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Implementation of Rotary Inertia Effect on the Free Vibration Response of Beams

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1 Technical Background

The effects of shear deformation and rotational inertia become significant when the span-depth ratio of the beam is relatively small. The free vibrational response of a beam including the effects of shear deformation and rotational inertia is given by [1]

$$EI\frac{\partial^4 v}{\partial x^4} + \bar{m}\,\frac{\partial^2 v}{\partial t^2} - \bar{m}\,r^2\,\frac{\partial^4 v}{\partial x^2 \partial t^2} + \frac{\bar{m}}{k'AG}\left(\bar{m}\,r^2\,\frac{\partial^4 v}{\partial t^4} - EI\frac{\partial^4 v}{\partial x^2 \partial t^2}\right) = 0 \qquad (1)$$

where E, and G denote the Young's modulus and shear modulus respectively; k' denotes the shear correction factor; $\bar{m}$ denotes the mass per unit length of the beam; I, A, and r denote the moment of inertia, area of cross-section and the radius of gyration respectively; and v denotes the transverse displacement of the beam.

Using the separation of variables approach, the transverse response $v(x,t)$ can be decomposed as

$$v(x,t) = \phi(x)\;Y(t) \qquad (2)$$

Assuming harmonic response in time $Y(t) = Y_0\;\sin\omega t$, Eq. [1] can be written as

$$EI\phi^{iv}(x) - \bar{m}\,\omega^2\,\phi(x) + \bar{m}\,r^2\,\omega^2\,\phi''(x) + \frac{\bar{m}\omega^2}{k'AG}\left[\bar{m}\,r^2\omega^2\,\phi(x) + EI\phi''(x)\right] = 0 \qquad (3)$$

Consider the case of a simply supported beam. In such a case, the vibrational mode shapes that satisfy the essential boundary conditions can be chosen as

$$\phi(x) = \Phi_0 \, \sin \frac{n\pi x}{L} \tag{4}$$

Substituting Eq. [4] into Eq. [3] and letting $a^4 = \frac{\bar{m}\omega^2}{EI}$, we have

$$\left(\frac{n\pi}{L}\right)^4 - a^4 - a^4\, r^2 \left(\frac{n\pi}{L}\right)^2 \left(1 + \frac{E}{k'G}\right) + a^4\, r^2 \left(a^4\, r^2\, \frac{E}{k'G}\right) = 0 \tag{5}$$

Solving the above equation, we get

$$a^4 = \frac{\left[-C_2 - (C_2^2 - 4\, C_1\, C_3)^{\frac{1}{2}}\right]}{2\, C_1} \tag{6}$$

where

$$C_1 = r^4\, \frac{E}{k'G} \tag{7}$$

$$C_2 = -\left[1 + \left(\frac{n\pi r}{L}\right)^2 \left(1 + \frac{E}{k'G}\right)\right] \tag{8}$$

$$C_3 = \left(\frac{n\pi}{L}\right)^4 \tag{9}$$

and

$$\omega_n^2 = \frac{EIa^4}{\bar{m}} \tag{10}$$

REMARK 1: It should be noted that the free vibrational response of a elementary beam model can be obtained simply by considering the first two terms of Eq. [1]. Consequently, the natural frequencies are obtained by considering the first two terms of Eq. [5] and are given by

$$a^4 = \left(\frac{n\pi}{L}\right)^4 \tag{11}$$

$$\omega_n^2 = n^4\, \pi^4 \left(\frac{EI}{\bar{m}\, L^4}\right) \tag{12}$$

REMARK 2: Free vibrational response of a simply supported beam including just the shear deformation effect is obtained by considering the first two terms and the second part of third term in Eq. [1]. Using the same terms in Eq. [5], the natural frequencies are obtained as

$$\omega_n^2 = \left(\frac{n\pi}{L}\right)^4 \left(\frac{EI}{\bar{m}}\right) \left[\frac{1}{1+\left(\frac{n\pi r}{L}\right)^2 \left(\frac{E}{k'G}\right)}\right] \tag{13}$$

REMARK 3: It is also to be noted that in general, an approximate expression for natural frequency including the effect of shear deformation and rotary inertia is given by

$$\omega_n^2 = \left(\frac{n\pi}{L}\right)^4 \left(\frac{EI}{\bar{m}}\right) \left[\frac{1}{1+\left(\frac{n\pi r}{L}\right)^2 \left(1+\frac{E}{k'G}\right)}\right] \tag{14}$$

REMARK 4: For the sake of completeness, the natural frequency in the axial vibration mode is given by (see page 325 of [1])

$$\omega_n = \frac{2n-1}{2}\,\pi\,\left(\frac{EA}{\bar{m}\,L^2}\right)^{\frac{1}{2}} \tag{15}$$

References

[1] R.W. Clough and Joseph Penzien. *Dynamics of Structures.* McGraw-Hill, 1989.

