PROPERTIES OF BICENTRIC CIRCLES FOR THREE-SIDED POLYGONS

THESIS

Presented to the Graduate Council of the University of North Texas in Partial Fulfillment of the Requirements

For the Degree of

MASTER OF ARTS

By

David J. Heinlein
Denton, Texas

August, 1998

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We define and construct bicentric circles with respect to three-sided polygons. Then using inherent properties of these circles, we explore both tangent properties, and areas generated from bicentric circles.

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CHAPTER 1

INTRODUCTION

1.1 The Outer and Inner Radius of a Triangle

From elementary geometry, we know that every triangle, T, has two circles associated with it. One which circumscribes T, and one which can be inscribed within T. Figure 1.1. Unless otherwise specified, we will assume that every triangle, T, has sides Z ≥ Y ≥ X.

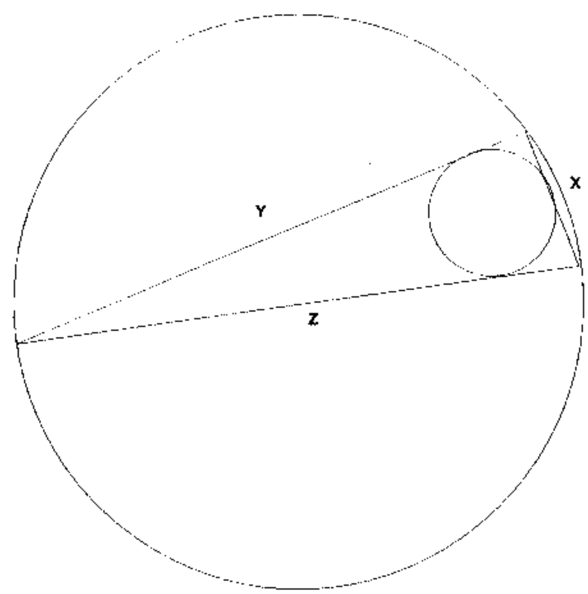

Figure 1.1

We now define the the outer and inner radii for T.

Definition 1.1.1 The Outer radius, R, is defined to be the radius of the circle, which circumscribes T. A common method for evaluating R, involves the Law of Sines.

method for evaluating R, involves the Law of Sines. For our purposes however, it would be convenient to find a formula for R which can be expressed as a function of side lengths X, Y, and Z only.

Proposition 1.1. Let T be a triangle with side lengths $Z \geq Y \geq X$. Then the Outer radius, R, is given by,

$$R = \frac{XYZ}{\sqrt{4X^2Y^2 - (X^2 + Y^2 - Z^2)^2}}.$$

Note: It can be shown that $R = \frac{XYZ}{A}$, where A is the area of T.

Proof:

We will begin this proof by making several observations about Figure 1.2.

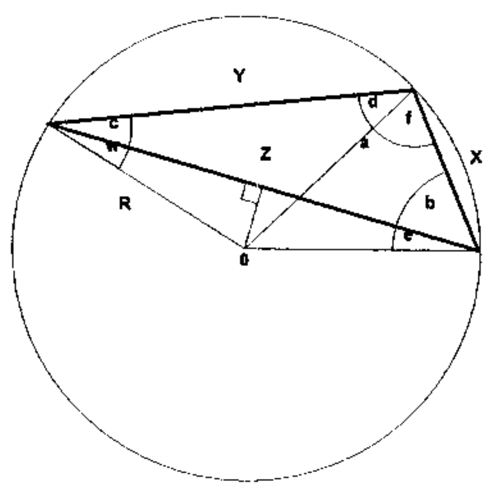

Figure 1.2

$$(1) \quad \cos\angle w = \frac{Z}{2R}$$

$$(2) \quad \angle w = \angle e, \quad \textit{Isosceles}$$

$$(3) \quad \angle c + \angle w = \angle d, \quad \textit{Isosceles}$$

$$(4) \quad \angle d + \angle f = \angle a$$

$$(5) \quad \angle e + \angle b = \angle f, \quad \textit{Isosceles.}$$

Adding (3) and (4), then rearranging we find,

$$(6) \quad \angle c + \angle w = \angle a - \angle f .$$

Using (2) and (5) we have,

$$(7) \quad \angle w + \angle b = \angle f .$$

Then by adding (6) and (7) we get:

$$2\angle w = \angle a - \angle c - \angle b ,$$

$$(8) \quad \angle w = \frac{\angle a - \angle c - \angle b}{2} .$$

Using the fact that

$$\angle a + \angle b + \angle c = 180° ,$$

we see that together with (8),

$$\angle w = \angle a - 90° .$$

Using (1) we find that,

$$(9) \qquad R = \frac{Z}{2\cos(\angle a - 90°)} .$$

Applying the subtraction formula for cosines, cos(A - B) = cos(A)cos(B) + sin(A)sin(B), to (9) we obtain,

$$(10) \qquad R = \frac{Z}{2\sin\angle a} .$$

Remark: Though we have assumed the center of our circle lies outside of our triangle, a similar proof shows we have the same equality, regardless of the position of the center.

Note: Repeating this method for the two other sides of T, along with the opposite angles, will establish the Law of Sines.

To find $\angle$ a, we use the Law of Cosines,

$$Z^2 = X^2 + Y^2 - 2XY\cos\angle a \, .$$

By solving we find,

$$\angle a = \arccos[\frac{X^2 + Y^2 - Z^2}{2XY}] \, .$$

Therefore by (10) we have,

$$(11) \qquad R = \frac{Z}{2\sin[\arccos\frac{X^2+Y^2-Z^2}{2XY}]} \, .$$

Finally by applying to (11), the trig identity

$$\sin[\arccos\xi] = \sqrt{1 - \xi^2}, \quad |\xi| \le 1,$$

our equation becomes,

$$R = \frac{Z}{2\sqrt{1 - (\frac{X^2 + Y^2 - Z^2}{2XY})^2}} \, .$$

Simplifying yields the desired result,

$$R = \frac{XYZ}{\sqrt{4X^2Y^2 - (X^2 + Y^2 - Z^2)^2}} .$$

When we restrict T to take a specific form, such as an Isosceles, Equilateral, or Right Triangle, we find,

<u>Isosceles</u>: Let base = B, let sides = S, then

$$R = \frac{S^2}{\sqrt{4S^2 - B^2}} .$$

<u>Equilateral</u>: Let sides = S, then

$$R = \frac{S\sqrt{3}}{3} .$$

<u>Right Triangle</u>: Let $X^2 + Y^2 = Z^2$, then

$$R = \frac{Z}{2} .$$

Definition 1.1.2 The Inner radius, denoted r, is defined to be the radius of the circle which is inscribed within T. Figure 1.3.

As with the case for R, we would also like to find a formula which will give r strictly in terms of the side lengths X, Y, Z.

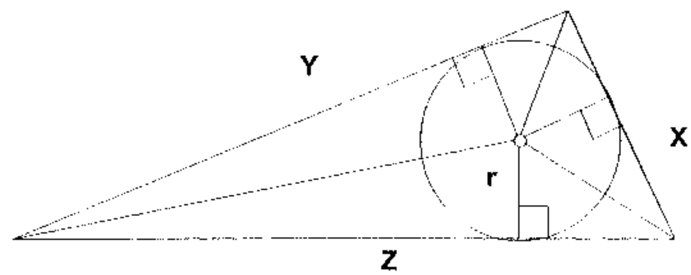

Figure 1.3

Proposition 1.2 Let T be a triangle with side lengths $Z \geq Y \geq X$. Then the Inner radius, r, is given by,

$$r = \frac{X+Y-Z}{2}\sqrt{\frac{Z^2-(X-Y)^2}{(X+Y)^2-Z^2}}.$$

Note: It can be shown the we also have, $r = \frac{2A}{(X+Y+Z)}$, where A is the area of T.

Proof: To prove this formula we will divide T into pieces then sum the areas of these pieces. Figure 1.4.

Connecting the center of the in circle to the vertices of T, we form three triangles, xsy, xsz, and zsy. Let A[T] = A[xyz], denote the area of T. So,

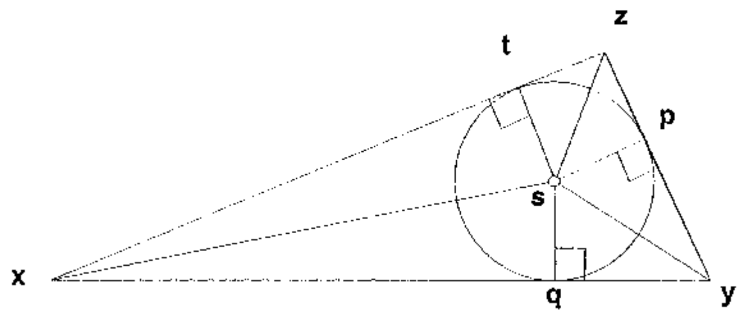

Figure 1.4

$$A[xyz] = A[xsy] + A[xsz] + A[ysz],$$

$$= \frac{(sq)(xy)}{2} + \frac{(st)(xz)}{2} + \frac{(sp)(yz)}{2},$$

$$= \frac{rZ}{2} + \frac{rY}{2} + \frac{rX}{2},$$

$$= \frac{r(X + Y + Z)}{2}.$$

Next, it can be shown that h, Figure 1.5 can be expressed as:

$$h = \sqrt{Y^2 - \left(\frac{Z^2 + Y^2 - X^2}{2Z}\right)^2}.$$

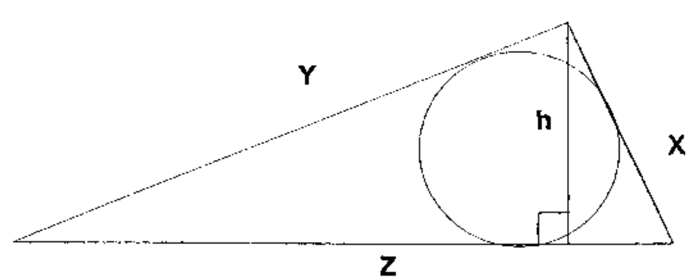

Figure 1.5

So,

$$A[T] = \frac{Z}{2}h, = \frac{Z}{2}\sqrt{Y^2 - \left(\frac{Z^2 + Y^2 - X^2}{2Z}\right)^2}.$$

Thus,

$$\frac{r(X + Y + Z)}{2} = \frac{Z}{2}\sqrt{Y^2 - \left(\frac{Z^2 + Y^2 - X^2}{2Z}\right)^2}.$$

Solving for r yields our desired result,

$$r = \frac{Z + Y - X}{2}\sqrt{\frac{X^2 - (Z - Y)^2}{(Z + Y)^2 - X^2}} .$$

The equation for the inner radius r, may also be further simplified when we restrict T to take a specific form, such as an Isosceles, Equilateral, or Right Triangle.

<u>Isosceles</u>: Let base = B, and sides = S, then

$$r = \frac{B}{2}\sqrt{\frac{2S - B}{2S + B}}$$

<u>Equilateral</u>: Let Sides = S, then

$$r = \frac{S\sqrt{3}}{6}$$

<u>Right Triangle</u>: Let $X^2 + Y^2 = Z^2$, then

$$r = \frac{X + Y - Z}{2}.$$

CHAPTER 2

THE CENTER DISTANCE

2.1 The Center Distance

In this chapter we discuss one of the fundamental relationships between a triangle T, and the inner and outer circles which are naturally associated with T.

Definition 2.1.1: Let C_O and C_I be nested circles of radii R and r respectfully. Let C_I lie entirely within C_O, then the Center Distance, denoted D_c, is defined to be the distance between the centers of C_O and C_I.

Theorem 2.1.2 Let C_O and C_I be nested circles having radii R and r respectfully, and suppose $r \leq \frac{1}{2}R$. Then there exists an Isosceles Triangle T with outer and inner radii R and r, if and only if $D_c^2 = R^2 - 2Rr$.

Proof: Suppose there exists a Isosceles triangle T with outer and inner radii R, and r.

Forming the right triangle as shown in figure 2.1, we have,

$$(r + R + D_c)^2 + (\frac{B}{2})^2 = S^2.$$

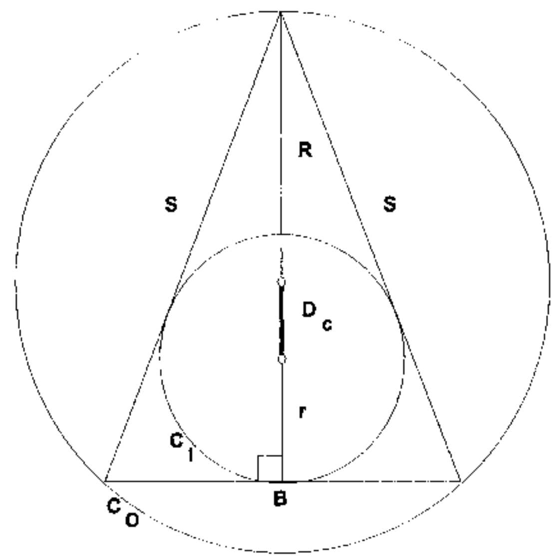

Figure 2.1

Expanding then solving the quadratic for D_c,

$$_c = \frac{-2(r + R) + \sqrt{4S^2 - B^2}}{2},$$

where we have taken the positive root.

Squaring both sides we have,

$$D_c^2 = \frac{4S^2 - B^2}{4} - \sqrt{4S^2 - B^2}(r+R) + (r+R)^2.$$

Replacing R and r, with S, B terms on the right,

$$D_c^2 = \frac{B^2}{4} - BS + \frac{B^2}{4} \cdot \frac{2S - B}{2S + B} + \frac{SB^2}{2S+B} + \frac{S^4}{4S^2 - B^2}.$$

Multiplying top and bottom of the right by $4(4S^2 - B^2)$, then simplifying,

$$D_c^2 = \frac{S^4 - 2BS^3 + B^2S^2}{4S^2 - B^2},$$

$$= \frac{S^4}{4S^2 - B^2} - \frac{BS^2(2S - B)}{(2S + B)(2S - B)}.$$

Rewriting we have,

$$D_c^2 = \frac{S^4}{4S^2 - B^2} - \frac{BS^2\sqrt{2S-B}}{\sqrt{4S^2-B^2}\sqrt{2S+B}}$$

$$= \frac{S^4}{4S^2 - B^2} - 2 \cdot \frac{S^2}{\sqrt{4S^2-B^2}} \cdot \frac{B}{2} \cdot \sqrt{\frac{2S-B}{2S+B}}.$$

Writing S and B in terms of R and r, we have,

$$D_c^2 = R^2 - 2Rr.$$

Next suppose that C_0 and C_I are nested circles of radii R and r respectfully, and that $D_c^2 = R^2 - 2Rr$. See Figure 2.2.

