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APPLICATION OF THE BOOTSTRAP TO THE ANALYSIS OF VIBRATION TEST DATA

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Structural dynamic testing is concerned with the estimation of system properties, including frequency response functions and modal characteristics. These properties are derived from tests on the structure of interest, during which excitations and responses are measured and Fourier techniques are used to reduce the data. The inputs used in a test are frequently random, and they excite random responses in the structure of interest. When these random inputs and responses are analyzed they yield estimates of system properties that are random variable and random process realizations. Of course, such estimates of system properties vary randomly from one test to another, but even when deterministic inputs are used to excite a structure, the estimated properties vary from test to test. When test excitations and responses are normally distributed, classical techniques permit us to statistically analyze inputs, responses, and system parameters. However, when the input excitations are non-normal, the system is nonlinear, and/or the property of interest is anything but the simplest, the classical analyses break down. The bootstrap is a technique for the statistical analysis of data that are not necessarily normally distributed. It can be used to statistically analyze any measure of input excitation or response, or any system property, when data are available to make an estimate. It is designed to estimate the standard error, bias, and confidence intervals of parameter estimates. This paper shows how the bootstrap can be applied to the statistical analysis of modal parameters.

INTRODUCTION AND MOTIVATION

Analysis of structural test data follows the sequence shown in Figure 1. Spectral densities and transfer functions are derived using averages of input and response Fourier spectra. Modal frequencies, dampings, and mode shapes are identified by fitting linear oscillatory models to the observed spectral properties. Bias, standard error, and confidence intervals for spectral estimators are described in Bendat and Piersol (1986). These statistics are based on the assumptions of stationary, Gaussian data, and on rather complex statistical derivations. In this paper we explore the accuracy of spectral estimators from another viewpoint. We utilize a recently derived statistical technique called the bootstrap (Efron, 1979) to directly estimate the error bounds associated with the computation of spectral densities and transfer functions from simulated test data. Assumptions of stationarity, linearity, or Gaussianity are not required. We compare the results that are obtained from bootstrap analysis to the theoretical estimates of standard error and confidence intervals for a linear system. We then use the bootstrap to compute standard error and confidence intervals for a strongly nonlinear oscillator and compare the results to theoretical values which are based on the assumptions of linearity and Gaussianity.

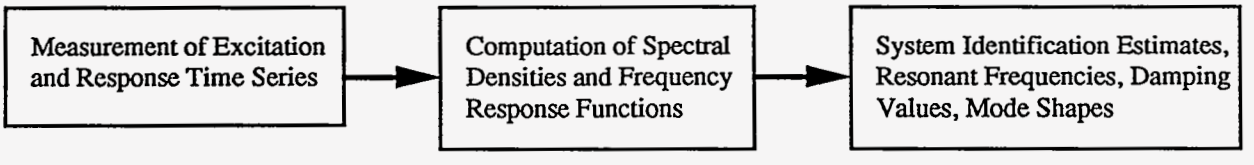

Figure 1. Analysis of structural test data.

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MASTER

THE BOOTSTRAP PROCEDURE

The objective of bootstrap analysis is to assess the accuracy of parameter estimates that are statistics of measured data by estimating standard error, confidence intervals, and bias. To perform a bootstrap analysis, we measure data from a random source and assume that the observed data represent the source. The source is assumed to have an unknown probability distribution. Each observed data point is assigned a probability of occurrence of 1/n, where n is the total number of data points measured. A bootstrap sample of the data is created by selecting at random, with replacement, n elements from the measured data set. This process is illustrated in Figure 2. The procedure is readily implemented using a uniform random number generator which selects, with equal probability, integer values in the range 1 to n. Sampling is done with replacement, so each bootstrap sample may have several occurrences of some data values and other data values may be absent.

$$F \rightarrow X = (x_1, x_2, \ldots, x_n)$$

$$\downarrow$$

$$\hat{F} \rightarrow X^* = (x_2, x_7, \ldots, x_4)$$

Creation of bootstrap sample accomplished through random selection among elements of X.

Figure 2. Obtaining a bootstrap sample.

In a bootstrap analysis, numerous bootstrap samples are created. The statistic of interest is computed from each bootstrap sample; the resulting quantities are known as bootstrap replicates of the statistic of interest. Standard error, confidence intervals, and bias of the statistic of interest are computed using standard techniques and formulas on the bootstrap replicates of the statistic of interest. For example, let B denote the number of bootstrap samples used in an analysis, and let $\hat{\theta}_b^*, b = 1, \ldots, B$, denote the bootstrap replicates of the statistic of interest. Then the standard error of the statistic of interest is estimated with

$$\hat{se} = \left[\frac{1}{B-1} \sum_{b=1}^{B} \left(\hat{\theta}_b^* - \hat{\theta}^* \right)^2 \right]^{1/2} \tag{1}$$

where $\hat{\theta}^* = \frac{1}{B} \sum_{b=1}^{B} \hat{\theta}_b^*$.

