CONTINUA AND RELATED TOPICS

THESIS

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This paper is a study of continua and related metric spaces. Chapter I is an introductory chapter. Irreducible continua and noncut points are the main topics in Chapter II. The third chapter begins with a few results on locally connected spaces. These results are then used to prove results in locally connected continua. Decomposable and indecomposable continua are dealt with in Chapter IV. Totally disconnected metric spaces are studied in the beginning of Chapter V. Then we see that every compact metric space is a continuous image of the Cantor set. A continuous map from the Cantor set onto [0,1] is constructed. Also, a continuous map from [0,1] onto [0,1]×[0,1] is built. Then an order preserving homeomorphism is constructed from a metric arc onto [0,1].

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CHAPTER I

PRELIMINARIES

Introduction

Irreducible continua and noncut points are the main topics in Chapter II. The Vietoris Topology is introduced to prove that in a compact Hausdorff space a net of nonempty closed sets has a convergent subnet. Chapter III begins with some results on locally connected spaces. Several of these results are then used to prove results in locally connected continua. Indecomposable and decomposable continua are dealt with in Chapter IV. Totally disconnected metric spaces are explored in the beginning of Chapter V. Then we see that every compact metric space is a continuous image of the Cantor set. A few examples of maps between continua are also exhibited in Chapter V.

Notation

The theorems presented in this paper can be found in the books listed in the bibliography. Only the proofs, examples and remarks given in this paper are original. It is assumed that the reader is familiar with the material presented in a beginning graduate topology course.

There is however some notation which needs clarification. We will let R^1 denote the real numbers and R^2 denote $R^1 \times R^1$.

Let (X,T) be a topological space and A,B nonempty subsets of X. Let $\|A\|$ stand for the number of elements in A if A is a finite set and let diam A denote the diameter of A. If $\{a_\alpha\}_{\alpha\in D}$ is a net in X then $a_\alpha \overset{c}{\underset{\infty}{\to}} a$ means that the net clusters to a. Similarly, if $\{a_n\}_{n=1}^{\infty}$ is a sequence in X which clusters to $a \in X$ we will write $a_n \overset{c}{\to} a$. Let b(A) denote the boundary of A. If $A \subseteq B$ then let $b^B(A)$ denote the boundary of A with respect to the relative topology on B. Similarly, $\overline{A}^B$ will denote the closure of A with respect to the relative topology on B. The sets A and B are separated sets if $A \cap \overline{B} = B \cap \overline{A} = \phi$. If A and B are separated sets such that their union is all of X then we will write $X = A \uplus B$.

DEFINITION 1.1. Let (X,T) be a topological space and $\{A_\alpha\}_{\alpha\in D}$ a net of sets in X. Then the <u>limit superior of $\{A_\alpha\}_{\alpha\in D}$</u> is $\{x \in X \mid$ if $U \in T$ and $x \in U$ then for every $\alpha \in D$ there exists a $\beta \geq \alpha$, $\beta \in D$, such that $U \cap A_\beta \neq \phi\}$. Let lim sup A_α denote the limit superior of $\{A_\alpha\}_{\alpha\in D}$.

DEFINITION 1.2. Let (X,T) be a topological space and $\{A_\alpha\}_{\alpha\in D}$ a net of sets in X. Then the <u>limit inferior of $\{A_\alpha\}_{\alpha\in D}$</u> is $\{x \in X \mid$ if $U \in T$ and $x \in U$ then there exists an $\alpha_0 \in D$ such that if $\alpha \in D$ and $\alpha \geq \alpha_0$ then $A_\alpha \cap U \neq \phi\}$. Let lim inf A_α denote the limit inferior of $\{A_\alpha\}_{\alpha\in D}$.

DEFINITION 1.3. Let (X,T) be a topological space and $\{A_\alpha\}_{\alpha\in D}$ a net of sets in X. Then A is said to be the <u>limit of $\{A_\alpha\}_{\alpha\in D}$</u> if and only if lim inf A_α = lim sup A_α = A.

DEFINITION 1.4. Let (X,T) be a topological space. A connected subset of X that is not properly contained in any other connected subset of X is a <u>component</u> of X.

CHAPTER II

CONTINUA

Irreducible Continua

DEFINITION 2.1. A <u>continuum</u> is a compact connected Hausdorff space.

EXAMPLE 2.2. Let X be the subspace of R^2 shown in Figure 1; then X is a continuum.

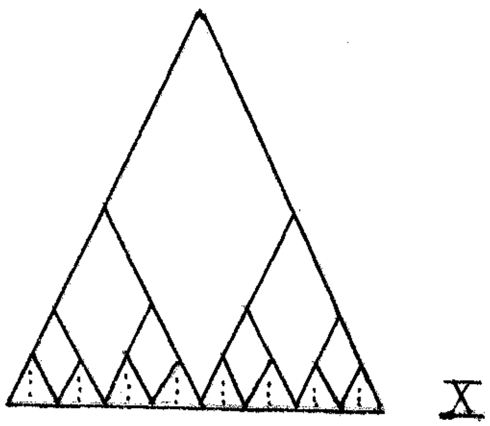

Figure 1 Continuum

LEMMA 2.3. Let X be a compact Hausdorff space and let $\underline{F}$ be a family of nonempty closed subsets of X with the property that if A, B $\in \underline{F}$ then there exists a C $\in \underline{F}$ such that $C \subseteq A \cap B$. Then the following two properties hold:

i) If U is an open set containing $\cap \underline{F}$, then there is some $F \in \underline{F}$ such that $F \subseteq U$.

ii) If in addition, each $F \in \underline{F}$ is connected then $\cap \underline{F}$ is nonempty, compact and connected.

PROOF. First index $\underline{F}$, say $\underline{F} = \{F_\alpha\}_{\alpha\varepsilon A}$. Pick an element in $\underline{F}$, say F_{α_1}. Let $D = \{\alpha\varepsilon A | F_\alpha \subseteq F_{\alpha_1}\}$ and direct D as follows:

$$\alpha < \beta \leftrightarrow F_\beta \subseteq F_\alpha .$$

Let U be an open subset of X such that $\cap \underline{F} \subseteq U$. Suppose $F_\alpha \not\subseteq U$ for all $\alpha\varepsilon A$; then for each $\alpha\varepsilon D$ there exists a $f_\alpha \varepsilon F_\alpha$ such that $f_\alpha \notin U$. Now we have a net $\{f_\alpha\}_{\alpha\varepsilon D}$ in F_{α_1} with F_{α_1} compact. Hence $f_\alpha \overset{c}{\to} y\varepsilon F_{\alpha_1}$, for some $y\varepsilon F_{\alpha_1}$.

Suppose $y \notin F_\alpha$, for some α in A; then $y\varepsilon(X - F_\alpha)$ which is open in X. Now, there exists $F_{\alpha_2} \subseteq F_\alpha \cap F_{\alpha_1}$ with $\alpha_2 \varepsilon D$ and there exists $\alpha_3 \geq \alpha_2$ such that $f_{\alpha_3} \varepsilon (X - F_\alpha)$. Thus, $F_{\alpha_3} \subseteq F_{\alpha_2} \subseteq F_\alpha$ and so $f_{\alpha_3} \varepsilon X - F_{\alpha_3}$; but this contradicts $f_{\alpha_3} \varepsilon F_{\alpha_3}$. Hence $y\varepsilon \cap \underline{F}$ and therefore $y \varepsilon U$. Thus, for each $\alpha\varepsilon D$ there exists a $\beta \geq \alpha$, $\beta \varepsilon D$, such that $f_\beta \varepsilon U$. This contradicts $f_\alpha \notin U$ for all $\alpha\varepsilon D$. Hence, $\cap \underline{F} \neq \emptyset$ and there exists $F \varepsilon \underline{F}$ such that $F \subseteq U$.

To prove part two we will use part one. Now assume each $F \varepsilon \underline{F}$ is connected. From the proof of part one we have that $\cap \underline{F} \neq \emptyset$. Since X is compact and $\cap \underline{F}$ is closed we have that $\cap \underline{F}$ is compact.

Suppose $\cap \underline{F}$ is not connected. Then let $\cap \underline{F} = K \uplus H$. Since X is compact and Hausdorff there exist open disjoint sets U and V such that $K \subseteq U$ and $H \subseteq V$. By part one there exists $F_1 \varepsilon \underline{F}$ such that $F_1 \subseteq U \cup V$. Since F_1 is connected, either $F_1 \subseteq U$ or $F_1 \subseteq V$. Without loss of generality suppose $F_1 \subseteq U$; then $\cap \underline{F} \subseteq U$ and therefore $H = \emptyset$. This contradicts $H \neq \emptyset$ and therefore $\cap \underline{F}$ is connected. #

DEFINITION 2.4. Let (X,T) be a topological space and A a nonempty subset of X. A subcontinuum K of X is irreducible about A if $A \subseteq K$ and no proper subcontinuum of K contains A.

EXAMPLE 2.5. Let $X = \{(x,y) \in R^2 \mid x^2 + y^2 < 16\}$ and $A = \{(x,y) \in X \mid x^2 + y^2 < 1\}$. Then $K = \{(x,y) \in X \mid x^2 + y^2 \leq 1\}$ is subcontinuum of X which is irreducible about A.

THEOREM 2.6. Let X be a continuum and A a nonempty subset of X. Then X contains a subcontinuum which is irreducible about A.

PROOF. Let $\underline{K} = \{K \subseteq X \mid K$ is a subcontinuum of X and $A \subseteq K\}$. $\underline{K} \neq \emptyset$ since $X \in \underline{K}$. Partial order $\underline{K}$ as follows:

$$K_\alpha \leq K_\beta \leftrightarrow K_\beta \subseteq K_\alpha .$$

Let $\underline{C}$ be a chain in $\underline{K}$; then $C_1 = \bigcap_{C \in \underline{C}} C$ is an element of $\underline{K}$. Thus $\underline{C}$ has an upper bound and therefore $\underline{K}$ has a maximal element, say K_1. Clearly K_1 is irreducible about A. #

DEFINITION 2.7. Let A and B be nonempty disjoint subsets of a space X. A subcontinuum K of X is irreducible from A to B if K intersects both A and B and if no proper subcontinuum of K intersects both A and B.

EXAMPLE 2.8. Let X be R^1, $A = \{x \in X \mid 0 \leq x \leq 1\}$, $B = \{x \in X \mid 5 \leq x \leq 25\}$ and $K = \{x \in X \mid 1 \leq x \leq 5\}$. Then K is continuum which is irreducible from A to B.

THEOREM 2.9. Let X be a continuum and A,B be nonempty disjoint closed subsets of X. Then X contains a subcontinuum which is irreducible from A to B.

PROOF. Let $\underline{K} = \{K \subseteq X \mid K$ is a continuum, $K \cap A \neq \emptyset$ and $K \cap B \neq \emptyset\}$. $\underline{K} \neq \emptyset$ since $X \varepsilon \underline{K}$. Partial order $\underline{K}$ as follows:

$$K_\alpha \leq K_\beta \leftrightarrow K_\beta \subseteq K_\alpha.$$

Let $\underline{C}$ be a chain in $\underline{K}$; then $C_1 = \bigcap_{C \varepsilon \underline{C}} C$ is a continuum. Suppose $C_1 \cap A = \emptyset$. Then there exists $C_2 \varepsilon \underline{C}$ such that $C_2 \subseteq X - A$; but this contradicts $C_2 \cap A \neq \emptyset$. Thus, $C_1 \cap A \neq \emptyset$ and similarly $C_1 \cap B \neq \emptyset$. Hence, by Zorns' lemma, there exists a maximal element K_1 in $\underline{K}$. Clearly K_1 is irreducible from A to B. #

Let X be a compact Hausdorff space and A, B nonempty disjoint closed subsets of X. If there does not exist a subcontinuum of X which intersects both A and B then, due to Theorem 2.9, X is not connected. Moreover, we can find two separated sets in X such that their union is all of X, A is contained in one of these sets and B is contained in the other. The next three lemmas are needed to prove this.

LEMMA 2.10. Let X be a compact Hausdorff space, x and y elements of X such that $x \neq y$, and $\{H_\alpha\}_{\alpha \varepsilon A}$ a collection of closed sets each containing x and y such that each H_α is not the union of two separated sets one containing x and the other y. If $\{H_\alpha\}_{\alpha \varepsilon A}$ is totally ordered by set containment, then $\bigcap_{\alpha \varepsilon A} H_\alpha$ is not the union of two separated sets one containing x and the other y.

PROOF. Suppose $\bigcap_{\alpha\in A} H_\alpha = M \uplus N$, where $x \in M$ and $y \in N$. $M = \overline{M}$ and $N = \overline{N}$ since $\bigcap_{\alpha\in\Lambda} H_\alpha$ is closed. Let U and V be open disjoint sets such that $M \subseteq U$ and $N \subseteq V$. From Lemma 2.3 we know that there exists H_{α_1} such that $H_{\alpha_1} \subseteq U \cup V$. Thus, $H_{\alpha_1} = (H_{\alpha_1} \cap U) \uplus (H_{\alpha_1} \cap V)$ with $x \in H_{\alpha_1} \cap U$ and $y \in H_{\alpha_1} \cap V$. This contradicts H_{α_1} not being the union of two separated sets one containing x and the other y. Hence, $\bigcap_{\alpha\in\Lambda} H_\alpha$ is not the union of two separated sets one containing x and the other y. #

LEMMA 2.11. Let X be a compact Hausdorff space and x,y elements of X such that $x \neq y$. If X is not the union of two separated sets one containing x and the other y, then X has a subcontinuum joining x to y.

PROOF. Let $\underline{H} = \{H \subseteq X \mid x,y \in H,\ H = \overline{H}$ and H is not the union of two separated sets one containing x and the other y$\}$. $\underline{H} \neq \emptyset$ since $X \in \underline{H}$. Partial order $\underline{H}$ as follows:

$$H_\alpha \leq H_\beta \leftrightarrow H_\beta \subseteq H_\alpha \quad .$$

Let $\underline{C}$ be a chain in $\underline{H}$. Then, from Lemma 2.10, $\bigcap_{C\in\underline{C}} C \in \underline{H}$ and therefore $\underline{H}$ has a maximal element, say H_1. If H_1 is connected we are done; so suppose $H_1 = M \uplus N$ with $x,y \in M$. $H_1 \in \underline{H}$ implies that $H_1 = \overline{H}_1$ and therefore $M = \overline{M}$. Thus, if M is connected we are done. Suppose $M = Q \uplus R$. If $x \in Q$ and $y \in R$, then $H_1 = Q \uplus (R \cup N)$ with $x \in Q$ and $y \in R \cup N$. This contradicts $H_1 \in \underline{H}$. Thus without loss of generality say $x,y \in Q$. $x,y \in Q$ and $H_1 \in \underline{H}$ imply that Q is in $\underline{H}$. Then $Q \subsetneq H_1$ and $Q \in \underline{H}$ contradict H_1 being maximal in $\underline{H}$.

Hence, M is connected and therefore M is a subcontinuum containing x and y. #

LEMMA 2.12. Let X be a compact Hausdorff space and let A and B be closed disjoint subsets of X. If for each pair a,b, with $a \in A$ and $b \in B$, there exist sets H, K such that $X = H \uplus K$ with $a \in H$ and $b \in K$, then $X = M \uplus N$ where $A \subseteq M$ and $B \subseteq N$.

PROOF. Let $a \in A$. For each $b \in B$ there exist sets H_b and K_b such that $X = H_b \uplus K_b$ with $a \in H_b$ and $b \in K_b$. Let $\underline{K} = \{K_b | b \in B\}$. Then $\underline{K}$ is an open cover for B and therefore $\underline{K}$ has a finite subcover, say $K_{b_1}, K_{b_2}, \ldots, K_{b_{n_a}}$. Do the above for each $a \in A$. Thus for each $a \in A$, $B \subseteq \bigcup_{i=1}^{n_a} K_{b_i}$ and $a \in H_{b_i}$, $i = 1,2,\ldots,n_a$. By construction $(\bigcup_{i=1}^{n_a} K_{b_i}) \cap (\bigcap_{i=1}^{n_a} H_{b_i}) = \emptyset$ for each $a \in A$. Also, $A \subseteq \bigcup_{a \in A} (\bigcap_{i=1}^{n_a} H_{b_i})$. Since A is compact there exists a finite set $\underline{F} = \{a_1,\ldots,a_m\} \subseteq A$ such that $A \subseteq \bigcup_{a \in \underline{F}} (\bigcap_{i=1}^{n_a} H_{b_i})$. Let $M = \bigcup_{a \in \underline{F}} (\bigcap_{i=1}^{n_a} H_{b_i})$ and $N = \bigcap_{a \in \underline{F}} (\bigcup_{i=1}^{n_a} K_{b_i})$. Clearly $A \subseteq M$ and $B \subseteq N$.

Let $a \in \underline{F}$; then $(\bigcup_{i=1}^{n_a} K_{b_i}) \cap (\bigcap_{i=1}^{n_a} H_{b_i}) = \emptyset$ and therefore $M \cap N = \emptyset$. Let $x \in X$. For each $a \in \underline{F}$ $X = H_{b_i} \uplus K_{b_i}$ for $i = 1,2,\ldots,n_a$; thus $x \in \bigcap_{i=1}^{n_a} H_{b_i}$ or $x \in \bigcup_{i=1}^{n_a} K_{b_i}$. Hence $x \in M$ or $x \in N$ and therefore $X = M \uplus N$. #

THEOREM 2.13. Let X be a compact Hausdorff space and let A,B be closed disjoint subsets of X such that no subcontinuum of X intersects both A and B. Then there exist closed disjoint sets M,N such that $X = M \cup N$ with $A \subseteq M$ and $B \subseteq N$.

PROOF. Suppose there do not exist closed disjoint sets M,N such that $X = M \cup N$ with $A \subseteq M$ and $B \subseteq N$; then there exist $a \in A$ and $b \in B$ such that X is not the union of two separated sets one containing a and the other containing b. Then, from Lemma 2.11, there exists a subcontinuum joining a to b. This contradicts there not being a subcontinuum of X intersecting both A and B. Hence there exist closed disjoint sets M,N such that $X = M \cup N$ with $A \subseteq M$ and $B \subseteq N$. #

Theorem 2.14 also comes from Lemma 2.11 and Lemma 2.12.

THEOREM 2.14. Let X be a Hausdorff space and let A,B be disjoint closed subsets of X. Let K be a subcontinuum of X which is irreducible from A to B. Then the sets $K - (A \cup B)$, $K - A$ and $K - B$ are connected.

