CYCLES AND CLIQUES IN STEINHAUS GRAPHS

DISSERTATION

Presented to the Graduate Council of the

University of North Texas in Partial

Fulfillment of the Requirements

For the Degree of

DOCTOR OF PHILOSOPHY

By

Daekeun Lim, B.S., M.S.

Denton, Texas

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In this dissertation several results in Steinhaus graphs are investigated. First under some further conditions imposed on the induced cycles in steinhaus graphs, the order of induced cycles in Steinhaus graphs is at most $\lfloor \frac{n+3}{2} \rfloor$. Next the results of maximum clique size in Steinhaus graphs are used to enumerate the Steinhaus graphs having maximal cliques. Finally the concept of jumbled graphs and Posa's Lemma are used to show that almost all Steinhaus graphs are Hamiltonian.

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CHAPTER 1

INTRODUCTION

How many edges must a graph of order n have if it is required to contain a path of length l? A cycle of length at least l? A complete graph K^l of order l? These questions are special cases of the so called forbidden subgraph problem: given a graph F, determine $\text{ex}(n; F)$, the maximal number of edges in a graph of order n not containing F. The forbidden subgraph problem is a prime example of the rather large family of extremal problems in graph theory.

Therefore, we may ask the following question which is quite a different approach by comparing the forbidden subgraph problem: for a given class of graphs of order n with a certain graph parameter, say the number of edges or the maximum degree, is at most some number f, then can we find a largest number r such that at least one graph in the given class contains a complete graph K^r of order r? a path of length r? a cycle of length r? etc. Thus these problems also could be considered examples of extremal problems which are quite different from forbidden subgraph problems.

A Steinhaus graph is constructed by a certain rule which is called the Steinhaus property. Unlike many classes of graphs, one can not so easily specify the number of edges in the Steinhaus graphs since the edges are determined by the Steinhaus property. In [Ha], Harborth finds the upper bound for the number of edges in a Steinhaus graph. By aid of this upper bound, people have been investigating the above problems in the class of Steinhaus graphs. For example, the upper bound for the order of cliques in Steinhaus graphs, the upper bound for diameter of Steinhaus graphs etc. In Chapter 2, we define Steinhaus graphs and present recent results in Steinhaus graph theory. In Chapter 3, we define a simple induced cycle in Steinhaus graphs and ask the above problem about the upper bound for the order of simple

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induced cycles in Steinhaus graphs. In [BrDeDu], Brigham, Deo and Dutton find the upper bound for the order of cliques in Steinhaus graphs. From this result, in Chapter 4 we give the number of Steinhaus graphs which have a clique of maximal order and list them. Moreover, we present more general results and discuss the same problems in the complements of Steinhaus graphs.

An important application of probability to mathematics occurs in the theory of random graphs. One of the basic questions in random graph theory is to determine the asymptotic proportion of graphs possessing a given property. Though the properties of random Steinhaus graphs have been investigeted by Brand and other authors, few results have been known by using the standard methods in random graph theory because Steinhaus graphs have complicated structure. But the recent work of Brand and Jackson [BrJa] gives rich results in the random Steinhaus graph theory. They show that the theory of random Steinhaus graphs is first order complete and identical with the first order theory of random graphs. There are many important properties possessed by almost all Steinhaus graphs which can not be expressed by a first order sentence for examples, cycles and clique numbers etc. In Chapter 2, we define quasi-random graphs, pseudo-random graphs and generalized Steinhaus graphs. In Chapter 5, we present the results of Brand, Thomason and other authors, and prove that almost all Steinhaus graphs are Hamiltonian.

CHAPTER 2

PRELIMINARIES

2.1 Introduction

The purpose of this chapter is to introduce basic concepts and results of graph theory (see [Be], [B2] and [Go]). In section 2.2 we give some basic concepts of graphs. In section 2.3 we define Steinhaus graphs and generalized Steinhaus graphs and give some results of the Steinhaus graph theory. Finally, in section 2.4 we state the two basic models of the theory of random graphs and state well known results in the random graph theory and define some pseudo-random graphs.

2.2 Basic concepts of graphs

Intuitively speaking, a graph is a set of points and a set of line segments joining some of the points. Formally, a *graph* G is an ordered pair of disjoint sets (V, E) such that E is a subset of unordered pairs of V. The set V is the set of vertices and E the set of edges. For the sake of convenience in most cases, we consider a graph with n vertices and take $V = \{1, 2, \cdots, n\}$ to be the vertex set. An *edge* $\{x, y\}$ is said to join the vertices x and y and is denoted by xy. If $xy \in E$, then x and y are adjacent and the vertices x and y are incident with the edge xy. As the terminology suggests, we do not usually think of a graph as an ordered pair, but as a collection of vertices some of which are joined by edges. If x is a vertex of a graph $G = (V, E)$, then instead of $x \in V$ we usually write $x \in G$. The *order* of G is the number of vertices; it is denoted by $|G|$. The same notation is used for the number of elements of a set: $|X|$ denotes the number of elements of the set X. The *size* of G is the number of edges; it is denoted by $e(G)$. The number of edges in a graph of order n is at least 0 and at

most $\binom{n}{2}$. A graph of order n and no edges is called an *empty graph* and is denoted by E^n; a *complete graph* K^n has order n and $\binom{n}{2}$ edges.

The set of vertices adjacent to a vertex $x \in G$ is said to be the *neighbor* of x and denoted by $\Gamma(x)$. The *degree* of x is $d(x) = |\Gamma(x)|$. We consider $\Gamma(U)$ to be the set of neighbors of the vertices in the subset U of V.

The *minimum degree* of the vertices of a graph G is denoted by $\delta(G)$ and the *maximum degree* of a graph G by $\Delta(G)$.

We say that a graph $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subset V$ and $E' \subset E$. In this case we write $G' \subset G$. If G' contains all edges of G that join two vertices in V' then G' is said to be the *subgraph induced by* V' or simply we say that G' is the induced subgraph of the graph G and is denoted by $G[V']$. We shall often construct new graphs from old ones by deleting some vertices. If $W \subset V$, then $G - W$ is the subgraph of G obtained by deleting the vertices in W and all edges incident with them. There is another method to construct new graphs from old ones. Consider a graph $G = (V, E)$ and form a new graph $\overline{G} = (V, \overline{E})$ where an edge $xy \in \overline{E}$ if and only if xy is not in E. We call $\overline{G}$ the complement of G.

