a relabeling symmetry is found for an incompressible, ideal fluid without internal energy, U.

We now proceed to show the connection of Ertel's theorem to Kelvin's circulation theorem. Integrating Eq. (29) over a volume, V, fixed in label space and contained in the domain, D, and using Gauss' divergence theorem gives

$$\frac{d}{dt}\oint_\Sigma s_0\,\nabla q^i \times \nabla \dot{q}_i \cdot d\sigma_0 = 0\,, \tag{36}$$

where Σ is the surface enclosing V and $d\sigma_0$ is the infinitesimal surface element. Now if V is chosen to be any volume sandwiched between parts of two surfaces of constant entropy separated by a small value, δs_0, the contribution to the integral from the sides is small and one has

$$\delta s_0 \frac{d}{dt} \int_S \nabla q^i \times \nabla \dot{q}_i \cdot d\sigma_0 = 0 , \tag{37}$$

where S is a part of any surface of constant s_0. Using the second of Eqs. (22) we can thus write for non-zero δs_0

$$\frac{d}{dt} \int_{\tilde{S}} \tilde{\nabla} \times v \cdot d\sigma = 0 . \tag{38}$$

In the above equation $d\sigma$ is an infinitesimal surface element in q space and the isentropic surface, S, which was fixed in label space is now considered to be an isentropic surface $\tilde{S}$ in q space which evolves in time but is made up of the same fluid elements. Equation (38) is Kelvin's circulation theorem and is true on surfaces of constant entropy.

For a homentropic fluid, or equivalently for barotropic flows, instead of Eq. (27) we simply get

$$\frac{\partial}{\partial t}(\nabla \dot{q}_i \times \nabla q^i) = 0 , \tag{39}$$

which implies

$$\frac{dQ_\tau}{dt} := \frac{d}{dt}\left(\frac{1}{\rho}\tilde{\nabla}\tau \cdot \tilde{\nabla} \times v\right) = 0 \tag{40}$$

for *any* advected quantity, $\tau(q,t) := \tau_0(a)$. It is thus quite clear that Kelvin's circulation theorem holds on *any* material surface for barotropic flows.

In the stability analysis of stationary fluid equilibria (in particular MHD) one often considers the second variation of energy functionals. As an example consider the potential energy functional:

$$W := \int_D \rho_0 [U(\rho, s) + \Phi(q)]\, d^3a , \tag{41}$$

The equilibrium q_e is considered to be an extremal point of W and the second variation is checked for definiteness at the equilibrium. Noting that W possesses the same symmetry as expressed earlier by Eq. (26) (but without any restriction on the time dependence since, here, the integral is only over space), leads to a generalized Bianchi identity:

$$\nabla \left(\frac{1}{\rho_0}\frac{\delta W}{\delta q^i}\right) \times \nabla q^i \cdot \nabla s_0 = 0 . \tag{42}$$

The functional derivatives of W, which are set to zero to obtain the extremal point, are thus not all independent of each other.

The existence of the symmetry also relates to spontaneous symmetry breaking and Goldstone's theorem, concepts of field theory (See e.g. [17]; in the context of noncanonical Hamiltonian theory see [18].) We describe this for static equilibria, but a more general development exists. For the potential energy functional the analogue of (15) is

$$\delta_* W = \int_D \frac{\delta W}{\delta q^i} \delta_* q^i \, d^3a \equiv 0 , \tag{43}$$

where $\delta_* q$ is given by Eq. (35). Taking a second variation of (43) yields

$$\delta_*^2 W = \int_D \left(\delta_* q^i \frac{\delta^2 W[q]}{\delta q^j \delta q^i} \cdot \delta q^j - \frac{\delta W}{\delta q^i} \frac{\partial \delta q^i}{\partial a^j} \delta_* a^j \right) d^3a \equiv 0 \,, \tag{44}$$

where this second δq is arbitrary and the dot indicates that the operator on the left acts on the quantity to the right. Evaluating (44) on an equilibrium point q_e yields

$$\delta_*^2 W_e = \int_D \delta q^j \frac{\delta^2 W[q_e]}{\delta q^i \delta q^j} \cdot \delta_* q_e^i \, d^3a \equiv 0 \,. \tag{45}$$