In one type of bootstrap analysis, the two-sided, $(1-\alpha)\times 100\%$ confidence intervals are obtained by sorting the bootstrap replicates of the statistic of interest, and identifying (or interpolating) the $(\alpha/2)\times 100\%$ percentile value and the $(1-\alpha/2)\times 100\%$ percentile value in the sorted list, and using the identified values as the limits of the confidence interval. Another more advanced method for confidence interval estimation is discussed in Efron and Tibshirani (1993).

The number of bootstrap samples, B, used in an analysis, ranges from 20 to several thousand. The standard error of a parameter estimate may be computed using 25 to 50 bootstrap samples. Accurate computation of the confidence intervals of an estimated parameter requires analysis of several thousand bootstrap samples.

Bootstrap sampling provides an optimal estimate of the probability density function which characterizes the data source given that our knowledge of the source is limited to the measured data. Computation of a statistic from the bootstrap samples simulates computation of the same statistic on samples drawn from the real world distribution. Properties of the "real world" distribution are estimated in the "bootstrap world" as illustrated in Figure 3.

DISCLAIMER

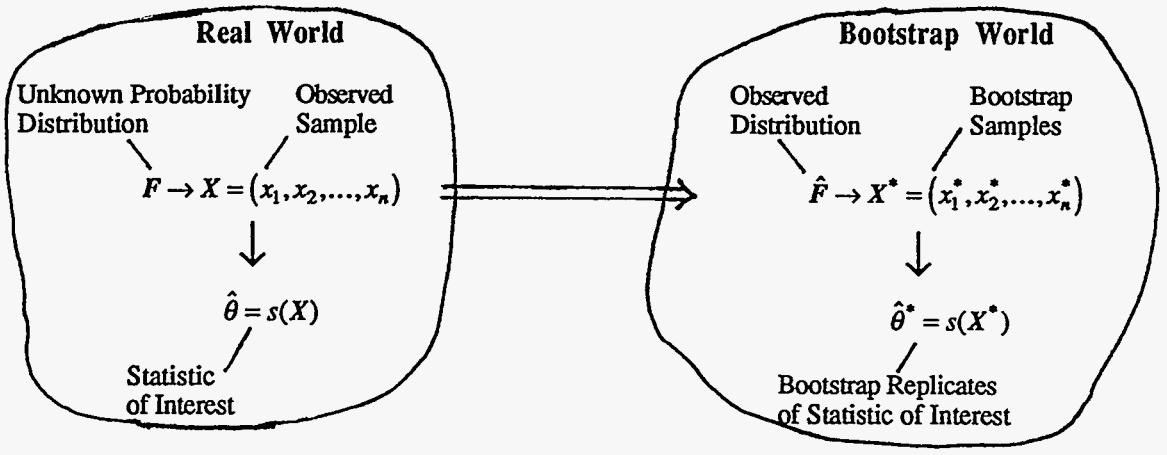

Figure 3. The bootstrap approximation to the real world. The observed distribution is our best estimate of the true distribution. The observed sample is X, and the statistic of interest $\hat{\theta} = s(X)$ can be computed based on this. In the bootstrap world the observed data are used to generate as many bootstrap samples X^* as we wish. Each bootstrap sample is used in the formula $\hat{\theta}^* = s(X^*)$ to compute a bootstrap replicate of the statistic of interest. The bootstrap replicates are used to analyze the standard error, confidence intervals and bias of the statistical estimator.

A BOOTSTRAP EXAMPLE

Consider a set of data drawn from a random source with the probability density function illustrated in Figure 4. One hundred data points are generated using a random source with this density.

We assume that the 100 points are characteristic of the source. The mean of the sampled points is 1.3440. Using the bootstrap procedure outlined above, we create 400 bootstrap samples of these 100 points. (Normally, each bootstrap sample contains as many points as are available in the original measured data set.) From each sample we compute the sample mean. The standard deviation of these sample means is 0.0660; this is the standard error of the mean estimate. The theoretical mean of this distribution is 1.3333. The 400 bootstrap replications of the original data also allow computation of confidence intervals on our estimated mean. The 99% confidence intervals on the mean are 1.1471 to 1.5073. The true mean lies well within these intervals. To further illustrate the typical bootstrap results, Table I shows the results of seven different realizations of the distribution. In each case, the true mean lies well within the confidence intervals indicated. Note that, as expected, a smaller number of points leads to a broader confidence interval.

