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UTILIZING GAUSS-HERMITE QUADRATURE TO EVALUATE UNCERTAINTY IN DYNAMIC SYSTEM RESPONSE

Richard V. Field, Jr.
Structural Dynamics and Control
Department 9234, MS 0439
Sandia National Laboratories
Albuquerque, NM 87185-0439

Thomas L. Paez
Experimental Structural Dynamics
Department 9119, MS 0557
Sandia National Laboratories
Albuquerque, NM 87185-0557

John R. Red-Horse
Structural Dynamics and Control
Department 9234, MS 0439
Sandia National Laboratories
Albuquerque, NM 87185-0439

ABSTRACT. Probabilistic uncertainty is a phenomenon that occurs to a certain degree in many engineering applications. The effects that this uncertainty has upon a given system response is a matter of some concern. Techniques which provide insight to these effects will be required as modeling and prediction become a more vital tool in the engineering design process. As might be expected, this is a difficult proposition and the focus of many research efforts. The purpose of this paper is to outline a procedure to evaluate uncertainty in dynamic system response exploiting Gauss-Hermite numerical quadrature. Specifically, numerical integration techniques are utilized in conjunction with the Advanced Mean Value (AMV) method to efficiently and accurately estimate moments of the response process. A numerical example illustrating the use of this analytical tool in a practical framework is presented.

NOMENCLATURE

AMV	Advanced Mean Value
$E[\cdot]$	Operator of mathematical expectation
$f_Y(y)$	Probability Density Function (PDF) of Y
$F_Y(y)$	Cumulative Distribution Function (CDF) of Y
FPI	Fast Probability Integration
$g(\cdot)$	Memoryless deterministic function
$g_L(\cdot)$	Linearized form of $g(\cdot)$
K	Number of terms in quadrature formula
m	Number of AMV runs
M	Moment to be calculated
n	Number of elements in random vector
N	Number of terms in CDF,PDF approximation
$P(\cdot)$	Probability of an event
$T(\cdot)$	Rosenblatt transform
x, y	Realizations of X, Y; function arguments
X, Y	Random variables
z	Realization of Z
Z	Standard normal random variable
β	Distance vector from origin to design point in standard normal space
Δ	Operator denoting a finite perturbation of associated variable
$\Phi(\cdot)$	CDF of a standard normal random variable
$\phi(\cdot)$	PDF of a standard normal random variable
μ_x, σ_X	Mean and standard deviation of random variable X

1. INTRODUCTION

Certain response characteristics of structural dynamic systems exhibit behavior that can only be quantified to within some level of uncertainty. An example of this phenomenon is found in manufacturing, where ensembles of nominally identical structures exhibit unit-to-unit variation. This manifests itself via important measures of engineering behavior, such as frequency response functions, modal frequencies, and mode shapes. Among the sources of this variability is uncertainty due to manufacturing tolerances. These uncertainties are often incorporated into system models as parametric quantities, such as material and geometrical properties.

A previous paper [5] developed a technique for the analysis of this class of uncertainty using a probabilistic approach where the system parameters are assumed to be random variables with known probability distributions. The technique is based on the AMV method, an approach that was developed specifically for application to system reliability analysis by Wu and Wirsching [7]. AMV is strongly motivated by the fact that the functional relationship mapping the random parameters to the response quantity of interest need not be known analytically. Finite element analysis proves to be an ideal application because this information is communicated via a limited number of function evaluations (finite element code runs, for example). The number of these evaluations required for execution of AMV is far fewer than is necessary for sampling-based probabilistic techniques.

Though the AMV approach is a very efficient approach for the computation of cumulative probabilities of measures of system response, it has certain shortcomings. Specifically, its output is cumulative distribution function (CDF) values, not the statistical characteristics of the output quantity, and they are given at abscissa locations which differ from those specified by the user. Therefore, it is not well suited to the direct approximation of CDFs and probability density functions (PDFs) at arbitrary abscissa locations and, as a result, is not well suited to the approximation of the moments of response variables.

In this paper, a technique for approximating a probability distribution based on arbitrary CDF data is presented. The approximation can be used to smooth estimates or to create a PDF estimate at arbitrary abscissa locations. This approxima-

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tion is then used with efficient quadrature rules to estimate statistical moments for the associated response random variables.

