SOME PROPERTIES OF METRIC SPACES

THESIS

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CHAPTER I

INTRODUCTION

Topology is a relatively modern branch of mathematics which consists of the study of sets of elements called points and, more specifically, which points are limit points of which sets. The limit point concept is the basic concept in point set topology. The results obtained from the general study of topology can be used to better understand specific topological spaces, such as the set of real numbers with its usual topology. The usual topology for the real numbers is the collection of all unions of open intervals of real numbers.

The study of metric spaces is closely related to the study of topology in that the study of metric spaces concerns itself, also, with sets of points and with a limit point concept based on a function which gives a "distance" between two points. In some topological spaces it is possible to define a distance function between points in such a way that a limit point of a set in the topological sense is also a limit point of the same set in a metric sense. In such a case the topological space is "metrizable". The real numbers with its usual topology is an example of a topological space which is metrizable, the distance function being the absolute value of the difference of two real numbers. Chapters II and III of this thesis attempt to classify, to a certain extent, what type of topological space is metrizable. Chapters IV and V deal with several properties of metric spaces and certain functions of metric spaces, respectively.

CHAPTER II

TOPOLOGICAL SPACES AND METRIC SPACES

Definition 1.1 If A and B are sets, then "A+B" means the union of A and B and "A.B" means the intersection of A and B. The empty set will be denoted by $\emptyset$.

Definition 1.2 A topology is a collection T of sets having the properties:

1) The intersection of any two elements of T is an element of T,

2) The union of any subset of T is an element of T.

Let X be the union of all the elements of T; then X is called the space of the topology T. The ordered pair (X,T) is called a topological space.

Definition 1.3 In the topological space (X,T) the elements of T are called open sets. If t is in T, the elements of t are called points.

Definition 1.4 If A is a subset of X and p is in X, the statement that p is a limit point of A means that every open set which contains p contains a point of A different from p.

Definition 1.5 If p is a point and U is a point set, the statement that U is a neighborhood of p means that there exists an open set V such that p is in V and V is a subset of U.

<u>Definition 1.6</u> If M is a point set, X-M is the set of all points which are in X, but not in M.

<u>Definition 1.7</u> The statement that the point set M is closed means X-M is open.

<u>Definition 1.8</u> If G is a collection of sets, the sets in G are mutually separated means that they are disjoint and no point of one is a limit point of any other.

<u>Definition 1.9</u> The statement that (X,T) is separable means that there exists a countable subset K of X such that every point of X is a point, or a limit point, of K.

<u>Theorem 1.1</u> The point set M is closed if and only if it contains all its limit points.

<u>Proof:</u> Suppose the point set M contains all its limit points. Suppose that p is a point not in M; then p is not a limit point of M. There exists an open set Vp such that p is in Vp and Vp contains no point of M. Then Vp is a subset of X-M. Therefore X-M is open because X-M can be written as the union of all such sets Vp for p in X-M and the union of any collection of open sets is open. Since X-M is open, M is closed.

Now suppose M is closed, then X-M is open and contains no point of M. Therefore if p is a limit point of M, it cannot be in X-M and so it must be in M.

<u>Definition 1.10</u> If (X,T) is a topological space, then a base for T is a subcollection A of T such that if p is in X and U is a neighborhood of p, then there is an element V of A such that p is in V and V is a subset of U.

<u>Theorem 1.2</u> If A is a collection of sets and X is the union of all the sets in A, then A is a base for a topology for X if and only if each point p of X, if U and V are elements of A containing p, then there is an element W of A such that p is in W and W is a subset of U.V.

<u>Proof:</u> Suppose A is a collection of sets and X is the union of all the sets in A. Suppose A is a base for a topology for X. Let p be in X and let U and V be elements of A which contain p. U and V are open sets and therefore U.V is an open set containing p. Then U.V is a neighborhood of p and from the definition of a base there is an element W of A such that p is in W and W is a subset of U.V.

Suppose that for each point p in X, if U and V are elements of A containing p, then there is an element W of A such that p is in W and W is a subset of U.V. Let T be the collection of all sets B such that for some subset A' of A, B is the union of all the sets in A'.

If T' is a subset of T, then the union of all the sets in T' is the union of a collection of sets of A and therefore is an element of T.

If U and V are elements of T, then there exists sets B and B' which are subsets of A such that U is the union of all sets in B and V is the union of all sets in B'. Then U.V is the same as the union of all sets b.b', where b is in B and b' is in B'. For some b in B and b' in B', consider b.b'$\neq\emptyset$. If p is in b.b', then there is a set C in A such that p is in C and C is contained in b.b'. Then b.b' is the union of all such sets C and therefore b.b' is the union of a subcollection of A and therefore is an element of T. Since U.V is the union of all such sets b.b' and the union of any subcollection of T is in T, then U.V is in T. Therefore T is a topology for X. Obviously, by the way T is defined, A is a base for T.

As a result of this theorem, it is easy to show that if (X,T) is a topological space and A is a base for T, then every open set is the union of some subcollection of A.

<u>Definition 1.11</u> The statement that (X,T) is completely separable means that there exists a countable base for T.

Separation Properties for Topological Spaces

TO If p and q are two points, there is an open set which contains one and not the other.

T1 If p and q are two points, there is an open set which contains p and not q.

T2 If p and q are two points, there exist two disjoint open sets, one containing p and the other q.(Hausdorff)

T3 If p is a point and k is a closed set not containing p, then there exist two disjoint open sets, one containing k and the other p.(Regular)

T4 If k and m are two disjoint closed sets, then there exist two disjoint open sets, one containing k, the other m.(Normal)

T5 If k and m are mutually separated sets, then there exist two disjoint open sets, one containing k, the other m.(Completely Normal)

Definition 1.12 Suppose M is a set and d is a real valued function whose domain is $\{(x,y) \mid x \text{ is in } M, y \text{ is in } M\}$ such that:

1. If x is in M and y is in M then $d(x,y) \geq 0$
2. If x is in M and y is in M then $d(x,y)=d(y,x)$.
3. If x, y are in M then $d(x,y)=0$ if and only if $x=y$.
4. If x, y, z are in M then $d(x,y)+d(y,z) \geq d(x,z)$. (triangle inequality)

then (M,d) is called a metric space, M is called the space and d is called the metric for the metric space.