Since (45) vanishes for arbitrary δq, it follows that

$$\frac{\delta^2 W[q_e]}{\delta q^i \delta q^j} \cdot \delta_* q_e^i \equiv 0 \,. \tag{46}$$

There are two ways to solve (46): either (i) $\delta_* q_e^i = -(\partial q_e^i / \partial a^j) \delta_* a^j = 0$, which implies that the equilibrium point has the same relabeling symmetry as W [a notably trivial case since $q_e(s_0(a))$], and no symmetry is broken, or (ii) $\delta_* q_e^i \neq 0$, which implies that $\delta^2 W[q_e]/\delta q^i \delta q^j$ has $\delta_* q_e$ as a null eigenvector, and symmetry is "spontaneously broken." Observe that $\delta_* q_e$ is a zero frequency eigenfunction of the linearized equations of motion written in Lagrangian variables.

Since relabeling is a symmetry group, it is obvious that one can make a finite displacement from the equilibrium point and remain on the same level set of W. This can be seen by iterating the above variational procedure. For example, the next variation of (44) gives

$$\delta_*^3 W = \int_D \left(\delta_* q^i \left[\frac{\delta^3 W[q]}{\delta q^k \delta q^j \delta q^i} \cdot \delta q^j \right] \cdot \delta q^k - 2 \frac{\partial \delta q^i}{\partial a^j} \delta_* a^j \frac{\delta^2 W[q]}{\delta q^k \delta q^i} \cdot \delta q^k \right) d^3a \equiv 0 \,, \tag{47}$$

which when evaluated on q_e yields

$$\delta_*^3 W_e = \int_D \delta_* q^i \left[\frac{\delta^3 W[q_e]}{\delta q^k \delta q^j \delta q^i} \cdot \delta q^j \right] \cdot \delta q^k \, d^3a \equiv 0 \,. \tag{48}$$

This procedure is analogous to Taylor expanding a potential energy function about an equilibrium of a finite system that lies in a trough. This was worked out explicitly to all orders for the special case of toroidal geometry in [19]. Although in terms of Lagrangian variables the equilibria that are connected by the relabeling transformation are distinct, it is evident by the definition of relabeling that in the Eulerian description these equilibria are identical.

4 Symmetry of the Eulerian variables

We now consider the Hamiltonian formulation of hydrodynamics (e.g. [8]). Expressed in Lagrangian variables the Hamiltonian has the form

$$H[\pi, q; a] := \int_D \mathcal{H}(\pi, q, \partial q, a) \, d^3a := \int_D \rho_0 \left[\frac{1}{2} \left(\frac{\pi}{\rho_0} \right)^2 + U(\rho_0/\mathcal{J}, s_0) + \Phi(q) \right] d^3a \,, \tag{49}$$

which together with the canonical Poisson bracket,

$$[F,G] = \int_D \left[\frac{\delta F}{\delta q} \cdot \frac{\delta G}{\delta \pi} - \frac{\delta G}{\delta q} \cdot \frac{\delta F}{\delta \pi} \right] d^3a\,, \tag{50}$$

produces the ideal fluid equations of motion. On making the transformation $\hat{a} = a + \delta a(a,t)$, $\Delta q := \hat{q}(\hat{a},t) - q(a,t) \equiv 0$ and $\Delta\pi := \hat{\pi}(\hat{a},t) - \pi(a,t) = (\delta a \cdot \nabla \rho_0)(\pi/\rho_0)$ (so that $\Delta(\pi/\rho_0) \equiv 0$) it is seen that, up to first order,

$$\hat{\mathcal{H}}(\hat{\pi}, \hat{q}, \hat{\partial}\hat{q}, \hat{a}) := \frac{\partial(a)}{\partial(\hat{a})} \mathcal{H}(\pi, q, \partial q, a) = \mathcal{H}(\hat{\pi}, \hat{q}, \hat{\partial}\hat{q}, \hat{a}) \tag{51}$$