A *path* is a graph P of the form

$$V(P) = \{x_0, x_1, \ldots, x_l\}, \quad E(P) = \{x_0 x_1, x_1 x_2, \ldots, x_{l-1} x_l\}.$$

This path P is usually denoted by $x_0 x_1 \ldots x_l$. The vertices x_0 and x_l are the end vertices and l is the length of P. We say that P is a path from x_0 to x_l or an x_0-x_l path. Most paths we consider are subgraphs of a given graph G. A *walk* W in G is an alternating sequence of vertices and edges, say $x_0, \alpha_1, x_1, \alpha_2, \ldots, \alpha_l, x_l$ where $\alpha_i = x_{i-1} x_i, 0 < i \leq l$. In accordance with the terminology above, W is an $x_0 x_l$ walk and is denoted by $x_0 x_1 \ldots x_l$; the length of W is l. This walk W is said to be a *trail* if all its edges are distinct. Note that a path is a walk with distinct vertices. A trail whose end vertices coincide (a closed trail) is said to be a *circuit*. If a walk $W = x_0 x_1 \ldots x_l$ is such that $l \geq 3, x_0 = x_l$ and the vertices $x_i, 0 < i \leq l$, are distinct

from each other and x_0 then W is said to be a *cycle*. For simplicity this cycle is denoted by $x_1 x_2 \ldots x_l$. The symbol P^l denotes an arbitrary path of length l.

A cycle containing all the vertices of a graph is said to be a *Hamilton cycle* of the graph and a *Hamilton path* of a graph is a path containing all the vertices of the graph. A graph containing a Hamilton cycle is said to be *Hamiltonian*.

Given two vertices x, y, their *distance* $d(x, y)$ is the minimum length of an x-y path. If there is no x-y path then $d(x, y) = \infty$. The *diameter* of a graph is the maximum of the distances between all pairs of vertices.

A graph is *connected* if for every pair $\{x, y\}$ of distinct vertices there is a path from x to y. Note that a connected graph of order at least 2 cannot contain an isolated vertex. A maximal connected subgraph is a *component* of the graph.

A *clique* of a graph is a maximal complete subgraph.

A graph G is said to be *bipartite* with vertex classes V_1 and V_2 if $V(G) = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ and each edge joins a vertex of V_1 to a vertex of V_2.

Let $G = (V, E)$ be a graph of order n. Let V be the set $\{1, 2, \ldots, n\}$. Consider the n by n matrix $A = (a_{i,j})$, where each row (and each column) of A corresponds to a distinct vertex of V. Let $a_{i,j}$ be equal to 1 if vertex i is adjacent to vertex j in G and $a_{i,j}$ be equal to 0 otherwise. Note that $a_{i,i} = 0$ for each $i = 1, 2, \ldots, n$. This matrix $A = (a_{i,j})$ is said to be the *adjacency matrix* of G. The adjacency matrix of G is clearly a symmetric (0, 1)-matrix, with zeros down the main diagonal. The adjacency matrix contains all the structural information about G and thus can be used as a representation for G.

2.3   Steinhaus graphs

Steinhaus graphs are named in honor of Hugo Steinhaus [Ste] who defined a triangle of plus (+) signs and minus (−) signs in terms of an initial row according to the following procedure. If a given row is of length k, then the following row is of length $k - 1$. Moreover, the ith entry of this row is + if the ith and $(i + 1)^{st}$ entries of

the preceding row are the same. Otherwise, this entry is $-$. Thus, for example, the triangle generated by $- - + - +$ is

$$
\begin{array}{ccccc}
- & - & + & - & + \\
& + & - & - & - \\
& & - & + & + \\
& & & - & + \\
& & & & -
\end{array}
$$

Since there are 2^n sequences of plus and minus signs of length n, there are 2^n triangles generated by these sequences and each such triangle has $\binom{n+1}{2}$ entries. Steinhaus asked if there exists for each n, such that $\binom{n+1}{2}$ is even (i.e. $n \equiv 0, 3 \pmod 4$), a triangle with the same number of plus and minus signs.

In [Ha], Harborth called these triangle Steinhaus triangles and answered Steinhaus' question with the following theorem.

Theorem 2.3.1 *For every $n \equiv 0, 3 \pmod 4$ there exists at least four Steinhaus triangles with $\frac{1}{2}\binom{n+1}{2}$ plus and minus signs.*

Harborth [Ha] also changed the notation by replacing $+$ with 0 and $-$ with 1 and hence a Steinhaus triangle is formed by addition mod 2. We illustrate this by replacing the $+$'s with 0's and $-$'s with 1's in the above example.

$$
\begin{array}{ccccc}
1 & 1 & 0 & 1 & 0 \\
& 0 & 1 & 1 & 1 \\
& & 1 & 0 & 0 \\
& & & 1 & 0 \\
& & & & 1
\end{array}
$$

Thus if $a_{i,j}$ is the entry in the ith row and j^{th} column of a Steinhaus triangle, then for $1 < i < j$,

$$
a_{i,j} \equiv a_{i-1,j-1} + a_{i-1,j} \equiv \sum_{k=0}^{i-1} \binom{i-1}{k} a_{1,j-k} \pmod 2.
$$

Any triangle of zeros and ones where $a_{i,j} \equiv a_{i-1,j-1} + a_{i-1,j} \pmod 2$ is said to have the Steinhaus property. Now we are ready to introduce Steinhaus graphs.

A Steinhaus matrix generated by the sequence of zeros and ones, $(a_{1,j})_{j=2}^{n}$ is a symmetric n by n $(0,1)$-matrix, $A = (a_{i,j})$ with the diagonal entries zero and the upper triangular part of A the Steinhaus triangle generated by $(a_{1,j})_{j=2}^{n}$. Thus if $(a_{1,j})_{j=1}^{n}$ is the first row of a Steinhaus matrix, then for $1 \leq i, j \leq n$,

$$a_{i,i} = 0, a_{i,j} = a_{j,i}$$

and for $1 < i < j \leq n$,

$$a_{i,j} \equiv a_{i-1,j-1} + a_{i-1,j} \equiv \sum_{k=0}^{i-1} \binom{i-1}{k} a_{1,j-k} \pmod{2}.$$

A graph G with n vertices is said to be a *Steinhaus* graph if the adjacency matrix of G is the Steinhaus matrix generated by a sequence $(a_{1,j})_{j=2}^{n}$ of zeros and ones and the matrix is said to be the *Steinhaus matrix* of G. The first row $(a_{1,j})_{j=1}^{n}$ in the Steinhaus matrix is said to be the *generating string* of the Steinhaus graph G and the graph G is said to be generated by the string $(a_{1,j})_{j=1}^{n}$. From now, we denote the Steinhaus triangle by $(a_{i,j})_{1 \leq i \leq j \leq n}$ if there is no confusion.