PROOF. First we will show that $K - (A \cup B)$ is connected. Suupose $K - (A \cup B)$ is not connected; then let R and S be subsets of X such that $K - (A \cup B) = R \uplus S$. Then $R \cup (K \cap A) \cup (K \cap B)$ and $S \cup (K \cap A) \cup (K \cap B)$ are closed subsets of K. Let $a \in K \cap A$ and $b \in K \cap B$; then K irreducible from A to B and Lemma 2.11 imply that $R \cup (K \cap A) \cup (K \cap B)$ is the union of two separated sets one containing a and the other containing b. A similar result follows for $S \cup (K \cap A) \cup (K \cap B)$. Hence, from Lemma 2.12, there exist sets M,N,P and Q such that $R \cup (K \cap A) \cup (K \cap B) = M \uplus N$,

$K \cap A \subseteq M$, $K \cap B \subseteq N$, $S \cup (K \cap A) \cup (K \cap B) = P \uplus Q$, $K \cap A \subseteq P$ and $K \cap B \subseteq Q$. Thus $K = (M \cup P) \cup (N \cup Q)$, $M \cup P = \overline{M \cup P}$ and $N \cup Q = \overline{N \cup Q}$. Let $z \in M$; then $z \notin N$ since $M \cap N = \emptyset$. If $z \in R$ then $z \notin Q$ and if $z \in K \cap A$ then $z \in P$. Hence $z \notin N \cup Q$. Let $p \in P$; then $p \notin Q$ since $P \cap Q = \emptyset$. If $p \in S$ then $p \notin N$ and if $p \in K \cap A$ then $p \in M$. Hence $p \notin N \cup Q$. Thus $(M \cup P) \cap (N \cup Q) = \emptyset$. This contradicts K being connected. Hence $K - (A \cup B)$ is connected.

Now suppose there exist subsets of X C and D such that $K - A = C \uplus D$. Let U and V be K - open sets such that $U \cap V = \emptyset$, $K \cap A \subseteq U$ and $K \cap B \subseteq V$. Without loss of generality let $K - (A \cup B) \subseteq C$. C and D are open in K since they are open in $K - A$. Hence $C \cup U$ and $D \cap V$ are open in K. $U \cap V = \emptyset$ and $C \cap D = \emptyset$ imply that $(C \cup U) \cap (D \cap V) = \emptyset$. Hence $K = (C \cup U) \uplus (D \cap V)$; contradicting the connectedness of K. Thus $K - A$ is connected. One can show that $K - B$ is connected using a similar argument. #

DEFINITION 2.15. Let X be a space, $x \in X$ and $Q_x = \{y \in X|$ there do not exist open disjoint sets U,V such that $X = U \cup V$ with $x \in U$ and $y \in V\}$. Then Q_x is called the quasi-component of X determined by x.

REMARK 2.16. Let X be a compact Hausdorff space and $x \in X$. Then, due to Lemma 2.11, for each $y \in Q_x$, $x \neq y$, there exists a subcontinuum K_y in X which joins x to y.

THEOREM 2.17. Let X be a space and $x \in X$. Then $Q_x = \bigcap \{U \subseteq X \mid U$ is both open and closed and $x \in U\}$.

PROOF. Let $z \in Q_x$ and $U \subseteq X$ such that U is both open and closed and such that $x \in U$. U both open and closed implies that $X = (X - U) \cup (U)$, where both U and $X - U$ are open and where $(X - U) \cap (U) = \emptyset$. $z \in Q_x$ and $x \in U$ imply that $z \in U$. Hence, $Q_x \subseteq \bigcap \{U \subseteq X \mid U$ is both open and closed and $x \in U\}$.

Let $\omega \in \bigcap \{U \subseteq X \mid U$ is both open and closed and $x \in U\}$. Suppose $\omega \notin Q_x$; then there exist open disjoint sets V, W such that $X = V \cup W$, $x \in V$ and $\omega \in W$. Then V is both open and closed and therefore $\omega \in V$. Hence $V \cap W \neq \emptyset$. This contradicts $V \cap W = \emptyset$. Thus $\omega \in Q_x$ and therefore $\bigcap \{U \subseteq X \mid U$ is both open and closed and $x \in U\} \subseteq Q_x$. #

THEOREM 2.18. Let X be a space and $x \in X$. Let C_x be the component of X containing x; then $C_x \subseteq Q_x$.

PROOF. Let $z \in C_x$ and suppose $z \notin Q_x$. Then there exist open disjoint sets U,V such that $X = U \cup V$, $x \in U$ and $z \in V$. C_x being connected implies that $C_x \subseteq U$ or $C_x \subseteq V$. $x \in U$ and $z \in V$ contradict $C_x \subseteq U$ or $C_x \subseteq V$. Hence $z \in Q_x$ and therefore $C_x \subseteq Q_x$.#

EXAMPLE 2.19. Consider the subspace X of R^2 shown below.

Figure 2 Supspace X of R^2.

Consider the two vertical line segments farthest to the right in X. They are of equal length and shorter than the reamining line segments. Let A be the upper one and B the lower. Choose $x \varepsilon X$ as shown.

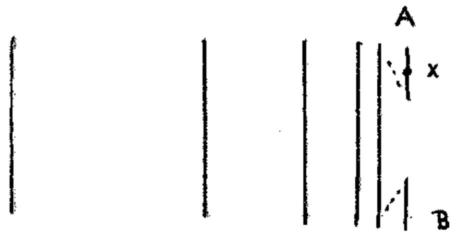

Figure 3 Clarification of A,B and x .

$C_x = A$ but $Q_x = A \cup B$. Hence $C_x \subsetneq Q_x$. Notice that X is not compact.

THEOREM 2.20. Let X be compact and T_2. Let $x \varepsilon X$; then $Q_x = C_x$.

PROOF. Due to Theorem 2.18 it is sufficient to show that $Q_x \subseteq C_x$. Let $q \varepsilon Q_x$. Then, due to Lemma 2.11, there exists a subcontinuum K joining x to q. $K \cup C_x$ is connected since $x \varepsilon K$ and $x \varepsilon C_x$. Thus $K \subseteq C_x$ and therefore $q \varepsilon C_x$. Hence $Q_x \subseteq C_x$. #

Noncut Points

THEOREM 2.21. If X is a continuum such that X has more than one point then X has at least two noncut points.

PROOF. Let X be a continuum such that X has more than one point. If every $x \varepsilon X$ is a noncut point then we are done.

So suppose X has a cut point. Let $x \in X$ be a cut point and let U_x, V_x be subsets of X such that $X - \{x\} = U_x \psi V_x$. Suppose each $y \in U_x$ is a cut point. For each $y \in U_x$ let U_y, V_y be subsets of X such that $x \in V_y$ and such that $X - \{y\} = U_y \psi V_y$. Then for each $y \in U_x, U_y \cup \{y\}$ is a continuum such that $U_y \cup \{y\} \subsetneq X - \{x\}$. Hence $U_y \cup \{y\} \subsetneq U_x$ for each $y \in U_x$. Set $\underline{U} = \{U_y \cup \{y\} \mid y \in U_x\}$ and then partially order $\underline{U}$ with set containment. Let $\underline{C}$ be a chain in $\underline{U}$; then $K = \bigcap_{\underline{C}} (U_y \cup \{y\})$ is a nonempty continuum with $K \subsetneq U_x$.

Let $k \in K$; then $U_k \cup \{k\} \subsetneq K$. To see this we will consider two cases. First suppose $k \in U_y$ for every $U_y \cup \{y\} \in \underline{C}$. Let $y \in U_x$ be such that $U_y \cup \{y\} \in \underline{C}$. If $y \in U_k$ then $V_k \cup \{k\} \subsetneq X - \{y\} = U_y \psi V_y$. $k \in U_y$ and $x \in V_k \cap V_y$ imply that $(V_k \cup \{k\}) \cap U_y \neq \emptyset$ and $(V_k \cup \{k\}) \cap V_y \neq \emptyset$. This contradicts the connectedness of $V_k \cup \{k\}$. Hence $y \notin U_k$ and therefore $U_k \cup \{k\} \subsetneq U_y$. Thus $U_k \cup \{k\} \subsetneq K$ when $k \in U_y$ for every $U_y \cup \{y\} \in \underline{C}$. Now suppose there exists a $y_0 \in X$ such that $U_{y_0} \cup \{y_0\} \in \underline{C}$ but $k \notin U_{y_0}$. Since $k \in K = \bigcap_{\underline{C}} (U_y \cup \{y\})$ we have that $k = y_0$ and therefore $U_k = U_{y_0}$. Let $\omega \in X$ be such that $\omega \in U_x$, $\omega \neq k$ and $U_\omega \cup \{\omega\} \in \underline{C}$. Then $k \in U_\omega$ and therefore $U_k \cup \{k\} \subsetneq U_\omega$. Hence $U_k \cup \{k\} \subsetneq K$ if there exists a $y_0 \in X$ such that $U_{y_0} \cup \{y_0\} \in \underline{C}$ but $k \notin U_{y_0}$.

Thus $U_k \cup \{k\}$ is an element of $\underline{U}$ such that $U_k \cup \{k\} \subsetneq K$. Therefore, by Zorns' lemma, $\underline{U}$ has a minimal element. Let S be the minimal element in $\underline{U}$; then S is a continuum such that

$S \subseteq \bigcap_{y \in U_x} (U_y \cup \{y\})$. Let $s \in S$. Then, since $U_y \cup \{y\} \subseteq U_x$ for each $y \in U_x$, $s \in U_x$. Thus $X - \{s\} = U_s \uplus V_s$, $s \in U_x$ and $x \in V_s$. Then $U_s \cup \{s\}$ connected, $U_s \cup \{s\} \subseteq X - \{x\}$ and $s \in U_x$ imply that $U_s \cup \{s\} \subseteq U_x$. Let $z \in U_s$; then $z \in U_x$ and therefore $X - \{z\} = U_z \uplus V_z$. Again $U_z \subsetneq U_x$. Now, $s \in S$ and $s \in X - \{z\}$ imply that $s \in U_z$. Thus $(V_s \cup \{s\}) \cap U_z \neq \emptyset$ and, since $x \in V_z \cap V_s$, $(V_s \cup \{s\}) \cap V_z \neq \emptyset$. $z \in U_s$ implies that $V_s \cup \{s\} \subseteq X - \{z\} = U_z \uplus V_z$. Thus $V_s \cup \{s\} \subset U_z$ or $V_s \cup \{s\} \subseteq V_z$. This contradicts $(V_s \cup \{s\}) \cap U_z \neq \emptyset$ and $(V_s \cup \{s\}) \cap V_z \neq \emptyset$. Hence there exists a noncut point in U_x. Similarly there exists a noncut point in V_x and therefore X has at least two noncut points. #

Vietoris Topology

THEOREM 2.22. Let (X,T) be a topological space and $S(X) = \{F \subseteq X \mid F = \overline{F} \text{ and } F \neq \emptyset\}$. For each $G \in T$ set $S(G) = \{F \in S(X) \mid F \subseteq G\}$ and $I(G) = \{F \in S(X) \mid F \cap G \neq \emptyset\}$. Let $\underline{S}_1 = \{S(G) \mid G \in T\}$, $\underline{S}_2 = \{I(G) \mid G \in T\}$ and let E be the topology on S(X) with $\underline{S} = \underline{S}_1 \cup S_2$ as a subbase. Let $U_1, U_2, \ldots, U_n$ be open sets in X and set $< U_1, U_2, \ldots, U_n > = \{F \in S(X) \mid F \subseteq \bigcup_{i=1}^{n} U_i$ and $F \cap U_i \neq \emptyset$ for all $i\}$. Then $\underline{B} = \{< U_1, U_2, \ldots, U_n > \mid U_i \in T$ for all i and n is a positive integer$\}$ is a base for E.

PROOF. Let n be a positive integer, $U_1, U_2, \ldots, U_n$ be open sets in X and $F \in < U_1, U_2, \ldots, U_n >$. Set

$V = [S(\bigcup_{i=1}^{n} U_i)] \cap [\bigcap_{i=1}^{n} I(U_i)]$. Then $V \in E$ and $F \in V \subseteq < U_1, U_2, \ldots, U_n >$.

Thus $<U_1, U_2, \ldots, U_n> \varepsilon$ E and therefore $\underline{B} \subseteq E$.

Let $F \varepsilon S(X)$. If $F \varepsilon S(G)$ for some $G \varepsilon T$ then $F \varepsilon <G> \subseteq S(G)$. If $F \varepsilon I(G)$ for some $G \varepsilon T$ then let $V \varepsilon T$ be such that $F \subseteq V$. Then $F \varepsilon <G,V> \subseteq I(G)$. Hence $\underline{B}$ is a base for E. #

The topology E on S(X), given in Theorem 2.22, is sometimes called the Vietoris Topology on S(X). The Vietoris Topology will enable us to prove that a net of closed nonempty sets in a compact T_2 space has a convergent subnet. This result will be used throughout the rest of the paper.

THEOREM 2.23. Let (X,T) be a T_1 space. Then S(X) is T_1.

PROOF. Choose two distinct points $F_1, F_2 \varepsilon S(X)$. Without loss of generality let $x_2 \varepsilon F_2 - F_1$. For each $x \varepsilon F_1$ let U_x be an open set in X such that $x \varepsilon U_x$ but $x_2 \notin U_x$. Set $U_1 = \bigcup_{x \varepsilon F_1} U_x$; then $F_1 \varepsilon <U_1>$ but $F_2 \notin <U_1>$. If $F_1 \not\subseteq F_2$, a similar argument shows that there exists a $U_2 \varepsilon T$ such that $F_2 \varepsilon <U_2>$ but $F_1 \notin <U_2>$. If $F_1 \subseteq F_2$ then $F_2 \varepsilon <U_1, X - F_1>$, but $F_1 \notin <U_1, X - F_1>$. Hence S(X) is T_1. #

THEOREM 2.24. Let (X,T) be a compact space; then (S(X),E) is also a compact space.

PROOF. Let $\underline{C}$ be a cover of S(X) by subbase elements. Set $S = \{G \varepsilon T \mid S(G) \varepsilon \underline{C}\}$ and $I = \{G \varepsilon T \mid I(G) \varepsilon \underline{C}\}$. If $I = \emptyset$ then $X \varepsilon S(G_0)$ for some $G_0 \varepsilon S$. Then $S(X) \subseteq S(G_0)$ and therefore S(X) is compact.

If $I \neq \emptyset$ and $X \subseteq \bigcup_{G \varepsilon I} G$ then I is an open cover for X. Thus there exist sets $G_1, G_2, \ldots, G_n \varepsilon I$ such that $X \subseteq \bigcup_{i=1}^{n} G_i$.

$X \subseteq \bigcup_{i=1}^{n} G_i$ implies that $S(X) \subseteq \bigcup_{i=1}^{n} I(G_i)$ and therefore $S(X)$ is compact.

Finally suppose $I \neq \emptyset$ and $X \nsubseteq \bigcup_{G\varepsilon I} G$. Then $X - \bigcup_{G\varepsilon I} G \varepsilon S(X)$. Therefore there exists a $G_0 \varepsilon S$ such that $X - \bigcup_{G\varepsilon I} G \varepsilon S(G_0)$. Hence $X = \left[\bigcup_{G\varepsilon I} G\right] \cup G_0$. X compact implies that there exist sets $G_1, G_2, \ldots, G_n \varepsilon I$ such that $X = \left[\bigcup_{i=1}^{n} G_i\right] \cup G_0$. Let $F \varepsilon S(X)$. $X = \left[\bigcup_{i=1}^{n} G_i\right] \cup G_0$ implies that if $F \nsubseteq G_0$ then $F \varepsilon I(G_i)$ for some i, $i = 1,2,3,\ldots,n$. Hence $S(X) \subseteq \left[\bigcup_{i=1}^{n} I(G_i)\right] \cup S(G_0)$ and therefore $S(X)$ is compact. #

THEOREM 2.25. Let (X,T) be a T_1 space and let $(S(X),E)$ be a compact space. Then (X,T) is a compact space.

PROOF. Let $\underline{C}$ be an open cover for X. Say $\underline{C} = \{C_\alpha | \alpha \varepsilon A\}$ for some index set A. Set $\underline{C}_E = \{I(C_\alpha) | C_\alpha \varepsilon \underline{C}\}$; then $\underline{C}_E$ is a cover of $S(X)$ by subbase elements. Hence there exist $I(C_{\alpha_1}), I(C_{\alpha_2}), \ldots, I(C_{\alpha_n})$, members of $\underline{C}_E$, such that $S(X) \subseteq \bigcup_{i=1}^{n} I(C_{\alpha_i})$. Choose $x \varepsilon X$. $\{x\}$ is a closed subset of X since X is T_1. Hence $\{x\} \varepsilon I(C_{\alpha_{i_0}})$ for some i_0, $1 \leq i_0 \leq n$. Thus $x \varepsilon C_{\alpha_{i_0}}$ and therefore $X \subseteq \bigcup_{i=1}^{n} C_{\alpha_i}$. Thus $\underline{C}$ has a finite subcover and therefore X is compact. #

THEOREM 2.26. Let (X,T) be a T_1 space. Then $(S(X),E)$ is a connected space if and only if (X,T) is a connected space.

PROOF. Let $S(X)$ be connected and suppose $X = R \uplus S$, for some sets R and S. Let $A = \{F \varepsilon S(X) | F \subseteq R\}$, $B = \{F \varepsilon S(X) | F \subseteq S\}$ and $C = \{F \varepsilon S(X) | F \cap R \neq \emptyset$ and $F \cap S \neq \emptyset\}$. $R \varepsilon A$, $S \varepsilon B$ and $X \varepsilon C$. Hence

$A \neq \emptyset$, $B \neq \emptyset$ and $C \neq \emptyset$.

Let $F \in \bar{A}$ and choose a net $\{F_\alpha\}_{\alpha \in D}$ in A which converges, in S(X), to F. For all $\alpha \in D$. $F_\alpha \notin \langle S \rangle$ and $F_\alpha \notin \langle R, S \rangle$. Hence $F \in A$ and therefore A is closed in S(X). Similarly B and C are closed in S(X). Hence $S(X) = A \uplus B \uplus C$, which contradicts S(X) connected. Thus X is connected.

Next let (X,T) be a connected space. For each positive integer n let $\underline{F}_n = \{F \in S(X) | F$ has less than or equal to n elements$\}$ and define a function g_n from $\prod_{i=1}^{n} X_i$ into $\underline{F}_n$, where $X_i = X$ for all i, by $g_n((x_1, x_2, \ldots, x_n)) = \{x_1, x_2, \ldots, x_n\}$.

Let n_0 be a positive integer and $(x_1, x_2, \ldots, x_{n_0}) \in \prod_{i=1}^{n_0} X_i$, where $X_i = X$ for all i. Choose open sets $U_1, U_2, \ldots, U_k$ in X such that $g_{n_0}((x_1, x_2, \ldots, x_{n_0})) \in \langle U_1, U_2, \ldots, U_k \rangle$. For each each j, $j = 1, 2, 3, \ldots, n_0$, let $\underline{V}_j = \{U_i | x_j \in U_i,\ i = 1, 2, 3, \ldots, k\}$. Then for each j, $j = 1, \ldots, n_0$, set $W_j = \cap \underline{V}_j$. Now, $(x_1, x_2, \ldots, x_{n_0}) \in \prod_{j=1}^{n_0} W_j$ which is open in $\prod_{i=1}^{n_0} X_i$ and $g_{n_0}(\prod_{j=1}^{n_0} W_j) \subseteq \langle U_1, U_2, \ldots, U_k \rangle$. Hence g_{n_0} is continuous and therefore $\underline{F}_{n_0}$ is connected. Thus for each positive integer n, $\underline{F}_n$ is connected.