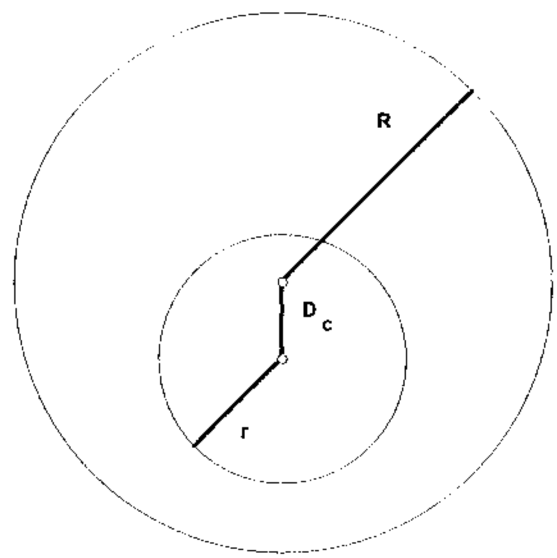

Figure 2.2

Define an Isosceles Triangle T' as follows: Let T' have base B' and sides S' satisfying,

$$B' = 2\sqrt{r(2R - r - 2D_c)}$$

and

$$S' = \sqrt{2R(R + r - D_c)}$$

Since $R > D_c$ for all $0 < r < \frac{1}{2}R$, both B' and S' are well defined. By making the substitution

$$\xi = R - D_c ,$$

B' and S' reduce to,

$$B' = 2\sqrt{r(2\xi - r)}, \quad and \quad S' = \frac{\xi\sqrt{r(2\xi - r)}}{\xi - r} .$$

We will suppose that T' has outer and inner radii R', and r'. Then show R' = R, and r' = r. Thereby establishing that there exists an isosceles triangle which has outer and inner radii R, and r, as shown in Figure 2.3.

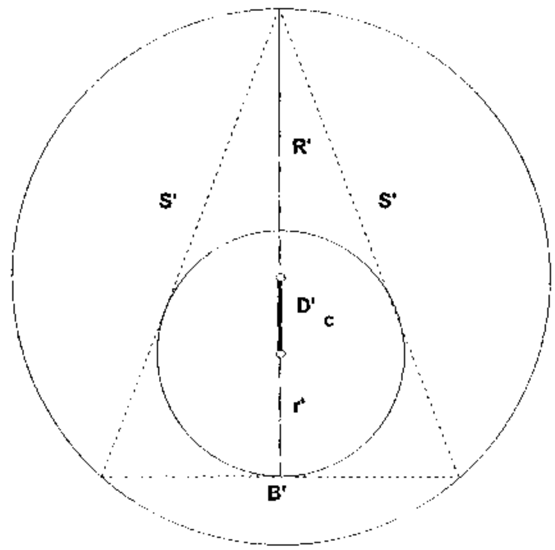

Figure 2.3

To show that the outer radius R' = R, we have that,

$$R' = \frac{S'^2}{\sqrt{4S'^2 - B'^2}}.$$

Substituting S' and B' into the equation yields

$$R' = \frac{2R(R + r + D_c)}{\sqrt{4(2R(R + r + D_c)) - 4r(2R - r - 2D_c)}}.$$

Since we have assumed that $D_c^2 = R^2 - 2Rr$, R' can be written as,

$$R' = \frac{R(R + r + D_c)}{\sqrt{(R + r + D_c)\,(R + r + D_c)}}\,.$$

Simplifying yields the desired results,

$$R' = R.$$

Now we must show that r' = r. By squaring the inner radius formula for T',

$$r'^2 = \frac{B'^2}{4}\left(\frac{2S' - B'}{2S' + B'}\right).$$

Substituting in S' and B' we have,

$$r'^2 = \frac{4\,(r(2\xi - r))}{4}\left(\frac{\dfrac{2\xi\sqrt{r(2\xi - r)}}{\xi - r} - 2\sqrt{r(2\xi - r)}}{\dfrac{2\xi\sqrt{r(2\xi - r)}}{\xi - r} + 2\sqrt{r(2\xi - r)}}\right).$$

Upon simplifying this reduces to,

$$r'^2 = r(2\xi - r)\left(\frac{2\sqrt{r(2\xi - r)}(\xi - \xi + r)}{2\sqrt{r(2\xi - r)}(\xi + \xi - r)}\right)$$

Simplifying yields

$$r'^2 = r(2\xi - r)\left(\frac{r}{2\xi - r}\right).$$

Simplifying and taking the square root of both sides establishes the desired results,

$$r' = r \ .$$

End of proof.

Theorem 2.1.3 Let T be a triangle with outer and inner radii R, r. Then $D_c^2 = R^2 - 2Rr$.

Proof: The proof of Theorem 2.2.3 closely parallels the proof of Theorem 2.2.2. Let T be arbitrary, see Figure 2.4.

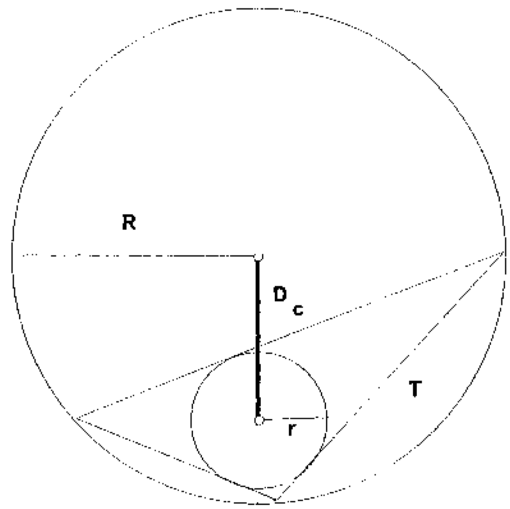

Figure 2.4

We want to show that there exists an Isosceles Triangle, T', with outer and inner radii R`, and r`, and that R' = R, r' = r. See Figure 2.5.

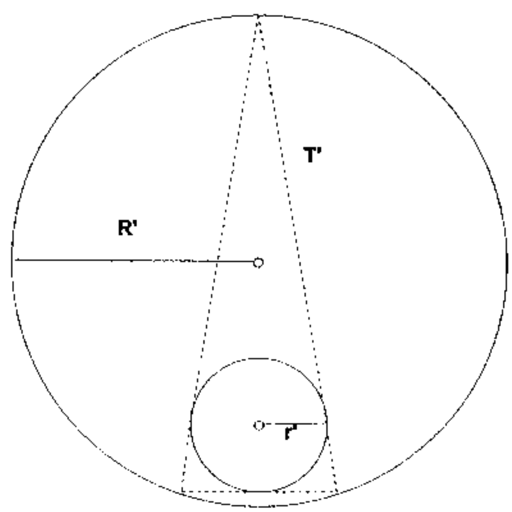

Figure 2.5

Define T' as follows. Let T' have base B', and sides S', such that,

$$B' = 2\sqrt{r(2R - r - 2\sqrt{R(R - 2r)})}$$

$$S' = \frac{\sqrt{r(2R - r - 2\sqrt{R(R - 2r)})}\,(R-\sqrt{R(R - 2r)})}{R - r - \sqrt{R(R - 2r)}}.$$

To show that the outer radius R' = R we make that substitution,

$$\phi = r + 2\sqrt{R(R - 2r)}.$$

Then B' and S' can be written as,

$$B' = 2\sqrt{r(2R - \phi)} \qquad S' = \frac{\sqrt{r(2R - \phi)}\,(2R + r - \phi)}{(2R - r - \phi)}$$

Substituting B' and S' into our Isosceles formula for R',

$$R' = \frac{S'^2}{\sqrt{4S'^2 - B'^2}},$$

we have,

$$R' = \frac{\left(\dfrac{r_\circ(2R_\circ - \phi)(2R_\circ - \phi + r_\circ)^2}{(2R_\circ - \phi - r_\circ)^2}\right)}{\sqrt{\dfrac{4r_\circ(2R_\circ - \phi)(2R_\circ - \phi + r_\circ)^2}{(2R_\circ - \phi - r_\circ)^2} - 4r_\circ(2R_\circ - \phi)}} .$$

Simplifying we have,

$$R' = \frac{(2R - \phi + r)^2}{4(2R - \phi - r)} .$$

Letting,

$$\phi = r + 2\sqrt{R(R - 2r)} ,$$

then simplifying yields,

$$R' = R.$$

To show that the inner radius r' = r, we square both sides of our formula for r',

$$r'^2 = \frac{B'^2}{4}\left(\frac{2S' - B'}{2S' + B'}\right).$$

Making the substitution,

$$\Psi = (R - \sqrt{R(R - 2r)}),$$

B' and S' can be written,

$$B' = 2\sqrt{r(2\Psi - r)}, \qquad S' = \frac{\Psi\sqrt{r(2\Psi - r)}}{\Psi - r}.$$

Substituting for B' and S' we have,

$$r'^2 = \frac{4(r(2\Psi - r))}{4}\left(\frac{\dfrac{2\Psi\sqrt{r(2\Psi - r)}}{\Psi - r} - 2\sqrt{r(2\Psi - r)}}{\dfrac{2\Psi\sqrt{r(2\Psi - r)}}{\Psi - r} + 2\sqrt{r(2\Psi - r)}}\right)$$

$$= r(2\Psi - r)\left(\frac{2\sqrt{r(2\Psi - r)}(\Psi - \Psi + r)}{2\sqrt{r(2\Psi - r)}(\Psi + \Psi - r)}\right).$$

Simplifying and taking the square root of both sides establishes the desired result,

$$r' = r.$$

Thus by Theorem 2.1.2, there exists an Isosceles triangle which and outer and inner radii, R, and r, therefore,

$$D_c^2 = R^2 - 2Rr.$$

CHAPTER 3

BICENTRIC CIRCLE CONSTRUCTION

3.1 The Center Radius Method

In addition to evaluating the distance between the centers of the circumscribed and inscribed circles for a triangle T, the Center Distance, D_c, is also useful in constructing triangles when both the inner and outer radius are known.

Clearly there is no difficulty with constructing triangles within a circle of radius R, nor is there any problem with constructing triangles around a circle of radius r. Suppose however we are asked to construct a triangle which has both a fixed inner radius r, and a fixed outer radius, R.

The problem encountered here is that we must be able to satisfy both the fixed inner and outer radius with the same triangle. The following method, the Center Radius Method, is useful in constructing such a triangle.

Center Radius Method

In chapter 2 we proved that for any triangle,

$$D_c^2 + r^2 = (R - r)^2.$$

Tools needed: 1. straight edge

2. compass

Step 1) Construct two perpendicular lines.

Step 2) Construct two concentric circles of radius R and r. We require that $r \leq ½R$ as shown in figure 3.1.

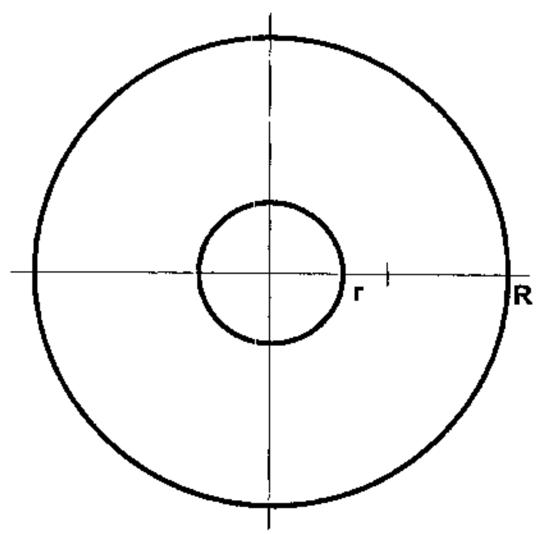

Figure 3.1

Step 3) Construct a circle with center at (0,r), having radius (R - r)as shown in Figure 3.2.

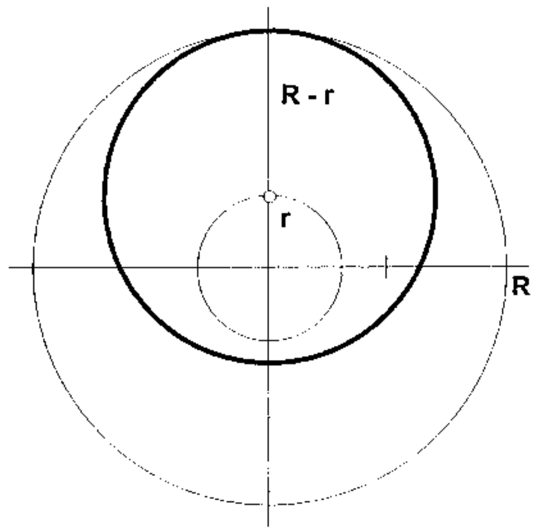

Figure 3.2

Step 4) Locate the point of intersection between this circle and the positive x-axis, label it p, as shown in Figure 3.3.

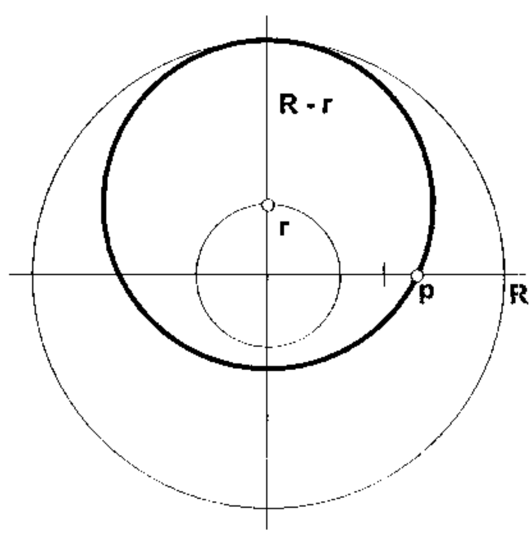

Figure 3.3

Step 5) Construct a circle of radius r centered at p. See Figure 3.4.

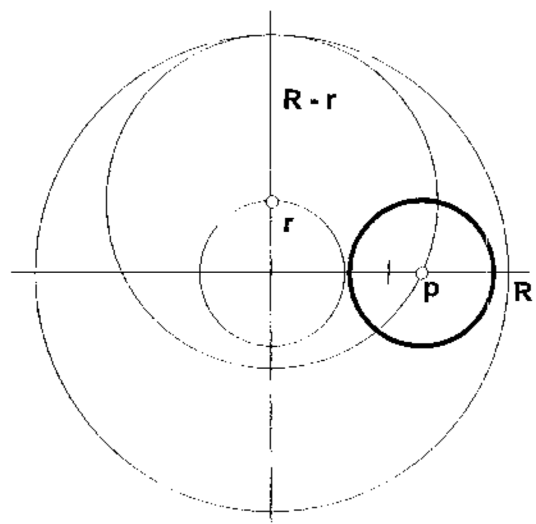

Figure 3.4

Step 6) By a standard right triangle we have, $P^2 + r^2 = (R-r)^2$. (Figure 3.5).

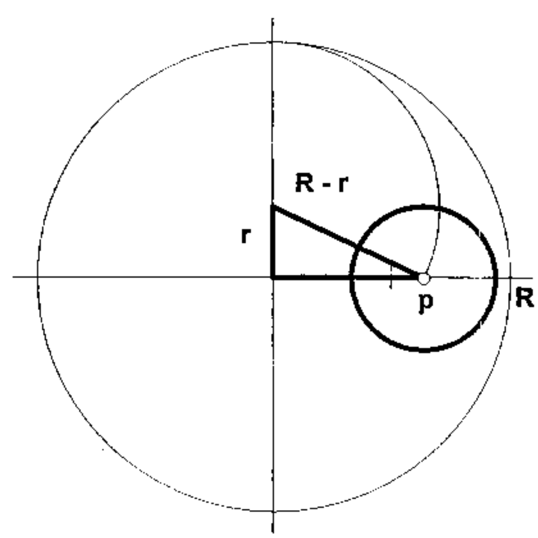

Figure 3.5

Step 7) The center of C_p, is placed a distance p from the center of C. (Figure 3.6).

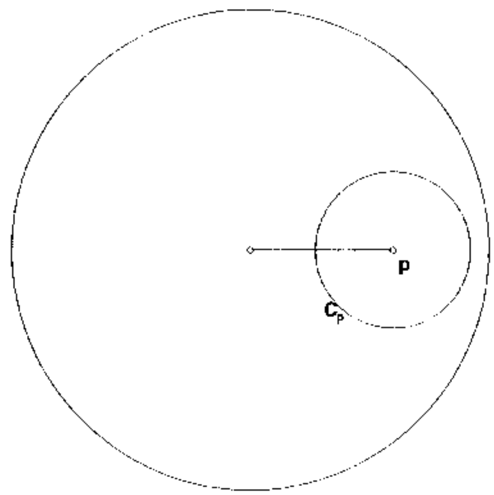

Figure 3.6

Step 8) Let q be an arbitrary point on C_p, and construct a line segment through points p and q, as shown in Figure 3.7.