In Section 2, a brief review of the AMV method is given. In Section 3, the methodology for establishing smoothed CDF approximations is developed, and the procedure for deriving the associated PDF is shown. Finally, in Section 4, standard quadrature methods are employed in conjunction with this approximation to estimate statistical moments. A simple numerical example is then presented, followed by concluding remarks.

2. THE ADVANCED MEAN VALUE (AMV) METHOD FOR PROBABILISTIC SYSTEM ANALYSIS

Let Y be a scalar random variable defined as follows:

$$Y = g(X), \tag{1}$$

where X is an n-variable random vector with arbitrary joint probability distribution, characterized by the joint PDF, $f_X(x)$, and $g(\cdot)$ is a deterministic function. The probability distribution of the random variable (r.v.) Y can be characterized with the CDF of Y, $F_Y(y)$, for various values of the arbitrary scalar y. By definition

$$F_Y(y) = P(Y \le y) = P(g(X) \le y) \tag{2}$$

$$= \int dx_1 \underset{g(x) \le y}{\cdots} \int dx_n f_X(x),$$

where the fact that the random vector X is n-dimensional has been used. The exact solution is an n-fold improper integral of the joint PDF $f_X(x)$ over a subset of the domain of definition of the underlying r.v. The integral can only be solved in closed form for a very limited number of cases, and many more general numerical approximations are quite expensive. In view of this, it is clear that less cumbersome, approximate solution approaches are particularly attractive.

For any value of y in Eq. (2), the AMV method approximates the solution in the following way. First, a linear approximation to the function in Eq. (1) is defined using a truncated Taylor series:

$$Y \cong g_L(X, \mu_X) = g(\mu_X) + \sum_{i=1}^{n} \left.\frac{\Delta g}{\Delta x_i}\right|_{\mu_X} (X_i - \mu_{X_i}), \tag{3}$$

where the $\mu_X = E[X]$ is the expected value of X, and $\Delta g / \Delta x_i$ are finite difference approximations to the partial derivatives. Evaluation of the coefficients in Eq. (3) requires $n + 1$ analysis runs, or function evaluations, which are used to evaluate the problem output measure of interest, Y, where n is the number of r.v. comprising the vector X. Note that it is this approximation step that circumvents the need for a closed-form expression for $g(\cdot)$. In addition, the explicit specification of a second argument in g_L indicates the point in design space about which the Taylor series is expanded.

Using Eq. (3), AMV approximates the problem stated in Eq. (2) with

$$P(Y \le y) \cong P(g_L(X, \mu_X) \le y) = P(g_L(X, \mu_X) - y \le 0). \tag{4}$$

At this stage, a second approximation is employed which systematically replaces the n-dimensional integral associated with Eq. (4) by a one-dimensional integral in a well-known probability space. This process is a so-called fast probability integration (FPI) technique, which transforms the random vector X into the space of uncorrelated, standard normal random variables, where a design point is subsequently found. This procedure is defined as follows. Let

$$z = T(x) \; x = T^{-1}(z) \tag{5}$$

denote the forward and inverse Rosenblatt transforms (see [6]). The forward transform uniquely maps realizations of the random vector X into the space of realizations of uncorrelated, standard normal random variables, Z, and the inverse transform performs the inverse mapping. The point z in the transformed space where $\|z\|$ is minimum subject to the constraint

$$g_L(T^{-1}(z), \mu_X) - y = 0 \tag{6}$$

is known as the design point for the problem, and is denoted z_1*. This point can be obtained rapidly because the linear approximation in Eq. (3) precludes the need for additional analysis code results. Let $\beta_1 = \|z_1*\|$.

