# On the Movement of a Liquid Front in an Unsaturated, Fractured Porous Medium, Part II. --- Mathematical Theory

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*ABSTRACT*

A simplified equation of motion is derived for the flow of liquid through an idealized one-dimensional fracture situated in an unsaturated imbibing porous medium. The equation is valid for the case where the matrix material has a much lower saturated conductivity than that of the fracture and the capillary tension in the matrix is sufficiently stronger than gravity. Asymptotic solutions and, in some cases, closed-form solutions are given for the motion of the liquid front in a parallel fracture system. With the introduction of natural time constants and dimensionless parameters, the flow behavior can be shown to possess various temporal flow regimes.

This work is part of the Nevada Nuclear Waste Storage (NNWSI) Project and is applicable to understanding some of the various physical parameters affecting liquid flow through a fracture in an unsaturated porous medium, and is particularly useful as a step in understanding the hydrological processes around a nuclear waste repository in an unsaturated environment as well as in other applications where unsaturated fracture flow conditions exist. The solutions are also relevant to numerical model verification.

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**Nomenclature**

Greek Symbols

- $\beta$ cosine of the angle of inclination from the vertical
- $\Gamma$ the gamma function
- $\lambda$ fracture storativity ratio, the initial unsaturated pore volume of the matrix relative to the volume of the fracture
- $\Omega$ function $\Omega(y)$ denoting the time at which the fracture front first reaches point $y$
- $\phi$ matrix porosity
- $\psi$ matrix capillary head
- $\sigma$ matrix diffusivity, or effective matrix diffusivity

Roman Symbols

- $a$ one-half the distance between adjacent parallel fractures
- $b$ one-half the fracture aperture width
- $C_{S_i}$ constant given by (4.9)
- $D$ diffusivity function
- $h$ distance of liquid front leading edge from the fracture entrance
- $K_f$ fracture-saturated hydraulic conductivity
- $K_m$ matrix-saturated hydraulic conductivity
- $k_r$ matrix relative permeability function
- $M$ kernel function given as inverse Laplace transform of expression given by (3.6)
- $p$ pressure in units of liquid head along the fracture
- $p_0$ pressure in units of liquid head at the fracture entrance
- $q$ specific volumetric flux into the matrix
- $q_f$ liquid velocity at the fracture entrance
- $q_I$ imbibition function into the matrix
- $S$ liquid saturation in the matrix
- $S_i$ initial liquid saturation in the matrix
- $t$ time
- $t_a$ fracture interference time scale, approximate time for matrix front to reach the no-flow boundary
- $t_b$ fracture storativity time scale, approximate time for cumulative matrix imbibition flux to become comparable to the volume in the fracture
- $u$ liquid velocity along the fracture
- $u_0$ liquid velocity at the fracture entrance
- $x$ coordinate distance normal to the fracture
- $y$ coordinate distance longitudinal to the fracture

$y_a$ longitudinal distance along fracture from the entrance where flow region III begins (Figure 7, [Nitao and Buscheck, 1989])

$y_b$ longitudinal distance along fracture from the entrance where flow region II begins

$z$ Laplace transform complex variable

$Z$ flow region length

## 1. Introduction

In this paper, we mathematically derive the conclusions that are described by Nitao and Buscheck [1989]. Under our simplifying assumptions, we show that the equations governing the flow down a one-dimensional fracture can be reduced to a single integro-differential equation in the fracture penetration. The asymptotic behavior of the solutions to this equation is shown to be directly related to the behavior of the Laplace transform of the matrix imbibition function. This function and its Laplace transform are derived for the case of a system of parallel fractures. For the case of constant boundary conditions, we demonstrate the existence of different flow regimes with the behavior of the solutions in each regime being described by its asymptotic expansion. In some cases a closed-form solution is derived. Some expressions for the solution are also given for the general problem with time-dependent boundary conditions.

## 2. Derivation of the Governing Equations

Using the simplifying physical assumptions given in Nitao and Buscheck [1989] we will derive the equations describing the movement of a liquid front in the fracture. We consider two separate types of boundary conditions at the entrance to the fracture: pressure head $p_0(t)$ and flux $u_0(t)$. It will be shown that, in each case, the governing equations reduce to a single equation for the location of the leading edge of the liquid front in the fracture with respect to the entrance to the fracture. This location will be referred to as the fracture penetration depth $h(t)$. These equations are integro-differential equations of the Volterra type [Burton, 1983].

In Nitao and Buscheck [1989] we saw that the following equations described the flow in the fracture and the matrix.

$$\frac{\partial u}{\partial y} = -\frac{1}{b} q_l(t - \Omega(y)) \tag{2.1}$$

$$u(y,t) = -K_f\left(\frac{\partial p}{\partial y} - \beta\right) \tag{2.2}$$

$$\frac{dh(t)}{dt} = u(h(t), t) \tag{2.3}$$

$$\Omega(h(t)) = t \tag{2.4}$$

We now reduce these equations to a single equation in the fracture penetration.

**Applied Flux Boundary Condition**

We first consider the problem where a given time-dependent flux $u_0(t)$ is applied to the opening of the fracture. The applicable boundary condition is

$$u(0,t) = u_0(t) \tag{2.5}$$

For this case, some care must be taken to see that the boundary condition is consistent with the assumptions of our derivation before applying the results for an applied flux boundary condition. Too great of a flux will create large pressure gradients in the fracture, thus violating the assumption of small gradients. Too small of a flux will result in a fracture front speed that is slower than the matrix fluxes which invalidates the assumption that the matrix streamlines are predominantly in a direction normal to the fracture.

We first integrate (2.1) from $y = 0$ to $y = h(t)$ and use (2.3) and (2.5). Making the change of variables $y = h(\xi)$ inside the integral, and using (2.4) we obtain

$$\frac{dh(t)}{dt} = u_0(t) - \frac{1}{b}\int_0^t q_l(t-\xi)\frac{dh}{d\xi}d\xi \tag{2.6}$$

which is the desired equation in $h(t)$.