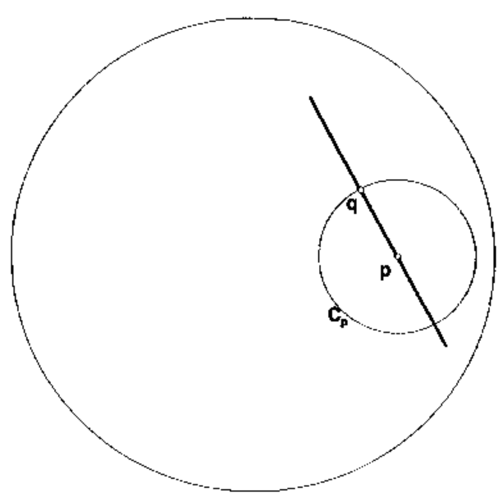

Figure 3.7

Step 9) Construct a perpendicular to this line segment which passes through q, and label the points of intersection with C as h, and k. See Figure 3.8.

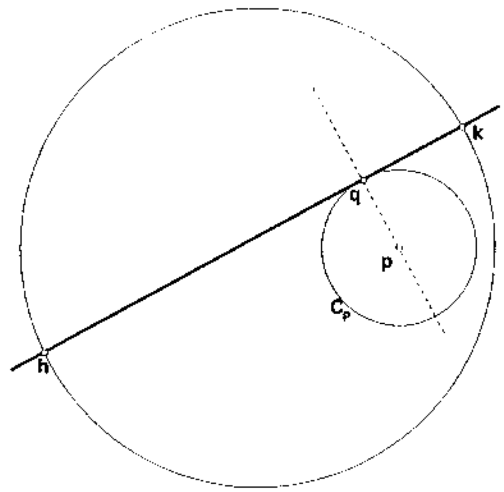

Figure 3.8

Step 10) Construct an arc of the circle whose center is h, and which passes through q. Label the point of intersection with C_p, s. See Figure 3.10.

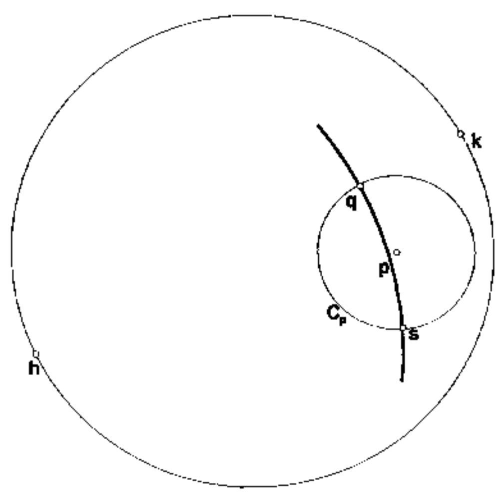

Figure 3.9

Step 11) Construct a line segment which passes through h, and s. Let j be the point of intersection with C, as shown in Figure 3.10.

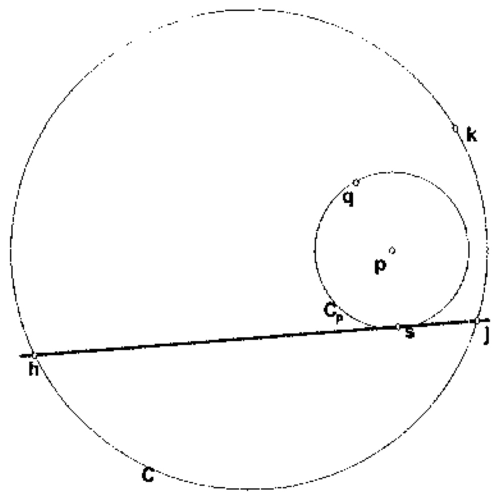

Figure 3.10

Step 13) Construct line segments hj and hk.

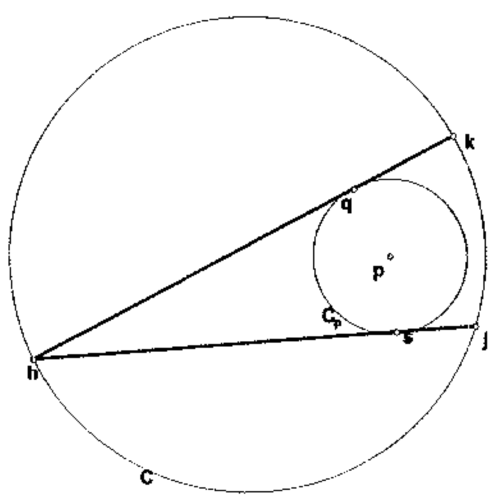

Figure 3.11

Since segments hj and hk are tangent to C_p we have that segment kj is also be tangent to C_p. See Figure 3.12.

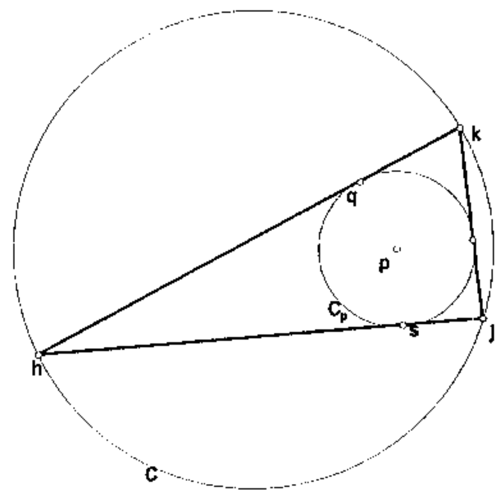

Figure 3.12

Thus our triangle is circumscribed by C and inscribes C_p. Now since for fixed R and r, our construction of T depends only on the point q, there exist an infinite number of triangles which share inner and outer circles C and C_p. See Figure 3.13.

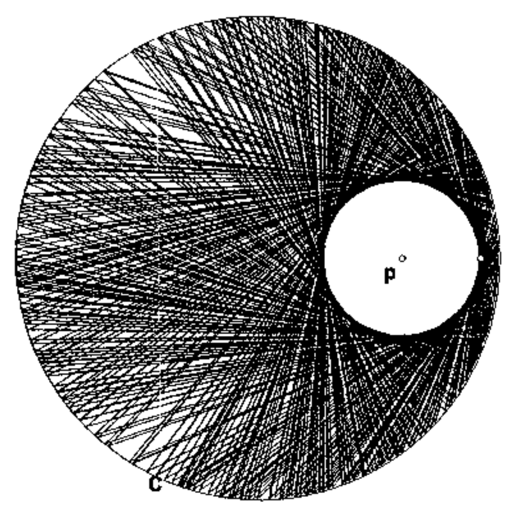

Figure 3.13

<u>3.2 The Center Distance Method</u>

Suppose we are given a circle C of radius R, and a point $p \in C$. The natural question to ask is; can we construct a circle centered at p, called C_p, such that C_p can be inscribed within a triangle, T, while T is inscribed within C? The answer is yes, and we will call this method of construction the <u>Center Distance Method</u>.

Tools needed: 1. straight edge

2. compass

Let C be a circle of radius R, and let $p \in C$, as shown in Figure 3.14.

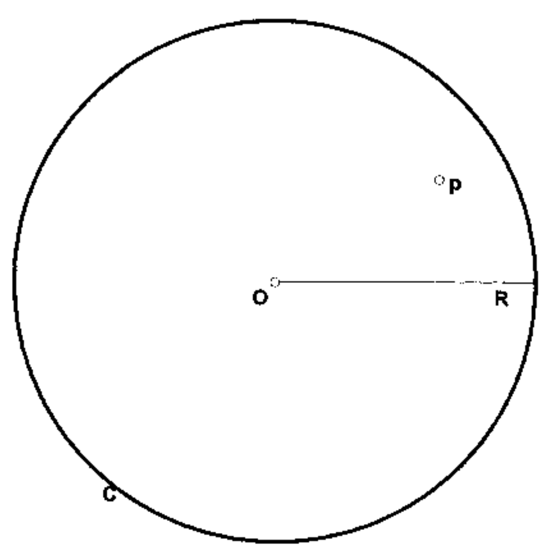

Figure 3.14

Step 1) Construct a diameter of C which passes through p, as shown in Figure 3.15.

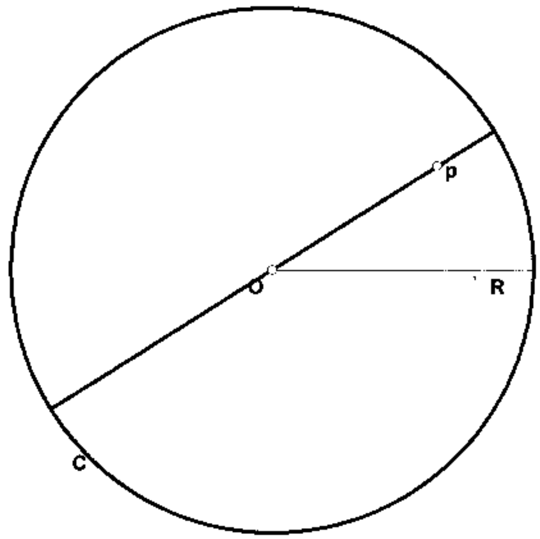

Figure 3.15

Step 2) Construct a perpendicular to this diameter which passes through O. See Figure 3.16.

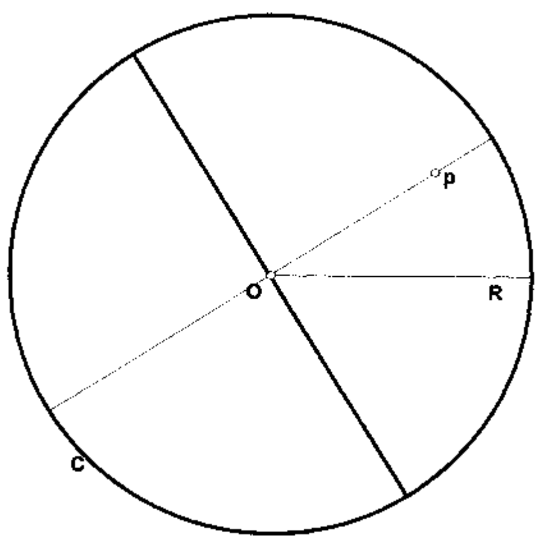

Figure 3.16

Step 3) Construct the line segment and midpoint, m, as shown in Figure 3.17.

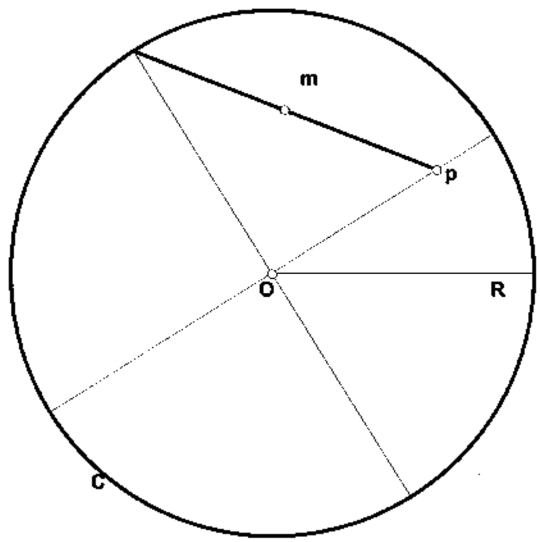

Figure 3.17

Step 4) Consruct a perpendicular bisector to this segment, and lable the point of intersection, r, as shown in Figure 3.18.

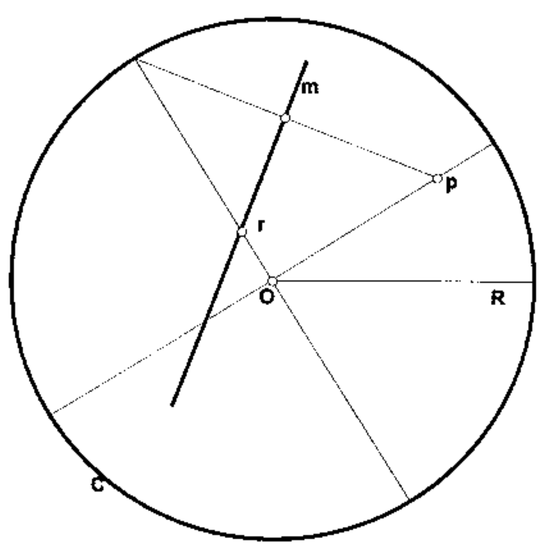

Figure 3.18

Step 5) Let $|r| = r$. Since through m passes a perpendicular bisector, we have by the SAS postulate, that $d(r,p) = R-r$. See Figure 3.19.

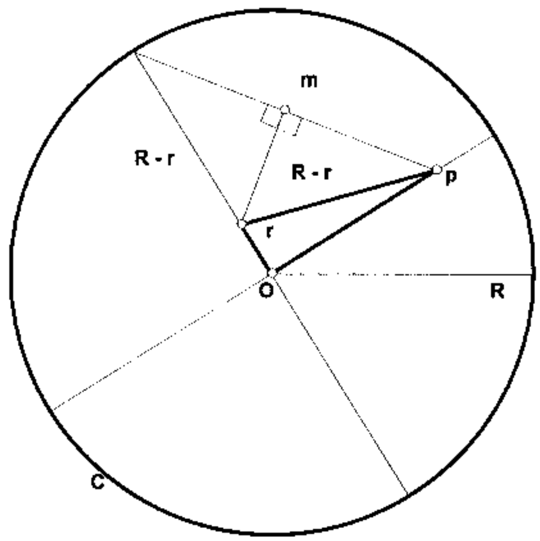

Figure 3.19

Step 6) Construct a circle of radius r, centered at p. See Figure 3.20.

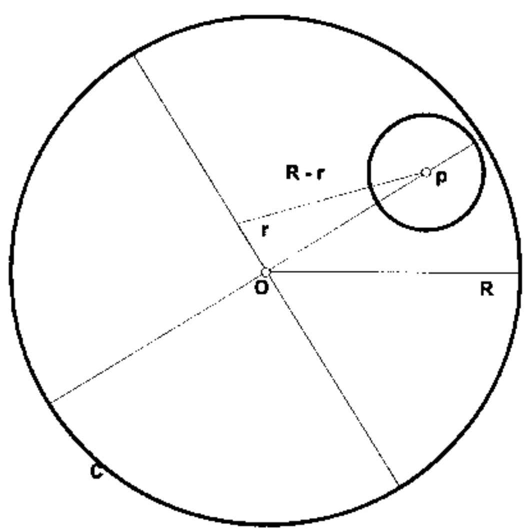

Figure 3.20

Now since $|p|^2 = R^2 - 2Rr$, this circle is properly spaced. Furthermore, since p was arbitrary, we can construct properly spaced circles about any point within C. See Figure 3.21.