Definition 1.13 If (M,d) is a metric space and p is in M, the statement that U is a spherical neighborhood of p means there exists a positive number r such that

$$U=\{x \mid x \text{ is in } M \text{ and } d(p,x)<r\}$$

<u>Theorem 1.3</u> If (M,d) is a metric space and $A=\{U \mid \text{for some point } p \text{ of } M, U \text{ is a spherical neighborhood of } p\}$, then A is a base for a topology T of M and under this topology if p is in M and K is a subset of M then p is a limit point of K if and only if for each positive number e there exists a point q of K such that $o<d(p,q)<e$.

<u>Proof:</u> Suppose (M,d) is a metric space and $A=\{U \mid \text{for some point } p \text{ of } M, U \text{ is a spherical neighborhood of } p\}$. Suppose p is in M and U and V are spherical neighborhoods containing p. There exists a point q and a positive number r such that $U=\{x \mid x \text{ is in } M \text{ and } d(q,x)<r\}$ and also, there exists a point y and a positive number r' such that

$$V=\{x \mid x \text{ is in } M \text{ and } d(y,x)<r'\}.$$

Since p is in U and in V, let $d(p,q)=e<r$ and $d(p,y)=e'<r'$. Then $r-e>o$ and $r'-e'>o$. Let r" be the smaller of the numbers $r-e$ and $r'-e'$. Then let $W=\{x \mid x \text{ is in } M \text{ and } d(p,x)<r''\}$. The point p is in W. Suppose t is a point in W. Then $d(p,t)<r''$. Now $d(q,t)\leq d(q,p)+d(p,t)<e+r''\leq e+r-e=r$. Therefore $d(q,t)<r$ and t is in U. Similarly t is in V and so t is in V.U and by Theorem 1.2, A is a base for a topology of M.

Suppose p is in M and K is a subset of M. First suppose p is a limit point of K and e is a positive number.

Let $U=\{x \mid x$ is in M and $d(p,x)<e\}$. Since U is an element of the base, it is an open set containing p and therefore it must contain some point q in K different from p then $0<d(p,q)<e$.

Now suppose that for each positive number e, there exists a point q of K such that $0<d(p,q)<e$. Suppose U is an open set containing p. Then there is a spherical neighborhood V such that p is in V and V is a subset of U. There exists a point q and a positive number r such that

$$V=\{x \mid x \text{ is in } M \text{ and } d(q,x)<r\}.$$

Let $d(p,q)=e'<r$, then $r-e'>0$. Let $r'=r-e'$ and

$$V'=\{x \mid x \text{ is in } M \text{ and } d(p,x)<r'\}.$$

Now V' is a subset of V and so is a subset of U and there is a point q in K such that $0<d(p,q)<r'$, so that q is in V' and therefore U contains a point q in K distinct from p and p is a limit point of K.

Definition 1.14 If (M,d) is a metric space, the topology for M whose base is the set of all spherical neighborhoods determined by the metric d is called the d-metric topology for M.

Definitions of Weaker Metrics

Suppose M is a set and d is a real valued non-negative function whose domain is $\{(x,y) \mid x \text{ is in } M \text{ and } y \text{ is in } M\}$:

1. If, for x, y, z in M, the following hold;

   (a) $d(x,y)=d(y,x)$

(b) $d(x,y)+d(y,z)\geq d(x,z)$

(c) $x=y$ implies $d(x,y)=0$

Then (M,d) is called a pseudo metric space.

2. If for x, y, z in M, the following hold;

(a) $d(x,y)=d(y,x)$

(b) $x=y$ implies $d(x,y)=o$

(c) $d(x,y)=o$ implies $x=y$

Then (M,d) is called a semi-metric space.

3. If, for x, y, z in M, the following hold;

(a) $d(x,y)+d(y,z)\geq d(x,z)$

(b) $x=y$ implies $d(x,y)=0$

Then (M,d) is called a Weak Metric Space.

Examples of Weaker Metric Spaces

Pseudo metric.--Let M be the set of all ordered pairs of real numbers. Now if p and q are points in M where $p=(x,y)$ and $q=(x',y')$, then define the metric d in this way: $d(p,q)=|x-x'|$. Obviously $d(p,q)=d(q,p)$. Now if $p=q$, then $x=x'$ and $d(p,q)=|x-x'|=|0|=0$. But if $d(p,q)=0$ that means $x=x'$, but $y=y'$ is not necessarily true. Therefore $d(p,q)=0$ does not imply $p=q$. If p, q, r are elements of M, and $p=(x,y)$, $q=(x',y')$, $r=(x'',y'')$, then $d(p,q)=|x-x'|$, $d(q,r)=|x'-x''|$. Then

$d(p,q)+d(q,r)=|x-x'|+|x'-x''|\geq|x-x'+x'-x''|=|x-x''|=d(p,r)$.

Therefore this metric satisfies the triangle inequality. Now suppose p and q are elements in M such that $p=(x,y)$

and $q=(x,y')$ where $y\neq y'$. Then $d(p,q)=0$ and any spherical neighborhood which contains p will also contain q. Therefore there does not exist an open set which contains one of the points and not the other, so that the d-metric topology does not have the T_0 property.

Semi-metric.--Let M be the set of all ordered pairs of real numbers. Suppose p and q are points of M, where $p=(x,y)$ and $q=(x',y')$. Then the metric d will be defined as follows:

(1) $d(p,q)=0$ if and only if $p=q$.

(2) If $x=y'$ and $x'=y$ and $x\neq x'$, then $d(p,q)=1$.

(3) For all other p and q in M, $d(p,q)=|x-y'|+|y-x'|$

To prove the symmetric property, suppose p and q are points in M. Obviously, if $p=q$, then $d(p,q)=d(q,p)$. Suppose property two holds, then $x=y'$, $y=x'$ and $x\neq x'$ implies that $x'=y$, $y'=x$ and $x\neq x'$. Therefore $d(q,p)=1=d(p,q)$. Suppose $d(p,q)=|x-y'|+|y-x'|$, then $d(q,p)=|x'-y|+|y'-x|$. Since $|x-y'|=|y'-x|$ and $|y-x'|=|x'-y|$ then $|x-y'|+|y-x'|=|x'-y|+|y'-x|$ and $d(p,q)=d(q,p)$. To show that this metric does not satisfy the triangle inequality, let $p=(1,-2)$, $q=(-1,0)$, $r=(2,-3)$. Then

$d(p,q)=|1|+|-1+2|=2$; $d(q,r)=|-1+3|+|2|=4$; $d(p,r)=|1+3|+|-2-2|=8$ and therefore $d(p,q)+d(q,r)$ is not greater than or equal to $d(p,r)$. Since by definition $d(p,q)=0$ if and only if $p=q$, this metric satisfies the properties of a semi-metric. Suppose $p=(5,0)$ and $q=(0,5)$ and $U=\{x|x \text{ is in } M \text{ and } d(q,x)<2\}$

and $V=\{x|x$ is in M and $d(p,x)<1\}$. Now $d(p,q)=1$ and so p is in U but q is not in V. It can easily be seen that $U.V=\{p\}$. Obviously there is not a spherical neighborhood W such that p is in W and W is a subset of U.V. Therefore the set of spherical neighborhoods defined by this metric does not satisfy the requirements for a topology on M.