Example 2.3.2 Consider the generating string $(a_{1,j})_{j=1}^{8} = (01010110)$. The Steinhaus matrix associated with this string is

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Let G be a Steinhaus graph with n vertices generated by the string $(a_{1,j})_{j=1}^{n}$. The *partner* of G, $P(G)$, is the Steinhaus graph generated by the string $(a_{n-i+1,n})_{i=1}^{n}$. For example, the partner of the graph generated by the string (01010110) is generated by the string (01010110) (see Example 2.3.2). A Steinhaus graph G is said to be *doubly symmetric* if the Steinhaus matrix of G is equal to the Steinhaus matrix of the

partner of G, $P(G)$. For example, the string (01010110) generates doubly symmetric graph (see Example 2.3.2).

Note that it follows from the Steinhaus property that if G is doubly symmetric, then the Steinhaus matrix of G is symmetric with respect to both diagonals; that is $a_{i,j} = a_{n-j+1,n-i+1}$ for $1 \leq i \leq n$, $1 \leq j \leq n$.

In [Mo], Molluzzo formed graphs from Steinhaus tringles. These, however, are the complements of what we have defined to be Steinhaus graphs.

Now we list two basic results of Steinhaus graphs.

(1)   All Steinhaus graphs with n vertices except the empty graph E^n are connected [My1].

(2)   Every Steinhaus graph with minimum degree at least three is two-connected [My2].

More on Steinhaus graphs can be found in [BrDu], [My3] and [My4].

Recently, Brand and Morton developed the following generalization of Steinhaus graphs. Let $s : N \to N - \{1\}$ be a function. A generalized Steinhaus triangle of order n and type s is the upper triangular portion of an $n \times n$ array $A = (a_{i,j})$ whose entries satisfies

$$a_{i,j} \equiv \sum_{r=0}^{s(i)-1} c_{r,i,j} a_{i-1,j-r} \pmod{2}$$

where $2 \leq i \leq n-1$, $i + s(i) - 1 \leq j \leq n$, $c_{r,i,j} \in \{0,1\}$ and $c_{s(i)-1,i,j} = 1$. As in the case of Steinhaus graphs, we define $a_{i,i} = 0$ for $1 \leq i \leq n$ and set $a_{i,j} = a_{j,i}$ for $i > j$. A graph with such an adjacency matrix is referred to as a generalized Steinhaus graph. Properties of generalized Steinhaus graphs are investigated in [BrMo].

2.4   Random graphs

The theory of the evolution of random graphs which grew from the paper of Erdös and Rényi, [ErRe2] (see [Bo3] and [Pa]), is a striking example of the use of the

probabilistic method in mathematics. We will not be concerned with the history of the theory but will only state the two basic models and some well known results. For more background and results see [Bo3], [Lu] and [Pa].

In the first model we consider, the sample spaces Ω_n consisting of all labeled graphs G of order n. Specifically, for each positive integer n and number $p = p(n)$ with $0 < p < 1$, the probability of a graph $G \in \Omega_n$ with m edges is given by

$$P(G) = p^m (1 - p)^{\binom{n}{2} - m}.$$

It is often convenient to view the set of pairs of vertices of G as a sequences of $\binom{n}{2}$ Bernoulli trials and consider p as the probability of an edge. This model of random graph theory is referred to as either Model A in [Pa], or $\mathcal{G}(n, p)$ in [Bo3].

In the second basic model, the sample spaces $\Omega_{n,m}$ consist of all labeled graphs G of order n and size $m = m(n)$, that is with $m(n)$ edges. In this model the probability of each graph G is given by

$$P(G) = \left(\binom{\binom{n}{2}}{m} \right)^{-1}.$$

This model is referred to as either Model B in [Pa] or $\mathcal{G}(n, M(n))$ where $M(n) = m$ in [Bo3]. This Model B seems to be much more difficult to analyse than Model A. Nevertheless, the two models are closely related, as indicated by the following theorem.

A set of graphs $\mathcal{A}$ is called convex if $G \in \mathcal{A}$ whenever G_1 and G_2 are in $\mathcal{A}$ and the subgraph relation $G_1 \subset G \subset G_2$ is satisfied.

Let Q be a property of graphs and consider the set $\mathcal{A}_n$ of graphs of order n that possess property Q. If $P(\mathcal{A}_n) \to 1$ as $n \to \infty$ then we say that almost every graph has property Q.

Theorem 2.4.1 ([Bo3], [Pa]) *We assume that $pn^2 \to \infty$ and $(1 - p)n^2 \to \infty$ where $p = p(n)$, with $0 < p < 1$, is the probability of an edge. Suppose $\mathcal{A}_n$ is a set of graphs of order n with property Q and $\varepsilon > 0$ is fixed. Furthermore, assume that if $m = m(n)$*

is any sequences of integers such that

$$(1 - \varepsilon)p\binom{n}{2} < q < (1 + \varepsilon)p\binom{n}{2},$$

then $P(\mathcal{A}_n) \to 1$ as $n \to \infty$ in model B, that is, almost every graph has property Q. Then also, in Model A, almost every graph has property Q. Now suppose $\mathcal{A}_n$ is a convex set, and in Model A almost every graph has the property Q. Then if we set $m = \lfloor p\binom{n}{2} \rfloor$, in Model B almost every graph has property Q.

From now, we will be using Model A only. A very useful property of graphs from which many results easily follow is property P_k.

Definition 2.4.2 ([BlHa]) *A graph $G = (V, E)$ has property P_k if whenever W_1, W_2 are disjoint sets of at most k vertices each then there is a vertex $z \in V - W_1 \cup W_2$ joined to every vertex in W_1 and none in W_2.*

In Model A, if p is fixed then almost every graph has P_k ([BlHa], [Pa]). From this it follows [BlHa] that if Q is any property giving a first order sentence, then either Q holds for almost every graph in Model A and Model B or fails for almost every graph in Model A and Model B.

In particular, in Model A with p and k fixed, we have [Pa]

- Almost every graph has diameter two.

- Almost every graph is k-connected.

- Almost every graph contains a subgraph of order k as an induced subgraph.

- Almost every graph is nonplanar.

- Almost every graph is locally connected.