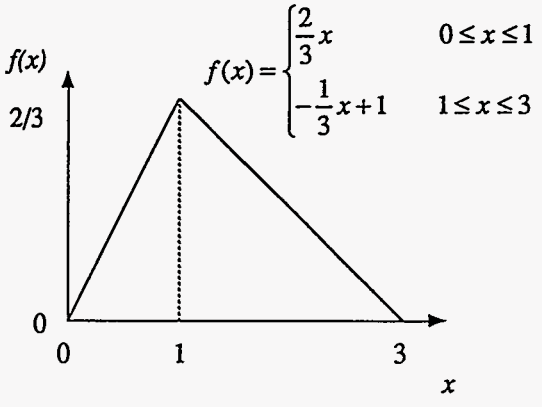

Figure 4. A random data source.

Table I.
Mean and confidence intervals for different realizations of the random data source shown in Figure 4.

NPTS Generated	Mean Estimate	Std. Err. Mn. Est.	Lower 99%C.I.	Upper 99%C.I.	True Mean
100	1.3440	0.0660	1.1471	1.5073	1.3333
100	1.3571	0.0612	1.1325	1.5250	1.3333
50	1.3274	0.0881	1.1042	1.5677	1.3333
25	1.3250	0.0898	1.0513	1.5287	1.3333

NPTS Generated	Mean Estimate	Std. Err. Mn. Est.	Lower 90%C.I.	Upper 90%C.I.	True Mean
25	1.3761	0.1181	1.1862	1.5898	1.3333
50	1.2419	0.0903	1.1046	1.3949	1.3333
100	1.3607	0.0635	1.2605	1.4706	1.3333

APPLICATION OF THE BOOTSTRAP TO THE ANALYSIS OF SPECTRAL ERRORS

We showed in the previous section that the bootstrap is a technique for the accuracy analysis of statistical estimators. It can be used to estimate standard error, and confidence intervals of statistical estimators. Spectral analysis involves the estimation of signal and system measures in the frequency domain, based on measured realizations of stationary random signals. We are frequently interested in estimating such quantities as the spectral density of a single signal, and the cross-spectral density, coherence, and frequency response function for a pair of signals. The bootstrap can be used to assess the accuracy of all these signal and system measures, and many more, regardless of the distribution of the underlying random signals.

In the following sections we write the general formulas for the estimation of autospectral density, cross-spectral density, and the single input/single output (SISO) frequency response function. Then we show how the bootstrap method can be used to develop estimates of the standard error and confidence intervals on all three.

BOOTSTRAP ANALYSIS OF AUTOSPECTRAL DENSITY

The estimator formula for the autospectral density of a stationary random process can be obtained via the method of maximum likelihood or directly, as in Bendat and Piersol (1986). It is usually based on the assumption that we have a single measured realization from a stationary random process $\{X(t), -\infty < t < \infty\}$. In the present analysis, we make no assumption regarding the probability distribution of the random process. The measured signal is sampled (temporally discretized) at a constant rate, and can be denoted $x_j, j = 0,...,N-1$. These values represent the signal at times $t_j = j\Delta t, j = 0,...,N-1$. Here Δt is the time increment between measured values of the sampled signal; N is the total quantity of points available for analysis. To estimate the spectral density, we form M data sequences from the original sampled signal, and we denote these $x_{jm}, j = 0,...,n-1, m = 1,...,M$. (The subscript j indexes time, and the subscript m indexes sequence number.) The first data sequence simply consists of the first n points in the original sampled signal. The second data sequence may start at point $n+1$ in the original sampled signal, or it may overlap the first sequence (to a limited extent). The third data sequence is defined similarly, etc. The final data sequence usually consists of the final n points in the original data sequence. If the data sequences overlap, then the overlap percentage is constant (or as close to a constant as possible). The M data sequences obtained from the original sampled signal form an ensemble of realizations of the original random process. When members of the ensemble overlap one another, the overlap is usually limited to about 50 percent. The reason is that this practice, known as overlap processing, starts to lose its effectiveness at this point. See Smallwood (1994) for more details.

The next step usually taken in autospectral density estimation is windowing of the signals in the ensemble. The reason for windowing is to limit leakage in the finite duration Fourier transform to be taken, later. Various windows and their characteristics are described in Harris (1978) and Nuttall (1981). Windowing is usually accomplished by multiplying each ensemble member by a temporal weighting, denoted $w_j, n = 0,...,n-1$. The windowed versions of the ensemble members are denoted $u_{jm}, j = 0,...,n-1, m = 1,...,M$.

In the next step, we compute the discrete Fourier transform (DFT) of each windowed member of the ensemble. The DFTs are denoted $U_{km}, k = 0,...,n-1, m = 1,...,M$. (The subscript k indexes frequency.) These represent the finite Fourier transforms of the windowed ensemble members at the frequencies $f_k = k\Delta f, k = 0,...,n/2$, where $\Delta f = 1/(n\Delta t)$. Because the DFT of a real-valued signal has certain symmetries about the index of the Nyquist frequency, $f_{n/2} = 1/(2\Delta t)$, we usually only consider the DFT up to the frequency index $k=n/2$. The DFT is usually accomplished using the fast Fourier transform (FFT) algorithm for reasons of efficiency. For a clear description of the FFT, see Stearns (1975).