Weak metric.--Let M be the set of all real numbers and define the metric d as follows:

If p,q are in M, $d(p,q)=q-p$ if $q>p$; $d(p,q)=0$ if $q\leq p$.

The only case where symmetry holds is when $p=q$ and $d(p,q)=0$ does not necessarily mean $p=q$. However, $p=q$ does imply $d(p,q)=0$. Suppose p,q and r are elements of M. Now if $d(p,r)=0$, then $d(p,q)+d(q,r)\geq d(p,r)$. Suppose $d(p,r)>0$, then $r>p$.

Case 1 Suppose $q\geq r$, then $d(p,q)=q-p\geq r-p$; then $d(p,q)+d(q,r)=q-p+d(q,r)\geq r-p+d(q,r)\geq r-p=d(p,r)$.

Case 2 Suppose $p<q<r$, then $d(p,q)=q-p, d(q,r)=r-q$, $d(p,r)=r-p$, $d(p,q)+d(q,r)=q-p+r-q=r-p=d(p,r)$.

Case 3 Suppose $q=p$, $d(p,q)+d(q,r)=0+d(q,r)=d(p,r)$

Case 4 Suppose $q<p$, then $d(p,q)=0$, $d(q,r)=r-q$, $d(p,r)=r-p$. Since $q<p$ then $-q>-p$ and $r-q>r-p$ then $d(p,q)+d(q,r)=r-q>r-p=d(p,r)$. Therefore the triangle inequality is satisfied.

The set of spherical neighborhoods defined by this metric is the set of all open half lines in the negative direction. Therefore if p and q are two points in M such that p<q, there is a spherical neighborhood which contains p and not q, but there does not exist a spherical neighborhood which contains q and not p, so that the d-metric topology will have the T_0 property, but not T_1.

Each of the above examples of weaker metric spaces is an example of the particular space it represents, and not an example of either of the other two types of metric spaces; with the exception that every pseudo metric space is also a weak metric space by definition. With respect to the spherical neighborhoods defined by the metrics, all three examples are both separable and completely separable.

CHAPTER III

METRIZATION

As was stated in Chapter II, not every topological space can be a metric space. The topological spaces which can also be metric spaces are said to be metrizable.

Definition 2.1 A topological space (X,T) is said to be metrizable if and only if there is a real valued function d whose domain is $\{(x,y) \mid x$ is in X and y is in X$\}$ and which satisfies the properties of a metric and such that if p is in X and K is a subset of X, then p is a limit point of K if and only if for each positive number e, there exists a point q of K such that $0<d(p,q)<e$.

Theorem 2.1 If (X,T) is a topological space such that there exists a point p of X such that $\{p\}$ has a limit point, then (X,T) is not metrizable.

Proof: Let p be a point of X such that q is a limit point of $\{p\}$. Suppose (X,T) is metrizable. Then there is a function d. Let $d(p,q)=e$. Obviously e cannot be zero because if e' is a positive number, $0<d(p,q)<e'$. Therefore $e>0$; but if $0<e'<e$ there is not a point p in $\{p\}$ such that $0<d(p,q)<e'$. This contradicts the assumption that (X,T) is metrizable and so the theorem is true.

Definition 2.2 If p is in X and $S=\{q_1,q_2,q_3,\ldots\}$ is a sequence of points in X, then p is a sequential limit point of S means that if $e>0$, there is a positive integer N such that if $n>N$, then $0<d(p,q_n)<e$. Then S is said to converge to p.

Theorem 2.2 If (X,T) is metrizable, K is a subset of X, and p is a limit point of K, then p is a sequential limit point of a sequence of distinct points of K different from p.

Proof: Since p is a limit point of K, then for $e>0$, there is a point q in K such that $0<d(p,q)<e$. Suppose $e=1$, then there is a point q_1 in K such that $0<d(p,q_1)<1$. If $e=1,\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{n},\ldots$, then for each number $\frac{1}{n}$, there is a point q_n such that $0<d(p,q_n)<\frac{1}{n}$. Obviously, the sequence

$$S=\{q_1,q_2,q_3,\ldots\}$$

converges to p. Now form a new sequence S' in the following manner:

$p_1 = q_1$

p_2 is the first point in S different from p_1.

p_3 is the first point in S different from p_1 and p_2.

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p_n is the first point different from $p_1,p_2,p_3,\ldots p_{n-1}$.

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Suppose there is a last point p_k in S'. Then there is a positive integer n such that if m>n, then $q_m = p_i$ for some $i \leq k$. Let $Q=\{d(p,p_n) | n=1,2,3,\ldots k\}$. Let e be the smallest number in Q. Then there is a positive integer r such that $\frac{1}{r}<e$ and there is a q_r such that $d(p,q_r)<\frac{1}{r}$. But

$$q_r \neq p_i \quad 1 \leq i \leq k$$

which contradicts the assumption that there is a last point in S'. Then S' is an infinite sequence of distinct points of K and since S' is a subset of S, then S' converges to p.

Theorem 2.3 The topological space (X,T), where X is the set of real numbers and $T=\{U | X-U \text{ is finite}\}$ is not metrizable.

Proof: Assume (X,T) is metrizable. Let $K=\{x | 0 \leq x \leq 1\}$. Suppose p is a point of X and U is an open set containing p. Since X-U is finite and K is infinite, there must be an infinite number of points of K which are not in X-U and which, therefore, are in U. Then U must contain some point of K different from p and p is a limit point of K. Let $d(5,6)=c$. Since 5 is a point of X, 5 is a limit point of K and by Theorem 2.2 there is a sequence of points $q_1, q_2, q_3, \ldots$ in K such that 5 is a sequential limit point of this sequence. Then there is a positive integer N such that if $n>N$, $d(5,q_n)<\frac{c}{2}$.