An important property possessed by almost every graph which can not be expressed by a first order sentence is given by the following theorem.

Theorem 2.4.3 ([Po2], [Bo2] p.140) *In Model A, with $p = c(\log n)/n$ almost every graph is Hamiltonian when c is somewhat larger than 1.*

Now let us give the definition of $(p, \alpha)-$jumbled graphs.

Definition 2.4.4 ([Th2]) *A graph is said to be (p, α)-jumbled if p, α are real numbers satisfying $0 < p < 1 \leq \alpha$ and if every induced subgraph H of G satisfies*

$$\left| \; e(H) - p\binom{|H|}{2} \; \right| \leq \alpha|H|.$$

where $e(H)$ is the number of edges in H.

Equivalently, if $d(H)$ is the average degree inside H we may say

$$\left| \; d(H) - p(|H| - 1) \; \right| \leq 2\alpha$$

holds for every induced sugraph H. We think of a (p, α)-jumbled graph as behaving somewhat like a random graph where each edge is chosen with probability p. Note that if G is (p, α)-jumbled then every induced subgraph is (p, α)-jumbled and the complement of G is also $(1 - p, \alpha)$-jumbled. Observe too that the clique number of G is at most $1 + 2\alpha(1 - p)^{-1}$ and the independence number is at most $1 + 2\alpha p^{-1}$ [Th2].

Here are some examples of jumbled graphs.

Example 2.4.5 ([Th2])

(1) let $G \in \mathcal{G}(n, p)$, that is, the edges of G are chosen with probability p. Then G is almost surely $(p, 2(pn)^{\frac{1}{2}})$-jumbled provided $pn \to \infty$ and $(1 - p)n \to \infty$.

(2) Choose a graph in $\mathcal{G}(n, p)$, select a subset X of the vertices, with $|X| = \lfloor (pn)^{\frac{1}{2}} \rfloor$, and join each pair of vertices in X. Then G is almost surely $(p, (pn)^{\frac{1}{2}})$-jumbled.

We close by mentioning a result of Chung, Graham and Wilson. In [CGW], the authors show that the equivalence of a set of graph properties possessed by almost all graphs in $\mathcal{G}(n, \frac{1}{2})$ in the sense that any graph possessing any of one of them must

possess all the others. Such graphs are called quasi-random. They followed in the spirit of the seminal paper of Thomason [Th2]. Let us list 7 equivalent properties which give a quasi-random graph ([CGW]). Each of the properties will contain occurences of the asymptotic "little-oh" notation $o(\cdot)$. Let $\{x_n\}$ and $\{y_n\}$ be sequence of real numbers. The little-oh notation is defined as usual:

$$x_n = o(y_n)$$

means that there is a sequence $\{k_n\}$ of positive terms such that $k_n \to 0$ and a constant N so that

$$|x_n| \le k_n y_n$$

for all $n \ge N$. For example, if $x_n = o(1)$, then $x_n \to 0$.

Let $G = \{V, E)$ denote a graph with vertex set V and edge set E. We use the notation $G(n)$ (and $G(n, e)$) to denote that G has n vertices (and e edges). For $X \subset V$, we let $e(X)$ denote the number of edges in the induced subgraph X of G. Further, if $G' = (V', E')$ is another graph, we let $N_G^*(G')$ denote the number of (labelled) occurrences of G' as an induced subgraph of G and $N_G(G')$ denote the number of occurrences of G' as a (not necessarily induced) subgraph of G. Let $A = (a_{v,v'})$ be the adjacency matrix of G. For $v, v' \in V$, define $s(v, v') = \{y \in V : a_{v,y} = a_{v',y}\}$.

P_1: For all graphs $M(s)$ on s vertices,

$$N_G^*(M(s)) = (1 + o(1))n^s 2^{-\binom{s}{2}}.$$

P_2: $e(G) \ge (1 + o(1))\frac{n^2}{4}$, $N_G(C_t) \le (1 + o(1))\frac{n}{2}^t$

where C_t is the cycle with t edges.

P_3: $e(G) \ge (1 + o(1))\frac{n^2}{4}$, $\lambda_1 = (1 + o(1))\frac{n}{2}$, $\lambda_2 = o(n)$

where λ_i's are the eigenvalues of A so that $|\lambda_1| \ge |\lambda_2| \ge \ldots \ge |\lambda_n|$.

P_4: For each subset $S \subset V$, $e(S) = \frac{1}{4}|S|^2 + o(n^2)$

P_5: For each subset $S \subset V$ with $|S| = \lfloor \frac{n}{2} \rfloor$, $e(S) = (\frac{1}{16} + o(1))n^2$

P_6: $\sum_{v,v'} |s(v,v') - \frac{n}{2}| = o(n^3)$.

P_7: $\sum_{v,v'} \left| |\Gamma(v) \cap \Gamma(v')| - \frac{n}{4} \right| = o(n^3)$

where $\Gamma(v)$ and $\Gamma(v')$ are the neighbors of v and v' respectively.

In Chapter 5, we will prove that almost all Steinhaus graphs are Hamiltonian. To do this, we will use P_4 among the properties in order to have a good condition on the number of edges in Steinhaus graphs.

CHAPTER 3

INDUCED CYCLES IN STEINHAUS GRAPHS

3.1 Preliminaries

Steinhaus graphs have several interesting properties which are not shared by all graphs. For examples,

(1). The diameter of all Steinhaus graphs with n vertices except P^n, E^n is at most $\lfloor \frac{1}{2}(n+2) \rfloor$ ([BDD]).

(2). The order of a large clique in any Steinhaus graph with n vertices is at most $\lceil \frac{1}{2}(n+3) \rceil$ ([BDD]).

(3). The larger component of the complement of a Steinhaus graph has diameter at most two ([Dy2]).

(4). A Steinhaus graph is bipartite if and only if it has no triangle ([Dy3]).