**Applied Pressure Head Boundary Condition**

We now consider the case where the opening to the fracture at the ground surface is some known function of time $p_1(t)$ and the pressure at the leading edge of the fracture front is kept at $p_2(t)$. The boundary condition is therefore

$$p(y=0,t) = p_1(t) \qquad p(y=h(t),t) = p_2(t) \tag{2.7}$$

This set of conditions can, for example, be used to incorporate a constant capillary pressure drop at the leading edge of the front, or to include the effects of a constant head of water at the entrance. Note that since the form of the equations depends only on gradients in $p$, the solutions with these boundary conditions are equivalent to those satisfying

$$p(y=0,t) = p_0(t) \qquad p(y=h(t),t) = 0 \tag{2.8}$$

where $p_0$ is defined as

$$p_0(t) = p_1(t) - p_2(t)$$

One must be careful that the magnitude of the pressure boundary condition $p_0$ is sufficiently small that the assumption of small pressure gradients in the fracture are satisfied.

We first solve (2.1) and (2.2) subject to (2.8). Substituting (2.2) into (2.1) we have

$$\frac{\partial^2 p}{\partial y^2} = \frac{1}{K_f\, b}\, q\,(y,t) \tag{2.9}$$

The solution to this equation that satisfies (2.8) can be shown to be

$$p\,(y,t) = (1 - \frac{y}{h})p_0(t) + \frac{1}{b\,K_f}\left[F\,(y,t) - \frac{y}{h}F\,(h,t)\right] \tag{2.10}$$

where we define

$$F\,(y,t) = \int_0^y\int_0^\eta q\,(\mu,t)d\mu\, d\eta \tag{2.11}$$

and where as before $h$ is the fracture penetration depth. From Darcy's Law, (2.2), we have that the fluid velocity at the leading edge of the liquid fracture front is given by

$$\frac{dh}{dt} = u\,(h,t) \tag{2.12}$$

$$= K_f\,(\beta + \frac{p_0(t)}{h}) - \frac{1}{b}\left[\frac{\partial F}{\partial y}(h,t) - \frac{1}{h}F\,(h,t)\right]$$

Using the change of variables of the form $\mu = h(\xi)$ in the same manner as we have done before, it can be shown that

$$\frac{\partial F}{\partial y}\,(h,t) = \int_0^t q_l\,(t-\xi)\,\frac{dh\,(\xi)}{d\xi}\,d\xi \tag{2.13}$$

Using a similar change of variables twice in the double integral $F\,(h,t)$ and performing an interchange in the order of integration we obtain

$$F\,(h,t) = h(t)\int_0^t q_l\,(t-\xi)\,\frac{dh(\xi)}{d\xi}\,d\xi - \int_0^t q_l\,(t-\xi)\,h(\xi)\,\frac{dh(\xi)}{d\xi}\,d\xi \tag{2.14}$$

Substituting these expressions into (2.12) and using (2.3) we finally obtain

$$h(t)\,\frac{dh(t)}{dt} = K_f\,(h(t)\beta + p_0(t)) - \frac{1}{b}\int_0^t q_l\,(t-\xi)\,h(\xi)\,\frac{dh\,(\xi)}{d\xi}\,d\xi \tag{2.15}$$

which is the desired equation for $h(t)$. Note that this equation is non-linear in contrast to that for the specified flux boundary condition. The solution must also satisfy the initial condition

$$h(0) = 0$$

since the penetration depth is taken to be zero at time zero. Note that in the special case when $p_0$ is identically zero, the trivial solution is one of the solutions to the problem, and a problem with non-uniqueness of the solutions may occur. It will be shown later that the non-trivial solution has the asymptotic expansion $h(t) \sim K_f t$, which can be used to start numerical solutions along the correct solution.

For the case of a horizontal fracture where gravity is not important we have

$$h(t)\frac{dh(t)}{dt} = K_f p_0(t) - \frac{1}{b}\int_0^t q_l(t-\xi)h(\xi)\frac{dh(\xi)}{d\xi}\,d\xi \tag{2.16}$$

We will later take advantage of the fact that this equation is linear in $h\,dh/dt$ and, as will be shown later, has a similar form to the equation (2.6) for the constant flux boundary value problem.

The net specific volumetric flux $q_f(t)$ at the opening to the fracture, per unit area of opening, is an important quantity. By utilizing the same algebraic manipulations as used above it can be shown to be given by

$$q_f(t) = \frac{dh}{dt} + \int_0^t \frac{1}{b} q_l(t-\xi)\frac{dh}{d\xi}\,d\xi \tag{2.17}$$

### 3. Techniques for Analytic and Asymptotic Solution

We now describe some general methods for obtaining analytic and asymptotic solutions of the integro-differential equation derived in the previous section. The case where a specified flux boundary condition is applied at the entrance to the fracture and the case where a specified pressure head exists, but with gravity neglected, can through renaming variables both be reduced to the following form

$$\frac{dg(t)}{dt} = f(t) - \frac{1}{b}\int_0^t q_l(t-\xi)\frac{dg(\xi)}{d\xi}\,d\xi \tag{3.1}$$

where the definition of $g(t)$ and $f(t)$ depends on the boundary condition and is given by