Let $J=\{q_n | d(5,q_n)<\frac{c}{2}\}$. Since J is infinite every point of X is a limit point of J and therefore 6 is a limit point of J and there is a point q in J such that $d(6,q)<\frac{c}{2}$. Then $d(6,q)+d(5,q)<\frac{c}{2}+\frac{c}{2}=c$ but $d(6,q)+d(5,q)=d(6,q)+d(q,5)\geq d(6,5)=c$ which is a contradiction; therefore the assumption is false and (X,T) is not metrizable.

Theorem 2.4 If (X,T) is metrizable, then it is completely normal. (T_5)

Proof: Suppose (X,T) is metrizable and K and M are subsets of X which are mutually separated. Let p be in K, then p is not a limit point of M. Then there is an $e_p>0$ so that if q is in M, $d(p,q)\geq e_p$. Let

$$U_p=\{x | x \text{ is in } X \text{ and } d(p,x)<\frac{e_p}{2}\}.$$

Let U be the union of all such sets U_p where p is in K. Now let q be in M. Then q is not a limit point of K and there is an $e_q>0$ such that if p is in K, $d(q,p)\geq e_q$. Let $V_q=\{x | x \text{ is in } X \text{ and } d(q,x)<\frac{e_q}{2}\}$. Let V be the union of all such sets V_q where q is in M. Then U and V are open sets which contain K and M respectively.

Assume $U.V\neq\emptyset$, then let s be in U.V. Since s is in U, there is a p in K such that s is in U_p and $d(s,p)<\frac{e_p}{2}$ and if q' is in M, $d(p,q')\geq e_p$. Also, since s is in V, there is a q in M such that s is in V_q and $d(s,q)<\frac{e_q}{2}$ and if p' is in K, $d(p',q)\geq e_q$. Let r be the larger of e_p and e_q. Then $d(s,p)=d(p,s)<\frac{r}{2}$ and $d(s,q)<\frac{r}{2}$. Then $d(p,s)+d(s,q)<\frac{r}{2}+\frac{r}{2}=r$, but $d(p,q)\geq r$ which gives

$$d(p,s)+d(s,q)\leq d(p,q)$$

which is a contradiction of the triangle inequality property. Therefore the assumption is false and $U \cdot V=\emptyset$ and (X,T) is completely normal.

Definition 2.3 If K is a set and A is a collection of sets, the statement that A covers K means that every element of K is in some element of A. If B is a subset of A which covers K, then B is called a subcover of K. If each element of A is an open set, then A is called an open cover of K.

Theorem 2.5 If (X,T) is a completely separable topological space and A is an open cover of X, then A contains a countable subcover of X.

Proof: Suppose (X,T) is completely separable and let $H=\{h_1,h_2,h_3,\dots\}$ be a countable base. Let A be an open cover of X. If p is in X, then there is an a in A such that p is in a and an h_i in H such that p is in h_i and h_i is a subset of a. Let $B=\{b_1,b_2,b_3,\dots\}$ be the set of all sets in H such that there is a U in A such that b_i is a subset of U. For each b_i, let

$V_i = \{U | U$ is in A and b_i is a subset of $U\}$.

Then using the Axiom of Choice, form a sequence

$$A' = \{U_1, U_2, U_3, \ldots\}$$

where U_i is from V_i. Obviously A' is a subset of A and is countable. Now suppose p is in X. There is a U in A such that p is in U and an h_i in H such that p is in h_i and h_i is a subset of U. Then h_i is in B and there is an integer k such that $h_i = b_k$. Then there is a U_k in A' which contains b_k and, therefore, contains p. Then A' is a cover of X and and the theorem is true.

If (X,T) is a separable topological space and A is an open cover of X, then it is not necessarily true that A contains a countable subcover of S. Suppose X is the set of real numbers and let H be the collection consisting of $\{\{0\}\} + \{U | U$ is $\{0$ and one irrational number$\}\}$ plus $\{U | U$ is an open interval of rational numbers not containing $0\}$. Suppose p is in X and U and V are elements of H containing p. Since there are not two different elements of H which contain the same irrational number, suppose p is rational. If p is zero, then U and V must be of the form $\{0\}$ or $\{0,x\}$ where x is an irrational number. Then $U.V = \{0\}$ and there is a W in H such that p is in W and W is a subset of U.V. If p is not zero, then U and V are both open intervals of rational numbers and, therefore, U.V is an open interval of rational numbers containing p. Therefore, H is a base for a topology T. Let K={all rational numbers}, then K is countable. Suppose p is in X and p is not rational. Then every open set which contains p must also contain zero, which is rational, and so p is a limit point of K. Therefore (X,T) is separable and H is an open cover of X, but since the set of irrational numbers is uncountable, H does not contain a countable subcover of X.

Theorem 2.6 If the separable topological space (X,T) is metrizable, then it is completely separable.

Proof: Let (X,T) be a separable topological space which is metrizable. Then there is a countable set K such that if q is in X, then q is in K or q is a limit point of K. If p is in K, let

$$Bp=\{U \mid U=\{x \mid x \text{ is in } X \text{ and } d(p,x)<\frac{1}{n}\} \text{ for } n=1,2,3,\ldots\}.$$

Let A be the union of all such sets Bp for p in K. Let A'={all spherical neighborhoods of all points in X}, then by Theorem 1.3 A' is a base for T. Suppose q is in X. Then there are two possibilities;

1. If q is in K, there is a U in A such that q is in U,
2. If q is not in K, then q is a limit point of K and there is a p in K such that $d(p,q)<\frac{1}{2}$.

Therefore there is a U in A such that q is in U. Then X is equal to the union of all sets in A.

Suppose p is in X and U and V are elements of A containing p. Since A is a subset of A', U and V are also elements of A' and so there is a W in A' such that p is in W and W is a subset of U.V. There is a point q and a positive number r such that $W=\{x|x$ is in X and $d(q,x)<r\}$. Let $d(p,q)=e$, then $r-e>0$. Either p is in K or p is a limit point of K. If p is in K, let n be a positive integer such that $\frac{1}{n}<r-e$, then let $W'=\{x|x$ is in X and $d(p,x)<\frac{1}{n}\}$. Then W' is in A and p is in W' and W' is a subset of U.V.

If p is not in K, then p is a limit point of K. Let m be a positive integer such that $\frac{1}{m}<\frac{r-e}{2}$. There is a point s in K such that $d(p,s)<\frac{1}{m}$. Let

$$W''=\{x|x \text{ is in X and } d(s,x)<\frac{1}{m}\}.$$

Then W" is a subset of W and W" is an element of A such that p is in W" and W" is a subset of U.V. Therefore A is a countable base for T.