Next, approximate the probability of the response as

$$P(Y \le y) \cong P(g_L(X, \mu_X) \le y) \cong P(Z \le \beta_1) = \Phi(\pm\beta_1), \tag{7}$$

where Z is a single standard normal r.v. This approximate solution is known as the "mean value" solution. (Note that we use the plus sign (+) in the final term in Eq. (7) when y is greater than the mean of $g_L(X, \mu_X)$, and the minus sign (-), otherwise.) Rather than using the approximation of Eq. (7), the point in the space of the original variables x that corresponds to the design point z_1* is updated using the inverse Rosenblatt transform

$$x_1* = T^{-1}(z_1*). \tag{8}$$

Based on this value, a new abscissa for the response is determined

$$y_1 = g(x_1*), \tag{9}$$

and yields the final AMV approximation

$$P(Y \le y_1) \cong \Phi(\pm\beta_1). \tag{10}$$

Two important facts must be noted about the result in Eq. (10). First, it does not yield the result sought in Eq. (2), the CDF of the random variable Y at the point y. Rather, it estimates the CDF of the random variable Y at y_1. This is due to the fact that the transformation and operations that lead from $g(X) = Y$ in Eq. (2) to the design point following Eq. (6) are different from those leading back from the design point to x_1^* and y_1 in Eqs. (8) and (9). Second, the fact that the result in Eq. (10) is only an approximation to the exact CDF of Y at y_1 is reemphasized. There are many reasons for this. The design point is not as accurate as it might be if it were obtained using a more time-consuming approach. More important, the probability evaluation in the normal space is based on a linear approximation of the design surface (the transformed constraint that reflects the condition $g(x) \le y$ in Eq. (2)).

3. CDF/PDF APPROXIMATION

The fact that the abscissa location y_1 differs, in general, from the point y, precludes approximation of the CDF or PDF at an arbitrary point. To overcome this difficulty, we propose to use an interpolating and extrapolating approximation first suggested by Wu and Burnside [8]. The form of the approximation is

$$F_Y(y) \cong \Phi\left[\sum_{j=0}^{N} c_j\left(\frac{y-a}{b}\right)^j\right], \tag{11}$$

where $\Phi(\cdot)$ is, as above, the CDF of a standard normal random variable, a is an arbitrary constant, b is an arbitrary nonnegative constant. The c_j, $j = 0, \ldots, N$, are constants that can be identified using a linear least squares approach.

Assume that the AMV approach has been used to approximate the CDF of Y at a collection of abscissa locations, y_i, $i = 1, \ldots, m$. Then the following relation holds.

$$\Phi^{-1}[F_Y(y_i)] \cong \sum_{j=0}^{N} c_j\left(\frac{y_i-a}{b}\right)^j, \; i = 1, \ldots, m. \tag{12}$$

If $N+1 < m$ then the c_j, $j = 0, \ldots, N$ can be identified using a linear least squares approach.

Once the c_j, $j = 0, \ldots, N$ are identified, Eq. (11) can be used to interpolate or extrapolate the CDF of Y at any value of y. Further, the PDF of Y can be approximated. It is the derivative of $F_Y(y)$ with respect to y

$$f_Y(y) \cong \phi\left[\sum_{j=0}^{N} c_j\left(\frac{y-a}{b}\right)^j\right]\sum_{j=1}^{N} \frac{jc_j}{b}\left(\frac{y-a}{b}\right)^{j-1}, \; -\infty < y < \infty, \tag{13}$$

where $\phi(\cdot)$ denotes the standard normal PDF function.

Some comments can be made about this approach. First, one should exercise caution when using Eq. (11) or Eq. (13) to extrapolate the CDF or PDF approximation, particularly since it is based, indirectly, on a polynomial approximation. Second, the choice of basis functions in Eq, (11) (*i.e.*, the powers of y) can cause ill-conditioning of the governing equations in the least squares problem [2]. To remedy this, basis functions that form an orthogonal system can be used. One example is the orthogonal set of Hermite polynomials, and will be the subject of future work.

4. NUMERICAL INTEGRATION TECHNIQUES

With this approximation to $f_Y(y)$, the moments of Y can be calculated directly utilizing some numerical integration scheme.

Consider a weighted quadrature formula of the form

$$\int_{\alpha}^{\beta} h(\tau)W(\tau)d\tau = \sum_{k=1}^{K} w_k h(\tau_k) + E_K(f), \tag{14}$$

where h is the function to be integrated, and W is a nonnegative continuous weighting function assumed integrable over $[\alpha, \beta]$. The w_k and τ_k terms, which depend on the type of weighting function W used, are discrete weights and nodes, respectively, for the kth term in the sum. $E_K(f)$ is the remainder or error involved in using this quadrature technique.