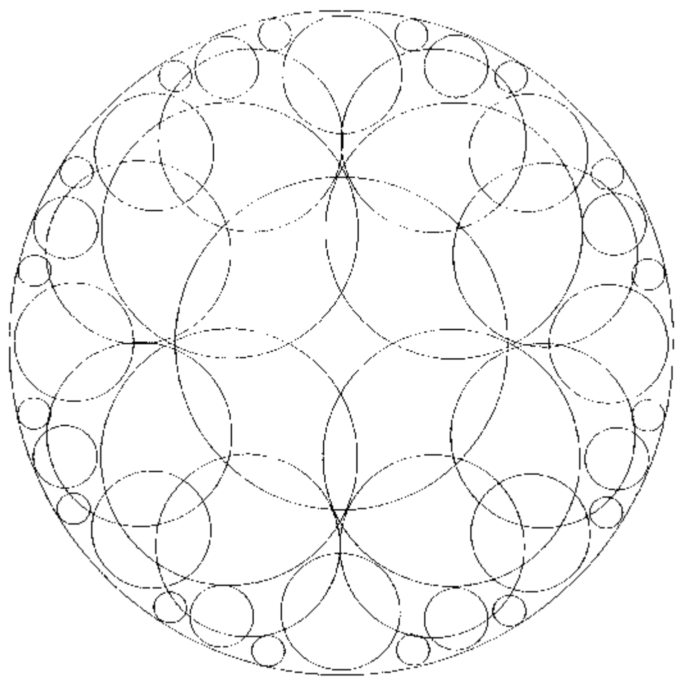

Figure 3.21

CHAPTER 4

TANGENTIAL CIRCLES

4.1 Introduction

In this section we define generated circles and explore the properties of properly spaced tangential circles. Unless otherwise specified all circles are assumed to be properly spaced. Let $C = \{(x,y) \mid x^2 + y^2 < R^2\}$, and let $p \in C$, i.e. $p = (p_1, p_2)$.

Definition 4.1.1 We say that a circle C_p is generated by a point $p \in C$, see Figure 4.1 if C_p is the circle given by,

$$(x-p_1)^2 + (y-p_2)^2 = r_p^2, \quad where \quad r_p = \frac{R^2 - |p|^2}{2R}.$$

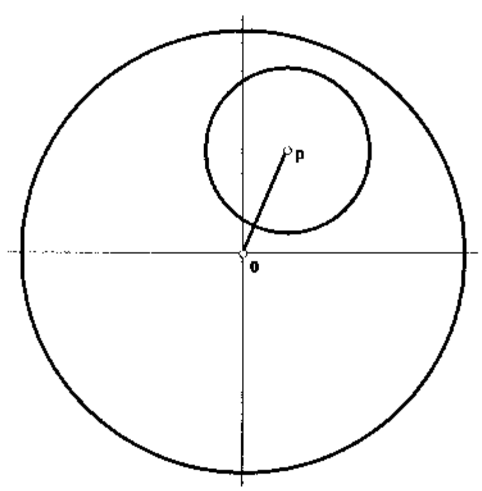

Figure 4.1

4.2 Tangential Circles

Definition 4.2.1 Let $p, q \in C$, and let C_p and C_q resp., be the circles generated by p and q. We say that the two circles, C_p and C_q are tangent within a circle of fixed radius R, if and only if $|p - q| = r_p + r_q$. This is shown in Figure 4.2.

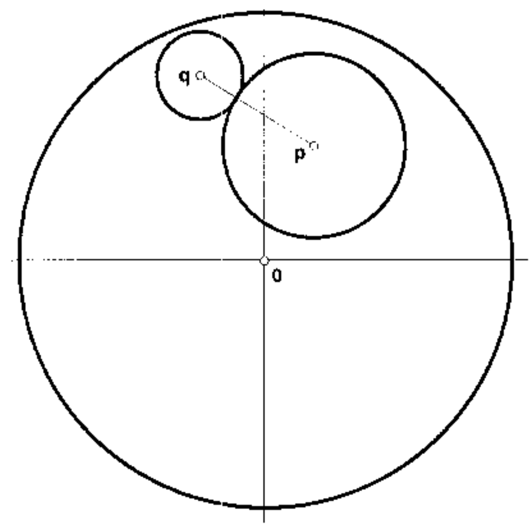

Figure 4.2

Definition 4.2.2 We define T_p as the set of all $q \in C$ s.t. C_q is tangent to C_p.

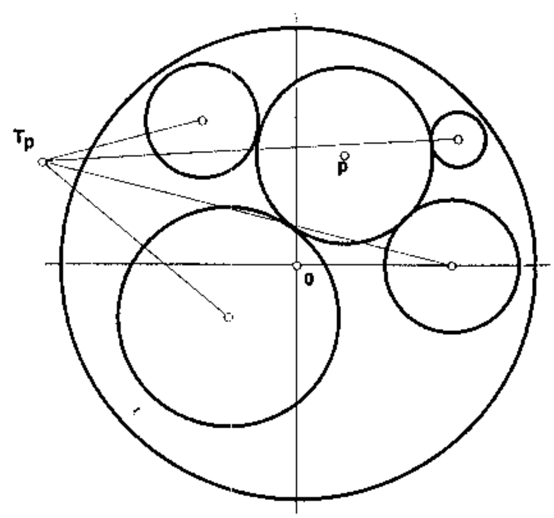

Figure 4.3

Proposition 4.2.3 Let $R > 0$. Let $p, q \in C$, and let C_p be the circle generated by p. If $q = (q_1, q_2)$, then

$$q \in T_p \quad \leftrightarrow \quad {q_2}^2 = 4R^2 - {q_1}^2 - |p|^2 - 2R\sqrt{3R^2 - 2|p|\,q_1}\,.$$

That is,

$$T_p \;=\; \{\, q \in C \mid {q_2}^2 = 4R^2 - {q_1}^2 - |p|^2 - 2R\sqrt{3R^2 - 2|p|\,q_1}\ \}.$$

Proof: Let $p \in C$, and let C_p be the circle generated by p. W.L.O.G. rotate p to lie along the positive x-axis, and let $q = (q_1, q_2) \in T_p$, as shown in Figure 4.4.

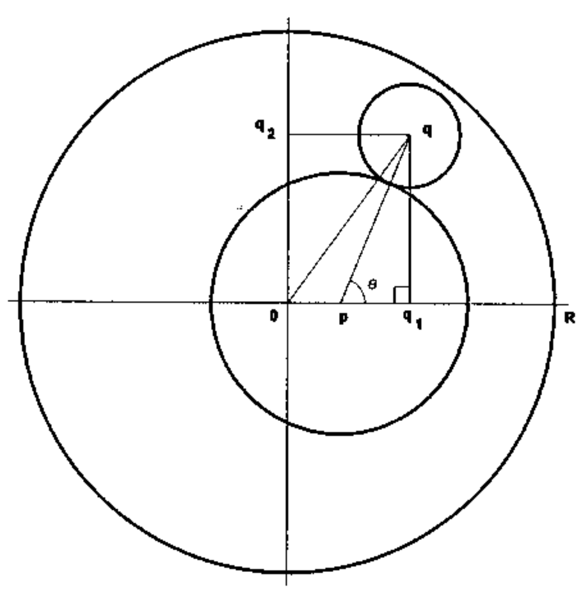

Figure 4.4

Then we have both,

$$\cos\theta = \frac{|q|^2 - p^2 - (r_p + r_q)^2}{2p(r_p + r_q)}$$

$$\cos\theta = \frac{q_1 - p}{r_p + r_q}.$$

Thus,

$$2p(q_1 - p) = |q|^2 - p^2 - (r_p + r_q)^2.$$

Since $|q|^2 = R^2 - 2Rr_q$,

$$r_q^2 + 2(R + r_p)\, r_q + (r_p^2 + 2pq_1 - R^2 - p^2) = 0.$$

Solving this quadratic in r_q,

$$r_q = -(R + r_p) + \sqrt{(R + r_p)^2 - (r_p^2 + 2pq_1 - R^2 - p^2)}.$$

Using the equalities,

$$r_q = \frac{R^2 - |q|^2}{2R}, \qquad |q|^2 = q_1^2 + q_2^2,$$

we have,

$$q_2^2 = 4R^2 - q_1^2 - p^2 - 2R\sqrt{3R^2 - 2pq_1}.$$

Thereby establishing the claim that,

$$T_p = \{ q \in C \mid q_2^{\,2} = 4R^2 - q_1^{\,2} - p^2 - 2R\sqrt{3R^2 - 2pq_1} \}.$$

With this result we are now ready to define the minimum and maximum tangential circles to a circle generated by an arbitrary point $p \in C$.

Definition 4.2.4 Let $R > 0$. Let $p \in C$, and let C_p be the circle generated by p, having radius r_p. Let $a \in T_p$. We say that C_a is the minimum tangential circle if $r_a \leq r_q$ for all $q \in T_p$. Similarly if $b \in T_p$, then C_b is the maximum tangential circle if $r_b \geq r_q$ for all $q \in T_p$.

The following Theorem establishes a fundamental result with respect to the minimum and maximum tangential circles.

Theorem 4.2.5 Let $R > 0$. Let $C = \{(x,y) | x^2 + y^2 < R^2\}$. Let $p \in C$, let C_p be the circle generated by p. Let $q \in T_p$. Let r_{min}, resp, r_{max} denote the minimum and maximum radii $\forall$ C_q. Let $a,b \in T_p$ s.t. $r_a = r_{min}$, and $r_b = r_{max}$, then a and b lie on the diameter of C which contains p.

Claim 1: If $q,t \in C$, then $|q| < |t|$ if and only if $r_t < r_q$.

The proof is trivial since,

$$r_q = \frac{R^2 - |q|^2}{2R} \quad \forall q \in C.$$

Proof: Let $p \in C$, and let C_p be the circle generated by p. W.L.O.G we may assume that through rotation $p = (p,0)$, where $0 \le p < R$. Let $a,b \in T_p$, let $a = (a,0)$, $a > 0$, let $b = (b,0)$, $b < a$.

Case 1: $p = 0$.

Let $q,t \in T$. So, $|g| = r_p + r_g$, $|t| = r_p + r_t$. Thus $|q| - |t| = r_q - r_t$. The claim here is that $r_q = r_t$. If not, say W.L.O.G. that $r_q > r_t$. Then by claim 1, $|q| < |t|$. This is a contradiction since, then $r_q - r_t > 0$, while $|q| - |t| < 0$. Therefore if $p = 0$, $r_q = r_t$, $\forall q, t \in T$, and $r_{min} = r_{max}$. Thus, $r_a = r_{min}$ and $r_b = r_{max}$, and a and b lie on the diameter of C which contains p. See Figure 4.5.

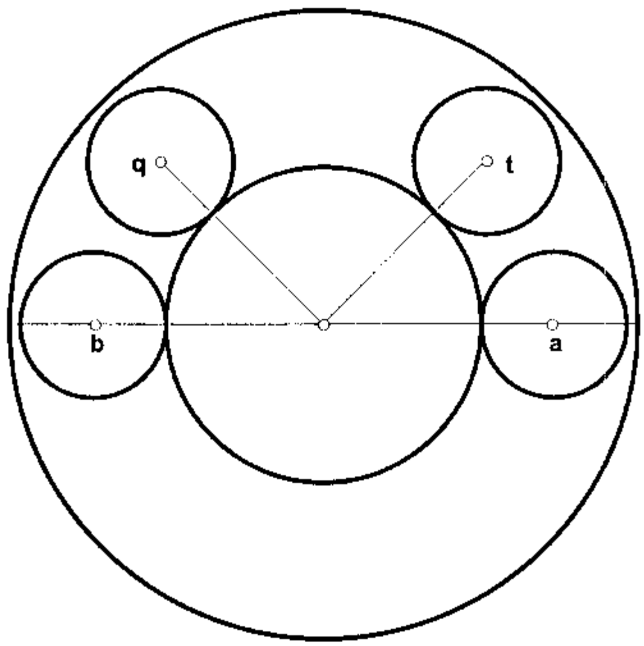

Figure 4.5

Case 2: $0 < p < R$.

Let α, θ represent the angles as shown in Figure 4.6. Due to symmetry about the diameter, we need only define $0 \le \alpha, \theta \le \pi$.

Let $p \in (0,R)$, let $q, t \in T_p$, let $\alpha = \alpha_q$, let $\theta = \theta_t$.

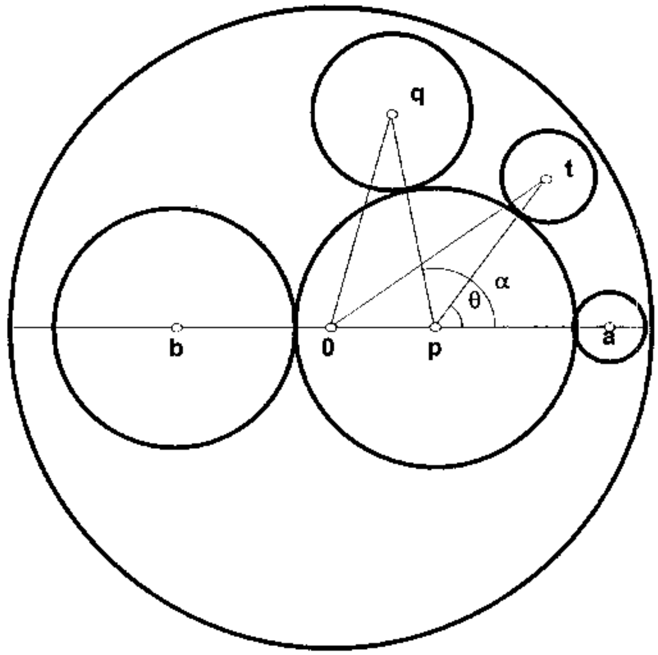

Figure 4.6

<u>Claim 2</u>: If $\alpha_q > \theta_t$, then $r_q > r_t$.

Proof: If $\alpha_q > \theta_t$, then $180° - \theta_t > 180° - \alpha_q$.

Since cos(x) is strictly decreasing on $[0,\pi]$,

we have, $\cos(180° - \theta_t) < \cos(180° - \alpha_q)$.

By the law of cosines we have,

$$\cos(180° - \theta_t) = \frac{p^2 + |p - t|^2 - |t|^2}{2p|p - t|}, \text{ and}$$

$$\cos(180° - \alpha_q) = \frac{p^2 + |p - q|^2 - |q|^2}{2p|p - q|}.$$

Thus,

$$\frac{p^2 + |p - t|^2 - |t|^2}{2p|p - t|} < \frac{p^2 + |p - q|^2 - |q|^2}{2p|p - q|}.$$

Since C_p is tangent to both C_q and C_t, we have that $|p-t| = r_p + r_t$, and $|p-q| = r_p + r_q$.

So,

$$\frac{p^2 + (r_p + r_t)^2 - |t|^2}{2p(r_p + r_t)} < \frac{p^2 + (r_p + r_q)^2 - |q|^2}{2p(r_p + r_q)},$$

The claim is that $r_q > r_t$. To show this suppose on the contrary that $r_q < r_t$.

<u>Note</u>: [We need not consider $r_g = r_t$ since we have assumed $\alpha_q > \theta_t$.]

By claim 1, $|q| > |t|$.

Thus,

$$\frac{p^2 + |p - t|^2 - |t|^2}{2p|p - t|} < \frac{p^2 + |p - q|^2 - |q|^2}{2p|p - q|}.$$

Since C_p is tangent to both C_q and C_t, we have that $|p-t| = r_p + r_t$, and $|p-q| = r_p + r_q$.

So,

$$\frac{p^2 + (r_p + r_t)^2 - |t|^2}{2p(r_p + r_t)} < \frac{p^2 + (r_p + r_q)^2 - |q|^2}{2p(r_p + r_q)},$$

The claim is that $r_q > r_t$. To show this suppose on the contrary that $r_q < r_t$.

<u>Note</u>: [We need not consider $r_q = r_t$ since we have assumed $\alpha_q > \theta_t$.]