for the same relabeling symmetry, viz. that given by Eq. (26). Thus for the same form of the Poisson bracket in the new variables, the form of the equations of motion is left unaltered under such a relabeling. The existence of this symmetry of the Hamiltonian density indicates that one may be able to obtain an alternative formulation of the dynamics in terms of variables which inherently possess this symmetry. This is indeed the case for the reduction (see e.g. [8], [9] and references therein) to Eulerian variables, which is conveniently represented by the following:

$$\rho(r,t) := \int_D \rho_0(a)\, \delta(r - q(a,t))\, d^3a\,, \tag{52}$$

$$\sigma(r,t) := \int_D \rho_0(a)\, s_0(a)\, \delta(r - q(a,t))\, d^3a\,, \tag{53}$$

$$M(r,t) := \int_D \pi(a,t)\, \delta(r - q(a,t))\, d^3a\,. \tag{54}$$

When one considers variations of the Eulerian variables ρ, σ, and M, that are *induced by relabeling*, we see that

$$\delta\rho = \int_D \nabla \cdot (\rho_0 \delta a)\, \delta(r - q(a,t))\, d^3a\,, \tag{55}$$

$$\delta\sigma = \int_D [s_0 \nabla \cdot (\rho_0 \delta a) + \rho_0\, \delta a \cdot \nabla s_0]\ \delta(r - q(a,t))\, d^3a\,, \tag{56}$$

$$\delta M = \int_D \frac{\pi}{\rho_0} \nabla \cdot (\rho_0 \delta a)\, \delta(r - q(a,t))\, d^3a\,. \tag{57}$$

The conditions for vanishing of these variations, together with the constraint, $\pi = \rho_0\, \dot{q}$, are the same as those of Eqs. (25). Thus the relabeling given by Eq. (26) is also a symmetry of the map from Lagrangian to Eulerian variables.

In the framework resulting from the reduction to Eulerian variables, we are naturally interested in functionals which can be expressed in terms of the Eulerian variables, $F[q,\pi] = \bar{F}[\rho, \sigma, M]$. Evidently, this is not possible for all $F[q,\pi]$. But note that $\bar{F}[\rho,\sigma,M]$ has the relabeling symmetry mentioned above since ρ, σ and M have it. Therefore, at the very least, one demands that $F[q, \pi]$ display the same symmetry. This consideration gives rise to a scheme for obtaining Casimir invariants, special invariants that arise in the Eulerian framework, from knowledge of the symmetry. Since the variation of F must vanish when the variations δq and $\delta\pi$ arise from the relabeling symmetry, δa, we demand

$$\delta F = \int_D \left[\frac{\delta F}{\delta q} \cdot \delta q \; + \frac{\delta F}{\delta \pi} \cdot \delta \pi \right] d^3a = 0\,. \tag{58}$$

It is clear that if there exists a functional, C, such that

$$\delta q = -\frac{\delta C}{\delta \pi} \quad \text{and} \quad \delta \pi = \frac{\delta C}{\delta q}\,, \tag{59}$$

its Poisson bracket with any F belonging to the class of functionals satisfying Eq. (58), vanishes. This will be the case when the Poisson bracket is expressed in terms of Eulerian, noncanonical variables [7] and therefore, by definition, C is a Casimir invariant. Obviously, Casimir invariants are constants of motion for any dynamics with a Hamiltonian that can be expressed in terms of Eulerian variables.

As might be expected from Eq. (28), and easily checked, the functional, C, defined by

$$C[q, \pi] := \int_D \epsilon_0(a)\, \nabla \left(\frac{\pi_i}{\rho_0} \right) \cdot \nabla q^i \times \nabla s_0 \, d^3a\,, \tag{60}$$

is the generator of the symmetry, i.e. it satisfies

$$[C, q^i] = -\frac{\delta C}{\delta \pi_i} = -\delta a \cdot \nabla q^i =: \delta q^i \tag{61}$$

and

$$[C, \pi_i] = \frac{\delta C}{\delta q^i} = \Delta \pi_i - \delta a \cdot \nabla \pi_i =: \delta \pi_i\,. \tag{62}$$

The Eulerian expression for the Casimir invariants, C, yields

$$C[\rho, s, v] = \int_D \rho\, f(Q_s)\, d^3q\,, \tag{63}$$

where f is arbitrary and $s(q,t) := \sigma(q,t)/\rho(q,t) = s_0(a(q,t))$.