More on the properties of Steinhaus graphs are found in [BrDu], [Dy5]. Also, it seems reasonable that the size of a large induced cycle in any Steinhaus graph with n vertices might have an upper bound similar to (2). In section 3.2, we investigate the maximum size of an induced cycle in Steinhaus graphs where the induced cycles are simple in the following sence:

Definition 3.1.1 *Let G be a Steinhaus graph. An induced cycle $C = x_1 x_2 \ldots x_l$ of G is said to be simple if $x_1 = 1$, $x_n = n$ and $x_i < x_j$ whenever $i < j$.*

Let G be a Steinhaus graph and C be a simple induced cycle in G. For our convenience, we decompose the Steinhaus graph G with the simple induced cycle C in G as follows: Let $\Phi_C = \{A_i : A_i \subset C\}$ be the partition of C such that each A_i is a

maximal subset of C which consists of consecutive vertices and the largest vertex in A_i is joined to the smallest vertex in A_{i+1}. Thus each induced subgraph A_i in G is a path. Let $\overline{\Phi}_C = \{B_j : B_j \subset G - C\}$ be the partition of $G - C$ such that each B_j is the largest subset of $G - C$ which consists of consecutive vertices. Then $\Phi = \Phi_C \cup \overline{\Phi}_C$ is said to be the *cover* of the Steinhaus graph G with the simple induced cycle C in G. Let us give an example of the cover of a Steinhaus graph with a simple induced cycle.

Example 3.1.2 Let G be the Steinhaus graph which is generated by the string 01000001111. Then $C = \{1, 2, 3, 4, 5, 11\}$ is a simple induced cycle of G. The cover of G with C is given by $A_1 = \{1, 2, 3, 4, 5\}$, $A_2 = \{11\}$ and $B_1 = \{6, 7, 8, 9, 10\}$.

In section 3.2, we will classify all simple maximal induced cycles in Steinhaus graphs. Before we begin the next section, let us mention the following simple lemma and some facts about Pascal's triangle.

Lemma 3.1.3 *Let G be a doubly symmetric Steinhaus graph with generating string $(a_{1,i})_{i=1}^{n}$. Then $a_{1,n} = 0$.*

Proof. If $a_{1,n} = 1$, then $a_{1,n-1} + a_{2,n} \equiv 1 \pmod{2}$. So G is not doubly symmetric. ∎

We now present some facts concerning Pascal's triangle modulo two that will be needed in section 3.2. The rows of the triangle are labelled $R_1, R_2, \ldots$ and so the kth element of R_n is $\binom{n-1}{k-1} \pmod{2}$ if $1 \le k \le n$. A Pascal triangle is said to be of *dimension* n if the triangle consists of the n rows $R_1, R_2, \ldots, R_n$ and is denoted by $(a_{i,j})_{1 \le j \le i \le n}$. More on the properties of Pascal triangles are found in [Dy5].

Example 3.1.4 Here we give the Pascal triangle of dimension 6.

$$
\begin{aligned}
R_1 &\to 1 \\
R_2 &\to 1\ \ 1 \\
R_3 &\to 1\ \ 0\ \ 1 \\
R_4 &\to 1\ \ 1\ \ 1\ \ 1 \\
R_5 &\to 1\ \ 0\ \ 0\ \ 0\ \ 1 \\
R_6 &\to 1\ \ 1\ \ 0\ \ 0\ \ 1\ \ 1
\end{aligned}
$$

Lemma 3.1.5 *Let* $(a_{i,j})$ *be the Pascal triangle of dimension* n. *If* $a_{n,j} = 1$ *for all* $j \geq \lfloor \frac{1}{3}(n+4) \rfloor$ *then* n *is a power of 2.*

Proof. We will use induction on n. Since $a_{n,j} \equiv \binom{n-1}{j-1} \equiv \binom{n-1}{n-j} \equiv a_{n,n-j+1}$ (mod 2), $a_{n,n-j} = 1$ for all $j \geq \lfloor \frac{1}{3}(n+4) \rfloor$. Let $n = 2^m + k$ for some $0 \leq k < 2^m$. We want to show that k is equal to 0. Suppose that k is greater than or equal to 1. Then the Pascal triangle of dimension k satisfies the condition in lemma. So k is a power of 2 by induction. Since $a_{n,j} = 1$ for $1 \leq j \leq n - \lfloor \frac{1}{3}(n+4) \rfloor + 1$, k is equal to 2^{m-1}. Then $a_{n,k+1} = 0$. This gives a contradiction since $k \leq \lfloor \frac{1}{3}(n+4) \rfloor$. This prove lemma. ∎

Lemma 3.1.6 *Let* $a_{n,j} = 1$ *for some* $1 < j < n$. *Then* n *is odd if and only if* $a_{n,j-1} = 0$ *and* $a_{n,j+1} = 0$.

Proof. Since $a_{n,j} = 1$, $\binom{n-1}{j-1}$ is odd. By Luscas' Theorem, we have the following fact:

fact. $\binom{n-1}{j-1}$ is odd if and only if if when $j - 1$ has a 1 as its i-th binary digit, so does $n - 1$.

Suppose that n is odd. Since $\binom{n-1}{j-1}$ is odd and $n - 1$ is even, $j - 1$ is even by the above fact. By applying the above fact to the 0-th binary digits of $j - 2, j$ and $n - 1$, $\binom{n-1}{j-2}$ and $\binom{n-1}{j}$ are even. So $a_{n,j-1} = a_{n,j+1} = 0$.

Conversely, suppose that n is even. So $n - 1$ is odd. Therefore $n - 1$ has 1 as its 0-th binary digit. Now, $j - 1$ has either 0 or 1 as its 0-th binary digit. If $j - 1$ has 0 as its 0-th binary digit, then j has 1 as its 0-th binary digit. So $\binom{n-1}{j}$ is odd by the above fact. If $j - 1$ has 1 as its 0-th binary digit, then $j - 2$ has 0 as its binary digit. So $\binom{n-1}{j-2}$ is odd also by the same fact. Thus either $a_{n,j-1} = 1$ or $a_{n,j+1} = 1$. ∎

3.2 Simple maximal induced cycles in Steinhaus graphs

Let G be a Steinhaus graph with n vertices and $(a_{i,j})$ be the Steinhaus matrix of G. Let C be a simple maximal induced cycle in G and $\Phi = \Phi_C \cup \overline{\Phi}_C$ be the cover of G with C. In particular, in C, the largest vertex in A_i is joined to the smallest vertex in A_{i+1}.

Now we give a series of lemmas in order to estimate the size of B_i in $\overline{\Phi}_C = \{B_i : i = 1, 2, \ldots, t\}$. Let α_i be the smallest vertex and β_i be the largest vertex in A_i respectively. Note that α_1 is the vertex 1 and β_i is joined to α_{i+1} for each $1 \leq i \leq t$. Also, $B_i = \{\beta_i + 1, \beta_i + 2, \ldots, \alpha_{i+1} - 1\}$. Let a_i be the size of A_i. Then $b_i = \alpha_{i+1} - \alpha_i - 1$ is the size of B_i respectively.