Let $\underline{F} = \{F \in S(X) | F \in \underline{F}_n$ for some positive integer n$\}$; then $\underline{F}$ is connected in S(X). Let $U_1, U_2, \ldots, U_m$ be open subsets of X and for each i, $i = 1, 2, \ldots, m$, let $x_i \in U_i$. Then $F = \{x_1, x_2, \ldots, x_m\} \in \underline{F}$ and $F \in \langle U_1, U_2, \ldots, U_m \rangle$. Hence $\underline{F}$ is a dense connected subset of S(X) and therefore S(X) is connected. #

Let (X,T) be a topological space and $\{A_\alpha\}_{\alpha\varepsilon D}$ a net in $S(X)$. $A_\alpha \overset{T}{\to} A$ means that the net of sets converges to A with respect to the topology T and $A_\alpha \overset{E}{\to} A$ means that the net converges to A in $S(X)$.

THEOREM 2.27. Let (X,T) be a compact T_2 space and $\{A_\alpha\}_{\alpha\varepsilon D}$ a net in $S(X)$. If there exists an $A \varepsilon S(X)$ such that $A_\alpha \overset{E}{\to} A$, then $A_\alpha \overset{T}{\to} A$.

PROOF. Let $\{A_\alpha\}_{\alpha\varepsilon D}$ be a net in $S(X)$ and $A\varepsilon S(X)$ such that $A_\alpha \overset{E}{\to} A$. Choose $x_0 \varepsilon A$ and let U be an open subset of X such that $x_0 \varepsilon U$. For each $x \varepsilon A$ let V_x be a T-open neighborhood of x; then $\underline{V} = \{V_x | x\varepsilon A\}$ is an open cover for A. Hence there exist sets $V_1, V_2, \ldots, V_n$ such that $V_i \varepsilon \underline{V}$ for $i = 1,2,\ldots,n$ and such that $A \subseteq \bigcup_{i=1}^{n} V_i$. Let $W = \langle U, V_1, V_2, \ldots, V_n\rangle$; then $A \varepsilon W$ and therefore there exists an $\alpha_0 \varepsilon D$ such that if $\alpha \geq \alpha_0$, $\alpha\varepsilon D$, then $A_\alpha \varepsilon W$. Hence, if $\alpha \varepsilon D$ and $\alpha \geq \alpha_0$ then $A_\alpha \cap U \neq \emptyset$. Thus $A \subseteq \lim\inf A_\alpha \subseteq \lim\sup A_\alpha$ in X.

Let $z \varepsilon X$ be such that $z \varepsilon \lim\sup A_\alpha$. Suppose $z \notin A$; then there exist disjoint T-open sets U and V such that $\{z\} \subseteq U$ and $A \subseteq V$. $A_\alpha \overset{E}{\to} A$ and $A \varepsilon \langle V\rangle$ imply that there exists an $\alpha_1 \varepsilon D$ such that if $\alpha\varepsilon D$ and $\alpha \geq \alpha_1$ then $A_\alpha \varepsilon \langle V\rangle$. Thus for $\alpha \geq \alpha_1$, $A_\alpha \subseteq V$. $z \varepsilon \lim\sup A_\alpha$ and $z \varepsilon U$ imply that there exists an $\alpha_2 \varepsilon D$, $\alpha_2 \geq \alpha_1$, such that $A_{\alpha_2} \cap U \neq \emptyset$. This contradicts $U \cap V = \emptyset$. Hence $z \varepsilon A$ and therefore $A \subseteq \lim\inf A_\alpha \subseteq \lim\sup A_\alpha \subseteq A$. Thus $A_\alpha \overset{T}{\to} A$. #

REMARK 2.28. Let X be a compact T_2 space and $\{A_\alpha\}_{\alpha\varepsilon D}$ a net in X such that $A_\alpha \neq \emptyset$ for all $\alpha\varepsilon D$. Then Theorem 2.24 and

Theorem 2.27 imply that $\{\overline{A}_\alpha\}_{\alpha \varepsilon D}$ has a convergent subnet. If X is a compact metric space and $\{A_n\}_{n=1}^{\infty}$ a sequence of closed nonempty sets in X then, similarly, $\{A_n\}_{n=1}^{\infty}$ has a convergent subsequence.

THEOREM 2.29. Let (X,T) be a compact T_2 space and $C(X) = \{F \varepsilon S(X) | F \text{ is a continuum}\}$. Then C(X) is closed in S(X).

PROOF. Let $x \varepsilon X$. Then $\{x\} \varepsilon C(X)$ and therefore $C(X) \neq \emptyset$. Let $F \varepsilon \overline{C(X)}$ and $\{F_\alpha\}_{\alpha \varepsilon D}$ a net in C(X) such that $F_\alpha \overset{E}{\to} F$. Then $F_\alpha \overset{T}{\to} F$ and therefore F is connected. Hence F is a continuum and therefore $F \varepsilon C(X)$. Thus C(X) is closed in S(X). #

EXAMPLE 2.30. Let $A = \{(x,y) \varepsilon R^2 | x^2 + y^2 = 1\}$, $D = \{(x,y) \varepsilon R^2 | x^2 + y^2 \leq 1\}$ and $Y = \{(x,y) \varepsilon R^2 | x^2 + y^2 \leq 1 + 2\pi\}$. Set $X = Y^0 - D^0$. Recall that $S(A) = \{F \subseteq A | F = \overline{F} \text{ and } F \neq \emptyset\}$ and $C(A) = \{F \varepsilon S(A) | F \text{ is a continuum}\}$. Let X* be the one point compactification of X. Thus $X^* = X \cup \{p\}$. Let $(x_1,y_1) \varepsilon A$ and choose $(x_2,y_2) \varepsilon A$ such that $(x_1,y_1) \neq (x_2,y_2)$; then let $K((x_1,y_1),(x_2,y_2))$ be the arc joining (x_1,y_1) to (x_2,y_2), moving in the clockwise direction, in A. Then $C(A) = \{(x_1,y_1) | (x_1,y_1) \varepsilon A\} \cup \{A\} \cup \{K((x_1,y_1),(x_2,y_2)) | (x_1,y_1) \varepsilon A, (x_2,y_2) \varepsilon A \text{ and } (x_1,y_1) \neq (x_1,y_1)\}$.

Let g be a function from C(A) into X* such that $g((x_1,y_1)) = (x_1,y_1)$, $g(A) = p$ and such that $g(K((x_1,y_1),(x_2,y_2)))$ is the point $(x_0,y_0) \varepsilon X$ such that (x_0,y_0) is on the line $y = (y_1/x_1)\, x$ and such that $d((x_1,y_1),(x_0,y_0))$ equals the length of $K((x_1,y_1),(x_2,y_2))$. Then g is a one to one and continuous map of C(A) onto X*.

CHAPTER III

LOCALLY CONNECTED CONTINUA

DEFINITION 3.1. Let (X,T) be a topological space and $x \in X$. X is <u>locally connected at x</u> if X has a neighborhood base at x of open connected sets. X is said to be locally connected if X is locally connected at each $x \in X$.

THEOREM 3.2. Let (X,T) be a topological space. Then X is locally connected if and only if each component of each open set is open.

PROOF. Suppose X is locally connected. Let Y be an open subset of X, C a component of Y and $y \in C$. Since X is locally connected there exists an open connected set B such that $y \in B \subseteq Y$. Thus $B \cup C$ is connected and therefore $B \cup C \subseteq C$. Hence $B \subseteq C$ and therefore C is open.

Next, suppose each component of each open set is open. Let $x \in X$ and for each open set U containing x let C_{U_x} be the component of U that contains x. Let $\underline{B}_x = \{C_{U_x} \mid U \text{ is open and } x \in U\}$; then $\underline{B}_x$ is a neighborhood base at x of open connected sets. Hence X is locally connected at x and therefore X is locally connected. #

REMARK 3.3. Let X be a locally connected compact space. Then $\underline{C} = \{C \mid C \text{ is a component of } X\}$ is an open cover for X of pairwise disjoint sets. Since X is compact $\underline{C}$ has only finitely many elements. Thus a compact locally connected space has only a finite number of components.

THEOREM 3.4. Let (X,T) be a topological space, Y a subset of X and $f:X\to Y$ a function from X onto Y. The set $T_f = \{G\subseteq Y \mid f^{-1}(G)\in T\}$ is the quotient topology on Y. If (X,T) is locally connected then (Y,T_f) is locally connected.

PROOF. Assume (X,T) is locally connected. Let $G\in T_f$ and consider some component C_G of G. For each $z\in f^{-1}(C_G)$ let $C_{f^{-1}(G)_z}$ be the component of $f^{-1}(G)$ which contains z. Since X is locally connected and f is continuous $C_{f^{-1}(G)_z}\in T$ for each $z\in f^{-1}(C_G)$. Hence $\bigcup_{z\in f^{-1}(C_G)} C_{f^{-1}(G)_z}\in T$. If we can show that

$f^{-1}(C_G) = \bigcup_{z\in f^{-1}(C_G)} C_{f^{-1}(G)_z}$ then, due to Theorem 3.2, we will be done.

Clearly $f^{-1}(C_G)\subseteq \bigcup_{z\in f^{-1}(C_G)} C_{f^{-1}(G)_z}$. Let $\omega\in f^{-1}(C_G)$; then $f(\omega)\in C_G$ and $f(\omega)\in f(C_{f^{-1}(G)_\omega})$. Since f is continuous and $C_G\cap f(C_{f^{-1}(G)_\omega})\neq\emptyset$ we have that $f(C_{f^{-1}(G)_\omega})\subseteq C_G$. Hence $\omega\in C_{f^{-1}(G)_\omega}\subseteq f^{-1}(f(C_{f^{-1}(G)_\omega}))\subseteq f^{-1}(C_G)$. Therefore $f^{-1}(C_G)\supseteq \bigcup_{z\in f^{-1}(C_G)} C_{f^{-1}(G)_z}$ and so (Y,T_f) is locally connected. #

DEFINITION 3.5. A space X is <u>connected im kleinen</u> at a point x if each open neighborhood U of x contains an open neighborhood V of x such that any pair of points of V lie in some connected subset of U.

Clearly if X is locally connected at x then X is connected im kleinen at x. Example 3.6 shows that the converse is not true.

EXAMPLE 3.6. The following subspace of R^2 is connected im kleinen at y but not locally connected at y.

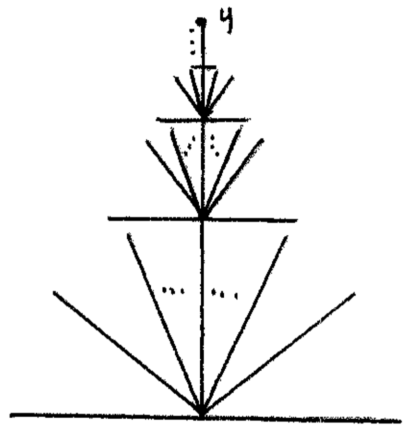

Figure 4 Subspace of R^2

THEOREM 3.7. Let (X,T) be a space such that X is connected im kleinen at x for all $x \in X$. Then X is locally connected.

PROOF. Let G be an open subset of X and C a component of G. From Theorem 3.2 it is sufficient to show that C is open. For each $x \in C$ let V_x be an open set such that $x \in V_x \subsetneq G$ and such that if $u,v \in V_x$ then there exists a connected set W such that $u,v \in W \subseteq G$. Fix $x \in C$ and for each $v \in V_x$ let W_v be a connected set such that $x,v \in W_v \subseteq G$. Then $\bigcup_{v\in V_x} W_v$ is connected and contains x. Hence $\bigcup_{v\in V_x} W_v \subseteq C$. Since $V_x \subseteq \bigcup_{v\in V_x} W_v$, we have that $V_x \subseteq C$. Thus for each $x \in C$,

$V_x \subsetneq C$. Hence $C \subsetneq \bigcup_{x \in C} V_x \subsetneq C$. Thus C is open and therefore X is locally connected. #

THEOREM 3.8. Let X be a metric continuum. Then X is locally connected if and only if for each $\varepsilon > 0$, X is the union of finitely many connected sets each of diameter less than ε.

PROOF. Let X be locally connected and $\varepsilon > 0$. For each $x \in X$ let V_x be an open connected set containing x with diameter less than ε. Then $\underline{V} = \{V_x \mid x \in X\}$ is an open cover for X and hence has a finite subcover. Let $\{V_{x_1}, \ldots, V_{x_n}\}$ be the finite subcover from $\underline{V}$; then $X = \bigcup_{i=1}^{n} V_{x_i}$. Hence X is the union of finitely many connected sets each of diameter less than ε.

Next, let X be such that for each $\varepsilon > 0$ X is the union of finitely many connected sets each of diameter less than ε. Suppose X is not locally connected. Then, from Theorem 3.7, there exists a $p \in X$ such that X is not connected im kleinen at p. Choose a neighborhood U of p such that if V is an open set with $p \in V \subsetneq U$ then there exists a $z \in V$ such that p and z do not lie together in any connected subset of U. Choose $\varepsilon > 0$ such that $S(p,\varepsilon) \subsetneq U$ and then let $C_1, C_2, \ldots, C_n$ be connected sets of diameter less than $\varepsilon/4$ such that $X = \bigcup_{i=1}^{n} C_i$. Without loss of generality suppose $p \in C_{j_0}$. Let $\omega \in C_{j_0}$. Then $d(p,\omega) \leq \varepsilon/4$ and therefore $\omega \in S(p,\varepsilon)$. Hence $C_{j_0} \subsetneq S(p,\varepsilon)$. Choose a positive integer n_1 such that $S(p,1/n_1) \subsetneq S(p,\varepsilon)$. Let $x_1 \in S(p,1/n_1)$ be such that x_1 does not lie together with p in any connected subset of

U. Then $x_1 \notin C_{j_0}$. For each positive integer n, $n \geq 2$, let $x_n \in S(p, 1/n_1 + n - 1)$ be such that x_n does not lie together with p in any connected subset of U. Hence $x_n \notin C_{j_0}$ for all $n \geq 1$. For some $k \neq j_0$, $1 \leq k \leq n$, C_k contains infinitely many of the x_n's. This and the fact that $x_n \to p$ imply that $p \in \overline{C}_k$. Hence $\overline{C}_k \not\subseteq U$. Thus let $q \in \overline{C}_k$ be such that $q \notin S(p,\varepsilon)$. Then $d(p,q) \geq \varepsilon$ and therefore diam $\overline{C}_k \geq \varepsilon$. This contradicts diam $C_k < \varepsilon/4$. Hence X is locally connected. #

For Theorem 3.14 we need to know that if X is a continuum and U is a proper open subset of X then every component of $\overline{U}$ intersects the boundary of U. Since every component of $\overline{U}$ is closed it is sufficient to show that the closure of every component of U intersects the boundary of U. Lemmas 3.9 and 3.10 show this.

LEMMA 3.9. Let C be a component of a compact Hausdorff space X and U an open set containing C. Then there exists an open set V such that $C \subseteq V \subseteq U$ and $b(V) = \emptyset$.

PROOF. Let $x \in C$; then the component of X containing x is C. Since X is compact and Hausdorff $C = Q_x$. Recall that Q_x is the quasicomponent of x. Thus $C = \bigcap\{V \mid V \text{ is open, } V \text{ is closed and } x \in V\}$. Hence, from Lemma 2.3, there exists a set V_0 such that that $C \subseteq V_0, x \in V_0, V_0 \subseteq U$, V_0 is open and V_0 is closed. Since V_0 is both open and closed $b(V_0) = \emptyset$. Hence V_0 is an open set containing C such that $b(V_0) = \emptyset$. #

LEMMA 3.10. Let U be a nonempty proper open subset of a continuum X and C a component of U. Then $\overline{C} \cap b(U) \neq \emptyset$.

PROOF. Suppose $\overline{C} \cap b(U) = \emptyset$; then $C = \overline{C}$. $\overline{C} \cap b(U) = \emptyset$ implies that C is a component of $\overline{U}$. To see this suppose that C is not a component of $\overline{U}$. Then the component of $\overline{U}$ containing C is a continuum which intersects b(U). From the proof of Theorem 2.9 we have that there exists a continuum K such that $K \subseteq \overline{U}$ and K is irreducible from C to b(U). Thus, from Theorem 2.14, $K - b(U)$ is connected and therefore $K - b(U) \subseteq C$. Hence $K \subseteq C \uplus b(U)$. This contradicts K being connected. Hence C is a component of $\overline{U}$ if $\overline{C} \cap b(U) = \emptyset$.

Since C is a component of $\overline{U}$ there exists a $\overline{U}$-open set V such that $C \subseteq V \subseteq U$ with $b^{\overline{U}}(V) = \emptyset$. Hence V is closed in X. V open in $\overline{U}$ implies that there exists a set W such that W is open in X and $V = W \cap \overline{U}$. Thus $V^0 = W^0 \cap (\overline{U})^0 = W \cap U$. $V \subseteq U$ and $V = W \cap \overline{U}$ imply that $V = W \cap U$. Hence $V = W \cap U$ and therefore V is both open and closed. This contradicts X being connected. Hence $\overline{C} \cap b(U) \neq \emptyset$. #

DEFINITION 3.11. A subcontinuum K of a continuum X is a <u>continuum of convergence</u> if there exists a sequence $\{K_n\}_{n=1}^{\infty}$ of pairwise disjoint continua such that the limit of $\{K_n\}_{n=1}^{\infty}$ is K and $K \cap K_n = \emptyset$ for all n.

THEOREM 3.12. Let X be a continuum. Then X is not the union of a countable (> 1) family of pairwise disjoint, nonempty closed sets.

PROOF. X is not the union of a finite (> 1) family of pairwise disjoint nonempty closed sets because X is connected. Suppose $X = \bigcup_{n=1}^{\infty} F_n$ where each F_n is closed and nonempty and where $F_n \cap F_m = \emptyset$ if $n \neq m$. Let U_1 be an open subset of X such that $F_1 \subseteq U_1$ and $\overline{U}_1 \cap F_2 = \emptyset$. Let U_2 be an open subset of X such that $F_2 \subseteq U_2$, $\overline{U}_2 \cap F_1 = \emptyset$ and $\overline{U}_2 \cap \overline{U}_1 \neq \emptyset$ (For example, take $U_2 = (\overline{U}_1)^c$). Assume that U_n has been constructed such that U_n is an open subset of X, $F_n \subseteq U_n$, $\overline{U}_n \cap (\bigcup_{j=1}^{n-1} F_j) = \emptyset$ and $\bigcap_{j=1}^{n} \overline{U}_j \neq \emptyset$. Let W_{n+1} be an open subset of X such that $F_{n+1} \subseteq W_{n+1}$ and $W_{n+1} \cap (\bigcup_{j=1}^{n} F_j) = \emptyset$. Let $z \varepsilon \bigcap_{j=1}^{n} \overline{U}_j$ and let V_{n+1} be an open subset of X such that $z \varepsilon V_{n+1}$ and $\overline{V}_{n+1} \cap (\bigcup_{j=1}^{n} F_n) = \emptyset$. Now let $U_{n+1} = W_{n+1} \cup V_{n+1}$; then $U_{n+1} \supseteq F_{n+1}$, $\overline{U}_{n+1} \cap (\bigcup_{j=1}^{n} F_j) = \emptyset$ and $\bigcap_{j=1}^{n+1} \overline{U}_j \neq \emptyset$. Let $\underline{F} = \{\overline{U}_n \mid n = 1,2,3,\ldots\}$. Since X is compact and any finite subset of $\underline{F}$ has a nonempty intersection we have that $\bigcap_{n=1}^{\infty} \overline{U}_n \neq \emptyset$. Let $q \varepsilon \bigcap_{n=1}^{\infty} \overline{U}_n$; then, since $X = \bigcup_{n=1}^{\infty} F_n$, there exists a positive integer n_1 such that $q \varepsilon F_{n_1}$. This contradicts $q \varepsilon \overline{U}_{n_1+1}$ since $\overline{U}_{n_1+1} \cap F_{n_1} = \emptyset$. Hence $X \neq \bigcup_{n=1}^{\infty} F_n$. #

REMARK 3.13. Let X be a continuum and K a subcontinuum of X. If K is a continuum of convergence then there is a sequence $\{K_n\}_{n=1}^{\infty}$ of pairwise disjoint continua such that $K_n \to K$ and $K \cap K_n = \emptyset$ for all n. Theorem 3.12 implies that $X \neq (\bigcup_{n=1}^{\infty} K_n) \cup K$.