Evidently, the Poisson bracket of a functional, $\bar{C}$, with *any* F also vanishes if

$$\frac{\delta \bar{C}}{\delta q} = 0 = \frac{\delta \bar{C}}{\delta \pi}\,. \tag{64}$$

This is true when the integrand of $\bar{C}$ is an arbitrary function of the labels and independent of q and π. There exists no Eulerian representation for most such $\bar{C}$'s; however

$$\bar{C}[\rho_0, s_0] := \int_D \rho_0\, \bar{f}(s_0)\, d^3a \tag{65}$$

does survive the Eulerianization, where $\bar{f}$ is arbitrary. A general expression for the Casimir invariants in Eulerian form is then given by

$$C[\rho, s, v] := \int_D \rho\, \mathcal{C}(s, Q_s)\, d^3q\,, \tag{66}$$

where $\mathcal{C}$ is an arbitrary function of both arguments.

In the noncanonical Hamiltonian formulation of the fluid, a Casimir has to satisfy the conditions:

$$\tilde{\nabla}\cdot\left(\rho\frac{\delta C}{\delta M}\right) = 0\,, \quad \frac{\delta C}{\delta M}\cdot\tilde{\nabla}\left(\frac{\sigma}{\rho}\right) = 0 \quad \text{and} \tag{67}$$

$$M_j\,\tilde{\nabla}\frac{\delta C}{\delta M_j} + \frac{\delta C}{\delta M}\cdot\tilde{\nabla}\left(\frac{M}{\rho}\right) + \rho\,\tilde{\nabla}\frac{\delta C}{\delta\rho} + \sigma\,\tilde{\nabla}\frac{\delta C}{\delta\sigma} = 0\,. \tag{68}$$

The equivalence of these conditions to the symmetry conditions, Eqs. (25), is seen when one notes that if C can be expressed as a functional of ρ, σ, and M, then

$$\frac{\delta C}{\delta\pi} = \frac{\delta C}{\delta M} \quad \text{and} \quad \frac{\delta C}{\delta q} = \mathcal{J}\left[\rho\,\tilde{\nabla}\frac{\delta C}{\delta\rho} + \sigma\,\tilde{\nabla}\frac{\delta C}{\delta\sigma} + M_i\,\tilde{\nabla}\frac{\delta C}{\delta M_i}\right]\,. \tag{69}$$

The use of Eqs. (61) and (62) then leads to Eqs. (67) and (68) when δa satisfies Eqs. (25) and vice versa. Note that for Casimirs satisfying Eq. (64), the conditions reduce down to

$$\frac{\delta\bar{C}}{\delta M} = 0 = \rho\,\tilde{\nabla}\frac{\delta\bar{C}}{\delta\rho} + \sigma\,\tilde{\nabla}\frac{\delta\bar{C}}{\delta\sigma}\,. \tag{70}$$

For barotropic flow, Eq. (40) is true for any advected τ. Therefore one can use Q_τ to generate yet another advected quantity, Q_{Q_τ} and so on; from one advected quantity we can generate an infinite family of advected quantities. Thus the Casimir has the form

$$C[\rho, \tau, v] = \int_D \rho\, f(\tau, Q_\tau, Q_{Q_\tau}, \cdots)\, d^3q\,, \tag{71}$$

where $f(\tau, Q_\tau, Q_{Q_\tau}, \cdots)$ is an arbitrary function of the arguments.