When considering most "real-world" systems, the output random variable is generally defined for all real values $-\infty < \tau < \infty$. Hence, Gauss-Hermite quadrature is a logical choice, because the weighting function is

$$W(\tau) = e^{-\tau^2}, \; -\infty < \tau < \infty. \tag{15}$$

The corresponding discrete weights and nodes on the right-hand side of Eq. (14) involve the orthogonal set of Hermite polynomials, described by

$$H_0(\tau) = 1, \; H_1(\tau) = 2\tau,$$

$$H_{k+1}(\tau) = 2\tau H_k(\tau) - 2kH_{k-1}(\tau), \; k = 1, 2, \ldots. \tag{16}$$

The weights are then given by

$$w_k = \frac{2^{K-1}K!\sqrt{\pi}}{K^2[H_{K-1}(\tau_k)]^2}, \tag{17}$$

where τ_k is the kth node of $H_K(\tau)$ [3].

To apply this numerical integration technique to compute the moments of $Y = g(X)$, form the Mth moment of Y as

$$E[Y^M] = \int_{-\infty}^{\infty} y^M f_Y(y)dy. \tag{18}$$

After careful study, it was determined that Eq. (14) converges quickly only if $h(\tau)$ has decayed to near zero for $|\tau| > 3$. For this reason, consider the Mth moment of a normalized variable

$$E[V^M] = \int_{-\infty}^{\infty} v^M f_V(v)dv, \; v = \frac{y-u}{w}, \tag{19}$$

where u is an estimate of the mean of Y, and w is used to ensure the integrand in Eq. (19) decays to zero sufficiently fast. Using Eq. (14), it can be shown that

$$E[V^M] = \int_{-\infty}^{\infty} \left\{ e^{v^2} w v^M f_Y(wv+u) \right\} e^{-v^2} dv = \tag{20}$$

$$\int_{-\infty}^{\infty} h(v)W(v)dv \cong \sum_{k=1}^{K} w_k h(v_k)$$

The inverse transformation of Eq. (19) can then be applied to this result to give the Mth central moment of Y.

Note that inherent in Eq. (14) is an assumption that $h(v)$ decays as e^{-v^2} for $v \to -\infty$, $v \to \infty$ [2]. Therefore, we can expect large remainder terms when considering PDFs that do not exhibit this characteristic.

5. EXAMPLE PROBLEM

This example illustrates the operation of AMV and the computation of moments based on the AMV approximation, for a simple function, and compares the results to analytically obtained results. Let

$$Y = g(X_1, X_2) = \frac{X_1}{X_2^2} \tag{21}$$

Graphically, this mapping is shown in Fig. 1. In addition, assume the random variables X_1 and X_2 to be independent and lognormally distributed, with means $\mu_{X_1} = \mu_{X_2}$ and standard deviations $\sigma_{X_1} = \sigma_{X_2}$.

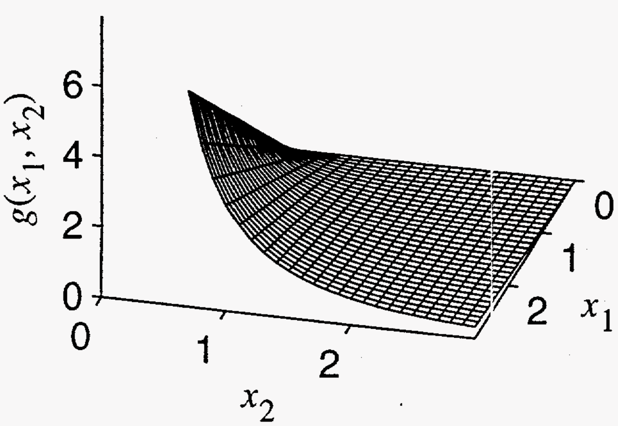

Figure 1: Scalar function of random vector, X.