THEOREM 3.14. Let X be a metric continuum such that X is not locally connected. Then there exists a $p \in X$ such that p is in some nondegenerate continuum of convergence.

PROOF. X is a metric continuum that is not locally connected. Hence, by Theorem 3.7, there exists a $p \in X$ such that X is not connected im kleinen at p. Let U be an open set such that $p \in U$ and such that if V is an open subset of U containing p then there exists a $y \in V$ such that y does not lie together with p in any connected subset of U. Let V_0 be an open subset of U such that $p \in V_0$ and $\overline{V}_0 \cap b(U) = \emptyset$. Choose a positive integer N_1 such that $S(p,1/N_1) \subseteq V_0$ and then choose $y_1 \in S(p,1/N_1)$ such that the component of $\overline{V}_0$ containing y_1 C_{y_1} is such that $C_{y_1} \cap \{p\} = \emptyset$. Now let W_1, V_1 be open sets such that $p \in W_1$, $C_{y_1} \subseteq V_1$ and such that $W_1 \cap V_1 = \emptyset$. Choose a positive integer N such that $S(p,1/N) \subseteq V_0 \cap W_1$ and then let $N_2 = \max\{N, N_1+1\}$. Let y_2 be such that $y_2 \in S(p,1/N_2)$ and such that the component of $\overline{V}_0$ containing y_2 C_{y_2} is such that $C_{y_2} \cap \{p\} = \emptyset$. $y_2 \in W_1$, $C_{y_1} \subseteq V_1$ and $W_1 \cap V_1 = \emptyset$ imply that $C_{y_1} \cap C_{y_2} = \emptyset$. Assume y_{n-1} has been chosen such that $y_{n-1} \in S(p,1/N_{n-1})$, the component of $\overline{V}_0$ containing y_{n-1} $C_{y_{n-1}}$ does not contain p and $C_{y_{n-1}} \cap (\bigcup_{m=1}^{n-2} C_{y_m}) = \emptyset$. Then let W_{n-1}, V_{n-1} be open disjoint sets such that $p \in W_{n-1}$ and $\bigcup_{i=1}^{n-1} C_{y_i} \subseteq V_{n-1}$. Choose a positive integer M such that $S(p,1/M) \subseteq V_0 \cap W_{n-1}$ and let $N_n = \max\{M, N_{n-1}+1\}$.

Let y_n be such that $y_n \in S(p,1/N_n)$ and such that the component of $\overline{V}_0$ containing y_n C_{y_n} is such that $C_{y_n} \cap \{p\} = \emptyset$. $y_n \in W_{n-1}$, $\bigcup_{i=1}^{n-1} C_{y_i} \subseteq V_{n-1}$ and $W_{n-1} \cap V_{n-1} = \emptyset$ imply that $C_{y_n} \cap (\bigcup_{m=1}^{n-1} C_{y_m}) = \emptyset$. Hence we have a sequence $\{C_{y_n}\}_{n=1}^{\infty}$ with each C_{y_n} a continuum. $C_{y_n} \cap b(V_0) \neq \emptyset$ for all n is a consequence of Lemma 3.10. Recalling Remark 2.28 we have that $\{C_{y_n}\}_{n=1}^{\infty}$ has a convergent subsequence $\{C_{y_{n_k}}\}_{k=1}^{\infty}$. Let $A \subseteq X$ be such that $C_{y_{n_k}} \to A$. Then A is a continuum. $p \in A$ since $y_{n_k} \to p$ and $A \subseteq U$ since $C_{y_{n_k}} \subseteq \overline{V}_0$ for all n_k. Hence for each n_k $C_{n_k} \cap A = \emptyset$. A is a nondegenerate continuum since $C_{n_k} \cap b(V_0) \neq \emptyset$ for each n_k. Hence p is in a nondegenerate continuum of convergence. #

DEFINITION 3.15. A metric continuum X is <u>semi-locally connected at a point $x \in X$</u> if for each open set U containing x, there is an open set V such that $x \in V \subseteq U$ and $X - V$ has only a finite number of components. X is said to be semi-locally if X is semi-locally connected at each $x \in X$.

DEFINITION 3.16. A metric continuum X is <u>regular at $x \in X$</u> if for each open set U containing x there is an open set V such that $x \in V \subseteq U$ and $b(V)$ is finite. X is said to be regular if X is regular at each $x \in X$.

REMARK 3.17. Recall the continuum X given in Example 2.2. X is both semi-locally connected and regular.

THEOREM 3.18. Let X be a metric continuum and $p \in X$ such that X is regular at p. Then X is locally connected at p and semi-locally connected at p.

PROOF. First suppose X is not locally connected at p. Then, from Theorem 3.14, p is in some nondegenerate continuum of convergence. Let K be a nondegenerate continuum of convergence containing p and $\{K_n\}_{n=1}^{\infty}$ a sequence of pairwise disjoint continua such that $K_n \to K$ with $K_n \cap K = \emptyset$ for all n. Pick $z \in K$ different than p and let $\varepsilon = d(z,p)$. Choose a positive integer N such that $1/N < \varepsilon/2$. Let V be an open set containing p such that $V \subsetneq S(p,1/N)$ and b(V) is finite. $p \in K$ implies that there exists a positive integer N_1 such that for $n \geq N_1$, $V \cap K_n \neq \emptyset$. $K_i \cap K_j = \emptyset$ for $i \neq j$ and b(V) finite imply that there is a $N_2 \geq N_1$ such that for $n \geq N_2$ $K_n \cap b(V) = \emptyset$. Hence for $n \geq N_2$, $K_n \subsetneq V$. This contradicts $z \in K$. Hence X is locally connected at p.

Now we will show that X is semi-locally connected at p. Let U be an open set containing p and let V be an open set such that $p \in V \subsetneq U$ and b(V) is finite. We will show that X - V has only a finite number of components. Let C be a component of X - V; then $C = \overline{C}$. If $C \cap b(V) = \emptyset$ then C is a component of $X - \overline{V}$ and therefore, due to Lemma 3.10, $\overline{C} \cap b(V) \neq \emptyset$. This contradicts $C = \overline{C}$. Hence $C \cap b(V) \neq \emptyset$. Since b(V) is finite, the number of components of $X - V$ is finite. Thus X is semi-locally connected at p. #

EXAMPLE 3.19. The subspace X shown in Figure 5 is locally connected at p but not semi-locally connected at p or regular at p.

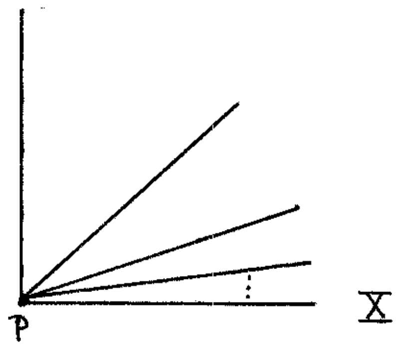

Figure 5. Subspace of R^2

Notice that X is not locally connected.

EXAMPLE 3.20. Let $X = \{(x,y) \in R^2 | x^2+y^2 \leq 1\}$ and $p = (0,0)$. Then X is locally connected and semi-locally connected. X is not regular at p. See Figure 6.

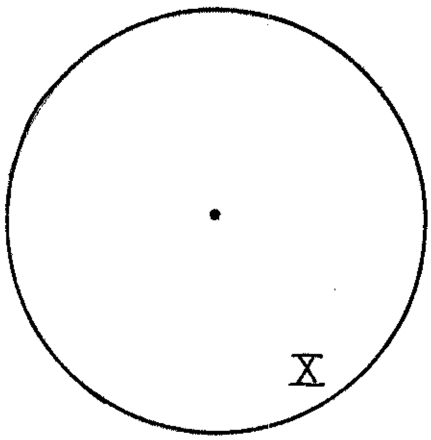

Figure 6 Subspace of R^2

THEOREM 3.21. Let X be a metric continuum. If X is locally connected then X is semi-locally connected.

PROOF. Let X be locally connected. Suppose there exists a $p \in X$ such that X is not semi-locally connected at p. Then there is an open set U containing p such that if W is an open set contained in U and containing p then $X - W$ has infintely many components. Choose $\varepsilon > 0$ such that $S(p,\varepsilon) \subsetneq U$. Let $V_1 = S(p,\varepsilon)$ and $V_2 = S(p,\varepsilon/2)$. Then both $X - V_1$ and $X - V_2$ have infinitely many components.

We need to show that there exist infinitely many components of $X - V_2$ which intersect $X - V_1$. Let $\underline{C} = \{C \subsetneq X \mid C$ is a component of $X - V_2$ and $C \cap (X - V_1) \neq \emptyset\}$. Suppose $\underline{C}$ has only finitely many elements. Then there is a positive integer n such that $\underline{C} = \{C_1, C_2, \ldots, C_n\}$. Let $x \in X - V_1$; then $x \in \bigcup_{i=1}^{n} C_i$. The set $\bigcup_{i=1}^{n} C_i$ is closed since each C_i is a component of the closed set $X - V_2$. Then $(X - \bigcup_{i=1}^{n} C_i) \cap U$ is an open set containing p and contained in U. Hence $X - [(X - \bigcup_{i=1}^{n} C_i) \cap U]$ has infinitely many components. However, $X - [(X - \bigcup_{i=1}^{n} C_i) \cap U] = \bigcup_{i=1}^{n} C_i \cup (X - U)$, $X - V_1 \subsetneq \bigcup_{i=1}^{n} C_i$ and $X - U \subsetneq X - V_1$ imply that $X - [(X - \bigcup_{i=1}^{n} C_i) \cap U] = \bigcup_{i=1}^{n} C_i$. This contradicts $X - [(X - \bigcup_{i=1}^{n} C_i) \cap U]$ having infinitely many components. Thus $\underline{C}$ has infinitely many elements.

Let $\{C_n\}_{n=1}^{\infty}$ be a sequence of components of $X - V_2$ such that $C_n \cap (X - V_1) \neq \emptyset$ for all n and $C_n \cap C_m = \emptyset$ if $n \neq m$. For each n we have that $C_n \cap b(V_1) \neq 0$ since $C_n \cap (X - V_1) \neq \phi$ and C_n is connected. Let $\{C_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{C_n\}_{n=1}^{\infty}$ and $K \subseteq X$ such that $C_{n_k} \to K$. Then there is a $z \in K \cap b(V_1)$ since $C_{n_k} \cap b(V_1) \neq \phi$ for all k. Let V be an open set containing z such that $V \cap \overline{V}_2 = \phi$. Choose an open connected set V_3 containing z and contained in V. We can do this because X is locally connected. Then $V_3 \cap C_{n_k} \neq \phi$ for all but finitely many k since $z \in V_3$ and $z \in K$. The fact $V_3 \cap \overline{V}_2 = \phi$ implies that $V_3 \subseteq X - V_2$. Now we have that $V_3 \subseteq C_{n_k}$ for all but finitely many k since V_3 is connected, $V_3 \subseteq X - V_2$, C_{n_k} is a component of $X - V_2$ for all k and $V_3 \cap C_{n_k} \neq \phi$ for infinitely many k. This contradicts $C_n \cap C_m = \phi$ for $n \neq m$. Hence X is semi-locally connected at p and therefore X is semi-locally connected. #

EXAMPLE 3.22. The converse of Theorem 3.21 is not true. The subspace X of R^2 shown in Figure 7 is semi-locally connected but not locally connected at x.

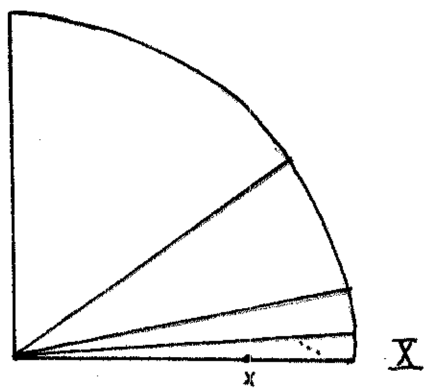

Figure 7 A Semi-locally Connected Subspace R^2.

DEFINITION 3.23. Let (X,T) be a topological space and choose two distinct points a,b in X. Let K be a subset of X. Then K is an arc joining a to b if K is a continuum containing a and b, $K - \{a\}$ is connected, $K - \{b\}$ is connected and if c is an element of K different from a and b then $K - \{c\} = U \uplus V$ with $a \in U$ and $b \in V$.

DEFINITION 3.24. Let (X,T) be a topological space, $\underline{U}$ a family of subsets of X and $a,b \in X$. A simple chain from $\underline{U}$ joining a to b is a finite sequence $U_1,\ldots,U_n$ of members of $\underline{U}$ such that $a \in U_1, b \in U_n$ and $U_i \cap U_j \neq \emptyset$ if and only if $|i-j| \leq 1$.

THEOREM 3.25. Let X be a locally connected metric continuum and choose two distinct points a,b in X. Then there is an arc K joining a to b.

PROOF. For each $x \in X$ let U_{1_x} be an open connected set containing x such that $\overline{U}_{1_x} \subseteq S(x,1)$. Then $X = \bigcup_{x \in X} U_{1_x}$ and therefore there exist $x_1,x_2,\ldots,x_{n_1}$ in X such that $\underline{U}_1 = \{U_{1_{x_1}},\ldots,U_{1_{x_{n_1}}}\}$ is a simple chain joining a to b. Let $U_{1_{x_i}} = U_{1,i}$ for each $i = 1,2,\ldots,n_1$; then $\underline{U}_1 = \{U_{1,1},U_{1,2},\ldots,U_{1,n_1}\}$. Set $U_1 = \bigcup_{i=1}^{n_1} U_{1,i}$.

For each $i = 1,2,3,\ldots,n_1$ and for each $x \in U_{1,i}$ let U_{2_x} be an open connected set containing x such that $\overline{U}_{2_x} \subseteq S(x,1/2) \cap U_{1,i}$. Then for each i, $i = 1,2,3,\ldots,n_1$, $\bigcup_{x \in U_{1,i}} U_{2_x} = U_{1,i}$. Choose $x_i \in U_{1,i-1} \cap U_{1,i}$ for $i = 2,3,4,\ldots,n_1$. Let $\underline{U}_{2,i}$ be a simple chain from $\{U_{2x} | x \in U_{1,i}\}$ joining x_i to x_{i+1} where $x_{n_1+1} = b$ and $x_1 = a$, for $i = 1,2,3,\ldots,n_1$. Now construct a simple chain $\underline{U}_2$ from $\bigcup_{i=1}^{n_1} \underline{U}_{2,i}$ joining a to b. If $U \in \underline{U}_2$ then $U \subseteq U_{1,i}$ for at most two i. Relabeling let $\underline{U}_2 = \{U_{2,1}, U_{2,2}, \ldots, U_{2,n_2}\}$ with $a \in U_{2,1}$, $b \in U_{2,n_2}$ and $U_{2,i} \cap U_{2,j} \neq \emptyset$ if and only if $|i-j| \leq 1$. Let $U_2 = \bigcup_{i=1}^{n_2} U_{2,i}$; then $\overline{U}_2 \subseteq U_1$.

Suppose that for $m \leq k$ a simple chain $\underline{U}_m = \{U_{m,1}, U_{m,2}, \ldots, U_{m,n_m}\}$, of open connected sets, from a to b has been formed such that $a \in U_{m,1}$, $b \in U_{m,n_m}$ and $U_{m,i} \cap U_{m,j} \neq \emptyset$ if and only if $|i-j| \leq 1$. Also assume $\overline{U}_{m+1} \subseteq U_m$, where $U_m = \bigcup \underline{U}_m$, and if $U \in \underline{U}_m$ then $U \subseteq U_{m-1,i}$ for at most two i. For each $i = 1,2,3,\ldots,n_k$ and for each $x \in U_{k,i}$, let $U_{k+1,x}$ be an open connected containing x such that $\overline{U}_{k+1,x} \subseteq S(x,1/k) \cap U_{k,i}$. Then $U_{k,i} = \bigcup_{x \in U_{k,i}} U_{k+1,x}$ for $i = 1,2,3,\ldots,n_k$. Choose $x_i \in U_{k,i-1} \cap U_{k,i}$ for $i = 2,3,\ldots,n_k$ and let $x_1 = a$ and $x_{n_k+1} = b$. Let $\underline{U}_{k+1,i}$ be a simple chain from $\{U_{k+1,i} | x \in U_{k,i}\}$ joining a to b for $i = 1,2,3,\ldots,n_k$. Now construct a simple chain $\underline{U}_{k+1}$ from $\bigcup_{i=1}^{n_k} \underline{U}_{k+1,i}$ joining a to b and let $U_{k+1} = \bigcup \underline{U}_{k+1}$. Relabeling, let $\underline{U}_{k+1} = \{U_{k+1,1}, \ldots, U_{k+1,n_{k+1}}\}$ with $a \in U_{k+1,1}$, $b \in U_{k+1,n_{k+1}}$ and $U_{k+1,i} \cap U_{k+1,j} \neq \emptyset$ if and only if $|i-j| \leq 1$. Then $\overline{U}_{k+1} \subseteq U_k$ and if $U \in \underline{U}_{k+1}$ then $U \subseteq U_{k,i}$ for at most two i.

Thus, $\{U_n\}_{n=1}^{\infty}$ is a sequence of open connected sets containing a and b such that $U_n \supseteq \overline{U}_{n+1}$ for all n. Let $K = \bigcap_{n=1}^{\infty} \overline{U}_n$; then, due to Lemma 2.3, K is a continuum containing a and b. We will show that K is an arc joining a to b.