5 Relabeling symmetry in MHD

The Lagrangian density for MHD [15] is given by

$$\mathcal{L}_{MHD} = \mathcal{L} - \frac{1}{2\mathcal{J}}\,\partial_j q^i\,\partial_k q_i\,B_0^j\,B_0^k\,, \tag{72}$$

where $\mathcal{L}$ is the fluid Lagrangian density given by Eq. (19) and $B_0^i(a)$ are components of the magnetic field as a function of the labels, e.g. the initial magnetic field.

Thus the MHD counterpart to Eq. (24) has the following additional terms due to a relabeling transformation:

$$\frac{1}{\mathcal{J}}\partial_j q^i\, \partial_k q_i \left[B_0^j\, B_0^k\, \partial_\ell \delta a^\ell + B_0^j\, \delta a^\ell\, \partial_\ell B_0^k - B_0^j\, B_0^\ell\, \partial_\ell \delta a^k\right] .$$

It can be verified that the above expression vanishes if δa is any function of the labels multiplying B_0. But we also require that the conditions obtained previously, Eqs. (25), be satisfied; this leads to overspecification and consequently there is no relabeling symmetry, δa, that satisfies all the requirements. A solution can, however, be found if one eliminates the second of Eqs. (25) by considering a barotropic flow, i.e. U and hence p depend only on the density, ρ. (A solution can also be found without imposing the restriction of barotropicity in the case where the entropy, s_0, is a flux label, i.e. $B_0 \cdot \nabla s_0 = 0$.) Then one has the symmetry

$$\delta a = \epsilon(x_0, y_0)\frac{B_0}{\rho_0} , \tag{73}$$

where $x_0(a)$ and $y_0(a)$ are flux labels. In other words, the initial magnetic field is expressible as $\nabla x_0 \times \nabla y_0$. However the existence of flux labels $x_0(a)$ and $y_0(a)$ is not crucial; if they do not exist one simply thinks of ϵ as an infinitesimal constant parameter.

For this symmetry, Noether's (first) theorem gives

$$\frac{\partial}{\partial t}\left(\dot{q}_j\, B_0 \cdot \nabla q^j\right) + \nabla \cdot \left[B_0 \left(\frac{\dot{q}^2}{2} - U - \rho\frac{dU}{d\rho} - \Phi\right)\right] = 0 . \tag{74}$$

Integrating over the domain and passing over to the Eulerian form using the relation, $B_0^k\, \partial_k = \mathcal{J}\, B^i\, \tilde{\partial}_i$, we get the conservation law

$$\frac{d}{dt}C[v, B] := \frac{d}{dt}\int_D v \cdot B\, d^3q = 0 , \tag{75}$$

where $C[v, B]$ is commonly referred to as cross helicity. Prior to this work conservation of cross helicity was derived from a Lagrangian symmetry involving Clebsch potentials and the polarization in [6]. (See also [20].)

The discussion in the previous section leads us to expect the existence of Casimirs, in the Hamiltonian formulation, which satisfy Eq. (64) and which may be expressible in terms of ρ, s, v, and B. It is easily verified that $B \cdot \tilde{\nabla}\tau/\rho = B_0 \cdot \nabla\tau_0/\rho_0$, where $\tau(q, t) := \tau_0(a)$ is an arbitrary advected quantity, and leads to the Eulerian expression:

$$C[\rho, s, B] := \int_D \rho\, g\left(s, \frac{B \cdot \tilde{\nabla} s}{\rho}, \frac{B \cdot \tilde{\nabla}}{\rho}\left(\frac{B \cdot \tilde{\nabla} s}{\rho}\right), \cdots\right) d^3q , \tag{76}$$

where g is an arbitrary function of its arguments. This form for the Casimirs is given in [21]; we obtain a more general expression next.