A closed-form expression for the CDF of Y,

$$F_Y(y) = \Phi\left[\frac{\ln y - c_1}{c_2}\right], \; y > 0, \tag{22}$$

was derived in [5]. It follows from this that the PDF of Y is

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \phi\left[\frac{\ln y - c_1}{c_2}\right]\left(\frac{1}{c_2 y}\right), \; y > 0, \tag{23}$$

where c_1 and c_2 depend on the numerical values for the mean and standard deviation of X_1 and X_2.

To derive closed-form expressions for the moments of Y, note that by definition, a random variable X is lognormal if $\ln X$ is normal. Assume $\ln X$ is normal with mean μ_X and standard deviation σ_X. The PDF of X is then given by [1]

$$f_X(x) = \frac{1}{\sqrt{2\pi}\zeta x} \exp\left[-\frac{1}{2}\left(\frac{\ln x - \lambda}{\zeta}\right)^2\right], \; 0 \le x < \infty, \tag{24}$$

where

$$\lambda = E(\ln X) = \ln \mu_X - \frac{1}{2}\zeta^2 \text{ and} \tag{25}$$

$$\zeta^2 = Var(\ln X) = \ln\left(1 + \frac{\sigma_X^2}{\mu_X^2}\right). \tag{26}$$

With this knowledge, it can be shown that

$$E[X^M] = \int_0^{\infty} x^M f_X(x)dx = \exp\left[\frac{1}{2}M(M\zeta^2 + 2\lambda)\right]. \tag{27}$$

The Mth moment of Y is then

$$E[Y^M] = E\left[\frac{X_1^M}{X_2^{2M}}\right] = E[X_1^M]E[X_2^{-2M}], \tag{28}$$

because X_1 and X_2 are independent.

The Mth central moment of Y can be expressed in terms of $E[Y^M]$ (see [4])

$$E[(Y-\mu_Y)^M] = \sum_{r=0}^{M} \binom{M}{r}(-1)^r \mu_Y^r E[Y^{M-r}]. \tag{29}$$

Using the analytical expressions of Eqs. (27) through (29), the first M central moments of Y can be found.

Shown in Table 1 are the first four central moments of Y for the simple example of Eq. (21), assuming $\mu_{X_1} = \mu_{X_2} = 1.0$ and $\sigma_{X_1} = \sigma_{X_2} = 0.1$. Also shown are the moment calcula-

Moment	Exact	Quadrature	Error, %
$\mu_Y = E[Y]$	1.0303	1.0293	0.09706
$E[(Y-\mu_Y)^2]$	5.4148e-2	5.4550e-2	0.7424
$E[(Y-\mu_Y)^3]$	8.6825e-3	8.6806e-3	0.02188
$E[(Y-\mu_Y)^4]$	1.1305e-2	1.1039e-2	2.353

Table 1: Central moments of $Y = g(X_1, X_2)$ with $\sigma_{X_1} = \sigma_{X_2} = 0.1$ using numerical quadrature technique.

tions using the Gauss-Hermite quadrature approach introduced in §4. Table 2 lists the identical information for the case of $\mu_{X_1} = \mu_{X_2} = 1.0$ and $\sigma_{X_1} = \sigma_{X_2} = 0.5$.

Results indicate that the numerical quadrature technique does a very good job estimating the first four moments for the case of $\sigma_{X_1} = \sigma_{X_2} = 0.1$; the error is much less than 5% for each of the first four moments. In fact, errors remain small for $M = 10$. When the variance in the underlying random variables is increased, however, the quadrature scheme can become highly inaccurate. The estimate of the mean of Y is adequate, but estimates of the higher-order moments are unacceptable. These discrepancies can be explained by examining the two major assumptions made while formulating the problem: (1) that the PDF approximation of Eq. (13) is able to adequately capture the behavior of a general distribution function, and (2) that the numerical quadrature formula of Eq. (14) is accurate for a general distribution function.

As illustrated in Figs. 2 and 3, when X_1 and X_2 exhibit little variance from the mean, the PDF approximation of §3 does an adequate job. In addition, the tails of $f_Y(y)$ decay rapidly. In contrast, with the variance of X_1 and X_2 much larger, the PDF approximation is poor. Also, note that the exact PDF of Y does not decay as e^{-y^2} near $y = 0$. With these observations is mind, we can expect poor performance using the quadrature technique to estimate moments of Y.