By claim 1, $|q| > |t|$. Using the inequality,

$$\frac{p^2 + (r_p + r_t)^2 - |t|^2}{2p(r_p + r_t)} < \frac{p^2 + (r_p + r_q)^2 - |q|^2}{2p(r_p + r_q)},$$

we have,

$$(r_p+r_q)p^2 + (r_p+r_q)(r_p+r_t)^2 - (r_p+r_q)|t|^2$$

$$< (r_p+r_t)p^2 + (r_p+r_t)(r_p+r_q)^2 - (r_p+r_t)|q|^2 ,$$

$$\Rightarrow \quad (r_q-r_t)p^2 + (r_p+r_q)(r_p+r_t)(r_t-r_q)$$

$$< (r_p+r_q)|t|^2 - (r_p+r_t)|q|^2,$$

$$\Rightarrow \quad (r_q-r_t)p^2 + r_t|q|^2 - r_q|t|^2 + (r_p+r_q)(r_p+r_t)(r_t-r_q)$$

$$< r_p(|t|^2 - |q|^2) .$$

Since by assumption $|q| > |t|$,

$$\Rightarrow \quad (r_q-r_t)p^2 + r_t|q|^2 - r_q|t|^2 + (r_p+r_q)(r_p+r_t)(r_t-r_q)$$

$$< 0,$$

$$\Rightarrow \quad r_t|q|^2 + (r_p+r_q)(r_p+r_t)(r_t-r_q)$$

$$< (r_t-r_q)p^2 + r_q|t|^2.$$

Since all circles are properly spaced, we have,

$$|q|^2 = R^2 - 2Rr_q,$$
$$p^2 = R^2 - 2Rr_p,$$
$$|t|^2 = R^2 - 2Rr_t.$$

Substituting in these equations and simplifying,

$$r_p^2 r_t - r_p^2 r_q + r_p r_t^2 - r_p r_q^2 + r_q r_t^2 - r_t r_q^2$$
$$< 2Rr_p r_q - 2Rr_p r_t,$$

$$\Rightarrow \quad r_p^2(r_t - r_q) + r_p(r_t^2 - r_q^2) + r_q r_t(r_t - r_q)$$
$$< 2Rr_p(r_q - r_t).$$

Here we arrive at our contradiction since, if $r_q < r_t$ then the L.H.S is greater than zero, while the R.H.S is less than zero, which is impossible.

Therefore if $\alpha_q > \theta_t$, then $r_q > r_t$.

Furthermore the Theorem is proved since $\pi \geq \alpha_q > \theta_t \geq 0$, implies that the maximum and minimum radii of all circles tangent to C_p occur when $\alpha = \pi$, and $\theta = 0$, respectfully. Thus $r_{min} = r_a$, and $r_{max} = r_b$, are s.t. a,b lie on the diameter of C which contains p.

Proposition 4.2.6 Let $R > 0$. Let $p \in C$, and let C_p be the circle generated by p, having radius r_p. Then if we let $d = 2R$, $h = R + r_p + |p|$, and $k = R + r_p - |p|$, the radii of the minimum and maximum tangential circles to C_p, r_{min} and r_{max}, are given by,

$$r_{min} = \sqrt{dh} - h, \quad and \quad r_{max} = \sqrt{dk} - k.$$

Proof:

We first consider r_{min}, using Figure 4.5. For convenience and W.L.O.G. rotate p to lie along the horizontal axis and let $r_{min} = r_a$. Let $D_a = a$, $|p| = p$. So we have,

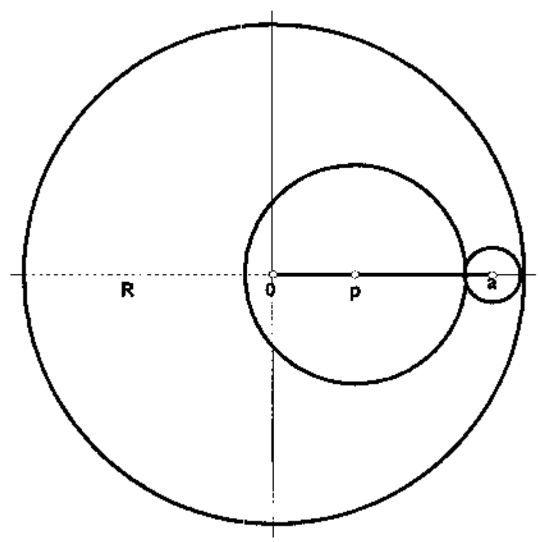

Figure 4.7

$$D_a = p + r_p + r_a.$$

Squaring both sides we see that,

$$D_a^2 = p^2 + 2pr_p + r_p^2 + 2pr_a + 2r_pr_a + r_a^2.$$

Since C_a is properly spaced we know that,

$$D_a^2 = R^2 - 2Rr_a.$$

Thus we have that,

$$R^2 - 2Rr_a = p^2 + 2pr_p + r_p^2 + 2pr_a + 2r_pr_a + r_a^2.$$

Rearranging we see that,

$$r_a^2 + (2R + 2r_p + 2p)\,r_a + p^2 + 2pr_p + r_p^2 - R^2 = 0.$$

Solving the quadratic for $\mathbf{r}_a$,

$$r_a = \frac{-2(R+r_p+p) + \sqrt{4(R+r_p+p)^2 - 4(p^2+2pr_p+r_p^2-R^2)}}{2}.$$

Simplifying we see that,

$$r_a = -(R+r_p+p) + \sqrt{2R(R+r_p+p)}$$

If we let d = 2R, and h = R + r_p + p , r_a = r_{min}, this equation reduces to,

$$r_{min} = \sqrt{dh} - h.$$

Now consider r_{max} as shown in figure 4.8.

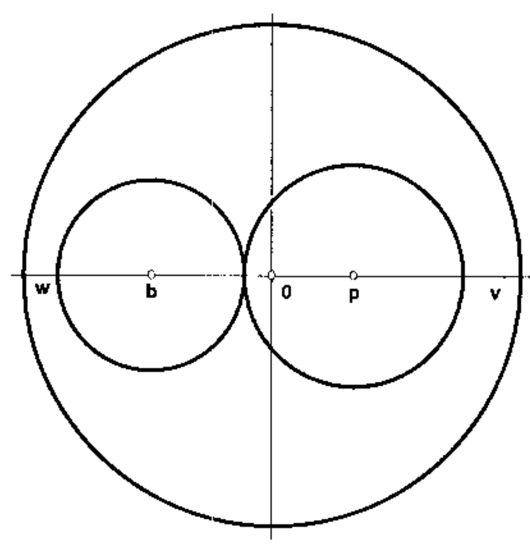

Figure 4.8

Let r_{max} = r_b, and let D_b = |b|. From this we can see,

$$p + r_p + v = R \quad \Rightarrow \quad v = R - r_p - p.$$

and,

$$w + r_b + |b| = R \quad \Rightarrow \quad w = R - r_b - |b|.$$

We also see that,

$$w + 2r_b + 2r_p + v = 2R.$$

Thus we may write,

$$[R - r_b - |b|] + 2r_b + 2r_p + [R - r_p - p] = 2R.$$

Simplifying we have,

$$|b| = r_b + r_p - p.$$

Squaring both sides,

$$b^2 = r_b^2 + 2r_p r_b + r_p^2 - 2pr_b - 2pr_p + p^2.$$

Since C_b is properly spaced,

$$b^2 = R^2 - 2Rr_b.$$

So we have,

$$r_b^2 + 2(R + r_p - p)\,r_b + (p^2 - 2pr_p - R^2) = 0$$

Solving the quadratic for r_b, then simplifying,

$$r_b = -(R + r_p - p) + \sqrt{2R\,(R + r_p - p)}$$

Again letting $d = 2R$, and $k = R + r_p - p$, and $r_b = r_{max}$, we see,

$$r_{max} = \sqrt{dk} - k.$$

Example 4.2.7 If T is an acute Isosceles triangle with sides Y and base X, show that $r_{min} = Y - h$. Where $h = R + D_c + r$.

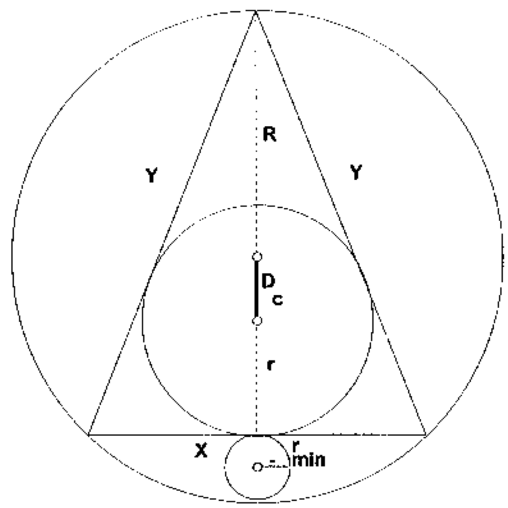

Figure 4.9

Solution: Forming the Right triangle as shown in Figure 4.9 we see,

(1) $$h^2 = Y^2 - \left(\frac{X}{2}\right)^2 \rightarrow h = \frac{1}{2}\sqrt{4Y^2 - X^2}.$$

The outer radius R, and r_{min} are given by,

(2) $$R = \frac{Y^2}{\sqrt{4Y^2 - X^2}}, \quad r_{min} = \sqrt{dh} - h.$$

Using (1) and (2), we have,

$$r_{\min} = \sqrt{\frac{2Y^2}{\sqrt{4Y^2 - X^2}} \cdot \frac{1}{2}\sqrt{4Y^2 - X^2}} - h$$

$$= \sqrt{Y^2} - h$$

$$= Y - h.$$

Example 4.2.8 Let $p \in C$. Let C_p be the circle generated by p. If the two tangent circles as shown in Figure 4.10 lie along the diameter of a circle of radius R, show the shaded area, A, can be expressed as,

$$A = \pi\left(10R^2 + 3r_p^2 - 2|p|^2 - 2\sqrt{d}\left[h^{3/2} + k^{3/2}\right]\right).$$

Where,

$p = |p|$, $d = 2R$,
$h = R + r_p + p$, and
$k = R + r_p - p$.

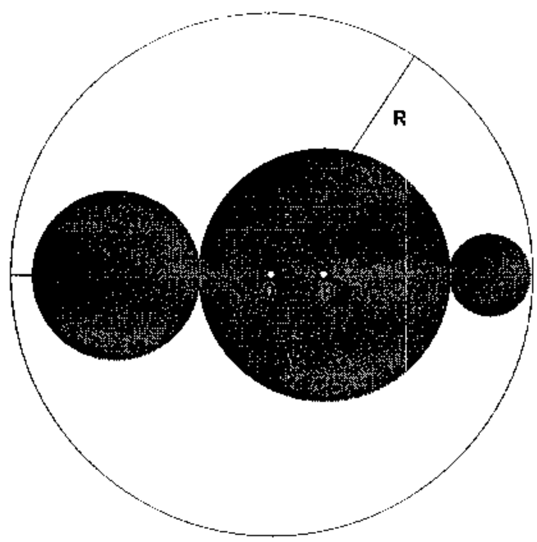

Figure 4.10

Solution:

Since all three tangent circles are centered along the diameter it must be the case that these circles are the minimum and maximum tangential circles to C_p.

By proposition 4.2.5, the radii of the minimum and maximum

By proposition 4.2.5, the radii of the minimum and maximum tangential circles are given by,

$$r_{\min} = \sqrt{dh} - h, \qquad r_{\max} = \sqrt{dk} - k.$$

The total area then is given by,

$$A = \pi(\sqrt{dk} - k)^2 + \pi(r_p)^2 + \pi(\sqrt{dk} - k)^2.$$

Combining and expanding each term we see,

$$= \pi(dk - 2k\sqrt{dk} + k^2 + r_p^2 + dh + 2h\sqrt{dh} + h^2).$$

Letting $h = R + r_p + p$, $k = R + r_p - p$, $d = 2R$,

$$A = \pi\Big(2R(R+r_p-p) - 2\sqrt{d}\,k^{3/2} + (R+r_p-p)^2 + r_p^2$$
$$+ 2R(R+r_p+p) - 2\sqrt{d}\,h^{3/2} + (R+r_p+p)^2\Big).$$

Simplifying yields our desired formula,

$$A = \pi\Big(10R^2 + 3r_p^2 - 2p^2 - 2\sqrt{d}\left[h^{3/2} + k^{3/2}\right]\Big).$$

CHAPTER 5

GENERATED AREAS

5.1 Introduction

Let $C = \{(x,y)|x^2 + y^2 < R^2\}$, let $p \in C$. Let C_p be the circle generated by p. Recall that C_p is *properly spaced* if and only if,

$$r_p = \frac{R^2 - |p|^2}{2R}.$$

Definition 5.1.1 Let $p \in C$, the area generated by p, denoted A(p), is given by the area of C_p.

We say that two points in C generate non-overlapping areas, when the following requirement is met.

Let $p = (p_1, p_2)$ and $q = (q_1, q_2)$ be two points in C. Then the distance from p to q is given by,

$$d = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}.$$

If p generates a circle of radius r_p, and q generates a circle of radius r_q, then the respective circles will *not* overlap if only if, $d \geq r_p + r_q$.

Thus if $S = \{ p_1, p_2, p_3, \ldots, p_n \}$ is any collection of points in C with non-overlapping areas then the total area, A(S) generated by S is,

$$A(S) = \sum_{i=1}^{n} \pi (r_{p_i})^2 .$$

5.2 The Diameter

In this section we evaluate the area generated by all points p, lying along a diameter of a circle, C, of radius R. See figure 5.1

For convenience we will choose the origin of the x-y plane to correspond to the center of C, and our diameter to lie along the x-axis from [-R, R].

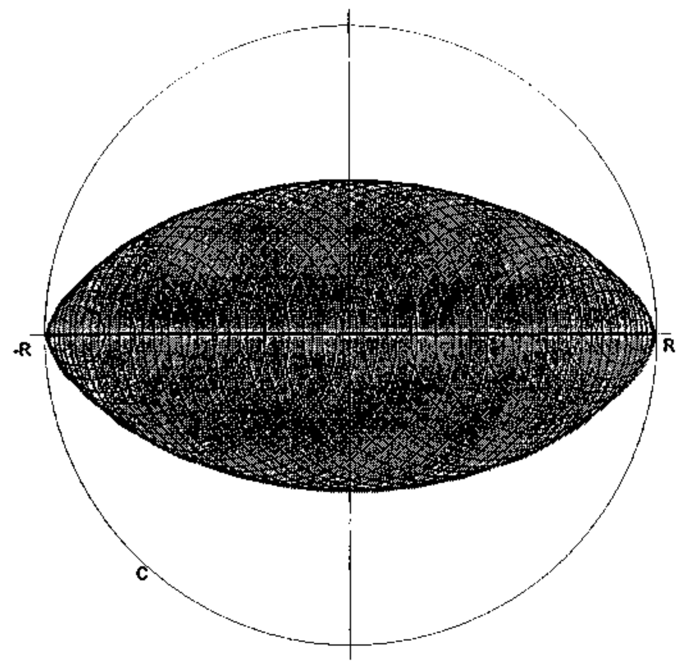

Figure 5.1

Note: The area is symmetric with respect to the four quadrants. This enables us to evaluate the area in say, quadrant I, then multiply this result by 4, to give us the total generated area of the diameter D, denoted A(D).

Restricting our attention to the first quadrant we see that the curve, f, is traced out by circles of decreasing radii as we go from 0 to R along the diameter, see Figure 5.2.