The Lagrange-Euler map for the magnetic field,

$$B^i(r, t) := \int_D B_0^j(a)\, \frac{\partial q^i}{\partial a^j}\, \delta(r - q(a, t))\, d^3a , \tag{77}$$

and its corresponding vector potential representation,

$$A_i(r,t;A_0,q) = \int_D A_{0j}(a)\frac{\partial a^j}{\partial q^i}\delta(r-q(a,t))\,\mathcal{J}\,d^3a\,, \tag{78}$$

lead to the conclusion that $A \cdot B/\rho = A_0 \cdot B_0/\rho_0$, within a gauge restriction. We note that in Eq. (78), one may add to $A_0(a)$, the gradient of a gauge, $\phi_0(a,t)$, which leads to a corresponding gauge choice, $\phi(r,t) := \phi_0(q^{-1}(r,t),t)$, for $A(r,t)$. But for the validity of $A \cdot B/\rho = A_0 \cdot B_0/\rho_0$, we must restrict the gauge to be advected, $\phi(r,t) := \phi_0(q^{-1}(r,t))$, which is equivalent to demanding that all explicit time dependence be removed from A_0. With this choice it can be seen that the vector potential in Eulerian coordinates satisfies the equation

$$\frac{\partial A}{\partial t} = v \times B - \tilde{\nabla}(A \cdot v)\,. \tag{79}$$

(This gauge choice and the corresponding invariant is discussed in [22].) Thus, more generally, the Casimir invariants are expressed by

$$C[\rho,s,A] := \int_D \rho\, g\left(s, \frac{A\cdot B}{\rho}, \frac{B\cdot\tilde{\nabla}s}{\rho}, \frac{B\cdot\tilde{\nabla}}{\rho}\left(\frac{B\cdot\tilde{\nabla}s}{\rho}\right), \frac{B\cdot\tilde{\nabla}}{\rho}\left(\frac{A\cdot B}{\rho}\right), \cdots\right) d^3q\,, \tag{80}$$

where B is understood to be an abbreviation for $\tilde{\nabla} \times A$. Operating within the restricted choice of gauges mentioned earlier, we note that the addition of a gauge, $A \to A + \tilde{\nabla}\phi$, changes $A \cdot B/\rho$ by the term $B \cdot \tilde{\nabla}\phi/\rho$, which is also advected. The numerical value of $C[\rho,s,A]$ thus depends on the gauge, but after the initial choice of the gauge has been made, it nevertheless is a constant of the motion. It is clear that magnetic helicity, $\int A \cdot B\, d^3a$, is a special case of this family of invariants.

For the barotropic case, the Casimir is written most generally as

$$C[\rho,v,A] = \int_D \rho\left[\frac{v\cdot B}{\rho} + f\left(\frac{A\cdot B}{\rho}, \frac{B\cdot\tilde{\nabla}}{\rho}\left(\frac{A\cdot B}{\rho}\right), \cdots\right)\right] d^3q\,, \tag{81}$$

where f is an arbitrary function of its argument. In the case where flux labels exist globally, the Casimir is given by

$$C[v,x,y] = \int_D f(x,y)\, v \cdot \nabla x \times \nabla y\, d^3q\,, \tag{82}$$

where f is an arbitrary function of the flux labels, $x(q,t) := x_0(a(q,t))$ and $y(q,t) := y_0(a(q,t))$.

6 Conclusions

We have described the consequences of Noether's theorems associated with the relabeling transformation for the ideal fluid and MHD. The action and Hamiltonian were seen to be invariant under such a transformation and it was seen that the same transformation was required for invariance of Eulerian variables. Consequently, the Hamiltonian is expressible entirely in terms of Eulerian variables, as can the Poisson bracket. This provides a way to understand the reduced Hamiltonian description of the fluid, in terms of the Eulerian variables, from the viewpoint of symmetries of the action. In addition Ertel's theorem, the Kelvin circulation theorem, cross and magnetic helicity, and other Casimir invariants, including a little known family of invariants in MHD, were discussed.

The formalism described is quite general and applies to a large class of ideal fluid models. More exotic fluids such as the Chew-Goldberger-Low model and gyroviscous fluids [2] possess a similar development.

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INSTITUTE FOR FUSION STUDIES
THE UNIVERSITY OF TEXAS AT AUSTIN
RLM 11.218
AUSTIN, TEXAS 78712-1060
U.S.A.