Moment	Exact	Quadrature	Error, %
$\mu_Y = E[Y]$	1.9531	1.8182	6.907
$E[(Y-\mu_Y)^2]$	7.8268	4.0612	48.11
$E[(Y-\mu_Y)^3]$	1.5845e+2	8.9474	94.35

Table 2: Central moments of $Y = g(X_1, X_2)$ with $\sigma_{X_1} = \sigma_{X_2} = 0.5$ using numerical quadrature technique.

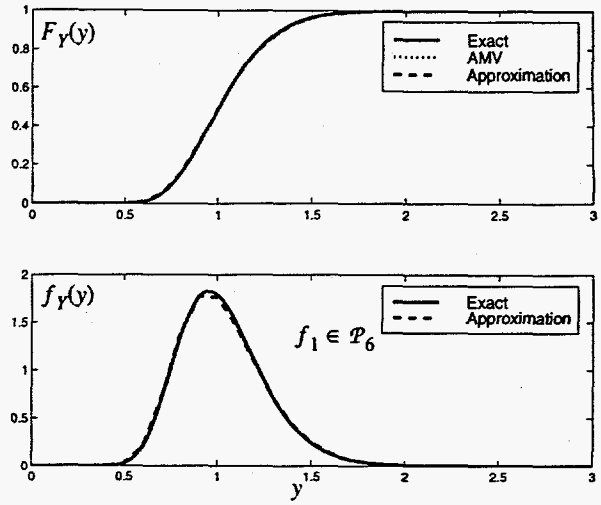

Figure 2: The CDF and PDF of Y with $\sigma_{X_1} = \sigma_{X_2} = 0.1$.

Some simplifying assumptions have been made to facilitate this example. First, the random variables X_1 and X_2 are specified independent to simplify the analytical expressions for the moments of Y; the quadrature method of §4 is not restricted to problems with independent underlying random variables. Second, as discussed in §1, AMV was developed to address problems where no explicit knowledge of the functional relationship $Y = g(X)$ exists. The Gauss-Hermite quadrature method to compute moments also functions in this situation, making it an ideal tool to be used in conjunction with finite element analysis to estimate the statistical nature of systems with uncertain parameters.

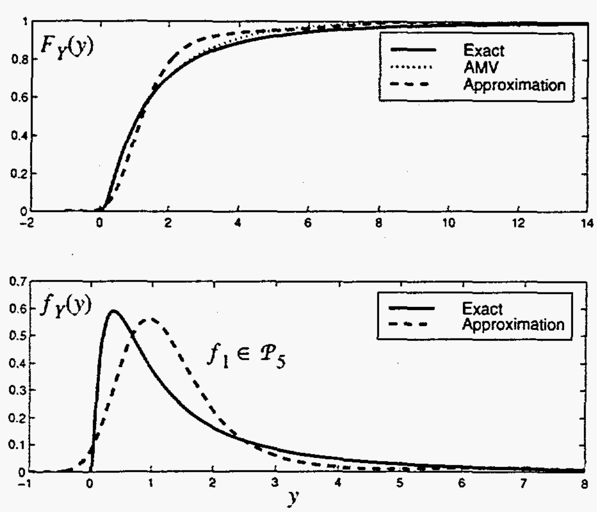

Figure 3: The CDF and PDF of Y with $\sigma_{X_1} = \sigma_{X_2} = 0.5$.

6. CONCLUSIONS AND FUTURE WORK

A technique for using Gauss-Hermite numerical quadrature coupled with the AMV method for the computation of the moments of the response of mechanical and other analytical systems has been developed. The method has been applied to an example problem in a practical framework with promising results. Furthermore, these techniques are efficient, accurate, and well-suited for use with general finite element analysis codes.

Some areas for future work include: (1) applying these methods to a system where the functional relationship between random parameters and response quantities is not explicitly known (*e.g.*, finite element analysis), (2) further investigating the convergence and error properties of Gauss-Hermite quadrature, and (3) considering alternative CDF approximation methods to the one presented in §3.

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