CHAPTER IV

COMPLETENESS AND TOTAL BOUNDEDNESS IN METRIC SPACES

Definition 3.1 If (M,d) is a metric space, then a subset E of M is totally bounded if and only if for each positive number e, there is a finite sequence $p_1, p_2, \ldots p_n$ of elements of E such that every element p in E is at a distance less than e from at least one element of the sequence.

Definition 3.2 If (M,d) is a metric space and $S=\{p_1, p_2, p_3, \ldots\}$ is a sequence in M, then S is a Cauchy sequence if and only if for each positive number e, there is a positive integer N such that if m,n are positive integers larger than N, then $d(p_m, p_n)<e$.

Theorem 3.1 Every infinite subset of a set E contained in a metric space (M,d) contains an infinite Cauchy sequence of different points if and only if the set E is totally bounded.

Proof: Suppose E is totally bounded and S is an infinite subset of E. If $e=\frac{1}{2}$, there is a p_1 such that an infinite number of elements of S are less than a distance of $\frac{1}{2}$ from p_1. Call the set of these elements

S_1. For $e=\frac{1}{2}\cdot\frac{1}{n}$, there is a p_n such that an infinite number of elements of S_{n-1} are at a distance less than $\frac{1}{2}\cdot\frac{1}{n}$ from p_n. If r,s are in S_n then $d(r,p_n)<\frac{1}{2}\cdot\frac{1}{n}$ and $d(p_n,s)<\frac{1}{2}\cdot\frac{1}{n}$. Since $d(r,s)\leq d(r,p_n)+d(p_n,s)<\frac{1}{2n}+\frac{1}{2n}=\frac{1}{n}$, then the distance between any two points of S_n is less $\frac{1}{n}$. Picking q_i in S_i such that $q_i\neq q_j$ if $i\neq j$, then $\{q_n\}$ $n\geq 1$ is a Cauchy sequence of distinct points.

Suppose every infinite subset of a set E contains a Cauchy sequence of different points and suppose E is not totally bounded. Then there exists an $e>0$ such that for any finite sequence of points, $p_1,p_2,p_3,\ldots p_n$, there is an element p of E such that $d(p,p_i)$, $i=1,n$, is greater than e. Suppose p_1 is a point in E. $S_1=\{p_1\}$ is a finite sequence and so there is a point p_2 in E such that

$$d(p_1,p_2)\geq e.$$

$S_2=\{p_1,p_2\}$ is a finite sequence, so there is a p_3 in E such that $d(p_1,p_3)\geq e$ and $d(p_2,p_3)\geq e$. For the $n^{\underline{th}}$ case $S_n=\{p_1,p_2,\ldots p_n\}$ is a finite sequence, so there exists a p_{n+1} such that $d(p_i,p_{n+1})\geq e$ for $i=1,2,\ldots n$. If the process is continued an infinite number of times then the union of all S_n, $n\geq 1$ is an infinite subset of E which cannot contain a Cauchy sequence. This contradicts the hypothesis and so E is totally bounded.

Definition 3.3 The statement that the topological space (X,T) is compact means that if G is an open cover of X, then some finite subset of G covers X. The topological space (X,T) is countably compact means that every infinite subset of X has a limit point.

Definition 3.4 A metric space (M,d) is complete if and only if every Cauchy sequence in M has a sequential limit point.

Lemma 3.1 A totally bounded metric space (M,d) is separable.

Proof: Suppose (M,d) is a totally bounded metric space. Then if e=1, there is a finite sequence

$$S_1=\{p_1, p_2, \ldots p_n\}$$

of points of M such that every point in M is at a distance less than one from some point of S_1. In general, if $e=\frac{1}{n}$ there is a finite sequence of points, S_n, such that every point in M is at a distance less than $\frac{1}{n}$ from some point of S_n. Consider the countable set K which is the union of all S_n for n=1,2,3,.... Suppose p is in M and p is not in K and suppose V is a neighborhood of p. Then there is a spherical neighborhood Up such that p is in Up and Up is a subset of V and for some q in M and positive number e, $Up=\{x|x$ is in M and $d(q,x)<e\}$. Suppose $d(p,q)=r$, then $e-r>0$ and there is a positive integer n such that $\frac{1}{n}<e-r$. Since there is a point of K at a distance less than $\frac{1}{n}$ from p, it must be in Up and therefore p is a limit point of K. Therefore (M,d) is separable.

Theorem 3.2 A metric space (M,d) is compact if and only if it is complete and totally bounded.

Proof: Suppose (M,d) is compact. Suppose e is a positive number. Let G be the set of all spherical neighborhoods with a radius less than e. Then G is an open cover of M and there is a finite subcollection G' of G such that G' covers M. For each g in G', there is a p in M such that there exists a positive number r such that $g=\{x|x$ is in M and $d(x,p)<r\}$. Then the finite sequence $p_1,p_2,\ldots p_n$, where p_i is from g_i in G', has the property that if p is in M, then p is in some g_i of G' and therefore $d(p_i,p)<e$ and so (M,d) is totally bounded.

Suppose $Q=\{p_1,p_2,\ldots\}$ is a Cauchy sequence of points of M. Since M is compact, it is countably compact. If $\{p|p$ is in $Q\}$ is infinite, then it has a limit point p. Suppose $e>0$; then there is a positive integer N such that if $m,n>N$, $d(p_m,p_n)<\frac{e}{2}$. Since $\{p_1,p_2,\ldots p_N\}$ is finite, p must be a limit point of $\{p_{N+1},p_{N+2},\ldots\}$ and so there is a point p_i, $i>N$ such that $d(p_i,p)<\frac{e}{2}$. Suppose $j>N$, then $d(p_j,p)\leq d(p_j,p_i)+d(p_i,p)<\frac{e}{2}+\frac{e}{2}=e$ and so Q converges to p. If $\{p|p \text{ is in } Q\}$ is finite, there is some integer N such that if $m>N$, then $p_m=p_N$ and Q converges to p_N.