*Case* 1. (*flux boundary condition*)

$$g(t) = h(t) \qquad f(t) = u_0(t) \tag{3.2}$$

*Case* 2. (*pressure head boundary condition but no gravity*, $\beta = 0$)

$$g(t) = \frac{1}{2} h(t)^2 \qquad f(t) = K_f\, p_0(t) \tag{3.3}$$

Equation (3.1) is linear and its solution $g(t)$ can be found by the taking the LaPlace transform. The Laplace transform of $g(t)$, through the use of the convolution theorem [Doetsch, 1974], can be shown to be

$$\hat{g}(z) = \hat{M}(z)\hat{f}(z) \tag{3.4}$$

Using the convolution theorem again, we have

$$g(t) = \int_0^t M(t-\xi) f(\xi)\, d\xi \tag{3.5}$$

where, here, $M$ is the inverse Laplace transform of a Laplace transform function given by

$$\hat{M}(z) = \frac{1}{z\left(1 + \frac{1}{b}\hat{q}_l(z)\right)} \tag{3.6}$$

with the $\hat{}$ 's denoting the Laplace transform operation. In the special case where the boundary condition is a constant in time, the function $f(t)$ will be a constant, say $f_0$, and the solution reduces to

$$g(t) = f_0 \int_0^t M(\xi)\, d\xi \tag{3.7}$$

Asymptotic forms for $g(t)$ can be most easily derived through looking at the asymptotic behavior of its corresponding Laplace transform. This was also the technique used by Philip [1968] in his study of infiltration into aggregated media. If the Laplace transform $\hat{g}(z)$ of a function $g(t)$ has the asymptotic expansion near $t = 0$ of the form

$$g(t) \sim t^{\nu} \sum_{k=0}^{n-1} a_k t^k + O(t^{n+\nu}) \qquad t \to 0 \tag{3.8}$$

then its Laplace transform for large $z$ has the expansion

$$\hat{g}(z) \sim \sum_{k=0}^{n-1} a_k \frac{\Gamma(\nu+k+1)}{z^{k+\nu+1}} \qquad z \to \infty \tag{3.9}$$

and vice versa [Doetsch, 1974]. Thus, the behavior at early time can be deduced from the behavior of the Laplace transform at infinity.

To determine the solution behavior at late times, if the Laplace transform $\hat{g}(z)$ has the expansion about its extreme singularity $z^*$ of the form

$$\hat{g}(z) \sim (z-z^*)^{-\nu} \sum_{k=0}^{n-1} A_k (z-z^*)^k \qquad z \to z^* \tag{3.10}$$

then $g(t)$ has the expansion

$$g(t) \sim t^{\nu-1} e^{z^* t} \sum_{k=0}^{n-1} \frac{B_k}{t^k} \qquad t \to \infty \tag{3.11}$$

where $B_k$ is defined as zero for $k$ such that $\nu - k$ is a negative integer but is otherwise given by

$$B_k = \frac{A_k}{\Gamma(\nu-k)}$$

The extreme singularity of a complex function is defined to be its singularity that has the largest real part. Some functions can have more than one extreme singularity, in which case there will be a sum of expansions of the form (3.11) for each one. We refer the reader to Doetsch [1974] for more details.

Using (3.3) we can obtain the behavior of $\hat{g}(z)$ in terms of of $M(z)$ and $\hat{f}(z)$ and can, therefore, derive the asymptotic expansions of $g(t)$ using the relationships we have just described. The asymptotic form for small time is given by the form for $\hat{M}(z)\hat{f}(z)$. For large time, one must find the extreme singularity of $g(z)$, that is, the singularity out of all those of either $\hat{M}$ or $\hat{f}$ that is the rightmost in the complex plane. The asymptotic behavior of the product $\hat{M}(z)\hat{f}(z)$ must then be found at this point. Note that the point is not necessarily the extreme singularity of both factors, although it will be of one them, so that, in general, the time domain behavior of $M(t)$ and $f(t)$ at infinity can not be necessarily used to deduce the asymptotic behavior of $g(t)$. Analysis in the Laplace domain is essential.

Since the Laplace transform function $\hat{M}$ depends on the transform of the imbibition function $\hat{q}_l$, the task of the following section will be to derive this function and its asymptotic expansions. Since the extreme singularity of $\hat{M}$ will turn out to be at $z = 0$, the behavior of $\hat{q}_l$ at this point will be important. Again, this behavior does not necessarily correspond to that of $q_l$ at $t \to \infty$ since the extreme singularity of its transform is not necessarily at the point of interest $z = 0$. For example, an exponentially decreasing imbibition function has its extreme singularity on the negative real axis, not at $z = 0$.

While the techniques given above apply to the problem with a pressure head boundary condition without gravity, the problem with gravity as given by (2.15) is non-linear and, therefore, can not be as readily nor as thoroughly treated. But, asymptotic solutions can still be obtained by making trial substitutions with various forms in $t$, equating like terms, and neglecting lower order terms as in done in standard perturbation theory [Nayfeh, 1973]. While this technique can give the leading terms of the expansion, in some cases, obtaining the higher order terms can cause problems. For example, in the case of constant boundary condition the higher terms can be shown to be negative powers in $t$ that go to infinity at $t = 0$ and, hence, the integral in (2.15) diverges when these terms are substituted. The way to avoid this problem is to perform the trial substitution in the Laplace transform domain instead of the time domain. By taking the Laplace transform of (2.15), we can relate the transform of the function $h$ to the transform of the function squared as

$$\widehat{h^2}(z) = K_f \hat{M}(z)\,(\beta \hat{h}(z) + \hat{p}_0(z)) \tag{3.12}$$

Let us restrict ourselves now to the case where $p_0$ is a constant so that $\hat{p}_0(z) = p_0/z$. The imbibition functions that we will encounter will be such that the resulting function $\hat{M}(z)$ will have an extreme singularity at $z = 0$. Therefore, trial asymptotic forms for the solution $\hat{h}(z)$ that also have an extreme singularity at this point are a likely choice. These functions turn out to be those that in the time domain increase as positive powers in time for large time. Their Laplace transform behavior at $z = 0$ is a power in $z$, and by expanding (3.12) in powers of $z$ and equating like terms, one can obtain an asymptotic expansion in $z$ and therefore in $t$. Since a mathematical proof as to the form of the expansion is not available, one must confirm the expansion using numerical methods of solution. Again, the Laplace transform behavior of $\hat{q}_I$ at $z = 0$ plays an important role in the analysis.

The asymptotic behavior of the specific flux $q_f$ into fracture as given by (2.17) can also be found using the Laplace transform. Its transform is given as

$$\hat{q}_f(z) = [1 + \frac{1}{b}\hat{q}_I(z)]\, z\, \hat{h}(z) \tag{3.13}$$

Therefore, the asymptotic behavior can be found directly from the asymptotic behavior of the Laplace transforms $\hat{q}_I(t)$ and $\hat{h}(t)$ near the extreme singularity $z = 0$.