Let us observe the following simple facts about strings in the Steinhaus triangle by using the above notations.

1. Since A_i is the path $\alpha_i \alpha_i + 1 \ldots \beta_i$, the string $(a_{\alpha_i,j})_{\alpha_i \leq j \leq \beta_i}$ is $(010 \ldots 0)$ for each i. Thus for all $\alpha_i \leq s \leq s' \leq \beta_i$,

$$a_{s,s'} = \begin{cases} 1 & \text{iff } s' = s + 1; \\ 0 & \text{otherwise.} \end{cases}$$

2. For each $1 \leq i \leq t$, the string, the transpose of $(a_{k,\alpha_{i+1}})_{\alpha_i \leq k \leq \beta_i}$, is the transpose of $(00 \ldots 01)$. Therefore, $(a_{\alpha_i,j})_{\alpha_{i+1} - a_i + 1 \leq j \leq \alpha_{i+1}}$ is $(10 \ldots 0)$.

3. Since β_i is joined to α_{i+1}, the string $(a_{\beta_i,j})_{\alpha_{i+1} \leq j \leq \beta_{i+1}}$ is $(10 \ldots 0)$.

4. Either a_1 or a_t is equal to 1. (Otherwise, the entries $a_{1,n-1}$ and $a_{1,n}$ are equal to 1 since $a_{1,n} = 1$, $a_{2,n} = 0$. Then C is not a cycle.)

We will use the above facts in the following lemmas, but we will not often mention these facts.

Lemma 3.2.1 *For each i, $b_i \geq \max\{a_i - 1, a_{i+1} - 1\}$.*

Proof. Without loss of generality, we assume that a_i is greater than or equal to a_{i+1} by considering its partner, $P(G)$, of G. Suppose that b_i is less than $\max\{a_i - 1, a_{i+1} - 1\}$. Consider the string in fact 2. So the entry $a_{\beta_i - b_i - 1, \beta_i}$ is equal to 1 by the Steinhaus property. Since $b_i < a_i - 1$, the entry $a_{\beta_i - b_i - 1, \beta_i}$ is in the subtriangle generated by the string $(a_{\alpha_i,j})_{\alpha_i \leq j \leq \beta_i}$. Therefore, $\beta_i - b_i - 1 + 1 = \beta_i$ by fact 1. We have $b_i = 0$, which gives a contradiction. ∎

Lemma 3.2.2 *If a_i is equal to a_{i+1} then*

$$b_i \geq \begin{cases} a_i & \text{if } a_i \text{ is a power of 2;} \\ a_i + 1 & \text{otherwise.} \end{cases}$$

Proof. First, b_i is at least $a_i - 1$ by Lemma 3.2.1.

Suppose that b_i is equal to $a_i - 1$. Then the string $(a_{\alpha_i,j})_{\alpha_i \leq j \leq \beta_{i+1}}$ in the α_i^{th} row in the Steinhaus triangle is clearly $(010\ldots0)$. Since b_i is equal to $a_i - 1$, the entry $a_{\alpha_{i+1}-1, \beta_{i+1}}$ is equal to 1. Since $(a_{\alpha_{i+1},j})_{\alpha_{i+1} \leq j \leq \beta_{i+1}}$ in the α_{i+1}^{th} row is equal to $(010\ldots0)$, the string $(a_{\alpha_{i+1}-1,j})_{\alpha_{i+1}-1 \leq j \leq \beta_{i+1}}$ is $(001\ldots1)$ by the Steinhaus property. But the triangle $(a_{k,j})_{\alpha_i \leq k \alpha_{i+1}-1, \beta_i+1 \leq j \leq k+a_i}$ is the Pascal triangle of dimension $2a_i - 1$. By Lemma 3.1.5, $2a_i - 1$ is a power of 2, which gives a contradiction.

Suppose that a_i is not a power of 2. Assume that b_i is equal to a_i.

Case 1. a_{α_i, β_i+1} is equal to 0.

If $a_{\alpha_{i+1}-1,\beta_{i+1}}$ is 0, $(a_{\alpha_{i+1}-2,j})_{\alpha_{i+1}-2 \leq j \leq \beta_{i+1}}$ in the α_{i+1}^{th} row is $(001\ldots1)$ by the Steinhaus property with the α_{i+1}^{th} row. By the same arguement in the above, $2a_i - 1$ is a power of 2. This gives a contradiction.

If $a_{\alpha_{i+1}-1,\beta_{i+1}}$ is 1, $(a_{\alpha_{i+1}-1,j})_{\alpha_{i+1}-1 \leq j \leq \beta_{i+1}}$ in the α_{i+1}^{th} row is $(001\ldots1)$ by the Steinhaus property. By the same arguement in the above, $2a_i$ is a power of 2, which gives a contradiction.

Case 2. a_{α_i, β_i+1} is equal to 1.

If $a_{\alpha_{i+1}-1,\beta_{i+1}}$ is equal to 0, then $2a_i$ is a power of 2 by the same arguement as in Case 1. This gives a contradiction.

Similarly, if $a_{\alpha_{i+1}-1,\beta_{i+1}}$ is equal to 1, then $2a_i + 1$ is a power of 2. This gives a contradiction also.

By combining both cases, we prove lemma. ∎

Lemma 3.2.3 *If* $|a_{i+1} - a_i|$ *is equal to 1, then*

$$b_i \geq \begin{cases} \max\{a_i, a_{i+1}\} & \text{if } \min\{a_i, a_{i+1}\} \text{ is a power of 2;} \\ \max\{a_i, a_{i+1}\} + 1 & \text{otherwise.} \end{cases}$$

Proof. Without loss of generality, we can assume that a_i is greater than a_{i+1}, by considering its partner, $P(G)$.

First, b_i is greater than or equal to a_{i+1} by Lemma 3.2.1.

Suppose that b_i is equal to a_{i+1}. Assume that a_{i+1} is not a power of 2. The string $(a_{\alpha_i,j})_{\alpha_i \leq j \leq \beta_{i+1}}$ in the α_i^{th} row is $(010\ldots010\ldots0)$ where $a_{\alpha_i,\beta_i+1} = 1$.

Case 1. $a_{\alpha_{i+1}-1,\beta_{i+1}}$ is equal to 0.