Numerical Example: Simply-Supported Beam

Geometric and material Properties:

E = 2e11; nu = 0.3; b = 2; h = 1; I = 2/12; A = 2; rho = 1e-4

shear factor = 5/6;

Output:

1) No rotary effect; L = 1000.0

Theoretical:

(frequency)		(radians)
mode number=	1	1.2741604E+02
mode number=	2	5.0966418E+02
mode number=	3	1.1467444E+03
mode number=	4	2.0386567E+03
mode number=	5	3.1854011E+03

Beam type = 11:

(frequency)		(radians)	(hertz)	(period)
mode number=	1	1.27416045E+02	2.02788934E+01	4.93123556E-02
mode number=	2	5.09664174E+02	8.11155726E+01	1.23280890E-02
mode number=	3	1.14674434E+03	1.82510030E+02	5.47915091E-03
mode number=	4	2.03865636E+03	3.24462238E+02	3.08202276E-03
mode number=	5	3.18539977E+03	5.06972119E+02	1.97249506E-03

Hughes-Liu Beam type:

(frequency)		(radians)	(hertz)	(period)
mode number=	1	1.27421122E+02	2.02797015E+01	4.93103906E-02
mode number=	2	5.09745441E+02	8.11285067E+01	1.23261236E-02
mode number=	3	1.14715605E+03	1.82575555E+02	5.47718449E-03
mode number=	4	2.03995885E+03	3.24669535E+02	3.08005492E-03
mode number=	5	3.18858375E+03	5.07478865E+02	1.97052541E-03

Beam type = 21:

(frequency)	(radians)	(hertz)	(period)

mode number=	1	1.27431741E+02	2.02813914E+01	4.93062818E-02
mode number=	2	5.09668098E+02	8.11161972E+01	1.23279941E-02
mode number=	3	1.14674609E+03	1.82510308E+02	5.47914258E-03
mode number=	4	2.03865735E+03	3.24462394E+02	3.08202127E-03
mode number=	5	3.18540040E+03	5.06972219E+02	1.97249467E-03

2) Including rotary inertia effect; L = 1000.0; 100 elements => Lel = L/100

Theoretical: r^2 = (Lel^2)/4

(frequency)		(radians)
mode number=	1	1.2740033E+02
mode number=	2	5.0941286E+02
mode number=	3	1.1454733E+03
mode number=	4	2.0346444E+03
mode number=	5	3.1756218E+03

Beam type = 11: r^2 = (Lel^2)/4

(frequency)		(radians)	(hertz)	(period)
mode number=	1	1.27400328E+02	2.02763920E+01	4.93184388E-02
mode number=	2	5.09412851E+02	8.10755733E+01	1.23341712E-02
mode number=	3	1.14547319E+03	1.82307721E+02	5.48523121E-03
mode number=	4	2.03464408E+03	3.23823663E+02	3.08810045E-03
mode number=	5	3.17562036E+03	5.05415677E+02	1.97856941E-03

Hughes-Liu Beam type: r^2 = I/A

(frequency)		(radians)	(hertz)	(period)
mode number=	1	1.27421070E+02	2.02796931E+01	4.93104108E-02
mode number=	2	5.09744602E+02	8.11283731E+01	1.23261439E-02
mode number=	3	1.14715180E+03	1.82574879E+02	5.47720479E-03
mode number=	4	2.03994540E+03	3.24667394E+02	3.08007524E-03
mode number=	5	3.18855084E+03	5.07473627E+02	1.97054575E-03

Beam type = 21: r^2 = (Lel^2)/4

(frequency)	(radians)	(hertz)	(period)

mode number=	1	1.27416026E+02	2.02788904E+01	4.93123628E-02
mode number=	2	5.09416777E+02	8.10761982E+01	1.23340761E-02
mode number=	3	1.14547494E+03	1.82307999E+02	5.48522285E-03
mode number=	4	2.03464507E+03	3.23823820E+02	3.08809896E-03
mode number=	5	3.17562099E+03	5.05415778E+02	1.97856902E-03