## 4. Imbibition Fluxes into the Matrix

Equations (2.6) and (2.15) both require knowledge of the imbibition flux $q_I$ from the fracture and into the matrix as a function of time. The behavior of its Laplace transform is important in that it determines the asymptotic behavior of the solutions. We will therefore derive in this section these imbibition functions under the assumptions stated in Nitao and Buscheck [1989] ---- the matrix streamlines are predominately in the direction normal to the fracture plane and the effect of gravity is negligible in the matrix. These imbibition functions will first be derived for the case of a single fracture with semi-infinite matrix blocks on both sides. Then, we consider the case of an infinite array of parallel and equally spaced fractures. The last case we consider is when the fractures are still parallel but not necessarily equally spaced from each other. Although the first two cases are included in the last case in the limit, their corresponding formulas will be useful in deriving simpler expressions.

Under our assumptions the flow in the matrix becomes one-dimensional, and the equation for the saturation field $S$ reduces to

$$\frac{\partial S}{\partial t} = \frac{\partial}{\partial x}\left[D(S)\frac{\partial S}{\partial x}\right] \tag{4.1}$$

where $D$ is the diffusivity function given by $D(S) = (K_m k_r / \phi) d\psi / dS$. A saturated boundary condition occurs at a point on the fracture face for time $t$ from the time the liquid fracture front first arrives at that point. Here, we will take the time origin to be at zero. Therefore, the boundary condition at the fracture face $x = 0$ of the matrix is

$$S(x{=}0,t) = 1 \qquad t \geq 0 \tag{4.2}$$

Additional boundary conditions will be present depending on the problem. The initial saturation in the matrix is assumed to be uniform

$$S(x,t{=}0) = S_i \qquad x \geq 0 \tag{4.3}$$

The imbibition flux at $x = 0$ is given by

$$q_I(t) = -\phi D(1)\frac{\partial S}{\partial x}(x{=}0,t) \tag{4.4}$$

### 4.1 Semi-Infinite Matrix

We now derive the one-dimensional imbibition flux into the end of a semi-infinite slab of matrix. The boundary conditions are

$$S(x=0,t) = 1 \qquad S(x=\infty,t) = S_i \tag{4.5}$$

The Boltzmann transformation [Marshall and Holmes, 1979, p. 115]

$$\eta = x/\sqrt{t} \tag{4.6}$$

can be shown to reduce (4.1) to an equation with only $\eta$ as the dependent variable; and, therefore, the solution can be shown to be of the form

$$S(x,t) = (1-S_i)F(x/\sqrt{t}) \tag{4.7}$$

where $F$ is a function that depends only on $S_i$ but not on $\phi$. Using (4.4) the imbibition flux is therefore equal to

$$q_I(t) = \phi(1-S_i)\,C_{S_i}\,t^{-1/2} \tag{4.8}$$

where we define the constant

$$C_{S_i} = \frac{\partial F}{\partial \eta}(\eta=0) \tag{4.9}$$

which, in general, depends on $S_i$. For constant diffusivity, $D(S) = \sigma$, it can be shown [Carslaw and Jaeger, 1959] that

$$C_{S_i} = \sqrt{\frac{\sigma}{\pi}} \tag{4.10}$$

By analogy, we define for non-constant $D(S)$, the "effective diffusivity" $\sigma$ as

$$\sigma = \pi C_{S_i}^{\,2} \tag{4.11}$$

so that (4.10) holds. For non-constant $D(S)$ the effective diffusivity is a function of the initial saturation $S_i$, unlike the constant $D(S)$ case. We have for the imbibition flux

$$q_I(t) = \phi(1-S_i)\sqrt{\frac{\sigma}{\pi t}} \tag{4.12}$$

A natural time constant that will arise is the time duration necessary, per unit longitudinal area of fracture, for the imbibition front to invade a volume equal to the void volume of the fracture. From (4.12) this time is on the order of

$$t_b = \frac{\pi(b/(1-S_i)\phi)^2}{\sigma} \tag{4.13}$$

The imbibition flux (4.8) can be rewritten in terms of this constant as

$$q_I(t) = \frac{b}{\sqrt{t\, t_b}} \tag{4.14}$$

The Laplace transform is given by

$$\bar{q}_I(t) = b\sqrt{\frac{\pi}{z\, t_b}} \tag{4.15}$$

### 4.2 Finite Matrix

We now derive an imbibition flux function for the case of an infinite array of parallel fractures having equal spacing $2a$. The line at $x = a$ will be a symmetry line and is assumed to be a no-flow boundary. The boundary conditions are then

$$S(x=0,t) = 1 \qquad \frac{\partial S}{\partial x}(x=a,t) = 0 \tag{4.16}$$

The initial condition is, as before,

$$S(x,t=0) = S_i \tag{4.17}$$

In order to derive the saturation field we will have to assume, in contrast to the semi-infinite case, that the diffusivity $D(S)$ can be approximated as being equal to the effective diffusivity $\sigma$. This value will ensure that the imbibition flux will be accurate at least until the imbibition front reaches the no-flow boundary. After that the imbibition flux will decline and will not make a significant contribution to the total imbibition flux occurring along the entire fracture wall.

We first introduce the time constant

$$t_a = \frac{a^2}{\sigma}\pi \tag{4.18}$$

which is the approximate time necessary for the imbibition front to reach the no-flow symmetry line between fractures. We will also use the time constant $t_b$ given by (4.13).

The solution found by using the Fourier series method is [Kirkham and Powers, 1972; Carslaw and Jaeger, 1959]

$$S(x,t) = \left[1 - 2(1-S_i)\sum_{n=0}^{\infty} \frac{1}{c_n} e^{-c_n{}^2 \pi t/t_a} \sin\frac{c_n x}{a}\right] \qquad (4.19)$$

where $c_n = (2n+1)\pi/2$.