Suppose (M,d) is complete and totally bounded. Suppose K is an infinite subset of M. By Theorem 3.1, since M is totally bounded, K contains a Cauchy sequence of distinct points. And since M is complete, the Cauchy sequence has a sequential limit point p and p is also a limit point of K. Therefore M is countably compact. Suppose A is an open cover which contains no finite subcover. Since (M,d) is completely separable, A contains a countable subcover $A'=\{A_1,A_2,A_3,\ldots\}$. Since no finite subset of A' covers M then there is a point p_1 which is not in A_1. Now p_1 is in some element of A', so let n_1 be the smallest integer such that A_{n_1} contains p_1. Consider the union of all A_k; $1\leq k\leq n_1$. It is an open set containing p_1 and since it is the union of a finite collection of sets in A', there is some point p_2 not in it. Let n_2 be the smallest integer such that p_2 is in A_{n_2}. Following this procedure a sequence of points $S=\{p_1,p_2,p_3,\ldots\}$ and a sequence of sets $B=\{A_1,A_{n_1},A_{n_2},A_{n_3},\ldots\}$ in A' can be formed. Since S is infinite, it has a limit point p such that p is not in S.* There is an open set U of A' such that p is in U. Since A' is countable, there is an integer m such that U is A_m which is in A'. Then there are only a finite number of sets in A' which precede A_m in the sequence A', and so A_m can contain only a finite number of points of the sequence S, say $p_1, p_2, \dots p_k$. But from the regular separation property there is an open set V which contains p and no point of $M-A_m$ and for each p_i, there is an open set V_i which contains p and not p_i. Then the intersection of V and all the V_i is an open set containing p and no point of S. This is a contradiction of the conclusion that the sequence S had a limit point and therefore the assumption that A does not contain a finite subcover is false and (M,d) is compact.

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*The reason that p p_k is that the union of A_n; $1 \le n \le n_k$, is an open set containing p_k and therefore its complement is closed and using the regular separation property there is an open set U containing p_k and no point of the closed set. Also for each p_j; $j<k$, there is an open set U_j containing p_k and not p_j; then the intersection of all the U_j and U is an open set containing p_k and no other point of S.

CHAPTER V

K-CONVEX SETS

The statement that two metric spaces (M,d) and (M',d') are isometric means that there is a function f which maps M onto M' in such a way that if p and q are points in M, then $d(p,q)=d'(f(p),f(q))$. If two metric spaces (M,d) and (M',d') are homeomorphic, then there is a natural way to define a new metric on M' so that the two spaces are isometric. That natural metric would be that the distance between two points in M' would be defined to be the distance between their inverse images in M. In general, there may be more than one way for two metric spaces to be isometric, but it seems likely that their metric functions would have a relatively simple relationship to each other. This chapter attempts to study the relationship between the metrics of two linear metric spaces which are homeomorphic, by looking at the set in the Euclidean plane made up of ordered pairs of numbers where the first number is the distance between two points in one metric space and the second number is the distance between the homeomorphic images of the two points in the second metric space.

Definition 4.1 A function is a set of ordered pairs such that no two ordered pairs have the same first element.

The domain of a function is the set of all first elements of the ordered pairs and the range of a function is the set of all second elements of the ordered pairs. If X and Y are sets, f is a function on X onto Y means X is the domain of f and Y is the range of f.

Definition 4.2 If f is a function, then f inverse (f^{-1}) is $\{(x,y) \mid (y,x) \text{ is in } f\}$.

Definition 4.3 If each of (X,T) and (Y,H) is a topological space, f is a function on X onto Y and p is in X, then "f is continuous at p" means that if U is a neighborhood of f(p), then there is a neighborhood V of p such that if q is in V, then f(q) is in U. The statement that f is continuous means if p is in X, then f is continuous at p.

Definition 4.4 If f is a function, f is a reversibly continuous function means that f is reversible and both f and f^{-1} are continuous. A homeomorphism is a reversibly continuous function. If f is a homeomorphism on X onto Y, then the topological spaces (X,T) and (Y,H) are said to be homeomorphic or topologically equivalent.

Lemma 4.1 If (X,T) and (Y,H) are topological spaces and f is a function on X onto Y, then f is continuous if and only if for each open set U in Y, f^{-1} (U) is open in X.

Proof: Suppose f is continuous. Suppose U is an open set in Y and f(p) is a point in U. Then there is an open set Vp of p such that if q is in Vp, then f(q) is in U.

Let V be the union of all such Vp for f(p) in U. Then V is open. Now $V=f^{-1}(U)$ because if p is in V then f(p) is in U and so p is in $f^{-1}(U)$ and if p is in $f^{-1}(U)$, then f(p) is in U and there is a Vp containing p which is a subset of V and so p is in V. Therefore $f^{-1}(U)$ is open.

Suppose for each open set U in Y, $f^{-1}(U)$ is open in X. Suppose p is in X and U is a neighborhood of f(p). Then there is an open set U' which is a subset of U such that f(p) is in U'. Then $f^{-1}(U')$ is open in X, and if q is in $f^{-1}(U')$ then f(q) is in U' which is a subset of U and therefore f is continuous at p. Since f has been shown to be continuous at the point p for any p in its domain, then f is continuous.

<u>Definition 4.5</u> A linear metric space is a metric space (M,d) with a relation "*" of ordered pairs of elements of M such that:

1. if x, y, z are in M and x*y and y*z, then x*z,
2. if x, y are in M, x*y and y*x, then x=y,
3. if x, y are in M, then x*y or y*x.
4. if x, y, z are in M, then x*y and y*z if and only if $d(x,y)+d(y,z)=d(x,z)$.

<u>Definition 4.6</u> The statement that a set S is separated means it is the union of two non-empty mutually separated sets. The statement that a set S is connected means it is not separated.

If (M,d) is a linear metric space homeomorphic to a subset of real numbers with its usual relative topology, and there is a homeomorphism f on (M,d) onto the metric space (M',d') and a is in M, then let

$$g_a = \{(x, d(a,x)) \mid x \text{ is in } M \text{ and } a \neq x\}$$

and let

$$g'_a = \{(f(x), d'(f(a), f(x))) \mid x \text{ is in } M \text{ and } a \neq x\}.$$

Let

$$h_a = \{(g_a(x), g'_a(f(x))) \mid x \text{ is in } M \text{ and } a \neq x\}.$$

<u>Theorem 4.1</u> If (M,d) is a linear metric space, then if a is in M, then g_a is a homeomorphism.

<u>Proof:</u> Since d is a function, g_a must be a function. To prove g_a is continuous, let x be a point in M such that $a \neq x$, let e be a positive number and let

$$U = \{p \mid p \text{ is a real number and } |g_a(x) - p| < e\}.$$

U is a neighborhood of $g_a(x)$ in the usual relative topology for the range of g_a. Let

$$U_x = \{q \mid q \text{ is in } M \text{ and } d(x,q) < e\}.$$

Now suppose q is in U_x, then $d(a,x) \leq d(a,q) + d(q,x) < d(a,q) + e$ and so $g_a(x) < g_a(q) + e$. Also, $d(a,q) \leq d(a,x) + d(x,q) < d(a,x) + e$, so that $g_a(q) < g_a(x) + e$. Then $g_a(q) - e < g_a(x) < g_a(q) + e$ and $|g_a(x) - g_a(q)| < e$ and $g_a(q)$ is in U. Therefore g_a is continuous at x and, so, is continuous.