3) Neglecting rotary effect; L = 5.0

Theoretical: (No shear effect)

(frequency)	(radians)
mode number=1	5.0966418E+06
mode number=2	1.4049485E+07
mode number=3	2.0386567E+07
mode number=4	4.2144988E+07
mode number=5	4.5869776E+07

Beam type = 11:

(frequency)		(radians)	(hertz)	(period)
mode number=	1	5.09664180E+06	8.11155735E+05	1.23280889E-06
mode number=	2	1.40494850E+07	2.23604499E+06	4.47218193E-07
mode number=	3	2.03865670E+07	3.24462291E+06	3.08202225E-07
mode number=	4	4.21449886E+07	6.70758326E+06	1.49084993E-07
mode number=	5	4.58697737E+07	7.30040122E+06	1.36978773E-07

Beam type = 21:

(frequency)		(radians)	(hertz)	(period)
mode number=	1	5.09664179E+06	8.11155734E+05	1.23280889E-06
mode number=	2	1.40494850E+07	2.23604499E+06	4.47218193E-07
mode number=	3	2.03865670E+07	3.24462291E+06	3.08202225E-07
mode number=	4	4.21449886E+07	6.70758326E+06	1.49084993E-07
mode number=	5	4.58697737E+07	7.30040122E+06	1.36978773E-07

Theoretical: (including shear effect)

(frequency)	(radians)
mode number=1	4.8536279E+06

mode number=2	1.4049485E+07
mode number=3	1.7165084E+07
mode number=4	3.3070992E+07
mode number=5	4.2144988E+07

Hughes-Liu Beam type:

(frequency)		(radians)	(hertz)	(period)
mode number=	1	4.85379031E+06	7.72504720E+05	1.29449047E-06
mode number=	2	1.40494850E+07	2.23604499E+06	4.47218193E-07
mode number=	3	1.71662641E+07	2.73209578E+06	3.66019378E-07
mode number=	4	3.30714783E+07	5.26348924E+06	1.89988039E-07
mode number=	5	4.21449886E+07	6.70758326E+06	1.49084993E-07

4) Including rotary inertia effect; L = 5.0; 100 elements => Lel = L/100

Theoretical: (No shear effect)

(frequency)	(radians)
mode number=1	5.0960131E+06
mode number=2	1.4049485E+07
mode number=3	2.0376514E+07
mode number=4	4.2144988E+07
mode number=5	4.5818930E+07

Beam type = 11:

(frequency)		(radians)	(hertz)	(period)
mode number=	1	5.09601314E+06	8.11055681E+05	1.23296097E-06
mode number=	2	1.40494850E+07	2.23604499E+06	4.47218193E-07
mode number=	3	2.03765140E+07	3.24302293E+06	3.08354280E-07
mode number=	4	4.21449886E+07	6.70758326E+06	1.49084993E-07
mode number=	5	4.58189277E+07	7.29230883E+06	1.37130780E-07

Beam type = 21:

(frequency)		(radians)	(hertz)	(period)
mode number=	1	5.09601313E+06	8.11055680E+05	1.23296097E-06
mode number=	2	1.40494850E+07	2.23604499E+06	4.47218193E-07

mode number=	3	2.03765140E+07	3.24302293E+06	3.08354280E-07
mode number=	4	4.21449886E+07	6.70758326E+06	1.49084993E-07
mode number=	5	4.58189277E+07	7.29230883E+06	1.37130780E-07

Theoretical: (including shear effect)

(frequency)	(radians)
mode number=1	4.7891067E+06
mode number=2	1.4049485E+07
mode number=3	1.6610713E+07
mode number=4	3.1736803E+07
mode number=5	4.2144988E+07

Hughes-Liu Beam type:

(frequency)		(radians)	(hertz)	(period)
mode number=	1	4.78924529E+06	7.62232061E+05	1.31193642E-06
mode number=	2	1.40494850E+07	2.23604499E+06	4.47218193E-07
mode number=	3	1.66116941E+07	2.64383323E+06	3.78238683E-07
mode number=	4	3.17370608E+07	5.05111011E+06	1.97976282E-07
mode number=	5	4.21449886E+07	6.70758326E+06	1.49084993E-07