Finally, we compute the modulus squared of each DFT in the ensemble, at each frequency index, and use the resulting quantities to form the autospectral density estimate for the random process. It is

$$\hat{G}_{XX}(f_k) = \frac{2Q\Delta t}{Mn} \sum_{m=1}^{M} |U_{km}|^2 \qquad k = 0,...,n/2 \tag{2}$$

where the quantity Q is a normalizing factor that takes into account the windowing of the original sampled signal. (When a Hanning window is used, $Q=8/3$.) This is the formula for autospectral density estimation used in most applications.

The bootstrap can be applied to the accuracy assessment of the autospectral density estimator, $\hat{G}_{XX}(f_k)$, one spectral line (frequency index) at a time, to evaluate standard error and confidence intervals. This is done in the following. Because the autospectral density estimate at each frequency index is simply the average of a sequence of numbers (the moduli squared of the DFTs of the windowed signal in the ensemble) multiplied times a normalizing factor, the bootstrap estimates of the standard error, etc., can be executed as described in the previous section. The only interpretation that needs to be made is: what is a bootstrap sample in the present application?

In a manner analogous to the way we defined a bootstrap sample for the simple average, we define a bootstrap sample associated with the autospectral density estimator by choosing a sample of size M at random, with replacement, from among the quantities, $|U_{km}|^2, m = 1,...,M$. Denote the bth bootstrap sample U_b; it is defined

$$U_b = \left(|U_{k_{1b}}|^2 \, |U_{k_{2b}}|^2 \, \text{K} \, |U_{k_{Mb}}|^2 \right) \qquad b = 1,...,B \tag{3}$$

where each of the indices $k_{mb}, m = 1,...,M, b = 1,...,B$, is a uniformly distributed discrete random digit from the interval [1,M], and we define B bootstrap samples. For example, when M=32, the first bootstrap sample might be $U_1 = \left(|U_7|^2 \, |U_{18}|^2 \, \text{K} \, |U_4|^2 \right)$.

Corresponding to the bootstrap samples, there are bootstrap replicates of the autospectral density estimate, denoted $\hat{G}_{XXb}(f_k), b = 1,...,B$, and defined

$$\hat{G}_{XXb}(f_k) = \frac{2Q\Delta t}{Mn} \sum_{m=1}^{M} |U_{k_{mb}}|^2 \qquad b = 1,...,B \tag{4}$$

Each bootstrap replicate of the autospectral density estimator is computed using an expression similar to Eq. (2), but in place of the entire sequence of quantities $|U_{km}|^2, m = 1,...,M$, we use the elements of the bth bootstrap sample defined in Eq. (3). The bootstrap replicates defined in Eq. (4) are now be used to establish the bootstrap estimates of standard error and confidence intervals on the autospectral density estimator, as described in a previous section. The standard error and confidence intervals are established at all frequencies as they were above for the frequency f_k.

BOOTSTRAP ANALYSIS OF CROSS-SPECTRAL DENSITY

The cross-spectral density between a pair of random processes is estimated in a manner completely analogous to the way in which the autospectral density was estimated. Let $\{X(t), -\infty < t < \infty\}$ and $\{Y(t), -\infty < t < \infty\}$ be a pair of random processes whose cross-spectral density we wish to estimate and assess. As in the previous analysis, we make no assumption regarding the probability distributions of the random processes; indeed, the two random processes need not have the same probability distribution. The measured signals are sampled at a constant rate, and denoted $x_j, y_j, j = 0,...,N-1$. To estimate the cross-spectral density, we form M data sequences from each of the original sampled signals, and denote these $x_{jm}, y_{jm}, j = 0,...,n-1, m = 1,...,M$, as we did to obtain the autospectral density estimate. The M data sequences obtained from each of the original sampled signals form an ensemble of joint realizations of the original random processes.

By analogy to the procedure for autospectral density estimation, the ensembles $x_{jm}, y_{jm}, j = 0,...,n-1, m = 1,...,M$, are windowed then FFTed to obtain $U_{km}, V_{km}, k = 0,...,n-1, m = 1,...,M$, respectively. These are used in the following formula to estimate the cross-spectral density.

$$\hat{G}_{XY}(f_k) = \frac{2Q\Delta t}{Mn} \sum_{m=1}^{M} U_{km} V_{km}^{*} \qquad k = 0,...,n/2 \tag{5}$$

where the asterisk superscript in the formula refers to the operation of complex conjugation.