Now suppose x and y are elements of M such that a*x and a*y and $d(a,x)=d(a,y)$. Suppose $x \neq y$, then either x*y or y*x. If x*y, then $d(a,x)+d(x,y)=d(a,y)$ and by substitution $d(a,x)+d(x,y)=d(a,x)$ which gives $d(x,y)=0$ and so $x=y$. The result is the same if the assumption y*x is made. Therefore g_a is reversible. To prove g_a^{-1} is continuous, let $g_a(x)$ be a point in the range of g_a, and, for some positive number e, let $U=\{p \mid p \text{ is in } M \text{ and } d(p,x)<e\}$. Now let

$V=\{g_a(y) \mid g_a(y) \text{ is in the range of } g_a \text{ and } |g_a(x)-g_a(y)|<e\}$. Suppose $g_a(y)$ is in V, then $g_a(y)-e<g_a(x)<g_a(y)+e$ and, so, $d(a,y)-e<d(a,x)$ and $d(a,x)<d(a,y)+e$. Since y is in M, then either x*y or y*x. If x*y, then $d(a,x)+d(x,y)=d(a,y)$ and, so $d(x,y)=d(a,y)-d(a,x)<e$. If y*x, then $d(a,y)+d(y,x)=d(a,x)$ and, so, $d(y,x)=d(a,x)-d(a,y)<e$. Then y is in U and g_a^{-1} is continuous and g_a is a homeomorphism.

Theorem 4.2 If M is connected, then g_a' is a homeomorphism.

Proof: The proof that g_a' is a continuous function is the same as the proof of g_a. To prove g_a' inverse is a function, let f(x) and f(y) be points in M' such that

$$d'(f(a),f(x))=d'(f(a),f(y)).$$

Suppose that $f(x) \neq f(y)$, so that $x \neq y$. Then either a*x*y or a*y*x. With no loss of generality, assume a*x*y. Since M is connected, then M' is also connected. Either $f(y)*'f(x)$ or $f(x)*'f(y)$, so suppose $f(x)*'f(y)$. Suppose U is a neighborhood of $f(y)$ which does not contain $f(x)$. Then there must be a neighborhood V of y such that if y' is in V, then $f(y')$ is in U. Then there is some point $r*y$ such that if $r*s*y$ then $f(x)*'f(s)$. Let

$$R=\{r \mid x*r*y,\ f(x)*'f(r) \text{ and if } r*s*y, \text{ then } f(x)*'f(s)\}.$$

Let r' be the greatest lower bound of R. If $r'\neq x$, then either $f(r')*'f(x)$ or $f(x)*'f(r')$. Suppose U' is a neighborhood of r'. Then there is some point of U'_x such that $f(x)*'f(z)$ and some point z' of U' such that $f(z')*f(x)$. Then, no matter whether $f(r')*'f(x)$, or $f(x)*'f(r)$, there is a neighborhood V' of $f(r')$ which does not contain $f(x)$ and so there will be no neighborhood of r' which maps into V'. This is a contradiction of the fact that f is continuous and so r' must be x and for every point p such that $x*p*y$, then $f(x)*'f(p)$. Similarly every point q such that $a*q*x$ is mapped into one side of x. Suppose both sets are mapped to the same side of x. Then there is a point p such that $p*x$ and $f(x)*'f(p)$ and a point q such that $x*q$ and $f(x)*'f(y)$. Suppose $f(q)*'f(p)$. Then there must be some point t such that $p*t*x$ and $f(t)=f(q)$. This is a contradiction that f is reversible and so either

(1) $f(a)*'f(x)*'f(y)$ or

(2) $f(y)*'f(x)*'f(a)$.

If (1) is true then

$$d'(f(a),f(x))+d'(f(x),f(g))=d'(f(a),f(y))$$

and so $d'(f(x),f(y))=0$ and $f(x)=f(y)$. If (2) is true the result is the same and so g_a' is reversible. The proof that g_a' inverse is continuous is the same as that of g_a^{-1}.

To show that if M is not connected, then g_a' may not be reversible, let M be the set of real numbers between 0 and 1 plus the set of real numbers between 2 and 3. Let

$$f(x)=x+2;\ 0\leq x\leq 1 \text{ and}$$

$$f(x)=x-2;\ 2\leq x\leq 3.$$

Then if $a=0$, $g_a'(1)=1$ and $g_a'(3)=1$ and so the ordered pairs (1,1) and (1,3) are in g_0' and g_0' is not reversible.

Theorem 4.3 If M is connected, then h_a is a homeomorphism.

Proof: Suppose $g_a(x)$ and $g_a(y)$ are elements of the domain of h_a such that $g_a(x)=g_a(y)$. Then $x=y$ and $f(x)=f(y)$ and $g_a'(f(x))=g_a'(f(y))$ so that h_a is a function. Since g_a and g_a' are reversible, h_a is also reversible. Since both g_a and g_a' are reversibly continuous, it readily follows that h_a is reversibly continuous. Therefore h_a is a homeomorphism.

Definition 4.7 The statement that a set M in the Euclidean plane is convex means that if $p=(x,y)$ and

$q=(x',y')$ are elements of M, then if $0 \leq t \leq 1$, then

$$r=(tx+(1-t)x',\ ty+(1-t)y')$$

is in M.

Definition 4.8 The statement that a set M in the Euclidean plane is k-convex means that M is connected and if p and q are points of M such that $p=(x,y)$, $q=(x,y')$ and, $y \leq y'' \leq y'$, then $r=(x,y'')$ is in M, and if p and q are points of M such that $p=(x,y)$, $q=(x',y)$ and $x \leq x'' \leq x'$ then $r=(x'',y)$ is in M.

Theorem 4.4 If (M,d) is a connected linear metric space, then the union of all h_a, for a in M, is a k-convex set.