The imbibition flux into the matrix from the fracture is therefore

$$q_I(t) = -\phi\sigma \frac{\partial S}{\partial x}(x=0, t) = \pi b \frac{\lambda}{t_a} \Lambda(\pi t / t_a) \qquad t > 0 \qquad (4.20)$$

where we define the function $\Lambda$ by

$$\Lambda(\xi) = 2\sum_{n=0}^{\infty} e^{-\left[\frac{2n+1}{2}\pi\right]^2 \xi} \qquad (4.21)$$

It can be shown (Appendix) that its Laplace transform $\hat{\Lambda}(z)$ is given by $\hat{\Lambda}(z) = \frac{1}{\sqrt{z}} \tanh\sqrt{z}$, and, hence,

$$\frac{1}{b}\hat{q}_I(t) = \lambda \frac{\tanh\sqrt{t_a z/\pi}}{\sqrt{t_a z/\pi}} \qquad (4.22)$$

This function has two separate expansions

$$\frac{1}{b}\hat{q}_I(z) \sim \lambda\sqrt{\frac{\pi}{z\, t_b}}, \quad t_a z >> 1 \qquad (4.23)$$

$$\frac{1}{b}\hat{q}_I(z) \sim \lambda\left(1 - \frac{1}{3}\frac{t_a z}{\pi}\right), \quad t_a z << 1 \qquad (4.24)$$

for large and small $z$. They will be used later to determine the fracture penetration at $t << t_a$ and $t >> t_a$, respectively.

In order to gain a perspective on the function $\Lambda$ from another direction, the solution to the saturation field by a different method, the method of images, gives

$$S(x,t) = \sum_{n=0}^{\infty} (-1)^n \left[f(x+2na, t) + f((2n+2)a - x, t)\right] \qquad (4.25)$$

where $f(x,t)$ is the solution [Carslaw and Jaeger, 1959] for the semi-infinite case given by

$$S(x,t) = 1 - \frac{2(1-S_i)}{\sqrt{\pi}} \int_0^{x/2\sqrt{\sigma t}} e^{-\xi^2} d\xi \qquad (4.26)$$

The resulting alternate expression for the imbibition flux is

$$\frac{1}{b} q_I(t) = \frac{1}{\sqrt{t\, t_b}} \left[1 + 2\sum_{n=1}^{\infty} (-1)^n e^{-\frac{n^2 t_a}{\pi t}}\right] \qquad t > 0 \qquad (4.27)$$

Therefore,

$$\frac{1}{b} q_I(t) \sim \frac{1}{\sqrt{t\, t_b}} + O\left[\frac{1}{\sqrt{\pi t / t_a}}\, e^{-\frac{1}{(\pi t / t_a)}}\right] \qquad t \ll t_a \tag{4.28}$$

This expansion expresses the fact that for small time the imbibition flux is approximately the same as for the semi-infinite matrix.

### 4.3 Unequally Sized Matrix Blocks

We now consider the effect of non-equal spacing on the imbibition flux. In Figure 3 of Nitao and Buscheck [1989] we have a fracture that is part of an array of fractures whose separations alternate between distances $2a_1$ and $2a_2$. The no-flow symmetry lines in the matrix are therefore $a_1$ from one side of the fracture and $a_2$ from the other. Each side is allowed to have different material properties $\phi_k$ and $\sigma_k$ ($k = 1, 2$), as well as initial saturation $S_{ik}$.

We introduce natural time constants analogous to those encountered for equidistant fracture systems.

$$t_{bk} = \frac{[2b/(1 - S_{ik})\phi_k]^2 \pi}{\sigma_k} \qquad t_{ak} = \frac{a_k^2}{\sigma_k}\pi \qquad k = 1, 2 \tag{4.29}$$

Note that there is a factor of two inside the brackets in the definition of $t_{bk}$ which is not present in $t_b$. The $t_{bk}$ refer to each of the matrix blocks, singly, draining the entire fracture width $2b$ by imbibition while $t_b$ refers to simultaneous imbibition into both matrix blocks. Let us now assume that

$$t_{a1} \ll t_{a2} \tag{4.30}$$

If the matrix diffusivity $\sigma$ were equal on both sides of the fracture, this assumption would correspond physically to an array of fractures with separations alternating between a small distance apart and a large distance apart.

At any given point along the fracture the imbibition flux $q_I(t)$ into the matrix, at time $t$ from start of imbibition, can be written as the sum of the flux into the two sides of the fracture. Using the expression (4.20) for the imbibition flux into a finite matrix slab, the flux from the two sides of the fracture into a half-fracture is

$$\frac{1}{b} q_I(t) = \pi\left[\frac{\lambda_1}{t_{a1}} \Lambda(\pi t / t_{a1}) + \frac{\lambda_2}{t_{a2}} \Lambda(\pi t / t_{a2})\right] \tag{4.31}$$

where the function $\Lambda(t)$ was defined in (4.21), and where we define

$$\lambda_k = \sqrt{\frac{t_{ak}}{t_{bk}}} \tag{4.32}$$

We now determine some asymptotic expansions for $q_l$. For early times such that $t << t_{a1}$, we have from (4.28) that

$$\frac{1}{b} q_l(t) \sim \frac{1}{\sqrt{t\, t_{b1}}} + \frac{1}{\sqrt{t\, t_{b2}}} = (t\, t_b)^{-1/2} \tag{4.33}$$

where $t_b$ is defined as the harmonic-root mean

$$\frac{1}{\sqrt{t_b}} = \frac{1}{\sqrt{t_{b1}}} + \frac{1}{\sqrt{t_{b2}}} \tag{4.34}$$

At intermediate times, $t_{a1} << t << t_{a2}$, we have from (4.20) and (4.28),

$$\frac{1}{b} q_l(t) \sim \pi \frac{\lambda_1}{t_{a1}} \Lambda(\pi t / t_{a1}) + (t\, t_{b2})^{-1/2} + O\left(\frac{1}{\sqrt{\pi t / t_{a2}}} e^{-\frac{1}{\pi t / t_{a2}}}\right) \tag{4.35}$$

The contribution of the imbibition flux into matrix block number 1 as given by the first term on the right decays exponentially with time while the second term, the flux into 2, is dominant since it decays as a power in time.