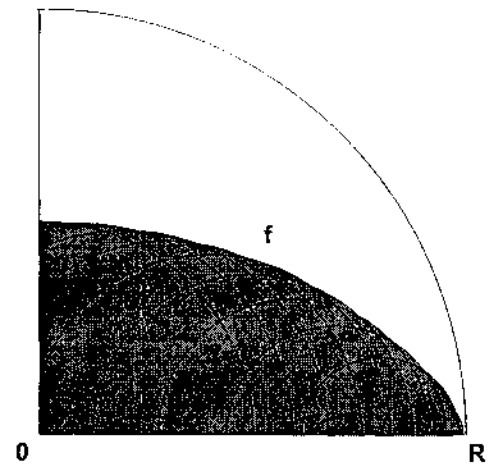

Figure 5.2

In order to evaluate this function in terms of area we must determine the height, h, of f at any point $x \in [0,R)$. Figure 5.3.

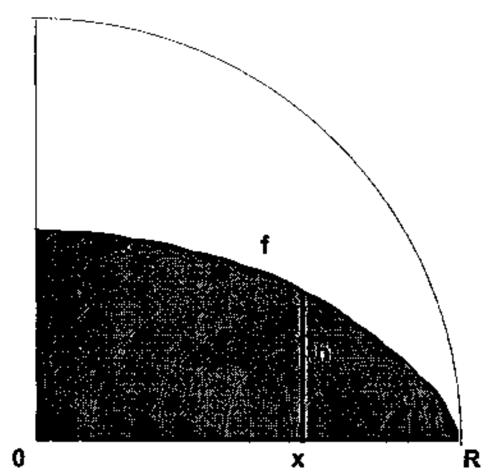

Figure 5.3

To find the h(x), we recall that f is composed of points which lie on circles whose centers lie along the positive x-axis. Therefore there exists a point $p \in [0,R)$, whose generated circle lies on f a distance h, from x. This is illustrated in Figure 5.4.

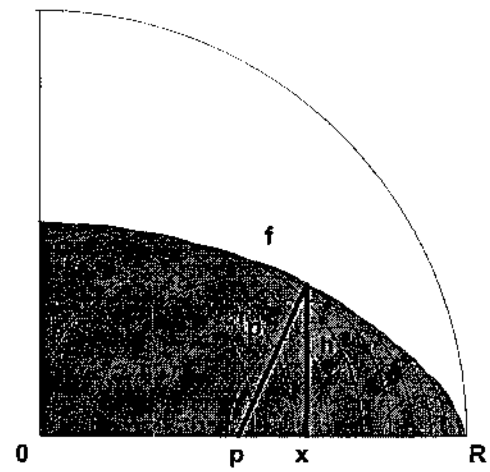

Figure 5.4

Recall that the radius of the circle with center at p, r_p is given by,

$$r_p = \frac{R^2 - p^2}{2R}.$$

So that for some point $p \in [0,R)$,

$$(x-p)^2 + h^2 = \left(\frac{R^2 - p^2}{2R}\right)^2.$$

Solving for h^2 we have,

$$h^2 = \left(\frac{R^2 - p^2}{2R}\right)^2 - (x-p)^2.$$

To maximize the height, h, we differentiate h with respect to p, then set the result equal to zero.

$$2h\frac{dh}{dp} = 2\left(\frac{R^2 - p^2}{2R}\right)\left(\frac{-2p}{2R}\right) - 2(x-p)(-1) = 0.$$

Note: For x ≠ 0, the maximum does not occur at the enpoints, since h'(x) > 0, and h'(x) < 0 if x > p. Thus the maximum of h occurs in (0, x).

Simplifying the right,

$$2h\frac{dh}{dp} = \frac{1}{R}(R^2 - p^2)(-p) + 2(x-p) = 0.$$

Dividing through by non-zero 2h,

$$\frac{1}{R}(R^2 - p^2)(-p) + 2(x-p) = 0.$$

Rearranging the right side shows,

$$p^3 - 3R^2p + 2R^2x = 0,$$

a cubic in p.

To solve this cubic, we note that since

$$\left(\frac{2R^2x}{2}\right)^2 + \left(\frac{-3R^2}{3}\right)^3 = R^4x^2 - R^6 < 0,$$

for all $0 \le x < R$, we have three distinct real solutions. Using Vieta's method of solving the *casus irreducibilis*,

$$\rho = \sqrt{\left(\frac{3R^2}{3}\right)^3} = R^3, \qquad \cos\varphi = \frac{\left(\frac{-2R^2x}{2}\right)}{\rho} = \frac{-R^2x}{R^3} = \frac{-x}{R}.$$

Thus,

$$\cos\varphi = \frac{-x}{R} \Rightarrow \varphi = \cos^{-1}\left(\frac{-x}{R}\right).$$

So the three possible solutions to the cubic equation are,

$$p_1 = 2R\cos\left(\frac{\varphi}{3}\right),$$

$$p_2 = 2R\cos\left(\frac{\varphi}{3} + \frac{2\pi}{3}\right),$$

$$p_3 = 2R\cos\left(\frac{\varphi}{3} + \frac{4\pi}{3}\right).$$

<u>Claim</u>: $p = p_3$.

Proof: Suppose on the contrary that $p = p_1$. For $0 \le x < R$, we have

$$-1 < -\frac{x}{R} \le 0$$

Since $\cos^{-1}(x)$ is decreasing on $(-1,0]$ we have,

$$\cos^{-1}0 \le \cos^{-1}(-\frac{x}{R}) < \cos^{-1}-1$$

$$(\#) \quad \Rightarrow \frac{1}{3}\cos^{-1}0 \le \frac{1}{3}\cos^{-1}(-\frac{x}{R}) < \frac{1}{3}\cos^{-1}-1$$

$$\Rightarrow \frac{\pi}{6} \le \frac{1}{3}\cos^{-1}(-\frac{x}{R}) < \frac{\pi}{3}$$

Since $\cos(x)$ is decreasing on $[0,\pi]$ we have,

$$\cos\frac{\pi}{3} < \cos(\frac{1}{3}\cos^{-1}(-\frac{x}{R})) \le \cos\frac{\pi}{6}$$

$$2R\cos\frac{\pi}{3} < 2R\cos(\frac{1}{3}\cos^{-1}(-\frac{x}{R})) \le 2R\cos\frac{\pi}{6}$$

$$R < p \le \sqrt{3}R$$

which is impossible since $p \in [0,R)$, so $p \ne p_1$.

Suppose $p = p_2$. Then by (#),

$$\frac{1}{3}\cos^{-1}0 + \frac{2\pi}{3} \le \frac{1}{3}\cos^{-1}(-\frac{x}{R}) + \frac{2\pi}{3} < \frac{1}{3}\cos^{-1}-1 + \frac{2\pi}{3}.$$

$$\Rightarrow \frac{5\pi}{6} \le \frac{1}{3}\cos^{-1}(-\frac{x}{R}) + \frac{2\pi}{3} < \pi.$$

On this interval cos(x) is decreasing, so

$$\Rightarrow \cos\pi < \cos(\frac{1}{3}\cos^{-1}(-\frac{x}{R}) + \frac{2\pi}{3}) \le \cos\frac{5\pi}{6}.$$

$$\Rightarrow 2R\cos\pi < 2R\cos(\frac{1}{3}\cos^{-1}(-\frac{x}{R}) + \frac{2\pi}{3}) \le 2R\cos\frac{5\pi}{6},$$

$$\Rightarrow -2R < p_2 \le -\sqrt{3}R.$$

Thus $p \neq p_2$.

The only possible solution then would be p_3.
From (#),

$$\frac{1}{3}\cos^{-1}0 + \frac{4\pi}{3} \le \frac{1}{3}\cos^{-1}(-\frac{x}{R}) + \frac{4\pi}{3} < \frac{1}{3}\cos^{-1}-1 + \frac{4\pi}{3},$$

$$\Rightarrow \frac{3\pi}{2} \le \frac{1}{3}\cos^{-1}(-\frac{x}{R}) + \frac{4\pi}{3} < \frac{5\pi}{3}.$$

Since cos(x) is increasing on $[\pi, 2\pi]$, we have,

$$\cos\frac{3\pi}{2} \leq \cos\left(\frac{1}{3}\cos^{-1}\left(-\frac{x}{R}\right) + \frac{4\pi}{3}\right) < \cos\frac{5\pi}{3},$$

$$0 \leq \cos\left(\frac{1}{3}\cos^{-1}\left(-\frac{x}{R}\right) + \frac{4\pi}{3}\right) < \frac{1}{2},$$

$$0 \leq 2R\cos\left(\frac{1}{3}\cos^{-1}\left(-\frac{x}{R}\right) + \frac{4\pi}{3}\right) < R,$$

$$0 \leq p_3 < R.$$

Thus, $p = p_3$ is the proper root, and so

$$\begin{aligned} p &= 2R\cos\left(\frac{\varphi}{3} + \frac{4\pi}{3}\right) \\ &= 2R\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{-x}{R}\right) + \frac{4\pi}{3}\right). \end{aligned}$$

The radius of the circle with center at p, r_p, is given by,

$$r_p = \left(\frac{R^2 - \left(4R^2\cos^2\left(\frac{1}{3}\cos^{-1}\left(\frac{-x}{R}\right) + \frac{4\pi}{3}\right)\right.}{2R}\right).$$

Therefore the height, h, of our curve at an arbitrary point x, is given by,

$$h(x) = \sqrt{\left(\frac{R^2-4R^2\cos^2\left(\frac{1}{3}\cos^{-1}\frac{-x}{R}+\frac{4\pi}{3}\right)}{2R}\right)^2-\left(x-2R\cos\left(\frac{1}{3}\cos^{-1}\frac{-x}{R}+\frac{4\pi}{3}\right)\right)^2}.$$

Thus the area under f is given by the definite integral,

$$A_1 = \int_0^R\sqrt{\left(\frac{R^2-4R^2\cos^2\left(\frac{1}{3}\cos^{-1}\frac{-x}{R}+\frac{4\pi}{3}\right)}{2R}\right)^2-\left(x-2R\cos\left(\frac{1}{3}\cos^{-1}\frac{-x}{R}+\frac{4\pi}{3}\right)\right)^2}\,dx$$

To solve this integral make the substitution,

$$\theta = \frac{1}{3}\cos^{-1}\left(\frac{-x}{R}\right) + \frac{4\pi}{3},$$

then,

$$d\theta = \frac{1}{3}\left(\frac{-1}{\sqrt{1-\left(\frac{-x}{R}\right)^2}}\right)\left(\frac{-1}{R}\right)dx = \frac{1}{3\sqrt{R^2 - x^2}}dx.$$

So,

$$dx = 3\sqrt{R^2 - x^2}\,d\theta.$$

Now,

$$\theta = \frac{1}{3}\cos^{-1}\left(\frac{-x}{R}\right) + \frac{4\pi}{3} \Rightarrow 3\theta = \cos^{-1}\left(\frac{-x}{R}\right) + 4\pi$$

$$\Rightarrow 3\theta - 4\pi = \cos^{-1}\left(\frac{-x}{R}\right)$$

$$\Rightarrow \cos(3\theta - 4\pi) = \frac{-x}{R}$$

$$\Rightarrow x = -R\cos 3\theta.$$

Thus

$$dx = 3\sqrt{R^2 - R^2\cos^2 3\theta}\,d\theta = 3R\sqrt{1-\cos^2 3\theta}\,d\theta = 3R\sin 3\theta\,d\theta.$$

Adjusting our limits of integration we see that,

$$\textit{If } x = R \textit{ then } \theta = \frac{5\pi}{3}.$$

$$\textit{If } x = 0 \textit{ then } \theta = \frac{3\pi}{2}.$$

So our integral becomes,

$$A_1 = \int_{\frac{3\pi}{2}}^{\frac{5\pi}{3}} \sqrt{\left(\frac{R - 4R\cos^2\theta}{2}\right)^2 - \left(-R\cos 3\theta - 2R\cos\theta\right)^2}\; 3R\sin 3\theta\, d\theta.$$

Simplifying we have,

$$A_1 = \frac{3R^2}{2}\int_{\frac{3\pi}{2}}^{\frac{5\pi}{3}} \sqrt{(1 - 4\cos^2\theta)^2 - 4(\cos 3\theta + 2\cos\theta)^2}\,\sin 3\theta\, d\theta$$

Using the trigonmetric identities:

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

and

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta,$$

Our integral can be writen as,

$$A_1 = \frac{3R^2}{2}\int_{\frac{3\pi}{2}}^{\frac{5\pi}{3}} \sqrt{(1 - 4\cos^2\theta)^2 - 4(4\cos^3\theta - \cos\theta)^2}\,(3\sin\theta - 4\sin^3\theta)\, d\theta$$

Simplifying we have,

$$A_1 = \frac{3R^2}{2}\int_{\frac{3\pi}{2}}^{\frac{5\pi}{3}} \sqrt{(1 - 4\cos^2\theta)^2(1 - 4\cos^2\theta)}\,(1 - 4\cos^2\theta)(-\sin\theta)\, d\theta$$

We can further simplify the integral as,

$$A_1 = \frac{3R^2}{2} \int_{\frac{3\pi}{2}}^{\frac{5\pi}{3}} \left(1 - 4\cos^2\theta\right)^{\frac{5}{2}} (-\sin\theta)\, d\theta$$

To solve we make the substitution $u = 2\cos\theta$. Adjusting the limits of integration,

$$If\ \theta = \frac{5\pi}{3}, \quad then\ u = 1,$$

and

$$if\ \theta = \frac{3\pi}{2}, \quad then\ u = 0.$$

Thus the integral can be represented by,

$$A_1 = \frac{3R^2}{2} \int_0^1 \left(1 - u^2\right)^{\frac{5}{2}} du\,.$$

Evaluating this integral,

$$A_1 = \frac{3R^2}{4}\left(\left[\frac{u}{6}(1-u^2)^{\frac{5}{2}} + \frac{5u}{24}(1-u^2)^{\frac{3}{2}} + \frac{5u}{16}\sqrt{(1-u^2)} + \frac{5}{16}\sin^{-1}u \right]\Bigg|_0^1 \right)$$

$$A_1 = \frac{3R^2}{4}\left(\left[0 + 0 + 0 + \frac{5\pi}{32} \right] - \left[0 + 0 + 0 + 0 \right] \right).$$

Finally multiplying this result by 4 to give us the total area, A(D), we have,

$$A(D) = \left(\frac{15}{32}\right)\pi R^2.$$

5.3 The Secant Line

A generalization of the previous diameter example leads us to examine the area generated by any secant line S, Figure 5.5 to a circle C, with a fixed radius R. We state the following as a proposistion.

Proposition 5.3.1 Let C be a circle of radius R, and let S be a secant line to C, having length λ. Then the area generated by S, is given by,

$$A(S) = \frac{R^2}{16}\left[\alpha\sqrt{1-\alpha^2}\,(14\alpha^2 + 3\alpha - 2) + (8\alpha^4 + 8\alpha^2 - 1)\sin^{-1}\alpha\right],$$

$$\alpha = \frac{\lambda}{2R}, \quad 0<\alpha<1.$$

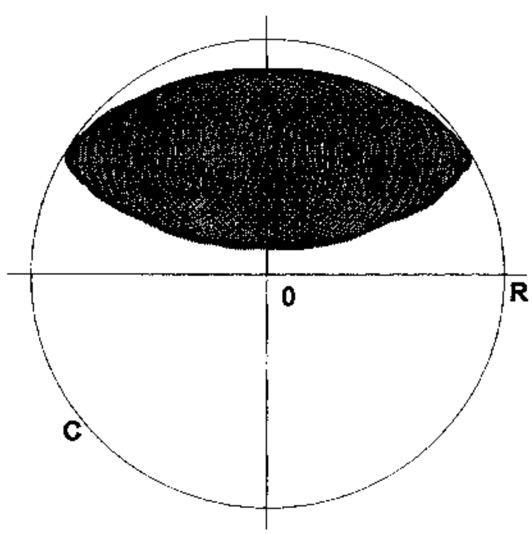

Figure 5.5

Introduction to proof:

The proof of proposition 5.2 closely parallels that of the diameter example in section 3, familiarization with that example is therefore useful. The main difference between the two is that instead of integrating along the x-axis from 0 to R, we will be integrating along a shifted x-axis from 0' to w, where w is half the length of S. A second notable difference is the lack of reducibility in the solution of the cubic equation, which is necessary to determine the height h, of our curve f, at an arbitrary point x lying along S.