Proof: Suppose p and q are points of the union of all h_a such that $p=(b,c)$ and $q=(b,c')$. Then there is an a in M and an x_a in M such that $b=g_a(x_a)$ and $c=g_a'(f(x_a))$ and an a' and $x_{a'}$ in M such that $b=g_{a'}(x_{a'})$ and

$$c'=g_{a'}'(f(x_{a'})).$$

Either $a*a'$ or $a'*a$, so suppose $a*a'$, then $x_a * x_{a'}$. Now if $a*r*a'$, then there is an x_r such that $x_a * x_r * x_{a'}$ and $g_r(x_r)=b$. Then for $a*r*a'$, let $h(r)=d'(f(r),f(x_r))$. Obviously h is a function. To prove h is continuous, suppose $a*r*a'$ and $e>0$. Let $U_r=\{h(s) \mid$ there is an s such that $a*s*a'$ and $h(s)=d(f(s),f(x_s))$ and $|h(r)-h(s)|<e\}$. Let $Q=\{f(x) \mid d'(f(x_r),f(x))<\frac{e}{2}\}$ and $T=\{f(x) \mid d'(f(r),f(x))<\frac{e}{2}\}$.

Now if $f(x)$ is in T and $f(y)$ is in Q, then

$$\left| d'(f(r),f(x_r))-d(f(x),f(y))\right| < e.$$

There is a neighborhood U of r such that if x is in U, then $f(x)$ is in T and a neighborhood V of x_r such that if x is in V, then $f(x)$ is in Q. There is a real number $t>0$ such that $U'=\{x|d(x,r)<t\}$ and U' is a subset of U and a real number $t'>0$ such that $V'=\{x|d(x_r,x)<t'\}$ and V' is a subset of V. Let t" be the smaller of t and t'. Then

$$V''=\{x|d(x_r,x)<t''\}$$

is a neighborhood of x_r and $U''=\{x|d(r,x)<t''\}$ is a neighborhood of r such that, if z is in U", then x_z is in V". Then U" is a neighborhood of r such that if r' is in U", then $x_{r'}$ is in V" and $f(r')$ is in T and $f(x_{r'})$ is in Q and

$$h(r')=d'(f(r'),f(x_{r'}))$$

and $\left|h(r)-h(r')\right|< e$, so that $h(r')$ is in U_r and h is continuous.

Suppose c" is a real number such that $c<c''<c'$ and $p'=(b,c'')$. Then h is a continuous function on the interval $[a,a']$ such that $h(a)=c$ and $h(a')=c'$ so that there must be an a" such that a*a"*a' and $h(a'')=c''=d'(f(a''),f(x_{a''}))$. Then $d(a,x_a)=b$ and so $p'=(b,c'')$ is in the union of all h_a for a in M. Then if p and q had the same ordinate and different abscissa, the proof is similar.

Suppose the union of all h_a is not connected. Then it is separated and, therefore, is the union of two non-empty mutually separated sets H and K. Suppose p is in H and q is in K such that $p=(b,c)$ and $q=(b',c')$. Then there is an a and an x_a such that $d(a,x_a)=b$ and $d'(f(a),f(x_a))=c$ and an a' and $x_{a'}$ such that $d(d,x_{a'})=b'$ and $d'(f(a'),f(x_{a'}))=c'$. Either $a*a'$ or $a'*a$, so suppose $a*a'$. Then $x_{a'}$ is in the domain of g_a and $g_a(x_{a'})=d(a,a')+g_{a'}(x_{a'})$. Since g_a is continuous, $g_a(a)=0<g_{a'}(x_{a'})$ and $g_a(x_{a'})>g_{a'}(x_{a'})$, then there is some x' such that $g_a(x')=g_{a'}(x_{a'})$. Since g_a is continuous, then $r=(g_a(x'),g_a'(x'))$ must also be in H. But since r and q have the same abscissa, every point between r and q must be in H and so q must be in H, which contradicts the statement that q was in K. Therefore the union of all h_a is connected and the theorem is true.

That the union of all h_a is not a convex set in the Euclidean plane can be seen from the following example. Let M be the set of real numbers from 0 to 6 with the usual distance function and let M' be the set of real numbers from 0 to 26 with the usual distance function. The homeomorphism f on M onto M' will be as follows:

i) $f(x)=4x$ for $0\leq x\leq 3$

ii) $f(x)=6(x-1)$ for $3\leq x\leq 4$

iii) $f(x)=4x+2$ for $4\leq x\leq 6$

Let $p=(3,12)$ and $q=(5,22)$ and $t=\frac{1}{2}$, then $r=(4,17)$. Then for the point 0, $g_0(3)=3$ and $g_0'(f(3))=12$ and $g_0(5)=5$ and $g_0(f(5))=22$. Now all the points a, such that $g_a(x_a)=4$, are between 0 and 2, but if $0\leq a\leq 2$ and $d(a,x_a)=4$ then $d'(f(a),f(x_a))=18$ so that $r=(4,17)$ is not in the union of all h_a and so it is not convex.

Theorem 4.5 If M is the union of two mutually separated sets H and K, where each of H and K is connected, then S, where S is the union of all h_a for a in M, is the union of three k-convex sets.

Proof: Considering the set H and the subset of S obtained using only points in H, by Theorem 4.4 that subset is a k-convex set. The same is true for the set K. Since M is linear and each of H and K is connected either every point of H is less than every point of K or *vice versa*. Suppose every point in H is less than every point in K. The only way to obtain more points in S is to pick an a in H and look at $g_a(x)$ for x in K. Consider the subset S' of S obtained in that way. Suppose p and q are points in S', where $p=(b,c)$ and $q=(b,c')$. Then there is an a in H and an x_a in K such that $d(a,x_a)=b$ and $d'(f(a),f(x_a))=c$ and an a' in H and $x_{a'}$ in K such that $d(a',x_{a'})=b$ and $d'(f(a'),f(x_{a'}))=c'$. Either $a*a'$ or $a'*a$, so suppose $a*a'$. Then $x_a*x_{a'}$. Since a and a' are in H, if r is between a and a' then r is in H and x_r is between x_a and $x_{a'}$. The proof from here is the same as Theorem 4.4.

Theorem 4.6 If M is the union of N mutually separated sets, $K_1, K_2, K_3, \ldots K_N$, where each K_i is connected, then S, where S is the union of all h_a for a in M, is the union of $\frac{N^2+N}{2}$ k-convex sets in the plane.

Proof: If there are N mutually separated sets, the first one will form a k-convex set with itself and with each of the N-1 sets which succeed it. The second will form a k-convex set with itself and with each of the N-2 sets which succeed it. Going on to the last set, it is easy to see the number of k-convex sets formed will be the sum of the first N positive integers, thus $\frac{N^2+N}{2}$.

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