Using (4.22) the Laplace transform is given by

$$\frac{1}{b} \hat{q}_l(z) = \lambda_1 \frac{\tanh \sqrt{t_{a1} z / \pi}}{\sqrt{t_{a1} z / \pi}} + \lambda_2 \frac{\tanh \sqrt{t_{a2} z / \pi}}{\sqrt{t_{a2} z / \pi}} \tag{4.36}$$

Each of the two terms is in the form of (4.22) so that they have expansions of the form (4.23) and (4.24). If we assume, without loss of generality, that $t_{a1} \leq t_{a2}$, then we have the following expansions

$$\frac{1}{b} \hat{q}_l \sim \sqrt{\frac{\pi}{t_b z}}, \qquad z\, t_{a_1} >> 1 \tag{4.37}$$

$$\frac{1}{b} \hat{q}_l \sim \lambda_1 \left(1 - \frac{1}{3} \frac{t_{a1} z}{\pi}\right) + \sqrt{\frac{\pi}{t_{b2} z}}, \qquad z\, t_{a_1} << 1, \quad z\, t_{a_2} >> 1 \tag{4.38}$$

$$\frac{1}{b} \hat{q}_l \sim \lambda - \frac{1}{3\pi} z\, (\lambda_1 t_{a1} + \lambda_2 t_{a2}), \qquad z\, t_{a_2} << 1 \tag{4.39}$$

These will correspond to the three flow periods $t << t_{a1}$, $t_{a1} << t << t_{a2}$, and $t_{a2} << t$, respectively, of the fracture penetration solution $h(t)$.

### 4.4 Imbibition Kernel Function

The imbibition kernel function function $M(t)$ given by (3.6) has been summarized in Table 1 for the fracture geometries we have just considered.

**Table 1. Laplace Transform of the Imbibition Kernel Function**

| | $\hat{M}(z)$ |
|---|---|
| semi-infinite matrix | $\dfrac{1}{\sqrt{z}\,(\sqrt{z} + \sqrt{\frac{\pi}{t_b}})}$ |
| finite matrix | $\dfrac{1}{z\,(1 + \lambda \frac{1}{\sqrt{t_a z/\pi}} \tanh \sqrt{t_a z/\pi})}$ |
| staggered fractures | $\dfrac{1}{z\,(1 + \frac{\lambda_1}{\sqrt{t_{a1} z/\pi}} \tanh \sqrt{t_{a1} z/\pi} + \frac{\lambda_2}{\sqrt{t_{a2} z/\pi}} \tanh \sqrt{t_{a2} z/\pi})}$ |

## 5. Non-Gravity Driven Flow

In this section we derive the solutions for the two types of boundary conditions, applied flux $u_0(t)$ and applied pressure head $p_0(t)$ with no gravity. Under the appropriate transformation of variables given by (3.2) and (3.3), the problems were shown to reduce to the same equation (3.1). In this section we will derive expressions for the solutions to this equation. Although the general time-dependent boundary condition will be considered first, we will be particularly interested in the asymptotic behavior of the solutions in the case where the boundary condition is kept at a constant value.

### 5.1 Time-Dependent Boundary Condition

The Laplace transforms $\hat{M}(z)$ of the imbibition kernel function $M(t)$ were given in Table 1 for various geometries. In the ca for a single fracture with semi-infinite matrix the inverse Laplace transform is known and is given by [Abrahamowitz and Stegun, 1964, p. 1024]

$$M(t) = e^{\pi t/t_b}\, erfc \sqrt{\pi t/t_b} \tag{5.1}$$

where $erfc$ is the complementary error function. Therefore, (3.5) can be used to give an expression for the general solution to $g(t)$, and the penetration depth can be found by (3.2) and (3.3) for the appropriate boundary condition.

For the case of a system of parallel fractures equally spaced, the kernel function can be found by taking the inverse transform of the expression for $\hat{M}(z)$ given in Table I by the method of residues [Doetsch, 1974]. It is given as an infinite sum

$$M(t) = \frac{1}{1+\lambda} + 2\lambda \sum_{n=1}^{\infty} \frac{1}{\zeta_n^2+\lambda+\lambda^2} e^{-\zeta_n^2 \pi t / t_a} \tag{5.2}$$

where the $\zeta_n$ are the roots of the equation

$$\lambda \tan \zeta_n = \zeta_n , \quad n = 1, 2, 3, \cdots \tag{5.3}$$

in ascending order. The $n$th term in the infinite sum corresponds to the exponentially decaying interference from the fracture that is the $n$th one away from the fracture of interest.

### 5.2 Constant Boundary Condition

We now consider the solutions to (3.1) in the case where the boundary condition $f(t)$ is equal to a constant $f_0$. Using (3.7), we find that the solution in the case of a single fracture with semi-infinite matrix is the integral of (5.1). Using various integral equalities we have that

$$g(t)/f_0 = \frac{t_b}{\pi} [e^{\pi t/t_b} erfc(\sqrt{\pi t / t_b}) - 1 + 2(t/t_b)^{1/2}] \tag{5.4}$$

For the case of a system of equidistant parallel fractures the solution is given by (3.7), which has as its Laplace transform $\hat{M}(z)/z$ where $\hat{M}$ is taken from Table I. We use the method of residues to take the inverse transform of this expression to obtain

$$g(t)/f_0 = \frac{1}{1+\lambda} t + \frac{\lambda t_a}{3\pi(1+\lambda)^2} - \frac{2\lambda t_a}{\pi} \sum_{n=1}^{\infty} \frac{1}{\zeta_n^2(\zeta_n^2+\lambda+\lambda^2)} e^{-\zeta_n^2 \pi t / t_a} \tag{5.5}$$

where the $\zeta_n$ are given by (5.3).