The bootstrap can be applied to the accuracy assessment of the cross-spectral density estimate. The first step is to define bootstrap samples of the products that appear in the sum of Eq. (5). The bth bootstrap sample can be defined

$$(UV)_b = \left((U_{k_{1b}} V^*_{k_{1b}})\,(U_{k_{2b}} V^*_{k_{2b}}) \text{K}\ (U_{k_{Mb}} V^*_{k_{Mb}})\right) \qquad b = 1,...,B \tag{6}$$

The bootstrap samples can be used to compute bootstrap replicates of the cross-spectral density. The bth bootstrap replicate of the cross-spectral density estimate is

$$\hat{G}_{XYb}(f_k) = \frac{2Q\Delta t}{Mn} \sum_{m=1}^{M} U_{k_{mb}} V^*_{k_{mb}} \qquad b = 1,...,B \tag{7}$$

The bootstrap replicates defined in Eq. (7) are now used on a frequency by frequency basis to establish the bootstrap estimates of standard error and confidence intervals on measures of the cross-spectral density estimator, as described in a previous section. Specifically, we may be interested in analyzing the magnitude and phase or the real and imaginary parts of the cross-spectral density. Or we may be interested in analyzing the joint confidence intervals of pairs of these quantities; the method for doing this will be presented in a later paper. The standard error and confidence intervals are established at all frequencies as they were above for the frequency f_k.

BOOTSTRAP ANALYSIS OF THE SISO FREQUENCY RESPONSE FUNCTION

We now consider estimation of the single input-single output (SISO) frequency response function. The formula used to obtain this estimator simply makes use of the estimators obtained above for the auto- and cross-spectral densities. We let $\{X(t), -\infty < t < \infty\}$ and $\{Y(t), -\infty < t < \infty\}$ represent input and response random processes. Then the SISO frequency response function estimator is the ratio between the response to input cross-spectral density and the input autospectral density. That is

$$\hat{H}_{YX}(f_k) = \frac{\hat{G}_{YX}(f_k)}{\hat{G}_{XX}(f_k)} = \frac{\sum_{m=1}^{M} V_{km} U^*_{km}}{\sum_{m=1}^{M} |U_{km}|^2} \qquad k = 0,...,n/2 \tag{8}$$

To perform a bootstrap analysis on the frequency response function we first form bootstrap samples of the summands in the numerator and denominator on the right side of Eq. (8). These are

$$(VU)_b = \left((V_{k_{1b}} U^*_{k_{1b}})\ (V_{k_{2b}} U^*_{k_{2b}}) \text{K}\ (V_{k_{Mb}} U^*_{k_{Mb}})\right), \quad U_b = \left(|U_{k_{1b}}|^2\, |U_{k_{2b}}|^2 \text{K}\ |U_{k_{Mb}}|^2\right) \qquad b = 1,...,B \tag{9}$$

As before, the indices $k_{mb}, m = 1,...,M, b = 1,...,B$, are discrete valued random numbers chosen uniformly on the interval $[1,M]$; note that the indices are the same for the two bootstrap samples. In these terms the bootstrap replicates of the frequency response function estimate are defined

$$\hat{H}_{YXb}(f_k) = \frac{\sum_{m=1}^{M} V_{k_{mb}} U^*_{k_{mb}}}{\sum_{m=1}^{M} |U_{k_{mb}}|^2} \qquad b = 1,...,B \tag{10}$$

The bootstrap replicates defined in Eq. (10) are now be used to establish the bootstrap estimates of standard error and confidence intervals on measures of the frequency response function estimator, as described in a previous section. We may be interested in analyzing the magnitude and phase or the real and imaginary parts of the frequency response function. The standard error and confidence intervals are established at all frequencies as they were above for the frequency f_k.

CLASSICAL FORMULAS FOR STANDARD ERROR AND CONFIDENCE INTERVALS

For purposes of comparison in some numerical examples to follow, we now present the classical formulas for the standard error and confidence intervals of auto- and cross-spectral density and the SISO frequency response function. The confidence intervals for the latter two functions are approximate. All the formulas in this section rely on the assumption that the signals under consideration come from a Gaussian source. The complete developments for the formulas are given, for example, in Bendat and Piersol (1986).