Choose an element z in K different from a and b. For $i = 1,2,3,\ldots$ let $P_i = \{U_{i,j} \varepsilon \underline{U}_i \mid z \notin U_{i,j}$ and if $z \varepsilon U_{i,k}$ for some k then $j < k\}$ and let $F_i = \{U_{i,j} \varepsilon \underline{U}_i \mid z \notin U_{i,j}$ and if $z \varepsilon U_{i,k}$ for some k then $j > k\}$. Set $W = \bigcup_{i=1}^{\infty} [(\bigcup P_i) \cap K]$ and $V = \bigcup_{i=1}^{\infty} [(\bigcup F_i) \cap K]$; then W and V are open in K and $W \cap V = \phi$. Clearly, $W \cup V \subseteq K - \{z\}$. We will show that $K - \{z\} \subseteq W \cup V$. Let $\omega \varepsilon K - \{z\}$ and set $\delta = d(\omega, z)$. Choose a positive integer i_0 such that $4/i_0 < \delta$. Consider $\underline{U}_{i_0}$ and suppose $z \varepsilon U_{i_0,j}$ for some j. $U_{i_0,j} \subseteq S(x, 1/i_0)$ for some $x \varepsilon U_{i_0-1}$. Thus $d(x,z) < 1/i_0$. Suppose $\omega \varepsilon U_{i_0,j}$; then $\delta = d(\omega,z) \leq d(\omega,x) + d(x,z) < 1/i_0 + 1/i_0 < \delta$. This is a contradiction. Hence $\omega \notin U_{i_0,j}$. Thus $\omega \varepsilon W \cup V$ and therefore $X - \{z\} = W \cup V$. Let $\alpha_1 = d(z,a)$, $\alpha_2 = d(z,b)$ and choose a positive integer n_1 such that $2/n_1 < \alpha_1$ and $2/n_1 < \alpha_2$. Then for all $i \geq n_1$ $a \varepsilon P_i$ and $b \varepsilon F_i$. Hence $a \varepsilon W$ and $b \varepsilon V$.

We still need to show that $K - \{a\}$ and $K - \{b\}$ are connected. If K is irreducible from a to b then, by Theorem 2.14, $K - \{a\}$ and $K - \{b\}$ are connected. Suppose K is not irreducible from a to b. Let K_1 be a continuum containing a and b such that $K_1 \subsetneq K$. Choose $q \varepsilon K - K_1$; then $K_1 \subseteq K - \{q\} = R \cup S$ for some sets R and S such that $a \in R$ and $b \in S$. K_1 connected implies that $K_1 \subseteq R$ or $K_1 \subseteq S$. This contradicts $a \in R$ and $b \in S$. Hence K is irreducible from a to b and therefore $K - \{a\}$ and $K - \{b\}$ are connected. Thus K is an arc joining a to b. #

Thus if X is a locally connected metric continuum and x,y two distinct points of X then there exists an arc joining x to y. Let $\varepsilon > 0$. Can we join x to y with an arc of diameter less than ε? Theorem 3.28 answers this question; but first we need two lemmas.

LEMMA 3.26. Let X be a compact metric space and let $U_1, U_2, \ldots, U_n$ be a finite open cover of X. Then there is a $\delta > 0$ such that if $A \subseteq X$ and the diameter of A is less than δ then $A \subseteq U_i$ for some $i = 1,2,3,\ldots,n$.

PROOF. Suppose that for all $\delta > 0$ there is a set A_δ such that the diameter of A_δ is less than δ, but $A_\delta \nsubseteq U_i$ for $i = 1,2,3,\ldots,n$. In particular for each positive integer n let A_n be a subset of X such that the diameter of A_n is less than $1/n$ and such that $A_n \nsubseteq U_i$ for all i. For each positive integer n let $a_n \in A_n$; then there exists a $z \in X$ such that $a_n \overset{c}{\to} z$. Without loss of generality suppose $z \in U_{k_0}$. Choose $\varepsilon > 0$ such that $S(z,\varepsilon) \subseteq U_{k_0}$ and let n_0 be a positive integer such that $1/n_0 < \varepsilon/2$. Since $a_n \overset{c}{\to} z$ we can choose a positive integer m greater than n_0 such that $a_m \in S(z,\varepsilon/2)$. Choose $a \in A_m$; then $d(z,a) \leq d(z,a_m) + d(a_m,a) < \varepsilon/2 + \varepsilon/2$. Hence $A_m \subseteq U_{k_0}$. This contradicts $A_n \nsubseteq U_i$ for all i and n. Thus there is a $\delta > 0$ such that if $A \subseteq X$ and the diameter of A is less than δ then $A \subseteq U_i$ for some $i = 1,2,3,\ldots,n$. #

LEMMA 3.27. Let X be a compact locally connected metric space. Then for each $\varepsilon > 0$, there is some $\delta > 0$ such that if $x,y \in X$ and $d(x,y) < \delta$, then there is an open connected set C such that the diameter of C is less than ε and $x,y \in C$.

PROOF. Let $\varepsilon > 0$ and for each $x \in X$ let S_x be an open connected set such that $x \in S_x \subseteq S(x, \varepsilon/4)$. Then the diameter of $S_x < \varepsilon/2$ for all $x \in X$. $\{S_x \mid x \in X\}$ is an open cover for X and therefore there exist $x_1, x_2, \ldots, x_n$ in X such that $X = \bigcup_{i=1}^{n} S_{x_i}$. By Lemma 3.26 we can choose a $\delta > 0$ such that if $A \subseteq X$ and the diameter of A is less than δ then $A \subseteq S_{x_i}$ for some $i = 1,2,\ldots,n$. Let $x,y \in X$ be such that $d(x,y) < \delta$; then $\{x,y\} \subseteq S_{x_{i_0}}$ for for some i_0. Hence if $d(x,y) < \delta$, there is an open connected set $C = S_{x_{i_0}}$ such that the diameter of C is less than ε and $x,y \in C$. #

THEOREM 3.28. Let X be a locally connected metric continuum. Then for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if x,y are in X with $d(x,y) < \delta$, then x can be joined to y by an arc of diameter less than ε.

PROOF. Let $\varepsilon > 0$ and choose $\delta > 0$ such that if x,y are in X and $d(x,y) < \delta$ then there is an open connected set C such that the diameter of C is less than ε and $x,y \in C$. Choose $x,y \in X$ such that $d(x,y) < \delta$. Let C be an open connected set, containing x and y, of diameter less than ε. Hence C is an open locally connected subset of X containing x and y of diameter less than ε. To construct an arc K joining x to y and contained in C follow the construction given in the proof of Theorem 3.25. Hence there exists an arc K of diameter less than ε joining x to y. #

REMARK 3.29. In the proof of Theorem 3.28 we can not simply apply Theorem 3.25 to the continuum $\overline{C}$ to get the arc K, since $\overline{C}$ may not be locally connected. Below is an example of a locally connected metric continuum X and an open connected subset of X whose closure is not locally connected.

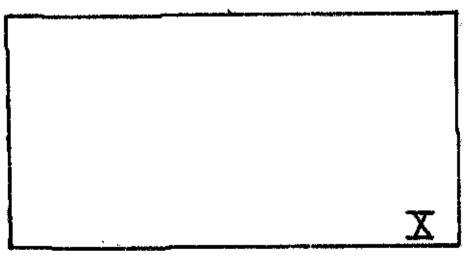

Figure 8 A Subcontinuum X of R^2.

Construct an open connected subset of X as shown in Figures 9, 10 and 11.

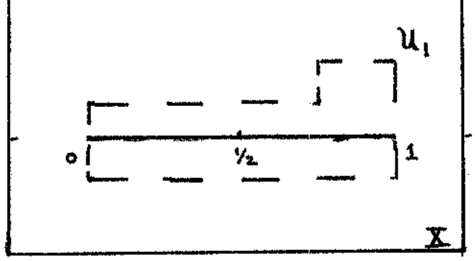

Figure 9 Construction of U_1.

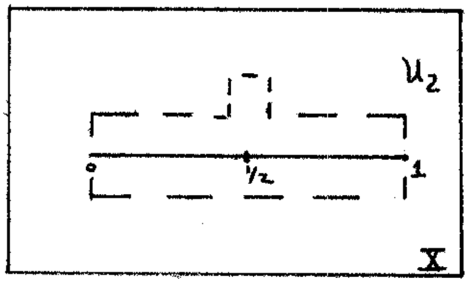

Figure 10 Construction of U_2.

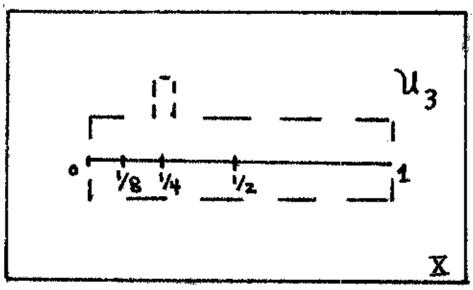

Figure 11 Construction of U_3.

Continue in this manner and let $U = \bigcup_{i=1}^{\infty} U_n$. Then U is an open connected subset of X whose closure is not locally connected.

CHAPTER IV

DECOMPOSABLE CONTINUA

DEFINITION 4.1. A nondegenerate continuum is <u>decomposable</u> if it is the union of two proper subcontinua. A nondegenerate continuum that is not decomposable is said to be <u>indecomposable</u>.

THEOREM 4.2. A continuum X is decomposable if and only if X contains a proper subcontinuum with interior points.

PROOF. Let X be a decomposable continuum and let K_1, K_2 be nonempty proper subcontinua such that $X = K_1 \cup K_2$. Suppose $\overset{0}{K_1} = \emptyset$; then $\overline{X - K_1} = X - \overset{0}{K_1} = X$. Choose $z \in K_1 - K_2$; then $z \in \overline{X - K_1}$. Let $\{x_d\}_{d \in D}$ be a net in $X - K_1$ such that $x_d \to z$. Since K_2 is compact, there is a $\omega \in K_2$ such that $x_d \overset{c}{\to} \omega$. We have that $z \neq \omega$ since $\omega \in K_2$ and $z \in K_1 - K_2$. Let U,V be open sets such that $\omega \in V$, $z \in U$ and $U \cap V = \emptyset$. Since $x_d \to z$ there exists a $d_0 \in D$ such that if $d \in D$ and $d \geq d_0$ then $x_d \in U$. However, $x_d \overset{c}{\to} \omega$ implies that there exists a $d_1 \geq d_0$ such that $x_{d_1} \in V$. This contradicts $U \cap V = \emptyset$. Hence $\overset{0}{K_1} \neq \emptyset$ and therefore X contains a proper subcontinuum with interior points.

Now suppose X contains a proper subcontinuum K_1 such that $\overset{0}{K_1} \neq \emptyset$. Then we have that $\overline{X - K_1} = X - \overset{0}{K_1} \neq X$ since $\overset{0}{K_1} \neq \emptyset$. Hence if $X - K_1$ is connected then $X = \overline{X - K_1} \cup K_1$ with $\overline{X - K_1}$ and K_1 proper subcontinua. Thus if $X - K_1$ is connected X is decomposable. Suppose $X - K_1$ is not connected. Let $X - K_1 = A \uplus B$ for some sets

A, B. Then $\overline{K_1 \cup A}$ and $\overline{K_1 \cup B}$ are connected sets and $X = \overline{K_1 \cup A} \cup \overline{K_1 \cup B}$. The fact $\overline{A} \cap B = \emptyset = B \cap \overline{A}$ implies that $\overline{K_1 \cup A}$ and $\overline{K_1 \cup B}$ are proper subcontinua. Hence X is decomposable. #

REMARK 4.3. From Theorem 4.2 we have that a continuum X is indecomposable if and only if every proper subcontinuum of X has an empty interior.

DEFINITION 4.4. Let X be a continuum and $p \in X$. Then $C_p = \{x \in X \mid$ there exists a proper closed connected subset of X containing both p and $x\}$ is the <u>composant of p</u> in X.

THEOREM 4.5. Every decomposable continuum is a composant of some one of its points.

PROOF. Let X be a decomposable continuum. Choose proper subcontinua K_1, K_2 such that $X = K_1 \cup K_2$. Since X is connected $K_1 \cap K_2 \neq \emptyset$. Let $p \in K_1 \cap K_2$; then the composant of p is X. #

THEOREM 4.6. Let X be a decomposable continuum and $a \in X$; then the composant of a is open in X.

PROOF. Let C_a be the composant of a. If $C_a = X$ we are done. Hence, suppose $C_a \subsetneq X$. Choose proper subcontinua M and N such that $X = M \cup N$. If $a \in M \cap N$ then $C_a = X$ and we are done. Thus, without loss of generality, suppose $a \in M - N$. Let $K = \{x \in X \mid X$ is irreducible from a to $x\}$. $K \neq 0$ since $C_a \subsetneq X$. $\overline{K} = \overline{K \cap M} \cup \overline{K \cap N}$. $M \subseteq C_a$ because $a \in M$. Therefore $M \cap K = \emptyset$. Hence $\overline{K} \subseteq N$. Now, suppose that for each open set U containing a $U \cap K \neq \emptyset$. Then $a \in \overline{K}$ and therefore $a \in N$. This contradicts $a \in M - N$. Thus there exists an open set U such that $a \in U \subseteq C_a$. Therefore C_a is open. #

THEOREM 4.7. Let X be a continuum; then every composant of X is connected.

PROOF. Let X be a continuum and $p \in X$. Suppose the composant of p C_p is such that $C_p = A \cup B$ for some sets A, B in X. Without loss of generality let $p \in A$. Choose $x \in B$. There exists a proper subcontinuum K such that $p, x \in K$ since $x \in C_p$. Thus $K \subseteq C_p$ and therefore $K \subseteq A$ or $K \subseteq B$. This contradicts $p \in A$ and $x \in B$. Hence C_p is connected and therefore every composant of X is connected. #

THEOREM 4.8. Let X be a continuum. Then every composant of the continuum X is dense in X.

PROOF. Let X be a continuum and suppose there exists a $p \in X$ such that the composant of p C_p is not dense in X. The fact $\overline{C}_p \subsetneq X$ implies that $\overline{C}_p \subseteq C_p$. Hence $C_p = \overline{C}_p$. Choose $x \in X - C_p$ and then let W, V be open disjoint sets such that $C_p \subseteq W$, $x \in V$ and $\overline{W} \cap V = \emptyset$. Let D be the component of W containing p. Then $\overline{D} \cap b(W) \neq \emptyset$ and therefore $C_p \cap b(W) \neq \emptyset$. This contradicts $C_p \subseteq W$. Hence $\overline{C}_p = X$ and therefore every composant of X is dense in X. #

THEOREM 4.9. If X is a metric continuum, then every composant of X is the union of countably many proper subcontinua of X.

PROOF. Let X be a metric continuum and $p \in X$. Let $\underline{B} = \{B_1, B_2, B_3, \ldots\}$ be a base for $X - \{p\}$ such that for each i, $i = 1, 2, 3, \ldots$, $\overline{B}_i \subseteq X - \{p\}$, and $B_i = S(x_i, \varepsilon_i)$ with $x_i \in X$ and $\varepsilon_i > 0$.

For each i, $i = 1,2,3,\ldots,$ let n_i be a positive integer such that $\overline{S(x_i,\varepsilon_i + 1/n_i)} \subseteq X - \{p\}$. Then for each i, $i = 1,2,3,\ldots,$ and j, $j = 0,1,2,3,\ldots,$ let $C_{i,j}$ be the component of $X - \overline{S(x_i,\varepsilon_i + \frac{1}{n_i+j})}$ which contains p. Let $C = \bigcup_{i=1}^{\infty} (\bigcup_{j=0}^{\infty} \overline{C_{i,j}})$; then C is the union of countably many proper subcontinua which contain p. Let C_p be the composant of p; then clearly $C \subseteq C_p$. Choose an element x of C_p different than p and let K be a proper subcontinuum containing p and x. Then choose $B_{i_0} \varepsilon \underline{B}$ such that $\overline{B}_{i_0} \subseteq X-K$. If $K \subseteq X-\overline{S(x_{i_0},\varepsilon_{i_0}+\frac{1}{n_{i_0}+j})}$ for some j then $x \varepsilon C$ and hence $C = C_p$. Thus suppose

$K \nsubseteq X-\overline{S(x_{i_0}, \varepsilon_{i_0}+\frac{1}{n_{i_0}+j})}$ for every j. Then

$K \cap b(S(x_{i_0},\varepsilon_{i_0}+\frac{1}{n_{i_0}+j})) \neq \emptyset$ for every j. For each j, $j = 0,1,2,\ldots,$ let $z_j \varepsilon K \cap b(S(x_{i_0},\varepsilon_{i_0}+\frac{1}{n_{i_0}+j}))$. Then there exists a $z \varepsilon \overline{B}_{i_0}$ such that $z_i \overset{\subset}{\to} z$. Hence $K \cap \overline{B}_{i_0} \neq \emptyset$. This contradicts $\overline{B}_{i_0} \subseteq X-K$. Thus $K \subseteq X-\overline{S(x_{i_0},\varepsilon_{i_0}+\frac{1}{n_{i_0}+j})}$ for some j and therefore $x \varepsilon C$ and $C = C_p$. Hence C_p is the union of countably many proper subcontinua. #

THEOREM 4.10. (Baire Category Theorem). If X is a compact T_2 space, then X is not the union of countably many closed sets each having empty interior.

THEOREM 4.11. Let X be an indecomposable continuum and $p,q \in X$. Let C_p be the composant of p and C_q the composant of q. Then $C_p = C_q$ or $C_p \cap C_q = \emptyset$.

PROOF. Suppose $C_p \cap C_q \neq \emptyset$. Choose $z \in C_p \cap C_q$ and let K_1, K_2 be proper subcontinua such that $z,p \in K_1$ and $z,q \in K_2$. X indecomposable implies that $K_1 \cup K_2$ is a proper subcontinuum of X. Let $\omega \in C_p$ and K_3 a proper subcontinuum such that $p,\omega \in K_3$. Then $K_1 \cup K_2 \cup K_3$ is a proper subcontinuum containing ω and q. Hence $C_p \subseteq C_q$. Similarly $C_q \subseteq C_p$. Thus if $C_p \cap C_q \neq \emptyset$ then $C_p = C_q$. #

REMARK 4.12. Let X be an indecomposable metric continuum. Then Theorem 4.2, Theorem 4.9, Theorem 4.10 and Theorem 4.11 imply that X has uncountably many pairwise disjoint composants. Thus every indecomposable metric continuum is irreducible between each two points of some uncountable set.

THEOREM 4.13. Let X be a metric continuum. Then X is indecomposable if and only if there exist three distinct points a,b,c such that X is irreducible between any two of these three points.

PROOF. Let X be an indecomposable metric continuum. Let $\underset{\sim}{C}$ be an uncountable subset of X such that X is irreducible between any two points of $\underset{\sim}{C}$. Choose three distinct points a,b,c from $\underset{\sim}{C}$; then X is irreducible between any two of these three chosen points.