The asymptotic expansions of the solution can be most easily derived by looking at the asymptotic expansions of its Laplace transform. From the convolution theorem we have

$$\hat{g}(z)/f_0 = \frac{1}{z} \hat{M}(z) \tag{5.6}$$

We consider the case of a system of fractures with staggered spacing. This case encompasses the other cases as limiting cases. From (4.37) to (4.39) the asymptotic expansions of $\hat{M}(z)$ and hence of $\hat{g}(z)$ can be found and are given in Table II. The corresponding expansion for $g(t)$ is also shown.

**Table II. Expansions for $\frac{1}{z}\hat{M}(z)$ and $g(t)/f_0$**

| Flow Period | $\frac{\hat{g}(z)}{f_0} = \frac{1}{z}\hat{M}(z)$ | $\frac{g(t)}{f_0}$ |
|---|---|---|
| I. | $z^{-2} - z^{-5/2}\sqrt{\pi/t_b}$ <br> $t_b^{-1} << z$ | $t - \frac{4}{3}\frac{1}{\sqrt{t_b}}t^{3/2}$ <br> $t << t_b$ |
| II. | $z^{-3/2}\sqrt{t_b/\pi} - z^{-1}t_b/\pi$ <br> $z_{a1}^{-1} << z << t_b^{-1}$ | $\frac{2}{\pi}(t_b t)^{1/2} - t_b/\pi$ <br> $t_b << t << t_{a1}$ |
| IIIa. | $z^{-3/2}\sqrt{t_{b2}/\pi} - z^{-1}(1+\lambda_1)t_{b2}/\pi$ <br> $t_{a2}^{-1} << z << t_{a1}^{-1}, z << t_{b2}^{-1}$ | $\frac{2}{\pi}(t_{b2}t)^{1/2} - (1+\lambda_1)t_{b2}/\pi$ <br> $t_b << t << t_{a1}, t_{b2} << t$ |
| IIIb. | $z^{-2}\frac{1}{1+\lambda} + z^{-1}\frac{\lambda_1 t_{a1} + \lambda_2 t_{a2}}{3\pi(1+\lambda)^2}$ <br> $z << t_{a2}^{-1}$ | $\frac{1}{1+\lambda}t + \frac{\lambda_1 t_{a1} + \lambda_2 t_{a2}}{3\pi(1+\lambda)^2}$ <br> $t_{a2} << t$ |
| IIIa.1 | $z^{-2}\frac{1}{1+\lambda_1} + z^{-1}\frac{\lambda_1 t_{a1}}{3\pi(1+\lambda_1)^2}$ <br> $t_{a2}^{-1} << z << t_{a1}^{-1}, z >> t_{b2}^{-1}$ | $\frac{1}{1+\lambda_1}t + \frac{\lambda_1 t_{a1}}{3\pi(1+\lambda_1)^2}$ <br> $t_b << t << t_{a1}, t_{b2} >> t$ |
| IIIa.2 | same as for III.a <br> $t_{a2}^{-1} << z << t_{b2}^{-1}$ | same as for III.a <br> $t_{b2} << t << t_{a2}$ |

## 6. Gravity Driven Flow

The case where there is gravity driven flow with a pressure head boundary condition possesses the governing equation given by (2.15). We will consider the solution when the boundary head is equal to some positive constant $p_0$.

**6.1 Asymptotic Solution**

The asymptotic expansions to (2.15) were given in Table II of Nitao and Buscheck [1989]. They were obtained by substituting the expansions for $\hat{M}(z)$ as given in Table II of this paper into a version of (3.12) given by

$$\hat{h}^2(z) = K_f \hat{M}(z)(\beta \hat{h}(z) + \frac{p_0}{z}) \qquad (6.1)$$

We then try the following functional form

$$\hat{h}(z) \sim \alpha z^{-\nu}$$

where $\nu > 0$. This assumption is equivalent to assuming that the form of the solution is

$$h(t) \sim \alpha t^{\nu-1}/\Gamma(\nu)$$

By equating the leading terms in $z$ one can find the value of $\alpha$ and $\nu$. The higher-order term can be added and its coefficient and power can be found. This process is straightforward if the boundary head $p_0$ is zero with respect to ambient. Otherwise, one expects that the effects of the boundary head will dominate at early times when the head of liquid in the fracture is small with respect to the boundary head. And, at later times, when the converse becomes true the effects of the gravity head will become dominant over the boundary head. Hence, the leading term in the expansion for (6.1) will depend on the relative magnitude of the head term $p_0(t)$ in (2.15) to the liquid column head term $\beta h(t)$. The time at which the boundary head dominates can be estimated by comparing the expansion for $\beta h$ based on zero boundary head with the term $p_0$. This determination was performed for each of the flow periods I through III as shown in Table II of Nitao and Buscheck [1989].

**6.2 Comparison with Numerical Solutions to the Integro-Differential Equation**

In order to confirm the asymptotic solutions for the case of a constant boundary condition with gravity, we have also found solutions to (2.15) numerically. Originally, we discretized the equation by the obvious procedure; the time derivatives were replaced by first-order differences and the integral with a sum. However, we found that the errors due to the differences inside the sum can accumulate, requiring very small time steps to be taken to maintain accuracy. A better method is to reduce the equation to a system of two equations while taking $h$ and $dh/dt$ as separate dependent variables. In the end, this method was refined by using $h$ and $v$ as the dependent variables where $v$ is defined as

$$v = \frac{1}{2}\frac{dh^2}{dt} - K_f \beta h \tag{6.2}$$

Equation (2.15) with $p_0$ zero is then equivalent to

$$v(t) = -\frac{1}{b}\int_0^t q_l(t-\xi)\,[v(\xi) + K_f \beta h(\xi)]\,d\xi \tag{6.3}$$