By Steinhaus property with $a_{\alpha_{i+1}-2,\beta_{i+1}} = 1$, $a_{\alpha_{i+1}-1,\beta_{i+1}} = 0$ and the α_{i+1}^{th} row in the Steinhaus triangle, the string $(a_{\alpha_{i+1}-2,j})_{\alpha_{i+1}-2 \leq j \leq \beta_{i+1}}$ is $(001\ldots1)$. Then $(a_{k,j})_{\alpha_i \leq k \leq \alpha_{i+1}-2, \beta_i+1 \leq j \leq k+a_i}$ is the Pascal triangle of dimension $a_{i+1} + b_i$ satisfying the condition in Lemma 3.1.5. So $a_{i+1} + b_i = 2a_{i+1}$ is a power of 2, which gives a contradiction.

Case 2. $a_{\alpha_{i+1}-1,\beta_{i+1}}$ is equal to 1.

Again, by the same arguement in the Case 1, $(a_{\alpha_{i+1}-1,j})_{\alpha_{i+1}-1 \leq j \leq \beta_{i+1}}$ in the $(\alpha_{i+1} - 1)^{th}$ row is $(001\ldots1)$. So $a_i + b_i = 2a_{i+1} + 1$ is a power of 2 by Lemma 3.1.5, which gives a contradiction.

By combining both cases, we prove lemma. ∎

From Lemma 3.2.2 and Lemma 3.2.3, we observe the followings:

Let $\{A_i : i = 1, 2, \ldots, t+1\} \cup \{B_i : i = 1, 2, \ldots, t\}$ be the cover of G with a simple induced cycle C with $|A_{t+1}| = 1$.

First, if b_i is greater than or equal to a_i for all $1 \leq i \leq t$ then it ic clear that the order of C is at most $\lfloor \frac{1}{2}(n+2) \rfloor$.

Second, if b_i is less then a_i for some i, we can not guarantee that the order of C is at most $\lfloor \frac{1}{2}(n+3) \rfloor$. It is the case from Lemma 3.2.1 and Lemma 3.2.3 that there exists i such that b_i is equal to either $a_i - 1 = a_{i+1}$ where a_{i+1} is a power of 2 or $a_i - 1$ where $a_{i+1} \leq a_i - 2$.

Thus we give better estimations regarding the second observation in the following two lemmas.

Lemma 3.2.4 *Suppose that a_i is equal to $a_{i+1} + 1$ and that b_i is equal to a_{i+1} for some i. Let a_{i+1} be a power of 2 which is greater than 1. Let k be the smallest number such that $k \geq i + 1$, $a_k \geq 2$ and for all $i + 1 \leq l \leq k - 1$*

$$a_l = a_{i+1}$$

and

$$a_k \neq a_{k-1}.$$

Then either

$$\sum_{l=i}^{k-1} a_l \leq \sum_{l=i}^{k-1} b_l$$

or

$$\sum_{l=i}^{k} a_l \leq \sum_{l=i}^{k} b_l.$$

Proof. First, since a_{i+1} is equal to a_{i+2} and a_{i+1} is a power of 2, by Lemma 3.2.2 we have $b_{i+1} \geq a_{i+1}$. Observe that if $b_{i+1} = a_{i+1} + 1$ then $b_{i+2} \geq a_{i+2} + 1$. By continuing this process, we have inequality

$$\sum_{l=i}^{k-2} a_l \leq \sum_{l=i}^{k-2} b_l + 1.$$

Suppose that we have the inequality

$$\sum_{l=i}^{k-1} a_l > \sum_{l=i}^{k-1} b_l.$$

Then from the above inequalities, we have $b_l = a_l$ for all $i + 1 \leq l \leq k - 2$. Since the a_l's are all the same and a power of 2, the string $(a_{\alpha_l,j})_{\alpha_l \leq j \leq \alpha_{l+1}}$ in the α_l^{th} row is $(010\ldots 0110\ldots 0)$ for $i + 1 \leq l \leq k - 1$. Then we have $b_{k-1} \geq a_{k-1}$. Otherwise, we have $b_{k-1} = a_{k-1} - 1$. Therefore the vector $(a_{\alpha_{k-1},j})_{\alpha_{k-1} \leq j \leq \alpha_k}$ in the α_{th} row is $(010\ldots 010\ldots 0)$, which is impossible by Lemma 3.1.6. Hence $b_{k-1} = a_{k-1}$. Moreover, we have $b_k \geq a_k$ by the same argument as above.

Next, we want to show that b_k is greater than or equal to $a_k + 1$, which gives the inequality

$$\sum_{l=i}^{k} a_l \leq \sum_{l=i}^{k} b_l.$$

Assume that b_k is equal to a_k.

Case 1. $a_k > a_{k-1}$.

First, if $a_k \geq a_{k-1} + 2$ then $b_{k-1} \geq a_{k-1} + 1$ by Lemma 3.2.1. This is impossible because $b_k = a_k$. Therefore, a_k must be equal to $a_{k-1} + 1$. So the entry a_{α_k-1,β_k} is 1. By applying the Steinhaus property to the α_k^{th} row in the Steinhaus triangle, the string $(a_{\alpha_{k-1}-1,j})_{\alpha_{k-1}-1 \leq j \leq \beta_{k-1}}$ in the $(\alpha_{k-1} - 1)^{th}$ row is $(01\ldots 1)$ because a_{k-1} is a power of 2. Since the string $(a_{l,\alpha_k})_{\alpha_{k-2} \leq l \leq \alpha_k - 1}$ in the α_k^{th} column is $(00\ldots 01)$, the entry $a_{\alpha_{k-1}-2,\beta_{k-1}}$ is equal to 0, which gives a contradiction by fact 3.

Case 2. $a_k < a_{k-1}$.

Since the string $(a_{\beta_{k-1},j})_{\beta_{k-1} \leq j \leq \alpha_{k+1}}$ in the β_{k-1}^{th} row is $(00\ldots 010\ldots 0)$ and a_k is less than a_{k-1}, we get the Pascal triangle $(a_{l,j})_{\beta_{k-1} \leq l \leq \alpha_k, j \leq k + a_{k-1}+1}$ of dimension $a_{k-1} + 2$ such that the entry a_{α_k,β_k+1} is 0. Since this entry a_{α_k,β_k+1} is in an even row in the above Pascal triangle, Lemma 3.1.6 implies that the entries a_{α_k,β_k+j} are either all 0's or all 1's where $j = 2, 3$. In both cases, $b_k \geq a_k + 1$. This gives a contradiction.