The sampling distribution of the spectral density estimator of a stationary random process is related to the chi square distribution. Specifically, the quantity $2M\left(\hat{G}_{XX}(f_k)/G_{XX}(f_k)\right)$ has a chi square distribution with $2M$ degrees of freedom, where M is the number of averages used to obtain the estimate and $G_{XX}(f_k)$ is the actual spectral density of the underlying random process. This fact can be used to compute confidence intervals for the autospectral density estimator. It is also related to the fact that the standard error of the autospectral density estimator is

$$se\left(\hat{G}_{XX}(f_k)\right)=\frac{G_{XX}(f_k)}{\sqrt{M}} \qquad k=0,...,n/2 \tag{11}$$

The standard errors of the magnitudes of the cross-spectral density and the frequency response function estimates are approximately

$$se\left(\left|\hat{G}_{XY}(f_k)\right|\right)=\frac{1}{\left|\gamma_{XY}(f_k)\right|\sqrt{M}}, \quad se\left(\left|\hat{H}_{YX}(f_k)\right|\right)=\frac{\left(1-\gamma_{YX}^2(f_k)\right)^{1/2}}{\left|\gamma_{YX}(f_k)\right|\sqrt{2M}} \qquad k=0,...,n/2 \tag{12}$$

where $\gamma_{XY}^2(f_k)$ is the ordinary coherence between the random processes. These estimators have sampling distributions that are very complicated, therefore, we usually use a normal assumption to develop their confidence intervals. Specifically, we assume that the estimators are unbiased with standard deviations given by the standard errors in Eq. (12).

The following section presents examples where the results presented here are compared to results obtained using the bootstrap.

NUMERICAL EXAMPLE
APPLICATION OF THE BOOTSTRAP TO LINEAR SYSTEM DATA

In applying the bootstrap technique to the estimation of spectral density and transfer functions, we consider two structural systems. These systems are simulated using the MATLAB Code "Simulink". The first system is a linear base-excited oscillator, the second a nonlinear hardening oscillator. The bootstrap estimates of confidence intervals compare accurately with those obtained from the classical formulas for the linear system, but differ from the classical estimates for the nonlinear system.

The oscillator is illustrated in Figure 5. Excitation is applied to the base mass M_0 and the response is measured on the top mass M_1. The equation governing this system is:

$$\ddot{x}_1+2\zeta\omega_n(\dot{x}_1-\dot{x}_0)+\omega_n^2(x_1-x_0)=0 \tag{13}$$

where $\omega_n=2\pi$ is the system natural frequency in rad/sec, $\zeta=0.02$ is the system damping factor, and dots denote differentiation with respect to time. The system is excited with Gaussian random noise, with a cutoff frequency of 8 Hertz. Both the input and response data are sampled at 20 samples/second. Figure 6 shows the estimated input excitation-response transfer function. As expected, there is a clean peak in the transfer function at 1.0 Hz., the theoretical system resonant frequency. A similar peak occurs in the spectral density of the response.

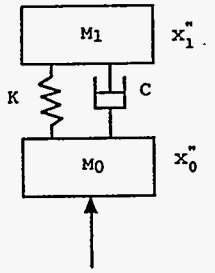

Figure 5. SDF oscillator.

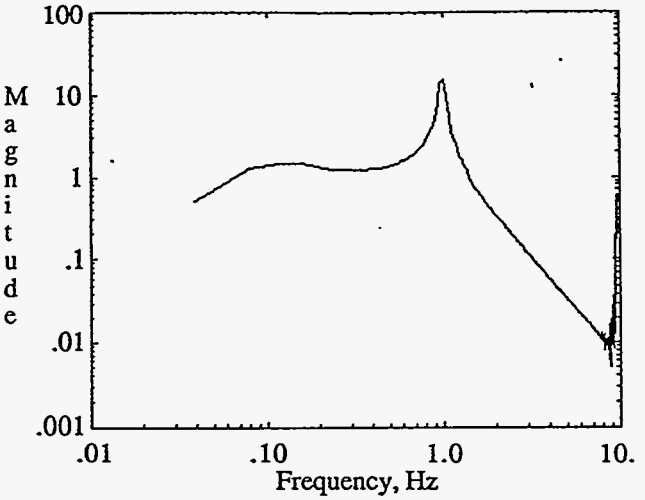

Figure 6. SDF system transfer function.

To estimate the standard deviation and confidence intervals of the spectral density of the linear system, we measure 20,000 point time series of excitation and response then break the data into 512 point blocks, discarding the last partially filled blocks. An overlap of 50%, or 256 points is used. We then form bootstrap samples using the 77 resulting blocks. As in the typical bootstrap, each bootstrap sample of the response consists of a random selection of the 77 input and corresponding response blocks. Altogether 2000 bootstrap samples are generated, and the spectral density and transfer function computed for each block set. The statistics on these 2000 estimates of the spectral density and the transfer function are then used to compute the appropriate confidence intervals. Theoretical classical intervals are readily computed for both the power spectrum and the transfer function. The classical confidence intervals on the spectral density are governed by the chi square distribution. These intervals are directly compared to the bootstrap intervals in Figure 7. Note that the bootstrap confidence intervals and the theoretical intervals are quite similar over the frequency range shown, as would be expected for a linear system.