This choice of variables eliminates the round-off error due to subtraction of the $K_f \beta h$ and the integral terms. These equations were discretized in time as follows, where the superscript refers to the time level,

$$\frac{1}{2}(h^{(n+1)})^2 = \frac{1}{2}(h^{(n)})^2 + \Delta t^{(n+1)}(K_f \beta h^{(n)} + v^{(n)}) \tag{6.4}$$

$$v^{(n+1)} = -\frac{1}{b}\sum_{k=1}^{n} q_l\,(t^{(n+1)} - t^{(k)})\left[v^{(k)} + K_f \beta h^{(k)}\right]\Delta t^{(k)} \tag{6.5}$$

$$- \frac{1}{b} q_l\,(\frac{1}{2}(t^{(n+1)} - t^{(n)}))\,\frac{1}{2}\left[v^{(n)} + v^{(n+1)} + K_f \beta (h^{(n)} + h^{(n+1)})\right]\Delta t^{(n+1)}$$

where

$$\Delta t^{(k)} = t^{(k)} - t^{(k-1)}$$

In Nitao and Buscheck [1989] comparisons were also made with solutions using a two-dimensional unsaturated flow simulator. These simulations will be discussed in more detail in a future report.

## 7. Fracture Influx Rate

The expression for the Laplace transform of the specific fracture influx rate $q_f$ was given by (3.13). It requires the behavior of the Laplace transform for the imbibition function $q_l$ and the fracture penetration $h(t)$ which have been found in the previous sections. Therefore, the asymptotic behavior of $q_f$ can be found, as shown in table V of Nitao and Buscheck [1989], using the techniques in section 3 that relate the asymptotic behavior of the Laplace transform to that of the original function in the time domain.

## 8. Matrix Flow Regions

We briefly explain how the lengths of the various matrix flow regions shown in Table VI of Nitao and Buscheck [1989] were derived. We first find the distances $y_a$ and $y_b$ (referring to Figure 7 of Nitao and Buscheck [1989]) of the flow regions from the fracture entrance as functions of time. To find $y_a(t)$ we note that the time elapsed from when the fracture front first hits a point $y^*$ is given as $t - \Omega(y^*)$. When this time increment equals $t_a$, the saturation front corresponding to this point has felt the no-flow boundary due to the neighboring fracture, according to the definition of $t_a$. Hence, the leading edge $y_a$ of the "saturated" flow region is at this point $y^*$. This may be expressed as

$$t - \Omega(y_a(t)) = t_a$$

Since $\Omega$ is the inverse function of the fracture penetration $h$, we have

$$y_a(t) = h(t - t_a).$$

Similarly, $y_b$ is given as $y_b(t) = h(t - t_b)$. Hence,

$$Z_1 = h(t) - y_b = h(t) - h(t - t_b) \tag{8.1}$$

$$Z_2 = y_b - y_a = h(t - t_b) - h(t - t_a) \tag{8.2}$$

By substituting the asymptotic expansions for $h(t)$ and dropping any higher-order terms one can obtain expansions for $Z_1$ and $Z_2$.

In particular, the entries in Table VI of Nitao and Buscheck [1989] were derived using the expansions for $h$ that are valid for $t >> t_a$. The requirement that the arguments of $h$ in (8.1) and (8.2) satisfy this condition on $t$ translates to having $t >> 2t_a$.

## 9. Conclusions

We have found that the unsaturated flow of a liquid front in a fracture can, under certain situations, be described by a single integro-differential equation whose solution gives the location of the front in the fracture. This equation can be most satisfactorily treated by using Laplace transform techniques. Various asymptotic approximations can be derived which are sufficient to characterize the physical processes of the system including various flow periods, regions in the matrix saturation field, and the liquid flux into the fracture. Closed-form solutions were derived for some types of boundary conditions.

The use of the Laplace transform has been found to be a very convenient device for the derivation of asymptotic solutions to our problem. The method is general enough to be applicable to other types of imbibition functions in addition to those treated in this report.

In the special case of a single fracture with semi-infinite matrix the analysis is applicable even when the matrix diffusivity is a non-constant function of saturation. In our more general analyses when the matrix is finite, the matrix diffusivity has to be approximated by a constant effective diffusivity which is defined in terms of the expression for the imbibition flux in a semi-infinite system. Comparisons with numerical simulations indicate that this approximation gives good results for the test cases considered. The reason for the agreement stems from the fact that the frontal movement in the fracture depends on the matrix imbibition flux along the fracture wall and not on the actual form of the saturation field. Either the imbibition flux along the fracture is (1) nearly the same as for a semi-infinite matrix because the matrix front has not yet felt the no-flow boundary with the neighboring fracture, (2) is very small because the the matrix block is almost saturated, or (3) has values intermediate to those in (1) and (2). In most cases, only a small amount of the net imbibition flux will be due to (2) and (3) which explains the applicability of the constant effective diffusivity approximation.

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**Appendix --- Laplace Transform of the Function $\Lambda(\xi)$**

It is well known that the Laplace transform of the exponential function $e^{-\alpha t}$ is given by $1/(z+\alpha)$. Thus,

$$\hat{\Lambda}(z) = 2\sum_{n=0}^{\infty} \frac{1}{z + ((2n+1)\pi/2)^2}$$

Integrating this function with respect to z starting from $z = 0$ gives

$$\int_0^z \hat{\Lambda}(\eta)\, d\eta = 2\sum_{n=0}^{\infty} \ln\left[1 + \frac{z}{((2n+1)\pi/2)^2}\right]$$

$$= 2\ln\prod_{n=0}^{\infty}\left[1 + \frac{z}{((2n+1)\pi/2)^2}\right]$$

Using the identity [Gradstehyn and Ryzhik, p. 37]

$$\cosh\zeta = \prod_{n=0}^{\infty}\left[1 + \frac{\zeta^2}{((2n+1)\pi/2)^2}\right]$$

we have

$$\int_0^z \hat{\Lambda}(\eta)\, d\eta = 2\ln\cosh\sqrt{z}$$

Taking the derivative,

$$\hat{\Lambda}(z) = \frac{1}{\sqrt{z}}\tanh\sqrt{z}$$