Preliminary results:

Let C be a circle of radius R, with it's center at the origin, and let S be a secant line to C having length λ as shown in Figure 5.6.

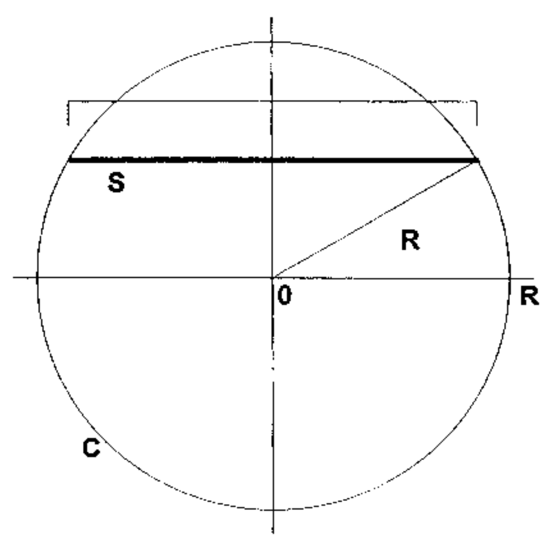

Figure 5.6

Let D_o be the minimum distance from O to S, let $\lambda = 2w$. See Figure 5.7, so we have,

(1) $$w^2 + D_o^2 = R^2$$

$$\mapsto \quad w^2 = R^2 - D_o^2.$$

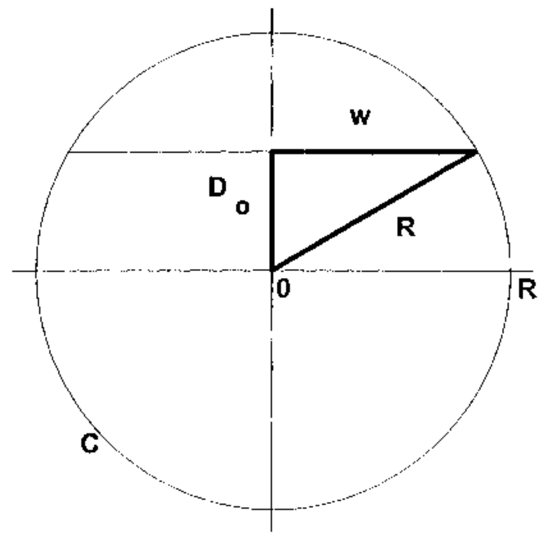

Figure 5.7

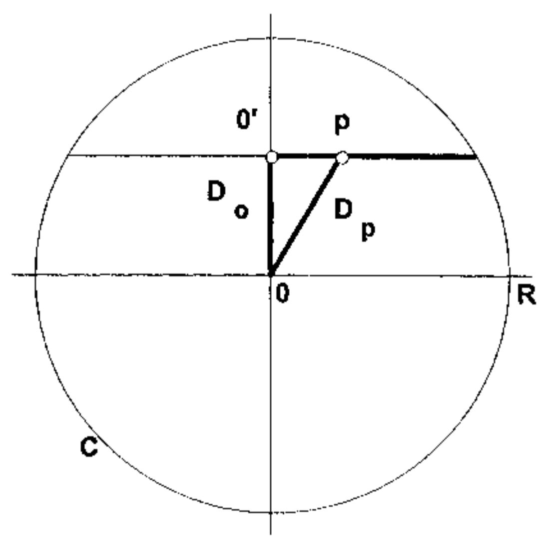

Figure 5.8

Next, let p be an arbitrary point on S, see Figure 5.8. Then forming a second right triangle we see that,

(2) $$D_o^2 + |0'-p|^2 = D_p^2.$$

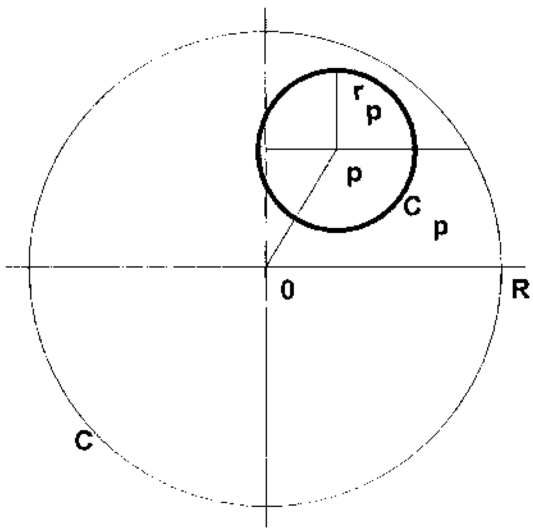

Figure 5.9

Let C_p be the circle generated by p. If D_p represents the Center distance between C and C_p, see figure 5.9 , then the radius of C_p, denoted r_p, is given by,

$$r_p = \frac{R^2 - D_p^2}{2R}.$$

Using (2) we have,

$$r_p = \frac{R^2 - (D_o^2 + |0' - p|^2)}{2R} = \frac{(R^2 - D_o^2) - |0' - p|^2}{2R}.$$

Using (1), we see r_p can be written as,

$$(3) \quad r_p = \frac{w^2 - |0' - p|^2}{2R}.$$

With these preliminary results stated, we are ready to prove Proposition 5.2.

Proof:

Let S be a secant line to a circle, C, of radius R. Rotate S parallel to the x-axis, with the center of C placed at the origin of the x-y plane as shown in Figure 5.10. Figure 5.11 shows the area generated by all properly spaced circles whose centers lie along S.

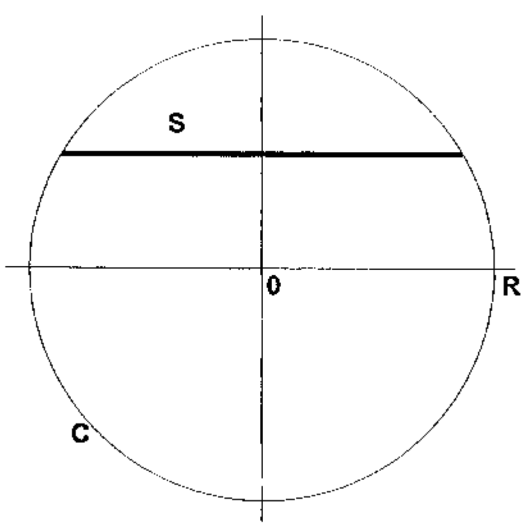

Figure 5.10

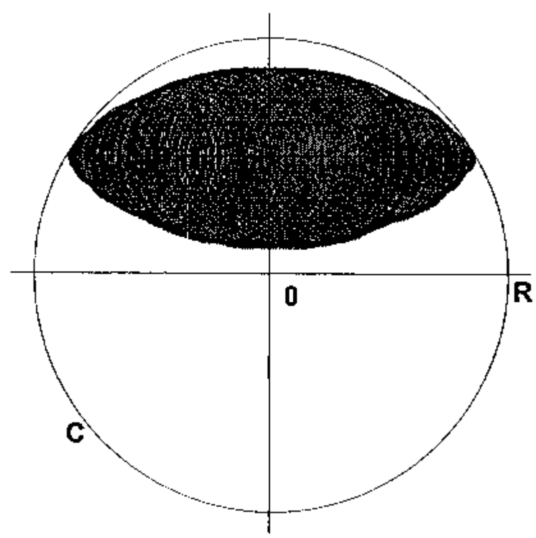

Figure 5.11

Due to the symmetry of the area generated the secant line, we will again be evaluating only the top right area, shown in Figure 5.12 then multiplying this by 4 to give us the total generated area, A(S).

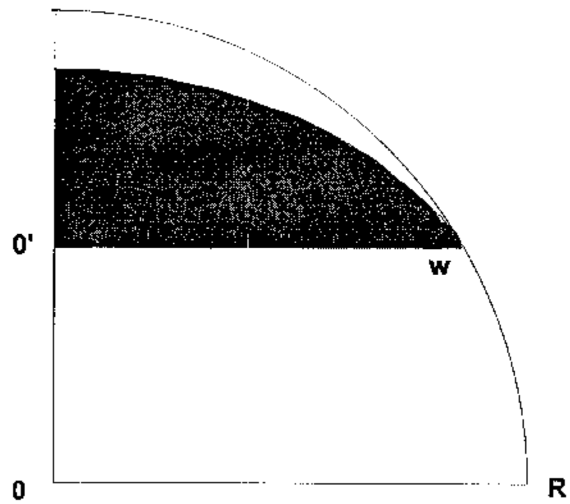

Figure 5.12

Let x be an arbitrary point along S. Here we will restrict x to lie in the first quadrant. Let f be the curve traced out by all circles with centers along w, as shown in Figure 5.13. Let h be the height above x, that is h = f(x) - 0'.

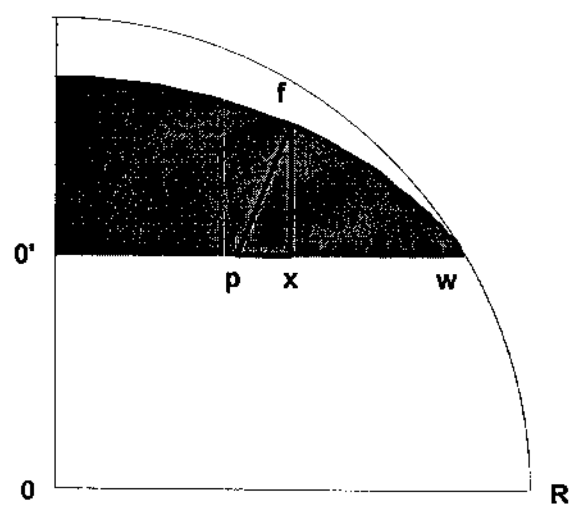

Figure 5.13

Then for some p on S, we have,

$$(4)\quad h^2 + |x-p|^2 = r_p^{\,2}$$

$$\mapsto \quad h = \sqrt{r_p^{\,2} - |x-p|^2}$$

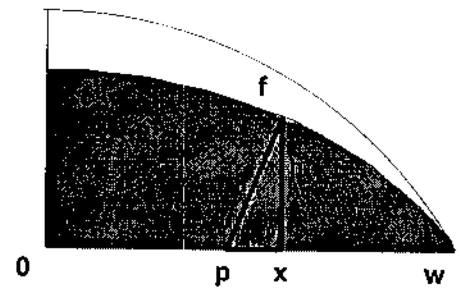

Figure 5.14

For convenience let 0' = 0. This will allow us to integrate f from 0 to w. This is illustrated in Figure 5.14, Also now we have,

$$|x-p| = (x-p)\,.$$

Using (3) and (4), h may be expressed as,

$$5)\qquad h^2 = \left(\frac{w^2 - p^2}{2R}\right)^2 - (x-p)^2\,.$$

To maximize h, we differentiate (5) with respect to p, then set the result equal to zero,

$$2h\frac{dh}{dp} = 2\left(\frac{w^2 - p^2}{2R}\right)\left(\frac{-2p}{2R}\right) - 2(x-p)(-1) = 0\,.$$

The left side being equal to zero only if,

$$p^3 - (w^2 + 2R^2)p + 2Rx = 0.$$

This cubic in p is found to be *casus irreducibilis* since,

$$\left(\frac{2R^2x}{2}\right)^2 + \left(\frac{-(w^2+2R^2)}{3}\right)^3 = R^4x^2 - \frac{1}{27}(w^2+2R^2)^3 < 0, \quad \forall x \in [0, w).$$

We therefore have three distinct real solutions.
So again we use Vieta's method for solving the casus irreducibilis,

$$\rho = \sqrt{\left(\frac{w^2 + 2R^2}{3}\right)^3}, \quad and \quad \cos\varphi = \frac{\left(\frac{-2R^2x}{2}\right)}{\rho}$$

$$\Rightarrow \quad \cos\varphi = \left(\frac{-R^2x}{\sqrt{\left(\frac{w^2 + 2R^2}{3}\right)^3}}\right).$$

Thus,

$$\varphi = \cos^{-1}\left(\frac{-R^2 x}{\sqrt{\left(\frac{w^2 + 2R^2}{3}\right)^3}}\right).$$

The three possible solutions to the cubic are,

$$p_1 = 2\rho^{\frac{1}{3}} \cos \frac{\varphi}{3},$$

$$p_2 = 2\rho^{\frac{1}{3}} \cos \left(\frac{\varphi}{3} + \frac{2\pi}{3}\right),$$

$$p_3 = 2\rho^{\frac{1}{3}} \cos \left(\frac{\varphi}{3} + \frac{4\pi}{3}\right).$$

A proof similiar to the one found in our diameter example shows the proper root to be p_3, thus $p = p_3$ and we have,

$$p = 2\sqrt{\left(\frac{w^2 + 2R^2}{3}\right)} \cos\left(\frac{1}{3}\cos^{-1}\left(\frac{-R^2 x}{\sqrt{\left(\frac{w^2 + 2R^2}{3}\right)^3}}\right) + \frac{4\pi}{3}\right).$$

Define h(x) = f(x). Then our area, A_1, is given by the definite integral,

$$A_1 = \int_0^w f(x)\,dx.$$

Substituting in for p, we have,

$$A_1 = \int_0^w \left[\left(\frac{w^2 - 4\left(\frac{w^2+2R^2}{3}\right)\cos^2\left(\frac{1}{3}\cos^{-1}\left(\frac{-R^2x}{\sqrt{\left(\frac{w^2+2R^2}{3}\right)^3}}\right)+\frac{4\pi}{3}\right)}{2R} \right)^2 - \left(x - 2\sqrt{\left(\frac{w^2+2R^2}{3}\right)}\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{-R^2x}{\sqrt{\left(\frac{w^2+2R^2}{3}\right)^3}}\right)+\frac{4\pi}{3}\right)\right)^2 \right]^{\frac{1}{2}} dx.$$

Letting w = αR, for some 0 < α < 1.

$$A_1 = \int_0^{\alpha R} \left[\left(\frac{\alpha^2 R^2 - 4R^2\left(\frac{\alpha^2+2}{3}\right)\cos^2\left(\frac{1}{3}\cos^{-1}\left(\frac{-x}{R\sqrt{\left(\frac{\alpha^2+2}{3}\right)^3}}\right)+\frac{4\pi}{3}\right)}{2R} \right)^2 \right.$$

$$\left. - \left(x - 2R\sqrt{\left(\frac{\alpha^2+2}{3}\right)}\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{-x}{R\sqrt{\left(\frac{\alpha^2+2}{3}\right)^3}}\right)+\frac{4\pi}{3}\right)\right)^2 \right]^{\frac{1}{2}} dx.$$