Now let X be a metric continuum containing three distinct points a,b,c such that X is irreducible between any two of these three points. Suppose X is decomposable. Choose proper subcontinua K_1, K_2 such that $X = K_1 \cup K_2$. Without loss of generality let $a \in K_1$; then $b,c \notin K_1$. Hence $b,c \in K_2$; but this contradicts X irreducible between b and c. Thus X is indecomposable. #

EXAMPLE 4.14. We will construct an indecomposable metric continuum. Let $X = \{(x,y) \in R^2 \mid x^2+y^2 \leq 10\}$. X is a locally connected metric continuum. Let $X^0 = \{(x,y) \in R^2 \mid x^2+y^2 < 10\}$ and choose three distinct points a,b,c in X^0. For $i = 1,2,3,\ldots,n_1$ let $C_{1,i}$ be an open connected subset of X with diameter less than 1/2 such that $\underline{C}_1 = \{C_{1,1}, C_{1,2}, \ldots, C_{1,n_1}\}$ is a chain joining a to c with $a \in C_{1,1}$, $c \in C_{1,n_1}$ and $b \in C_{1,j}$, $1 < j < n_1$, for exactly one j. We will call $\underline{C}_1$ a chain joining a to c through b. For $i = 1,2,3,\ldots,n_2$ let $C_{2,i}$ be an open connected set with diameter less than 1/4 such that $\underline{C}_2 = \{C_{2,1}, \ldots, C_{2,n_2}\}$ is a chain joining a to b with $a \in C_{2,1}$, $b \in C_{2,n_2}$, $c \in C_{2,j}$, $1 < j < n_2$, for exactly one j. We will call $\underline{C}_2$ a chain joining a to b through c. Each $C_{2,i}$, $i = 1,2,\ldots,n_2$, also has the property that $\overline{C}_{2,i} \subsetneq C_{1,k}$ for some k. Let $\underline{C}_3 = \{C_{3,1}, \ldots, C_{3,n_3}\}$ be a chain joining c to b such that $C_{3,i}$ is an open connected set of diameter less than 1/8, $c \in C_{3,1}$, $b \in C_{3,n_3}$ and $a \in C_{3,j}$, $1 < j < n_3$, for exactly one j. We will call $\underline{C}_3$ a chain joining c to b through a. Each $C_{3,i}$ is also such that $\overline{C}_{3,i} \subsetneq C_{2,k}$ for some k. For each $i = 1,2,3,\ldots,n_4$ let $C_{4,i}$ be an open connected set of diameter less than 1/16 such that $\underline{C}_4 = \{C_{4,1}, C_{4,2}, \ldots,$

$C_{4,n_4}\}$ is a chain joining a to c with $a \in C_{4,1}$, $c \in C_{4,n_4}$ and $b \in C_{4,j}$, $1 < j < n_4$, for exactly one j. Again each $C_{4,i}$ is such that $\overline{C}_{4,i} \subseteq C_{3,k}$ for some k. For each positive integer $n \geq 5$ let $\underline{C}_n$ be a chain connecting a,b and c with the following properties:

i) If $C_{n,i} \in \underline{C}_n$ then diam $C_{n,i} < 1/2^n$.

ii) Each $C_{n+1,i} \in \underline{C}_{n+1}$ is such that $\overline{C}_{n+1,i} \subseteq C_{n,k}$ for some k.

iii) If $n \equiv 1 \bmod 3$ then $\underline{C}_n$ is a chain joining a to c through b.

iv) If $n \equiv 2 \bmod 3$ then $\underline{C}_n$ is a chain joining a to b through c.

v) If $n \equiv 0 \bmod 3$ then $\underline{C}_n$ is a chain joining c to b through a.

Let $K_m = \bigcup_{i=1}^{n_m} \overline{C}_{m,i}$ for each positive integer m. Then $K = \bigcap_{m=1}^{\infty} K_m$ is a continuum containing a,b and c. See figure 12.

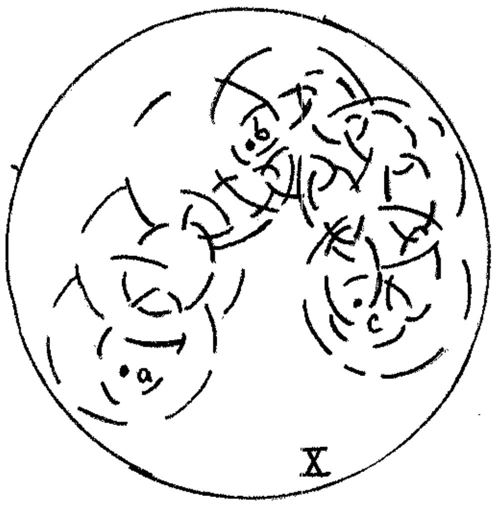

Figure 12 Clarification of K

Let B be a proper subcontinuum of K such that $a,b \varepsilon B$. Suppose $c \notin B$. Set $\delta = d(B,c)$ and choose a positive integer m_0 such that $1/2^{m_0} < \delta/2$. Let $m_1 \geq m_0$ be such that $\underline{C}_{m_1}$ is a chain joining a to b. Choose $C_{m_1,j} \varepsilon \underline{C}_{m_1}$ such that $c \varepsilon C_{m_1,j}$. The diameter of $C_{m_1,j}$ is less than $1/2^{m_1}$ and hence is less than $\delta/2$. Thus $C_{m_1,j} \cap B = \emptyset$. This contradicts the connectedness of B. Hence $c \varepsilon B$. Therefore if B is a proper subcontinuum of K containing any two of a,b,c then B contains the third.

Now, let A be a subcontinuum of K containing a,b and c. We will show that $A = K$. Suppose $A \subsetneq K$; then let $x \varepsilon K - A$. Let $\delta_1 = d(A,x)$ and choose a positive integer m_3 such that $1/2^{m_3} < \delta_1/2$. Choose $C_{m_3,j} \varepsilon \underline{C}_{m_3}$ such that $x \varepsilon C_{m_3,j}$. $C_{m_3,j} \cap A = \emptyset$ since the diameter of $C_{m_3,j}$ is less than $\delta_1/2$. This contradicts $a,b,c \varepsilon A$. Hence $A = K$.

We now have that K is irreducible between any two points of the set {a,b,c}. Hence, by Theorem 4.13, K is an indecomposable metric continuum.

THEOREM 4.15. Let X be a metric continuum. Then X has only one, only three or uncountably many composants.

PROOF. Let X be an indecomposable metric continuum. Then, due to Remark 4.12, X has uncountably many composants.

Now let X be a decomposable metric continuum and choose proper subcontinua K_1 and K_2 such that $X = K_1 \cup K_2$. Let $z_1 \varepsilon K_1$ and let C_{z_1} be the composant of z_1. Suppose $C_{z_1} \subsetneq X$ and suppose there exists a $z_2 \varepsilon K_1$ such that $z_1 \neq z_2$ and the composant of z_2 C_{z_2} is a proper subset of X. Let $\omega \varepsilon C_{z_2}$ and let K_ω be a proper subcontinuum containing ω and z_2. Then $K_\omega \subseteq C_{z_2}$. $K_1 \subseteq C_{z_2}, z_1, z_2 \in K_1$ and $K_\omega \subseteq C_{z_2}$ imply that $K_1 \cup K_\omega$ is a proper subcontinuum of X. Hence $\omega \in C_{z_1}$ and therefore $C_{z_2} \subseteq C_{z_1}$. Similarly $C_{z_1} \subseteq C_{z_2}$. Hence X has less than or equal to three composants.

We are still assuming that X is decomposable. If for all $x \in X$ the composant of x C_x is all of X then X has exactly one composant. There exists a $p_0 \in X$, from Theorem 4.4, such that the composant of p_0 C_{p_0} is all of X. Recall that $X = K_1 \cup K_2$. Suppose there exists a $p_1 \in K_1$ such that the composant of p_1 C_{p_1} is not all of X. Then if $y \in K_1$ is such that the composant of y C_y is not all of x, from the above, $C_y = C_{p_1}$. Choose $p_2 \in X - C_{p_1}$; then $p_1 \notin C_{p_2}$ and therefore $C_{p_1} \neq C_{p_2}$. Hence if there is a p_1 such that $C_{p_1} \subsetneq X$, then X has exactly three composants. #

THEOREM 4.16. Let X be an indecomposable metric continuum and $A \subseteq X$ such that A is the union of countably many proper subcontinua of X. Then X - A is connected.

PROOF. Let $A = \bigcup_{n=1}^{\infty} K_n$ where for each n K_n is a proper subcontinuum. Since X is indecomposable $K_n^0 = \emptyset$ for all n. Hence $A \neq X$ and therefore $X - A \neq \emptyset$. For each n, n = 1,2,3,..., choose $k_n \in K_n$ and let C_{k_n} be the composant of k_n. Then fo each n, $K_n \subseteq C_{k_n}$ and therefore $A \subseteq \bigcup_{n=1}^{\infty} C_{k_n}$. Thus $X - \bigcup_{n=1}^{\infty} C_{k_n} \subseteq X - A$. Since X has uncountably many pairwise disjoint composants, $X - \bigcup_{n=1}^{\infty} C_{k_n} \neq \emptyset$. Let $\omega \in X - \bigcup_{n=1}^{\infty} C_{k_n}$ and let C_ω be the composant of ω. Then $C_\omega \subseteq X - A$. Suppose $X - A = R \cup S$ for some sets R and S. Then, without loss of generality, let $C_\omega \subseteq R$. Since every composant is dense in X we have that $\overline{C}_\omega = X$. This contradicts $S \neq \emptyset$. Hence $X - A$ is connected. #

THEOREM 4.17. Let X be an indecomposable continuum, $a \in X$ and $K = \{x \in X \mid X$ is irreducible from a to $x\}$. If $K \neq \emptyset$, then $\overline{K} = X$.

PROOF. Let $x \in K$, C_x be the composant of x and C_a be the composant of a. Then $C_x \cap C_a = \emptyset$ and therefore $C_x \subseteq K$. We now have that $\overline{K} = X$ since $C_x \subseteq K$ and $\overline{C}_x = X$. #

THEOREM 4.18. Let X be a metric continuum. Then X is decomposable if and only if some composant of X is open.

PROOF. First suppose X is a decomposable metric continuum. Then, from Theorem 4.6, every composant is open. Hence there exists an open composant.

Now we will show that if X is an indecomposable metric continuum then no composant of X is open. Let X be an indecomposable metric continuum. Suppose there exists a $p \in X$ such that the composant of p C_p is open. Theorem 4.9 and Theorem 4.16 imply that $X - C_p$ is connected. Hence $X - C_p$ is a proper subcontinuum. Let a,b and c be distinct points of X such that X is irreducible between any two. Since $X - C_p$ is a proper subcontinuum no two of a,b,c are in $X - C_p$. Hence, without loss of generality, let $a,b \in C_p$. Let C_a be the composant of a and C_b be the composant of b; then, from Theorem 4.11,

$C_a = C_b = C_p$. This contradicts X irreducible between a and b. Hence C_p is not open and therefore no composant of X is open. #

CHAPTER V

CONTINUOUS MAPPINGS

DEFINITION 5.1. Let (X,T) be a topological space and $A \subseteq X$. A is said to be perfect if and only if each point of A is a limit point of A.

DEFINITION 5.2. Let (X,T) be a topological space. X is said to be totally disconnected if and only if the components in X are single points in X.

To show that any two totally disconnected perfect compact metric spaces are homeomorphic we need the following two lemmas.

LEMMA 5.3. Let X be a perfect compact totally disconnected T_2 space and U an open subset of X. Let n be any positive integer; then $U = U_1 \cup U_2 \cup \ldots \cup U_n$ for some choice of nonempty disjoint open sets $U_1, U_2, \ldots, U_n$.

PROOF. Clearly the result holds when $n = 1$. It is sufficient to prove the result for $n = 2$ since for $n > 2$ simply reapply case $n = 2$ until the desired number of open nonempty disjoint sets is obtained. Let $p \in U$; then $U \neq \{p\}$ since X is perfect. Choose $q \in U$ such that $q \neq p$. Then, due to Theorem 2.20, the quasicomponent of p is simply $\{p\}$. Hence, there exist open disjoint sets R, S such that $q \in R$, $p \in S$ and $X = R \cup S$. Let $U_1 = R \cap U$ and $U_2 = S \cap U$. Then $U = U_1 \cup U_2$ and

U_1, U_2 are nonempty disjoint open sets. #

LEMMA 5.4. Let X be a totally disconnected compact metric space. Choose $x \in X$ and let U be an open set that contains x. Then there exists an open set V such that $x \in V \subseteq U$ and $b(V) = \emptyset$.

PROOF. The component of X that contains x is $\{x\}$ since X is totally disconnected. Theorem 2.17, X compact and Theorem 2.20 imply that $\{x\} = \bigcap\{V \mid V \text{ is open, } V \text{ is closed and } x \in V\}$. Lemma 2.3 implies that there exists a set V_0 such that V_0 is open, V_0 is closed and $x \in V_0 \subseteq U$. Hence V_0 is an open set containing x such that $b(V_0) = \emptyset$ and $V_0 \subseteq U$. #

THEOREM 5.5. Any two totally disconnected perfect compact metric spaces are homeomorphic.

PROOF. Let (X,T) and (Y,S) be totally disconnected perfect compact metric spaces. For each $x \in X$ let S_x be such that $x \in S_x$, diam $S_x < 1$ and such that S_x is both open and closed. We can do this because of Lemma 5.4. Then $\{S_x \mid x \in X\}$ is an open cover for X and therefore has a finite subcover. Let $S_{x_1}, S_{x_2}, \ldots, S_{x_{n_1}}$ be a finite subcover and then let $U_{1,1} = S_{x_1}$, $U_{1,2} = S_{x_2} - S_{x_1}, \ldots, U_{1,n_1} = S_{x_{n_1}} - \bigcup_{i=1}^{n_1-1} S_{x_i}$. We are assuming that $S_{x_i} \nsubseteq S_{x_j}$ for all $i \neq j$. Set $\underline{U}_1 = \{U_{1,1}, U_{1,2}, \ldots, U_{1,n_1}\}$. Thus $\underline{U}_1$ is an open-closed cover for X made up of nonempty pairwise disjoint sets of diameter less than one.

Construct an open-closed cover $\underline{V}_1$ for Y made up of non-empty pairwise disjoint sets of diameter less than one in a manner similar to the above construction. We can assume that $\|\underline{U}_1\| = \|\underline{V}_1\|$ because of Lemma 5.3. Let $\underline{V}_1 = \{V_{1,1}, V_{1,2}, \ldots, V_{1,n_1}\}$.

For each j, $j = 1,2,\ldots,n_1$, and for each $x \in U_{1,j}$ let W_x be an open-closed set containing x such that diam $W_x < 1/2$ and $W_x \subseteq U_{1,j}$. Then $\{W_x \mid x \in X\}$ is an open cover for X and hence has a finite subcover. Let $W_{x_1}, W_{x_2}, \ldots, W_{x_{n_2}}$ be a finite subcover and set $U_1 = W_{x_1}, U_2 = W_{x_2} - W_{x_1}, \ldots, U_{n_2} = W_{x_{n_2}} - \bigcup_{i=1}^{n_2-1} W_{x_i}$. Again, we are assuming $W_{x_i} \not\subseteq W_{x_j}$ for all $i \neq j$. Now, for each j, $j = 1,2,3,\ldots,n_1$, let $B_{1,j} = \{U_i \mid U_i \subseteq U_{1,j}, i = 1,2,3,\ldots,n_2\}$. Relabeling let $B_{1,1} = \{U_{2,1}, U_{2,2}, \ldots, U_{2,m_1}\}$, $B_{1,2} = \{U_{2,m_1+1}, \ldots, U_{2,m_2}\}, \ldots, B_{1,n_1} = \{U_{2,m_{n_1-1}+1}, \ldots, U_{2,m_{n_1}}\}$. Let $\underline{U}_2 = \{U_{2,1}, \ldots, U_{2,m_1}, U_{2,m_1+1}, \ldots, U_{2,m_2}, U_{2,m_2+1}, \ldots, U_{2,m_3}, \ldots, U_{2,m_{n_1-1}+1}, \ldots, U_{2,m_{n_1}}\}$. Then $\underline{U}_2$ is an open-closed cover for X made up of nonempty pairwise disjoint sets with diameter less than 1/2. Construct an open-closed cover $\underline{V}_2$ for Y made up of nonempty pairwise disjoint sets with diameter less than 1/2 in a manner similar to the construction of $\underline{U}_2$. We can assume that $\|\underline{V}_2\| = \|\underline{U}_2\|$ because of Lemma 5.3. Moreover, we can assume that for $j = 1,2,\ldots,n_1$ $\|B_{1,j}\| = \|D_{1,j}\|$ where $D_{1,j} = \{V_i \in \underline{V}_2 \mid V_i \subseteq V_{1,j}, i = 1,2,\ldots,n_2\}$. We will label $\underline{V}_2$ in a manner

similar to $\underline{U}_2$. Let $\underline{V}_2 = \{V_{2,1},\ldots,V_{2,m_1},V_{2,m_1+1},\ldots,V_{2,m_2},$
$V_{2,m_2+1},\ldots,V_{2,m_3},\ldots,V_{2,m_{n_1-1}+1},\ldots,V_{2,m_{n_1}}\}$.

Continuing in this manner we then have sequences $\{\underline{U}_k\}_{k=1}^{\infty}$ and $\{\underline{V}_k\}_{k=1}^{\infty}$ of open-closed covers for X and Y respectively. For each k we have that $||\underline{U}_k|| = ||\underline{V}_k||$. Each $\underline{U}_k$ and $\underline{V}_k$ is made up of nonempty pairwise disjoint sets of diameter less than $1/2^k$. Also, for each k if $U_{k,j} \subseteq U_{k-1,i}$ for some i,j then $V_{k,j} \subseteq V_{k-1,j}$.

Let $x \in X$; then for each $k = 1,2,3,\ldots,$ there is a unique U_{k,j_k} such that $x \in U_{k,j_k}$ with $U_{k,j_k} \in \underline{U}_k$. Moreover, $U_{1,j_1} \supseteq U_{2,j_2} \supseteq U_{3,j_3} \supseteq \ldots,$ $\bigcap_{k=1}^{\infty} U_{k,j_k} = x$ and $V_{1,j_1} \supseteq V_{2,j_2}, \supseteq V_{3,j_3} \supseteq \ldots$. Define $g(x)$ $\bigcap_{k=1}^{\infty} V_{k,j_k}$. Do the above for each $x \in X$; then g is a function from X into Y. Clearly g is onto and one to one.

Choose $x \in X$ and suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence in X such that $a_n \to x$. For each k let U_{k,j_k} contain x; then $x = \bigcap_{k=1}^{\infty} U_{k,j_k}$ and $g(x)$ $\bigcap_{k=1}^{\infty} V_{k,j_k}$. Let $\varepsilon > 0$ and choose a positive integer k_0 such that $U_{k_0,j_{k_0}} \subseteq S(x,\varepsilon)$ and $V_{k_0,j_{k_0}} \subseteq S(g(x),\varepsilon)$. The fact $a_n \to x$ implies that there exists a positive integer N such that if $n \geq N$ then $a_n \in U_{k_0,j_{k_0}}$. Thus for $n \geq N, g(a_n) \in V_{k_0,j_{k_0}} \subseteq S(g(x),\varepsilon)$. Hence $g(a_n) \to g(x)$ and therefore g is continuous. Since X is compact and Y is Hausdorff g is a homeomorphism. #

THEOREM 5.6. Every compact metric space is a continuous image of the Cantor set.