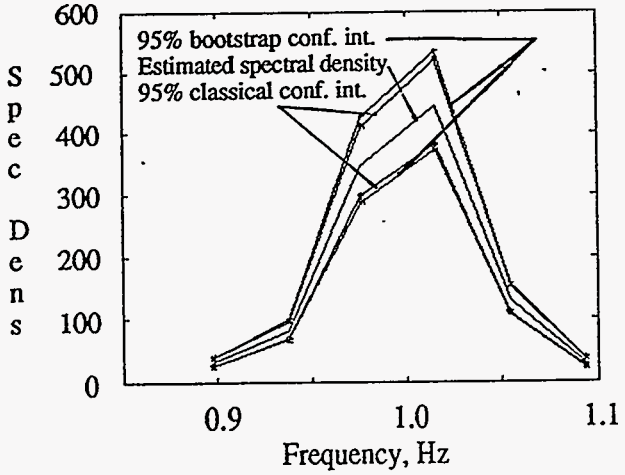

Figure 7. Confidence intervals for the spectral density of the linear oscillator.

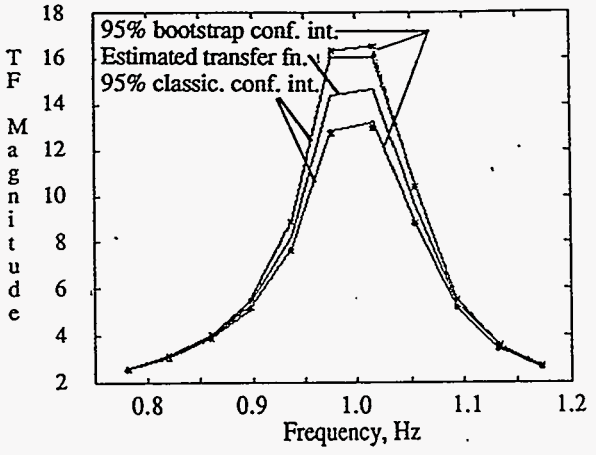

Figure 8. Confidence intervals of the linear system transfer function.

In a manner analogous to that used for the spectral density, the confidence intervals for the transfer function of the linear system are computed. These intervals are shown in Figure 8. Classical confidence intervals are approximated from the known values of the transfer function and coherence at each frequency (Bendat and Piersol, 1986) and the normal assumption. The 95% confidence intervals equal the estimated value plus or minus 1.6445 times the standard error at each frequency. The bootstrap and theoretical confidence intervals are quite similar throughout the frequency range shown, as would be expected for a linear system.

NUMERICAL EXAMPLE
APPLICATION OF THE BOOTSTRAP TO NONLINEAR SYSTEM DATA

We have shown that the bootstrap procedure gives reasonable estimates for the confidence intervals on the spectral density and the transfer function for a linear oscillator. In the linear case, we fortunately have reliable theoretical results to benchmark the bootstrap computations. In the case of a nonlinear oscillator, confidence interval estimates can readily be computed using the bootstrap. Theoretical confidence intervals are difficult to compute since the system response is non-Gaussian.

The nonlinear system of interest is described by the equation :

$$\ddot{x}_1 + 2\zeta\omega_n(\dot{x}_1 - \dot{x}_0) + \omega_n^2(x_1 - x_0) + 1000 \times \omega_n^2(x_1 - x_0)^3 = 0 \qquad (14)$$

where $\omega_n = 2\pi$ is the system natural frequency in rad/sec, $\zeta = 0.02$ is the system damping factor, and dots denote differentiation with respect to time. The system formulation is identical to that used for the linear base excited system in Figure 5, with the addition of the cubic term. The magnitude of the driving acceleration $\ddot{x}_0$ is adjusted to produce a cubic force that is approximately equal in magnitude to the force generated by the linear stiffness term.

The estimated autospectral density of the response $\ddot{x}_1$ is illustrated in Figure 9. The effect of the nonlinearity is evident in the increase in frequency in the first peak from the 1.0 Hertz observed in the linear system to 1.80 Hz. in Figure 9. Substantial third harmonic response is also evident near 6 Hz. Using the bootstrap technique described for the linear system, we form 77 windowed blocks of data, and sample, with replacement, 77 of these blocks at random 2000 times, producing 2000 bootstrap samples. Each bootstrap sample is used for the computation of spectral density and transfer functions, producing 2000 bootstrap replications of transfer function and spectral density. Confidence intervals are computed at each frequency using these 2000 data sets. The results are illustrated in Figures 10 and 11. Confidence intervals for the response autospectral density are shown in Figure 10, which contrasts the classical 95% confidence intervals with the 95% confidence intervals obtained from the bootstrap analysis. The bootstrap confidence intervals are significantly wider then the classical confidence intervals, indicating that the nonlinear behavior leads to more than the expected degree of uncertainty in the spectral computations. This result is intuitively appealing since the variation in the spectral response covers a greater amplitude range than observed in the linear case as the cubic stiffness is a major determinant of the response during some time periods and nearly absent in other time periods.