By letting,

$$k = \sqrt{\frac{\alpha^2 + 2}{3}},$$

our integral becomes,

$$A_1 = \int_0^{\alpha R} \left[\frac{R^2}{4}\left(\alpha^2 - 4k^2 \cos^2\left(\frac{1}{3}\cos^{-1}\left(\frac{-x}{Rk^3}\right)+\frac{4\pi}{3}\right)\right)^2 \right.$$

$$\left. - \left(x - 2Rk\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{-x}{Rk^3}\right)+\frac{4\pi}{3}\right)\right)^2 \right]^{\frac{1}{2}} dx.$$

To solve, make the substitution,

$$\theta = \frac{1}{3}\cos^{-1}\left(\frac{-x}{Rk^3}\right) + \frac{4\pi}{3}.$$

Then we see,

$$x = -Rk^3\cos 3\theta \quad \Rightarrow \quad dx = 3Rk^3\sin 3\theta\, d\theta.$$

Adjusting our limits of integration,

$$\textit{If} \;\; x = 0, \quad \textit{then} \;\; \theta = \frac{3\pi}{2}.$$

$$\textit{If} \;\; x = \alpha R, \quad \textit{then} \;\; \theta = \frac{1}{3}\cos^{-1}\left(\frac{-\alpha}{k^3}\right) + \frac{4\pi}{3}.$$

Our integral becomes,

$$A_1 = \int_{\frac{3\pi}{2}}^{\frac{1}{3}\cos^{-1}\left(\frac{-\alpha}{k^3}\right)+\frac{4\pi}{3}} \left[\sqrt{\frac{R^2}{4}\left(\alpha^2 - 4k^2\cos^2\theta\right)^2 - \left(-Rk^3\cos 3\theta - 2Rk\cos\theta\right)^2}\right.$$

$$\left.\left(3Rk^3\sin 3\theta\right)\right]\, d\theta.$$

Simplifying under the radical we have,

$$A_1 = \frac{3R^2}{2} \int_{\frac{3\pi}{2}}^{\frac{1}{3}\cos^{-1}\left(\frac{-\alpha}{k^3}\right)+\frac{4\pi}{3}} \left[\sqrt{(\alpha^2 - 4k^2\cos^2\theta)^2 - 4k^2\,(k^2\cos 3\theta + 2\cos\theta)^2}\right.$$

$$\left.(k^3 \sin 3\theta)\right] d\theta.$$

Using the trignometric identities,

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

and

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$= (1 - 4\cos^2\theta)(-\sin\theta),$$

we have,

$$A_1 = \frac{3R^2}{2} \int_{\frac{3\pi}{2}}^{\frac{1}{3}\cos^{-1}\left(\frac{-\alpha}{k^3}\right)+\frac{4\pi}{3}} \left[\sqrt{(\alpha^2 - 4k^2\cos^2\theta)^2 - 4k^2\,(4k^2\cos^3\theta - 3k^2\cos\theta + 2\cos\theta)^2}\right.$$

$$\left.(k^2 - 4k^2\cos^2\theta)(-k\sin\theta)\right] d\theta.$$

Simplifying,

$$A_1 = \frac{3R_o^2}{2} \int_{\frac{3\pi}{2}}^{\frac{1}{3}\cos^{-1}\left(\frac{-\alpha}{k^3}\right)+\frac{4\pi}{3}} \left[\sqrt{(\alpha^2-4k^2\cos^2\theta)^2-4k^2(4k^2\cos^3\theta-\cos\theta(3k^2-2))^2}\right.$$

$$\left.(k^2-4k^2\cos^2\theta)(-k\sin\theta)\right] d\theta.$$

Letting $3k^2 - 2 = \alpha^2$ under the radical, then simplifying,

$$A_1 = \frac{3R^2}{2} \int_{\frac{3\pi}{2}}^{\frac{1}{3}\cos^{-1}\left(\frac{-\alpha}{k^3}\right)+\frac{4\pi}{3}} \left[\sqrt{(\alpha^2-4k^2\cos^2\theta)^2-4k^2\cos^2\theta(\alpha^2-4k^2\cos^2\theta)^2}\right.$$

$$\left.(k^2-4k^2\cos^2\theta)(-k\sin\theta)\right] d\theta.$$

Simplifying,

$$A_1 = \frac{3R^2}{2} \int_{\frac{3\pi}{2}}^{\frac{1}{3}\cos^{-1}\left(\frac{-\alpha}{k^3}\right)+\frac{4\pi}{3}} \left[\sqrt{(\alpha^2-4k^2\cos^2\theta)^2(1-4k^2\cos^2\theta)}\right.$$

$$\left.(k^2-4k^2\cos^2\theta)(-k\sin\theta)\right] d\theta.$$

Making the substitution,

$$u = 2k\cos\theta \Rightarrow du = -2k\sin\theta \, d\theta,$$

and adjusting our limits of integration,

$$\textit{If } \theta = \frac{3\pi}{2}, \quad \textit{then } u = 0.$$

$$\textit{If } \theta = \frac{1}{3}\cos^{-1}\left(\frac{-\alpha}{k^3}\right)+\frac{4\pi}{3}, \quad \textit{then } u = \alpha.^{*}$$

* [The proof that this equality holds for all α, such that $0 < \alpha < 1$ can be found in appendix 1, on page 88.]

So we have,

$$A_1 = \frac{3R^2}{4}\int_0^{\alpha}\sqrt{(\alpha^2-u^2)^2(1-u^2)}\,(k^2-u^2)\;du.$$

Simplifying,

$$A_1 = \frac{3R^2}{4}\int_0^{\alpha}(\alpha^2-u^2)\,(k^2-u^2)\sqrt{(1-u^2)}\;du.$$

Upon expanding we see,

$$A_1 = \frac{3R^2}{4}\left[\alpha^2 k^2 \int_0^{\alpha} \sqrt{(1-u^2)}\,du - (\alpha^2+k^2)\int_0^{\alpha} u^2\sqrt{(1-u^2)}\,du + \int_0^{\alpha} u^4\sqrt{(1-u^2)}\,du\right].$$

Solving each integral we have,

$$A_1 = \frac{3R^2}{4}\left[\frac{\alpha^2 k^2}{2}\left(u\sqrt{1-u^2} + \sin^{-1} u \Big|_0^{\alpha}\right)\right.$$

$$- \frac{\alpha^2+k^2}{8}\left(u\sqrt{1-u^2} - 2u\sqrt{(1-u^2)^3} + \sin^{-1} u \Big|_0^{\alpha}\right)$$

$$\left. + \left(\frac{u}{16}\sqrt{1-u^2} - u\left(\frac{u^2}{6} + \frac{1}{8}\right)\sqrt{(1-u^2)^3} + \frac{1}{16}\sin^{-1} u \Big|_0^{\alpha}\right)\right].$$

Evaluating each integral solution from 0 to α, then combining like terms we have,

$$A_1 = \frac{3R^2}{4}\left[\alpha\sqrt{1-\alpha^2}\left(\frac{\alpha^2 k^2}{2} - \frac{1}{8}(\alpha^2+k^2) + \frac{\alpha}{16}\right)\right.$$

$$+ \alpha\sqrt{(1-\alpha^2)^3}\left(\frac{1}{4}(\alpha^2+k^2) - \frac{1}{24}(4\alpha^2+3)\right)\Bigg)$$

$$\left. + \left(\frac{\alpha^2 k^2}{2} - \frac{1}{8}(\alpha^2+k^2) + \frac{1}{16}\right)\sin^{-1}\alpha\right].$$

By letting,

$$k^2 = \frac{\alpha^2 + 2}{3},$$

we have,

$$A_1 = \frac{3R^2}{4}\left[\alpha\sqrt{1-\alpha^2}\left(\frac{\alpha^2}{2}\left(\frac{\alpha^2+2}{3}\right) - \frac{1}{8}\left(\alpha^2+\left(\frac{\alpha^2+2}{3}\right)\right) + \frac{\alpha}{16}\right)\right.$$

$$+ \sqrt{(1-\alpha^2)^3}\left(\frac{1}{4}\left(\alpha^2+\left(\frac{\alpha^2+2}{3}\right)\right) - \frac{1}{24}(4\alpha^2+3)\right)\Bigg)$$

$$\left. + \left(\frac{\alpha^2}{2}\left(\frac{\alpha^2+2}{3}\right) - \frac{1}{8}\left(\alpha^2+\left(\frac{\alpha^2+2}{3}\right)\right) + \frac{1}{16}\right)\sin^{-1}\alpha\right].$$

Simplifying and multiplying by 4 yields the desired result,

$$A(S) = \frac{R^2}{16}\left[\alpha\sqrt{1-\alpha^2}\,(14\alpha^2 + 3\alpha - 2) + (8\alpha^4 + 8\alpha^2 - 1)\sin^{-1}\alpha\right].$$

Where,

$$\alpha = \frac{\lambda}{2R}.$$

APPENDIX

APPENDIX 1

If $0 < \alpha < 1$, then

$$\alpha = 2\sqrt{\frac{\alpha^2+2}{3}}\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{-\alpha}{\left(\frac{\alpha^2+2}{3}\right)^{\frac{3}{2}}}\right) + \frac{4\pi}{3}\right).$$

<u>Proof</u>:

We begin with the following equality,

$$\frac{3}{\alpha^2 + 8} = \frac{3}{\alpha^2 + 8} .$$

Multiplying top and bottom of the left by (α^2+2), then multiplying top and bottom of the right by $(1-\alpha^2)^2$, then rearranging the right.

$$\frac{3(\alpha^2+2)}{(\alpha^2+2)(\alpha^2+8)} = \frac{3(1-\alpha^2)^2}{(\alpha^2+2)^3 - 27\alpha^2} .$$

Rewriting both sides we have,

$$\frac{1}{\left(\frac{\alpha^2+2}{3}\right)\left(\frac{\alpha^2+8}{\alpha^2+2}\right)} = \frac{1}{9}\frac{(1-\alpha^2)^2}{\left(\frac{\alpha^2+2}{3}\right)^3 - \alpha^2} .$$

Dividing both sides by

$$\left(\frac{\alpha^2+2}{3}\right)^2,$$

then taking the square roots of both sides,

$$\frac{1}{\left(\frac{\alpha^2+2}{3}\right)^{\frac{3}{2}}\sqrt{\frac{\alpha^2+8}{\alpha^2+2}}} = \frac{1}{3}\frac{(1-\alpha^2)}{\left(\frac{\alpha^2+2}{3}\right)\sqrt{\left(\frac{\alpha^2+2}{3}\right)^3 - \alpha^2}} .$$

Multiplying both sides by 2/3, and top and bottom of the right by

$$\left(\frac{\alpha^2+2}{3}\right)^2,$$

we have,

$$\frac{2}{\sqrt{\frac{\alpha^2+8}{\alpha^2+2}}}\cdot\frac{1}{3\left(\frac{\alpha^2+2}{3}\right)^{\frac{3}{2}}} = \frac{2}{9}\frac{\left(\frac{\alpha^2+2}{3}\right)^2(1-\alpha^2)}{\left(\frac{\alpha^2+2}{3}\right)^3\sqrt{\left(\frac{\alpha^2+2}{3}\right)^3-\alpha^2}} .$$

Rewriting both sides,

$$\frac{1}{\sqrt{\dfrac{4\alpha^2+8-3\alpha^2}{4\,(\alpha^2+2)}}}\cdot\frac{\left(\dfrac{\alpha^2+2}{3}\right)^{-\frac{1}{2}}}{3\left(\dfrac{\alpha^2+2}{3}\right)}$$

$$=\frac{1}{3}\,\frac{\left(\dfrac{\alpha^2+2}{3}\right)^{\frac{3}{2}}}{\sqrt{\left(\dfrac{\alpha^2+2}{3}\right)^3-\alpha^2}}\cdot\frac{\left(\dfrac{\alpha^2+2}{3}\right)^{\frac{1}{2}}\cdot\dfrac{2}{3}\,(1-\alpha^2)}{\left(\dfrac{\alpha^2+2}{3}\right)^3}.$$

Further manipulation yields,

$$\frac{1}{\sqrt{\dfrac{4\,(\alpha^2+2)-3\alpha^2}{4\,(\alpha^2+2)}}}\cdot\frac{\left(\dfrac{\alpha^2+2}{3}\right)^{-\frac{1}{2}}(\frac{2}{3})}{2\left(\dfrac{\alpha^2+2}{3}\right)}$$

$$=\frac{1}{3}\cdot\frac{1}{\sqrt{\dfrac{\left(\dfrac{\alpha^2+2}{3}\right)^3-\alpha^2}{\left(\dfrac{\alpha^2+2}{3}\right)^3}}}\cdot\frac{\left(\dfrac{\alpha^2+2}{3}\right)^{\frac{1}{2}}\cdot\left(\dfrac{\alpha^2+2}{3}-\alpha^2\right)}{\left(\dfrac{\alpha^2+2}{3}\right)^3}.$$

This is equivalent to,

$$\frac{d}{d\alpha}\sin^{-1}\left(\frac{\alpha}{2\left(\frac{\alpha^2+2}{3}\right)^{\frac{1}{2}}}\right) = \frac{1}{3}\frac{d}{d\alpha}\sin^{-1}\left(\frac{\alpha}{\left(\frac{\alpha^2+2}{3}\right)^{\frac{3}{2}}}\right).$$

Taking the integrals of both sides,

$$\sin^{-1}\left(\frac{\alpha}{2\left(\frac{\alpha^2+2}{3}\right)^{\frac{1}{2}}}\right) = \frac{1}{3}\sin^{-1}\left(\frac{\alpha}{\left(\frac{\alpha^2+2}{3}\right)^{\frac{3}{2}}}\right).$$

Taking the sine of both sides and using the fact that,

$$\cos\frac{3\pi}{2} = 0, \quad \textit{and} \quad \sin\frac{3\pi}{2} = -1,$$

we may write,

$$\frac{\alpha}{2\sqrt{\frac{\alpha^2+2}{3}}} = \cos\frac{3\pi}{2}\cos\left(\frac{1}{3}\sin^{-1}\left(\frac{\alpha}{\left(\frac{\alpha^2+2}{3}\right)^{\frac{3}{2}}}\right)\right)$$

$$- \sin\frac{3\pi}{2}\sin\left(\frac{1}{3}\sin^{-1}\left(\frac{\alpha}{\left(\frac{\alpha^2+2}{3}\right)^{\frac{3}{2}}}\right)\right).$$

Using the difference of angles formula for the cosine, we have,

$$\frac{\alpha}{2\sqrt{\dfrac{\alpha^2+2}{3}}} = \cos\left(\frac{3\pi}{2} + \frac{1}{3}\sin^{-1}\left(\frac{\alpha}{\left(\dfrac{\alpha^2+2}{3}\right)^{\frac{3}{2}}}\right)\right).$$

Using the trigonometric identity,

$$\sin^{-1}(\theta) = \cos^{-1}(-\theta) - \frac{\pi}{2},$$

we see that,

$$\frac{\alpha}{2\sqrt{\dfrac{\alpha^2+2}{3}}} = \cos\left(\frac{3\pi}{2} + \frac{1}{3}\left(\cos^{-1}\left(\frac{-\alpha}{\left(\dfrac{\alpha^2+2}{3}\right)^{\frac{3}{2}}}\right) - \frac{\pi}{2}\right)\right).$$

Rewriting yields our desired result,

$$\alpha = 2\sqrt{\frac{\alpha^2+2}{3}}\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{-\alpha}{\left(\dfrac{\alpha^2+2}{3}\right)^{\frac{3}{2}}}\right) + \frac{4\pi}{3}\right).$$

BIBLIOGRAPHY

[1] Euclid, with translation by Sir Thomas L. Heath, *The Elements*, Dover, New York, New York, 1956.

[2] Ross Honsberger, Mathematical Gems II, Mathematical Association of America, United States, 1976.