PROOF. Let X be a compact metric space and let C be the Cantor set. Construct a finite open-closed cover $\underline{U}_1$ for C made up of nonempty pairwise disjoint sets of diameter less than one. See the proof of Theorem 5.5 for the construction of $\underline{U}_1$. Let $\underline{V}_1$ be a finite open cover of X, made up of nonempty sets with diameter less than one. By repeating elements in $\underline{V}_1$ and using Lemma 5.3 we can assume that $\|\underline{U}_1\| = \|\underline{V}_1\|$. Let $\underline{U}_1 = \{U_{1,1}, \ldots, U_{1,n_1}\}$ and $\underline{V}_1 = \{V_{1,1}, \ldots, V_{1,n_1}\}$.

For each j, $j = 1,2,3,\ldots,n_1$, let $\underline{B}_{1,j}$ be a finite open-closed cover of $U_{1,j}$ such that if $B \in \underline{B}_{1,j}$ then $B \neq \emptyset$, $B \subseteq U_{1,j}$ and diam $B < 1/2$. Also, for each j, $j = 1,2,3,\ldots,n_1$, let $\underline{D}_{1,j}$ be a finite open cover of $V_{1,j}$ such that if $D \in \underline{D}_{1,j}$ then $D \neq \emptyset$, $D \subseteq V_{1,j}$ and diam $D < 1/2$. We can do this since $\overline{V}_{1,j}$ is compact for all j. By repeating elements in $\underline{D}_{1,j}$ and using Lemma 5.3 we can assume that $\|\underline{B}_{1,j}\| = \|\underline{D}_{1,j}\|$ for each j. Let $\underline{U}_2 = \{B \subseteq C \mid B \in \underline{B}_{1,j}$ for some j$\}$ and $\underline{V}_2 = \{D \subseteq X \mid D \in \underline{D}_{1,j}$ for some j$\}$. Label the elements of $\underline{U}_2$ and $\underline{V}_2$ such that $\underline{U}_2 = \{U_{2,1}, U_{2,2}, \ldots, U_{2,n_2}\}$ $\underline{V}_2 = \{V_{2,1}, V_{2,2}, \ldots, V_{2,n_2}\}$ and such that if $U_{2,i} \subseteq U_{1,j}$ then $V_{2,i} \subseteq V_{1,j}$.

Continue in this manner to construct finite covers for X and C. We have sequences $\{\underline{U}_k\}_{k=1}^{\infty}$ and $\{\underline{V}_k\}_{k=1}^{\infty}$ with the following properties:

i) $\|\underline{U}_k\| = \|\underline{V}_k\|$ for all k.

ii) For each k, $\underline{U}_k$ is an open-closed cover of C made up of nonempty pairwise disjoint sets of diameter less than $1/2^k$.

iii) For each k, $\underline{V}_k$ is an open cover of X made up of nonempty sets of diameter less than $1/2^k$.

iv) For each k let $\underline{U}_k = \{U_{k,1}, U_{k,2}, \ldots, U_{k,n_k}\}$ and $\underline{V}_k = \{V_{k,1}, \ldots, V_{k,n_k}\}$. If $U_{k,i} \subseteq U_{k-1,j}$ for some k,i and j then $V_{k,i} \subseteq V_{k-1,j}$.

Let $c \in C$; then for each k, $k = 1,2,3,\ldots$, there is a unique $U_{k,j_k} \in \underline{U}_k$ containing c. Furthermore, $U_{1,j_1} \supseteq U_{2,j_2} \supseteq U_{3,j_3} \supseteq \ldots$, $c = \bigcap_{k=1}^{\infty} U_{k,j_k}$ and $\overline{V}_{1,j} \supseteq \overline{V}_{2,j} \supseteq \overline{V}_{3,j} \supseteq \ldots$. let $g(c) = \bigcap_{k=1}^{\infty} \overline{V}_{k,j_k}$. Do the above for each $c \in C$; then g is a function from C into X. Clearly g is onto. However, g may not be one to one.

Let $\{c_n\}_{n=1}^{\infty}$ be a sequence in C and $c \in C$ such that $c_n \to c$. Choose $\varepsilon > 0$ and suppose $c \in \bigcap_{k=1}^{\infty} U_{k,j_k}$ where $U_{k,j_k} \in \underline{U}_k$ for all k. Then, by Lemma 2.3, there exists a positive integer k_0 such that $c \in U_{k_0,j_{k_0}} \subseteq S(c,\varepsilon)$ and $g(c) \in V_{k_0,j_{k_0}} \subseteq S(g(c),\varepsilon)$. The fact $c_n \to c$ implies that there exists a positive integer m_2 such that if $n \geq m_2$ then $c_n \in U_{k_0,j_{k_0}}$. Thus, for $n \geq m_2$ $g(c_n) \in V_{k_0,j_{k_0}}$ and therefore $g(c_n) \to g(c)$. Hence g is a continuous map of C onto X.#

EXAMPLE 5.7. We will construct a continuous function from the Cantor set onto I where $I = [0,1] \subseteq R^1$.

Let $C_0 = [0,1]$ and construct C_1 by removing the interval $(1/3,2/3)$ from C_0. Hence, $C_1 = [0,1/3] \cup [2/3,1]$. For each positive integer n, $n \geq 2$, construct C_n from C_{n-1} by removing the open middle third of each of the intervals in C_{n-1} from C_{n-1}. Then $C = \bigcap_{n=0}^{\infty} C_n$ is the Cantor set.

For each n, $n = 0,1,2,\ldots$, let $D_n = \{m/2^n \mid 0 \leq m \leq 2^n\}$. Let $B_0 = \{0,1\}$ and define a function g_0 from B_0 onto D_0 as follows:

$$g_0(x) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x = 1. \end{cases}$$

Recall that $C_1 = [0,1/3] \cup [2/3,1]$ and then let $B_1 = \{0,1/3,2/3,1\}$. Define a function g_1 from B_1 onto D_1 as follows:

$$g_1(x) = \begin{cases} 0, & \text{if } x = 0 \\ 1/2, & \text{if } x = 1/3,\ 2/3 \\ 1, & \text{if } x = 1 \end{cases}.$$

Thus, $g_1|_{B_0} = g_0$ and if $x,y \in B_1$ are such that $|x-y| \leq 1/3$ then $|g_1(x) - g_1(y)| \leq 1/2$. Recall that $C_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1]$ and then let $B_2 = \{0,1/9,2/9,1/3,2/3,7/9,8/9,1\}$. Define a function g_2 from B_2 onto D_2 as follows:

$$g_2(x) = \begin{cases} 0, & \text{if } x = 0 \\ 1/4, & \text{if } x = 1/9, 2/9 \\ 1/2, & \text{if } x = 1/3, 2/3 \\ 3/4, & \text{if } x = 7/9, 8/9 \\ 1 & \text{if } x = 1 \end{cases}.$$

Then $g_2|_{B_1} = g_1$ and if $x, y \in B_2$ are such that $|x-y| \leq 1/3^2$ then $|g_2(x) - g_2(y)| \leq 1/2^2$. For each $n \geq 0$, C_n is the union of 2^n disjoint closed intervals. For $n \geq 3$ let $B_n = \{x \in C |\ x$ is an endpoint of one of the closed intervals contained in $C_n\}$; then $||B_n|| = 2^{n+1}$ for all n. Assume that for each n, $3 \leq n \leq k$, a function g_n from B_n onto D_n has been constructed such that $g_n|_{B_{n-1}} = g_{n-1}$. Also, let $C_{n-1} = [a_1, b_1] \cup [a_2, b_2] \cup \ldots \cup [a_{2^{n-1}}, b_{2^{n-1}}]$ with $a_i < b_i$ for all i and $b_i < a_{i+1}$ for all $i \leq 2^{n-1}-1$ and $D_{n-1} = \{d_1 = 0/2^{n-1}, d_2 = 1/2^{n-1}, \ldots, d_{2^{n-1}+1} = 2^{n-1}/2^{n-1}\}$ then $g_n(a_i + 1/3(b_i - a_i)) = g_n(a_i + 2/3(b_i - a_i)) = (d_i + d_{i+1})/2$ for $i = 1, 2, 3, \ldots, 2^{n-1}$.

Let $C_k = [a_1, b_1] \cup [a_2, b_2] \cup \ldots \cup [a_{2^k}, b_{2^k}]$ with $a_i < b_i$ for all i and $b_i < a_{i+1}$ for all $i \leq 2^k - 1$ and $D_k = \{d_1 = 0/2^k, d_2 = 1/2^k, \ldots, d_{2^k+1} = 2^k/2^k\}$. Define a function g_{k+1} from B_{k+1} onto D_{k+1} such that $g_{k+1}|_{B_k} = g_k$ and $g_{k+1}(a_i + 1/3(b_i - a_i)) = g_{k+1}(a_i + 2/3(b_i - a_i)) = (d_i + d_{i+1})/2$ for $i = 1, 2, 3, \ldots, 2^k$. Thus if $x, y \in B_{k+1}$ and $|x-y| \leq 1/3^{k+1}$ then $|g_{k+1}(x) - g_{k+1}(y)| \leq 1/2^{k+1}$.

For each $x \in \bigcup_{n=0}^{\infty} B_n$ there is an integer n_x such that $x \in B_{n_x}$ and $x \notin \bigcup_{i<n_x} B_i$. Let $g(x) = g_{n_x}(x)$ for each $x \in \bigcup_{n=0}^{\infty} B_n$. Then g is a function from $\bigcup_{n=0}^{\infty} B_n$ onto $\bigcup_{n=0}^{\infty} D_n$ such that if $x,y \in \bigcup_{n=0}^{\infty} B_n$ and $|x-y| \leq 1/3^k$, for some k, then $|g(x)-g(y)| \leq 1/2^k$.

Let $B = \bigcup_{n=0}^{\infty} B_n$; then B is dense in C. Choose $x \in C - B$ and let $\{b_n\}_{n=1}^{\infty}$ be a sequence in B such that $b_n \to x$ and $b_n \leq b_{n+1}$ for all n. Let $f(x) = \lim_{n\to\infty} g(b_n)$. Do the above for each $x \in C-B$ and if $x \in B$ let $f(x) = g(x)$. Thus, f is a function from C into I.

We need to show that f is well defined on C-B. Let $x \in C-B$ and let $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ be sequences in B such that $c_n \leq x$ for all $x, d_n \leq x$ for all x, $c_n \to x$, $d_n \to x$, $c_n \leq c_{n+1}$ for all n and $d_n \leq d_{n+1}$ for all n. Suppose $c = \lim_{n\to\infty} g(c_n) \neq \lim_{n\to\infty} g(d_n) = d$. Without loss of generality let $c < d$. Set $\varepsilon = d-c$ and choose a positive integer N such that $1/2^N < \varepsilon/2$. Let $[0,1] = [a_1=0, b_1 = 1/2^N] \cup [a_2 = 1/2^N, b_2 = 2/2^N] \cup \ldots \cup [a_{2^N+1} = 1-1/2^N, a_{2^N+1} = 1]$; then $|b_j - a_j| = 1/2^N$ for all j. Say $c \in [a_j, b_j]$ and $d \in [a_k, b_k]$; then $j \neq k$ and $|j-k| > 1$. Let ε_1 and ε_2 be as follows:

$$\varepsilon_1 = \begin{cases} b_j - c & \text{if } c \neq b_j \\ 1/2^{N+1}, & \text{if } c = b_j \end{cases} \quad \text{and} \quad \varepsilon_2 = \begin{cases} d - a_k, & \text{if } d \neq a_k \\ 1/2^{N+1}, & \text{if } d = a_k. \end{cases}$$

Choose a positive integer N_1 such that for $n \geq N_1$: $|x - c_n| < 1/(2)(3^N)$, $|x - d_n| < 1/(2)(3^N)$, $|f(c_n) - c| < \varepsilon_1$ and $|f(d_n) - d| < \varepsilon_2$. Recall that C_N is the union of 2^N disjoint closed subintervals of $[0,1]$. Let $C_N = [a_1, b_1] \cup [a_2, b_2] \cup \ldots \cup [a_{2^N}, b_{2^N}]$ and choose $n \geq N_1$. Without loss of generality suppose $x \in [a_2, b_2]$; then $c_n, d_n \in [a_2, b_2]$ since $|x - c_n| < 1/2(3^N)$ and $|x - d_n| < 1/(2)(3^N)$. Hence $|f(c_n) - f(d_n)| \leq 1/2^N$. However, $|f(c_n) - c| < \varepsilon_1$ and $|f(d_n) - d| < \varepsilon_2$ imply that $|f(c_n) - f(d_n)| > 1/2^N$. This is a contradiction. Hence $c = d$ and therefore f is well defined on $C - B$.

Let $x \in C$ and $\varepsilon > 0$. Choose a positive integer N such that $1/2^N < \varepsilon$ and $\delta > 0$ such that $\delta < 1/3^N$. Let $y \in C$ be such that $|x-y| < \delta$; then $|f(x) - f(y)| \leq 1/2^N < \varepsilon$. Hence f is continuous.

The function f takes B onto $\bigcup_{n=0}^{\infty} D_n$ since g takes B onto $\bigcup_{n=0}^{\infty} D_n$. Let $x \in [0,1] - \bigcup_{n=0}^{\infty} D_n$. Choose a sequence $\{d_n\}_{n=1}^{\infty}$ in $\bigcup_{n=0}^{\infty} D_n$ such that $d_n \leq x$ for all n and $d_n \to x$. For each n let $c_n \in B$ be such that $f(c_n) = d_n$; then $\{c_n\}_{n=1}^{\infty}$ is a bounded sequence. Hence there exists a $c \in C$ and a subsequence $\{c_{n_k}\}_{k=1}^{\infty}$ of $\{c_n\}_{n=1}^{\infty}$ such that $c_{n_k} \to c$. We now have that $f(c_{n_k}) \to f(c)$ since f is continuous. Hence $f(c) = x$. Thus f is a continuous map from C onto I.

EXAMPLE 5.8. In example 5.7 we mapped C onto I continuously. Now let's map I onto $I \times X$ continuously. Let I denote $[0,1]$ in R^1. We will construct a continuous map h from I onto $I \times I$. In the construction of h we will use Theorem 5.6.

Let g be a continuous map from the Cantor set onto $I \times I$. Theorem 5.6 tells us that such a g exists. Let C be the Cantor set. The function g is uniformly continuous since C is compact. Recall the construction of C. Let $C_0 = [0,1]$, $C_1 = [0,1/3] \cup [2/3,1]$ and $C_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3, 7/9] \cup [8/9,1]$. In general, for each positive integer n construct C_n from C_{n-1} by removing the open middle third of each of the intervals in C_{n-1} from C_{n-1}. Then $C = \bigcap_{n=0}^{\infty} C_n$. Let $R_1 = [1/3,2/3]$, $R_2 = [1/9,2/9]$, $R_3 = [7/9,8/9]$, $R_4 = [1/27,2/27]$, $R_5 = [7/27,8/27]$, $R_6 = [19/27,20/27]$, $R_7 = [25/27,26/27], \ldots$. Define $h(x) = g(x)$ for all $x \in C$. For each positive integer n say $[a_n,b_n] = R_n$. So, for example, $a_1 = 1/3, b_1 = 2/3$, $a_2 = 1/9, b_2 = 2/9$, $a_3 = 7/9$, $b_3 = 8/9$, and $a_4 = 1/27$. Let $x \in I - C$. Then $x \in R_{n_0} = [a_{n_0},b_{n_0}]$ for exactly one positive integer n_0. If $g(a_{n_0}) = g(b_{n_0})$ then let $h(x) = g(a_{n_0})$. If $g(a_{n_0}) \neq g(b_{n_0})$ let h(x) be as follows:

$$h(x) = \left[\frac{g(a_{n_0}) - g(b_{n_0})}{a_{n_0} - b_{n_0}}\right] x + g(b_{n_0}) - b_{n_0}\left[\frac{g(a_{n_0}) - g(b_{n_0})}{a_{n_0} - b_{n_0}}\right] .$$

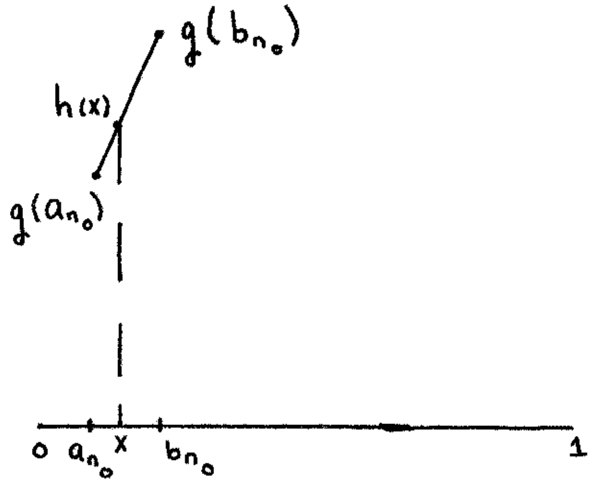

Figure 13 Clarification of h(x)

Do the above for each $x \varepsilon I - C$. Then h is a function from I onto $I \times I$ such that $h|_C = g$. The function h is onto $I \times I$ since g is onto $I \times I$.

We now must show that h is continuous. Let $x \varepsilon [0,1]$ and $\varepsilon > 0$. If $x \varepsilon R_n^0$ for some n then clearly h is continuous at x. Hence, suppose $x \varepsilon C$. Since g is uniformly continuous we can choose $\delta > 0$ such that if $c_1, c_2 \varepsilon C$ and $|c_1 - c_2| < \delta$ then $|g(c_1)-g(c_2)| < \varepsilon/2$. There exist only finitely many n such that diam $R_n \geq \delta$. We need to consider two cases. First suppose that for all n diam $R_n < \delta$. Let $y \varepsilon I$ be such that $|x-y|<\delta$. If $y \varepsilon C$ then $|h(x)-h(y)| = |g(x)-g(y)| < \varepsilon/2$. as desired. If $y \varepsilon I - C$ then $y \varepsilon [a_{k_0}, b_{k_0}] = R_{k_0}$ for exactly one k_0. The diameter of R_{k_0} less than δ implies that $|g(a_{k_0})-g(b_{k_0})| < \varepsilon/2$. Without loss of generality suppose $x < y$. Then

$|h(x)-h(y)| \leq |h(x)-h(a_{k_0})|+|h(a_{k_0})-h(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.
Hence h is continuous at x if diam $R_n < \delta$ for all n.

Now, suppose diam $R_n \geq \delta$ for $n = 1,2,3,\ldots,k$. $R_F = \{R_n | R_n \cap (x, x+\delta) \neq \emptyset$ and diam $R_n \geq \delta\}$ and $R_p = \{R_n | R_n \cap (x-\delta, x) \neq \emptyset$ and diam $R_n \geq \delta\}$. Set $a = \min\{a_n | R_n \in R_F\}$ if $R_F \neq \emptyset$ and $b = \max\{b_n | R_n \in R_p\}$ if $R_p \neq \emptyset$. Define α as follows:

$$\alpha = \begin{cases} \min\{a-x,\ x-b\}, & \text{if } R_F \neq \emptyset \text{ and } R_p \neq 0 \\ \min\{\delta, x-b\}, & \text{if } R_F = \emptyset \text{ and } R_p \neq \emptyset \\ \min\{\delta, a-x\}, & \text{if } R_p = \emptyset \text{ and } R_F \neq \emptyset \\ \delta, & \text{if } R_p = \emptyset \text{ and } R_F = \emptyset \end{cases}$$

Notice that $\alpha \leq \delta$. Recall that $x \in C$. Let $y \in I$ be such that $|x-y| < \alpha$. If $y \in C$ then $|h(x)-h(y)| = |g(x)-g(y)| < \varepsilon/2$ as desired. If $y \notin C$ then $y \in R_{k_1} = [a_{k_1}, b_{k_1}]$ for exactly one k_1. We have that $[a_{k_1}, b_{k_1}] \cap (x-\delta, x+\delta) \neq \emptyset$ since $|x-y| < \alpha \leq \delta$. The facts $[a_{k_1}, b_{k_1}] \cap (x-\delta, x+\delta) \neq \emptyset$ and $|x-y| < \alpha$ imply that diam $R_{k_1} < \delta$. Without loss of generality suppose $x < y$. Then $|h(x)-h(y)| \leq |h(x)-h(a_{k_1})| + |h(a_{k_1})-h(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.
Hence h is a continuous map from I onto $I \times I$.