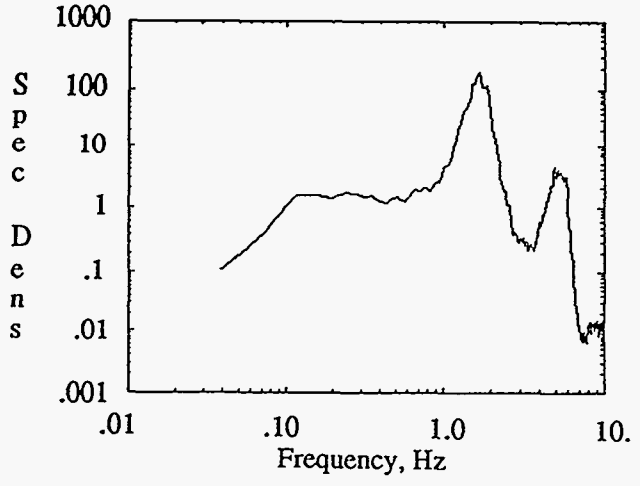

Figure 9. Estimated response autospectral density for the nonlinear oscillator.

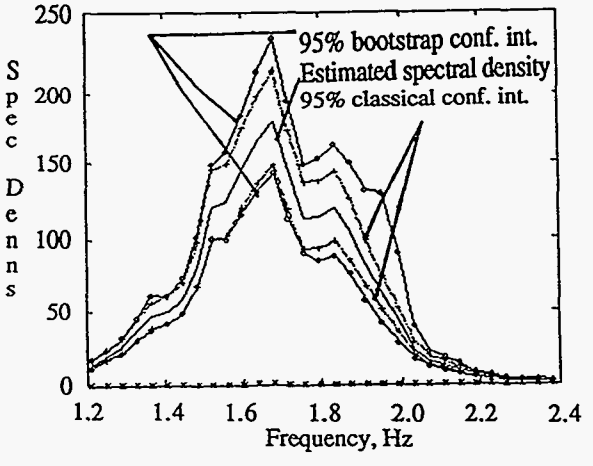

Figure 10. Confidence intervals for the autospectral density of the cubic oscillator near the primary resonant frequency.

Classical confidence intervals for the magnitude of the autospectrum are readily calculated (Bendat and Piersol, 1986). These confidence intervals are based on assumptions of Gaussian, stationary data and assume a chi squared distribution of the resultant spectral level at each frequency, just as in the case of the linear system above. The bootstrap confidence intervals are significantly broader than those predicted theoretically based on Gaussian data.

The estimated transfer function, bootstrap confidence intervals, and classical confidence intervals (based on the magnitude of the coherence function) for the cubic oscillator are illustrated in Figure 11. In contrast to the results obtained for the spectral density, the bootstrap confidence intervals are narrower than those predicted by the normal distribution-based theory. It appears, based on this example, that the low coherence in the vicinity of the resonance leads to excessively broad error bounds on the transfer function. The effects that leakage from adjacent blocks and nonlinearity have in this case is a topic for further study.

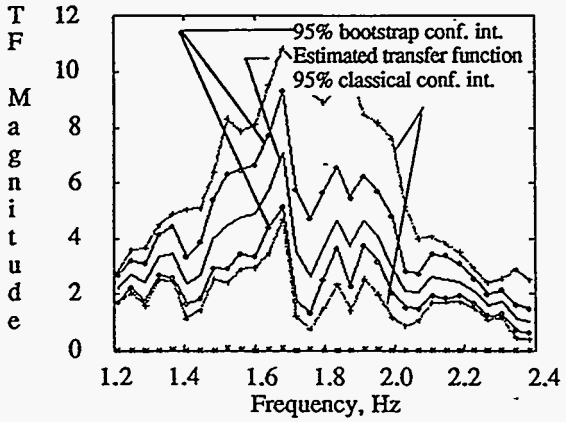

Figure 11. Confidence intervals of the cubic system transfer function.

CONCLUSIONS

The bootstrap approach provides an interesting alternative to traditional statistical analysis in numerous situations involving real-world data. We have shown, for a linear system driven by Gaussian noise, that one effective bootstrap technique uses a uniform random sample of windowed, indexed data blocks. When this procedure is used, the bootstrap results compare very accurately with those obtained from classical formulas. In contrast, for a strongly nonlinear hardening (cubic) oscillator, the examples show that the bootstrap results differ substantially from the results of classical analysis. Of course, this is to be expected because the classical results rely on Gaussian assumptions and the system response is nonGaussian. The bootstrap does not rely on assumptions of Gaussian distribution in the excitation or response data. The primary difficulty in application of the bootstrap to analysis of statistical accuracy is that it is very computer intensive.

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