Let X be a metric space. Choose two distinct points in X and let K be an arc joining a to b. We will define a linear order on K and define a function from K onto [0,1] that is one to one, onto, continuous and order preserving.

LEMMA 5.9. Let X be a metric space and a,b be two distinct points in X. Let K be an arc joining a to b in X. Define $<$ on K as follows:

i) Let $x,y \in K$. Then $x < y$ if and only if $K - \{x\} = U_a \uplus V_b$ with $a \in U_a$ and both y and b in V_b.

ii) $a < x$ for all $x \neq a$.

iii) $x < b$ for all $x \neq b$.

iv) $x = y$ if x and y are the same elements.

Then $<$ is a linear order on K.

PROOF. First we need to show that $<$ is well defined. We must show that if $x \in K$ and $K - \{x\} = U_{1,a} \uplus V_{1,b} = U_{2,a} \uplus V_{2,b}$ with $a \in U_{1,a} \cap U_{2,a}$ and $b \in V_{1,b} \cap V_{2,b}$ then $U_{1,a} = U_{2,a}$ and $V_{1,b} = V_{2,b}$. Both $U_{1,a} \dot{\cup} \{x\}$ and $V_{2,b} \dot{\cup} \{x\}$ are connected sets since K is connected. Then $K = U_{1,a} \cup V_{2,b} \dot{\cup} \{x\}$, since K is irreducible between a and b. Similarly, $K = U_{2,a} \cup V_{1,b} \dot{\cup} \{x\}$. Hence $U_{2,a} \cup V_{1,b} \dot{\cup} \{x\} = U_{1,a} \cup V_{2,b} \dot{\cup} \{x\}$. Then $U_{1,a} = U_{2,a}$ and $V_{1,b} = V_{2,b}$ since $U_{1,a} \cap V_{1,b} = \emptyset$ and $U_{2,a} \cap V_{2,b} = \emptyset$. Thus $<$ is well defined.

Clearly $x \leq x$ for all $x \in K$. Suppose $x,y \in K$ with $x < y$. Then y is not less than x. We need only consider x,y different than a,b. Suppose $y < x$; then $K - \{x\} = U_{a_x} \uplus V_{b_x}$ and $K - \{y\} = U_{a_y} \uplus V_{b_y}$ with $y \in V_{b_x}$ and $x \in V_{b_y}$. The set $U_{a_x} \dot{\cup} \{x\}$ is connected and does not contain y; hence $U_{a_x} \cup \{x\} \subseteq V_{b_y}$. The fact $a \in U_{a_x} \cap U_{a_y}$ contradicts $U_{a_y} \cap V_{b_y} = \emptyset$. Thus $y \nless x$. Hence if $x \leq y$ and $y \leq x$ then $x = y$.

Let $x,y,z \in K$ and suppose $x \leq y$ and $y \leq z$. We need to show that $x \leq z$. We need only consider x,y,z different than a,b with $x < y$ and $y < z$. The other cases are clear. The fact $x < y$ implies that $K \sim \{x\} = U_{a_x} \uplus V_{b_x}$ with $y \in V_{b_x}$. $K \sim \{y\} = U_{a_y} \uplus V_{b_y}$ with $z \in V_{b_y}$ since $y < z$. Suppose $z \in U_{a_x}$; then $a \in U_{a_y} \cap U_{a_x}$ and $U_{a_x} \cap V_{b_y} \neq \emptyset$. This contradicts the connectedness of $U_{a_x} \cup \{x\}$. Hence $z \in V_{b_x}$ and therefore $x < z$.

Let $x,y \in K$. We need to show that $x \leq y$ or $y \leq x$. We need only consider x,y different than a,b and $x \neq y$. The other cases are clear. $K \sim \{x\} = U_{a_x} \uplus V_{b_x}$ and $K \sim \{y\} = U_{a_y} \uplus V_{b_y}$. If $y \in V_{b_x}$ then $x < y$ and we are done. So, suppose $y \in U_{a_x}$. Then $V_{b_x} \cup \{x\} \subsetneq V_{b_y}$. Thus $x \in V_{b_y}$ and therefore $y < x$. Now we have that $<$ is a linear order on K. #

LEMMA 5.10. Let K be a metric arc joining a to b and T be the metric topology on K. Let $T_{\leqslant}$ be the order topology on K. A base for $T_{<}$ is $\underline{B} = \{[a,x) \mid x \in K \text{ and } x \neq b\} \cup \{(y,b] \mid y \in K \text{ and } y \neq a\} \cup \{(x,y) \mid x,y \in K\}$. Then $T = T_{<}$.

PROOF. First we will show that $T_{\leqslant} \subseteq T$. Choose $x \in K$ different from a or b. Then $K \sim \{x\} = U_{a_x} \uplus V_{b_x}$ with $a \in U_{a_x}$ and $b \in V_{b_x}$. Let $z \in [a,x)$; then $z \lessdot x$. $K \sim \{z\} = U_{a_z} \uplus V_{b_z}$ with $a \in U_{a_z}$ and $b \in V_{b_z}$. The fact $z \lessdot x$ implies that $x \in V_{b_z}$.

Thus $U_{a_z} \cup \{z\} \subseteq U_{a_x}$ and therefore $z \in U_{a_x}$. Hence $[a,x) \subseteq U_{a_x}$. Choose $\omega \in U_{a_x}$ different than a. $K - \{\omega\} = U_{a_\omega} \uplus V_{b_\omega}$ with $a \in U_{a_\omega}$ and $b \in V_{b_\omega}$. Suppose $x \in U_{a_\omega}$. Then $V_{b_\omega} \cap V_{b_x} \neq \emptyset$ and $V_{b_\omega} \cap U_{a_x} \neq \emptyset$. This contradicts $V_{b_\omega} \cup \{\omega\}$ being connected. Hence $x \in V_{b_\omega}$ and therefore $\omega < x$. Thus $U_{a_x} \subseteq [a,x)$. Now, $U_{a_x} = [a,x)$ for all $x \neq a,b$. Similarly $(y,b] = V_{b_y}$ for all $y \neq, a,b$. If $x \neq y$ and x,y are different than a,b then $(x,y) = U_{a_x} \cap V_{b_y}$. The sets U_{a_x} and V_{b_y} are open for all x,y in K. Hence $T_< \subseteq T$.

Let $U \in T$. Suppose $a \in U$. If there exists a $z \in K$ such that $[a,z) \subseteq U$ then $U \in T_<$. Suppose for all $z \in K$ we have that $[a,z) \not\subseteq U$. Then $[a,z) \cap (K - U) \neq \emptyset$ for all $z \in K$. Let $H = \bigcap_{\substack{z \in K \\ z \neq a}} [[a,z] \cap (K - U)]$; then $H \neq \emptyset$ since K is compact. However, $H \subseteq \bigcap_{\substack{z \in K \\ z \neq a}} [a,z] = a$ and $H \subseteq K - U$. This contradicts $a \in U$. Hence there exists a $z \in K$ such that $[a,z) \subseteq U$. Thus $U \in T_<$. Similarly if $b \in U$ then there exists a $z \in K$ such that $(z,b] \subseteq U$ and if $a,b \notin U$ then there exist $x,y \in K$ such that $(x,y) \subseteq U$. Thus $T \subseteq T_<$. Now, we have that $T = T_{<_1}$. #

LEMMA 5.11. Let K be a metric arc joining a to b. Choose two distinct points x,y in K. Without loss of generality suppose $x < y$; then there exists a $z \in K$ such that $x < z < y$.

PROOF. First suppose $x \neq a,b$ and $y \neq a,b$. Suppose there does not exist a $z \in K$ such that $x < z < y$. $K - \{x\} = U_{a_x} \psi V_{b_x}$ and $K - \{y\} = U_{a_y} \psi V_{b_y}$ with $a \in U_{a_x} \cap U_{a_y}$, $b \in V_{b_x} \cap V_{b_y}$ and $y \in V_{b_x}$. Then $U_{a_x} \cup \{x\} \subsetneq U_{a_y}$. Thus $x \in U_{a_y}$ and so $V_{b_y} \cup \{y\} \subseteq V_{b_x}$. Let $H = U_{a_x} \cup \{x\}$ and $M = V_{b_y} \cup \{y\}$; then $H = [a,x]$, $M = [y,b]$ and $H \cap M = \phi$. $K = H \psi M$ since there does not exist a $z \in K$ such that $x < z < y$. This contradicts K being connected. Hence there is a $z \in K$ such that $x < z < y$.

Now suppose $x = a$ and $y \neq a,b$. Suppose there does not exist a $z \in K$ such that $a < z < y$. Then $K - \{y\} = U_{a_y} \psi V_{b_y} = \{a\} \psi V_{b_y}$. Thus $\{a\}$ is both open and closed. This is a contradiction. Hence there exists a $z \in K$ such that $a < z < y$. Similarly if $y = b$ and $x \neq a,b$ then there exists a $z \in K$ such that $x < z < b$. If $x = a$ and $y = b$ then there exists a $z \in K$ such that $x < z < y$ since $\{a,b\} \subsetneq K$. #

THEOREM 5.12. If K is a metric arc joining a to b then there is an order preserving homeomorphism from K onto I.

PROOF. Let K be a metric arc joining a to b and let $D = \{x_1, x_2, x_3, \dots\}$ be a countable dense subset of K such that $x_i \neq x_j$ for $i \neq j$ and $x_i \neq a,b$ for all i. Set $x_0 = a$ and $x_\infty = b$. Let $g(a) = 0$, $g(b) = 1$ and $g(x_1) = 1/2$. Let $g(x_2)$ be as follows:

$$g(x_2) = \begin{cases} 3/4, & \text{if } x_2 > x_1 \\ 1/4, & \text{if } x_2 < x_1 \end{cases}$$

Assume that $g(x_k)$ has been defined for all $k \leq n-1$. Let $x_{p_n} = \max \{x_j \mid x_j < x_n,\ j = 0,1,2,\ldots,n-1\}$ and $x_{f_n} = \min \{x_j \mid x_j > x_n,\ j = 0,1,2,\ldots,n-1,\infty\}$. Then let $g(x_n) = (g(x_{p_n}) + g(x_{f_n}))/2$. Let $g(y) = \text{glb}\{g(x_n) \mid x_n > y\}$ for each $y \in K - D$. We will show that g is a one to one, onto, continuous and order preserving map from K into $I = [0,1]$.

Now, let's show that g preserves order. We will first use induction on the positive integers to show that g preserves order on D. We will show the following by induction:

Let n be a positive integer. Then

(i) If j is such that $1 \leq j \leq n-1$ and $x_j < x_n$ then $g(x_j) < g(x_n)$.

(ii) If j is such that $1 \leq j \leq n-1$ and $x_j > x_n$ then $g(x_j) > g(x_n)$.

Clearly the result holds for $n = 1$ and $n = 2$. Assume the result holds for all $n \leq k$. By definition, $g(x_{k+1}) = (g(x_{p_{k+1}}) + g(x_{f_{k+1}}))/2$ with $a \leq x_{p_{k+1}} < x_{k+1} < x_{f_{k+1}} \leq b$. Using the induction hypothesis we have that $g(x_{p_{k+1}}) < g(x_{f_{k+1}})$. Hence $g(x_{p_{k+1}}) < g(x_{k+1}) < g(x_{f_{k+1}})$. Applying the induction hypothesis and transitivity to $x_{p_{k+1}}$ and $x_{f_{k+1}}$ we have the desired result.

Hence g preserves order on D.

Let $x,y \in K$ be such that $x,y \notin D$. Without loss of generality suppose $x < y$. Recall that $g(x) = \text{glb}' \{g(x_n) \mid x_n > x\}$ and $g(y) = \text{glb}' \{g(x_n) \mid x_n > y\}$. There exists an $x_{n_0} \in D$ such that $x < x_{n_0} < y$ since D is dense in K. Thus $g(x) < g(x_{n_0})$. Suppose $g(y) < g(x_{n_0})$; then there exists a $x_{n_1} \in D$ such that $y < x_{n_1}$ and $g(x_{n_1}) < g(x_{n_0})$. However, $g(x_{n_1}) < g(x_{n_0})$ implies that $x_{n_1} < x_{n_0}$. Thus $x_{n_1} < x_{n_0} < y < x_{n_1}$. This is a contradiction. Hence $g(x) < g(x_{n_0}) \leq g(y)$ and therefore $g(x) < g(y)$. Thus, g preserves order on K.

Let $B_0 = \{0,1\}$ and for each positive integer let $B_n = \{0/2^n, 1/2^n, 2/2^n, \ldots, 2^n/2^n\}$. Set $B = \bigcup_{n=0}^{\infty} B_n$. We will show that g takes D onto B. Clearly, for each $b \in B_0 \cup B_1$ there exists a $d_b \in D$ such that $g(d_b) = b$. Consider $B_2 = \{0, 1/4, 1/2, 3/4, 1\}$. Let $H_2 = \{x_n \in D \mid x_n > x_1\}$ and let k_1 be the smallest positive integer such that $x_{k_1} \in H_2$. Then $x_1 = x_{p_{k_1}}$ and $b = x_{f_{k_1}}$. Thus $g(x_{k_1}) = 3/4$. Let $R_2 = \{x_n \in D \mid x_n < x_1\}$. Let k_2 be the smallest positive integer such that $x_{k_2} \in R_2$. Then $x_1 = x_{f_{k_2}}$ and $0 = x_{p_{k_2}}$. Thus, $g(x_{k_2}) = 1/4$. Hence if $b \in B_0 \cup B_1 \cup B_2$ then there exists a $d_b \in D$ such that $g(d_b) = b$. Assume that if $b \in \bigcup_{i=1}^{n} B_i$ then there exists a $d_b \in D$ such that $g(d_b) = b$. Consider B_{n+1}. $B_n = \{0/2^n, 1/2^n, 3/2^n, \ldots, 2^n-1/2^n, 2^n/2^n\}$ and $B_{n+1} = \{0/2^{n+1}, 1/2^{n+1}, 2/2^{n+1}, \ldots, 2^{n+1}-1/2^{n+1}, 2^{n+1}/2^{n+1}\}$.

We have that $\{0/2^{n+1}, 2/2^{n+1}, 4/2^{n+1}, \ldots, 2^{n+1}/2^{n+1}\} = B_n$; hence if $b \varepsilon \{0/2^{n+1}, 2/2^{n+1}, \ldots, 2^{n+1}/2^{n+1}\}$ then there is a $d_b \varepsilon D$ such that $g(d_b) = b$. Let k_3 be such that $g(x_{k_3}) = 2/2^{n+1}$ and let $R_3 = \{x_n \mid x_n < x_{k_3}, n > k_3\}$. Clearly $k_3 \neq 0$ and $R_3 \neq \emptyset$. Let k_4 be the smallest positive integer such that $x_{k_4} \varepsilon R_3$. Then $x_{k_3} = x_{f_{k_4}}$ and $x_{p_{k_4}} = a$. Thus $g(x_{k_4}) = 1/2^{n+1}$. Using a similar argument one can show that there is a $d \varepsilon D$ such that $g(d) = 2^{n+1} - 1/2^{n+1}$. Consider $4/2^{n+1}$. We have that $g(0) = a$ and $g(x_{k_3}) = 2/2^{n+1}$. Let $x_{k_5} \varepsilon D$ be such that $g(x_{k_5}) = 4/2^{n+1}$. Set $R_4 = \{x_n \varepsilon D \mid x_{k_3} < x_n < x_{k_5}$ and $n > k_3, k_5\}$. Clearly $R_4 \neq \emptyset$. Let k_6 be the smallest positive integer such that $x_{k_6} \varepsilon R_4$. Then $x_{p_{k_6}} = x_{k_3}$ and $x_{f_{k_6}} = x_{k_5}$. Hence $g(x_{k_6}) = 3/2^{n+1}$. Similarly there exist elements in D which map onto $5/2^{n+1}, 7/2^{n+1}, \ldots, (2^{n+1}-3)/2^{n+1}$. Thus for each $b \varepsilon B_{n+1}$ there is a $d_b \varepsilon D$ such that $g(d_b) = b$. Therefore, g takes D onto B.

We can now show that g takes K onto [0,1]. Let $z \varepsilon [0,1] - B$ and let $\{b_i\}_{i=1}^{\infty}$ be a sequence in B such that $b_i > b_{i+1}$ for all i and $b_i \to z$. For each i let $d_i \varepsilon D$ be such that $g(d_i) = b_i$. Then, because g preserves order, $d_i > d_{i+1}$ for all i. There exists a $\omega \varepsilon K$ such that $d_i \overset{c}{\to} \omega$ since K is compact. We have that $g(\omega) = \text{glb}\{g(x_n) \mid x_n \geq \omega\}$ whether $\omega \varepsilon D$ or $\omega \varepsilon K-D$. Suppose $g(\omega) < z$; then there is a $x_{m_0} \varepsilon D$ such that $x_{m_0} > \omega$ and $g(\omega) < g(x_{m_0}) < z$. Then $x_{m_0} < d_i$ for all i since $g(x_{m_0}) < z$. So, $\omega < x_{m_0} < d_i$ for all i. Thus, $K - \{x_{m_0}\} = U_{a_{x_{m_0}}} \psi V_{b_{x_{m_0}}}$ with

$\omega \varepsilon U_{a_{x_{m_0}}}$ and $d_i \varepsilon V_{b_{x_{m_0}}}$ for all i. This contradicts $d_i \overset{c}{\to} \omega$.

Hence $g(\omega) \geq z$. Suppose $g(\omega) > z$; then $g(\omega) > g(d_{i_0})$ for some i_0. This contradicts $\omega < d_i$ for all i. Hence $g(\omega) = z$ and so g takes K onto [0,1].

The function g is one to one since g is onto [0,1] and g preserves order. Let c,d be two distinct points in [0,1]. Then $g^{-1}((c,d)) = (g^{-1}(c),g^{-1}(d))$, $g^{-1}((0,d)) = (a,g^{-1}(d))$ and $g^{-1}((c,1)) = (g^{-1}(c),b)$. However, $(g^{-1}(c),g^{-1}(d))$, $(a,g^{-1}(d))$ and $(g^{-1}(c),b)$ are open in K. Hence g is a continuous map.

We now have that g is a continuous, onto, one to one and order preserving map from K to [0,1]. The function g is a homeomorphism since K is compact and [0,1] is T_2. #

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