By combining both cases, we prove lemma. ∎

Lemma 3.2.5 *Suppose that a_i is greater than or equal to $a_{i+1} + 2$ and that b_i is equal to $a_i - 1$. Let k be the smallest number such that $k \geq i + 2$, for all $i + 1 \leq l \leq k - 1$*

$$a_{l-1} \geq a_l$$

and

$$2 \leq a_{k-1} \leq a_k - 1.$$

Then either

$$\sum_{l=i}^{k-1} a_l \leq \sum_{l=i}^{k-1} b_l$$

or

$$\sum_{l=i}^{k} a_l \leq \sum_{l=i}^{k} b_l.$$

Proof. First, by Lemma 3.2.1, we have $b_{i+1} \geq a_{i+1} - 1$. Moerover, $b_{i+1} \geq a_{i+1}$ by the following argument. If $b_{i+1} = a_{i+1} - 1$ then we have $a_{\alpha_{i+1},\beta_{i+1}} = 0, a_{\alpha_{i+1},\beta_{i+1}+1} = 1$ and $a_{\alpha_{i+1},\beta_{i+1}+2} = 0$. But the entries are in the Pascal triangle $(a_{l,j})_{\alpha_i \leq l \leq \alpha_{i+1}, j \leq l + a_{i+1}}$ of dimension $a_i + b_i + 1$. But by Lemma 3.1.6, $a_i + b_i + 1 = 2a_i$ must be odd, which gives a contradiction.

Next, if b_j is equal to a_j for some $i + 1 \leq j \leq k - 2$, then $b_{j+1} \geq a_{j+1}$ by the same argument as above. Therefore, by continuing this process we have inequality

$$\sum_{j=i}^{k-2} a_l \leq \sum_{l=i}^{k-2} b_l + 1.$$

Since $a_k \geq a_{k-1} + 1$, we have $b_{k-1} \geq a_{k-1}$ by Lemma 3.2.1. Therefore, we have inequality

$$\sum_{l=i}^{k-1} a_l \leq \sum_{l=i}^{k-1} b_l + 1.$$

If $a_k \geq a_{k-1} + 2$ then we have $b_{k-1} \geq a_{k-1} + 1$ by Lemma 3.2.1. Therefore, we have inequality

$$\sum_{l=i}^{k-1} a_l \leq \sum_{l=i}^{k-1} b_l$$

and this inequality gives the proof of theorem. So we assume that a_k is equal to $a_{k-1} + 1$. Suppose that we have the inequality

$$\sum_{l=i}^{k-1} a_l > \sum_{l=i}^{k-1} b_l.$$

We want to show that b_k is greater than or equal to $a_k + 1$, which gives the inequality

$$\sum_{l=i}^{k} a_l \leq \sum_{l=i}^{k} b_l.$$

Assume that b_k is equal to a_k. Then by the ineqality in the above, i.e.

$$\sum_{j=i}^{k-1} a_j \leq \sum_{j=i}^{k-1} b_j + 1,$$

we have $b_l = a_l$ for all $i + 1 \leq l \leq k - 1$. Therefore, the string $(a_{\alpha_l,j})_{\alpha_l \leq j \leq \alpha_{l+1}}$ in the α_l^{th} row is $(010\ldots0110\ldots0)$ for each l. Since the entry $a_{\alpha_k-1,\beta_k} = 0$, the string $(a_{\alpha_k-1,j})_{\alpha_k-1 \leq j \leq \beta_k}$ in the $(\alpha_k - 1)^{th}$ row is $(001\ldots1)$ by the Steinhaus property along with the α_k^{th} row. Thus we have the Pascal triangle $(a_{m,j})_{\alpha_k-1 \leq m \leq \alpha_k-1, m \leq j}$ of dimension a_k whose a_k^{th} row is $(11\ldots1)$. Hence a_k is a power of 2. Since $|a_k - a_{k-1}|$ is equal to 1 and b_{k-1} is equal to a_{k-1}, we conclude that a_{k-1} is a power of 2. This gives a contradiction.

This proves that b_k is at least $a_k + 1$. ∎

Now we prove the following theorem.

Theorem 3.2.6 *Let G be a Steinhaus graph with n vertices and let C be a simple maximal induced cycle in G. Then the order of C is less than or equal to $\lfloor \frac{n+3}{2} \rfloor$.*

Proof. Let $\{A_i \; : \; i = 1, 2, \ldots, t+1\} \cup \{B_i \; : \; i = 1, 2, \ldots, t\}$ be the cover of the Steinhaus G with the simple induced cycle C.

Without loss of generality, we assume that a_1 is greater than or equal to a_l by considering its partner $P(G)$. Then a_l is equal to 1 otherwise the entry $a_{1,n-1}$ is equal to 1. So C contains a cycle of length 3, which gives a contradiction.

It is enough to show that

$$\sum_{i=1}^{t} a_i \leq \sum_{i=1}^{t} b_i + 2$$

because this inequality gives

$$
\begin{aligned}
2(\sum_{i=1}^{t+1} a_i) &\leq \sum_{i=1}^{t+1} a_i + \sum_{i=1}^{t} b_i + 3 \\
&= n + 3.
\end{aligned}
$$

Since the order of the simple induced cycle C is $\sum_{i=1}^{t+1} a_i$, we have the inequality

$$|C| \leq \lfloor \frac{1}{2}(n+3) \rfloor.$$

This prove theorem.

Sublemma. Let a_i be equal to 2 and b_i be equal to 1. Let i_0 be the smallest number such that $i_0 \geq i+1$, $a_{i_0} \geq 2$ and $a_j = 1$ for all $i+1 \leq j \leq i_0 - 1$. Then

$$\sum_{j=i}^{i_0-1} a_j \leq \sum_{j=i}^{i_0-1} b_j.$$

Proof of Sublemma. Consider the subtriangle generated by string $(a_{\alpha_i,k})_{\alpha_i \leq k \leq \beta_{i_0}}$ in the Steinhaus triangle of G. Note that if $b_j \leq 2$ for all $i \leq j \leq i_0 - 1$, then the generating string in the above subtriangle is $(0110\ldots0)$ by the Steinhaus property. Thus for $\alpha_i \leq s \leq \beta_{i_0}$, the pair $\{a_{s,s+1}, a_{s,s+2}\}$ is

$$(a_{s,s+1}, a_{s,s+2}) = \begin{cases} (0,1) & \text{iff } s - \alpha_i \text{ is odd;} \\ (1,1) & \text{iff } s - \alpha_i \text{ is even.} \end{cases}$$

Assume that we have the inequality

$$\sum_{j=i}^{i_0-1} a_j > \sum_{j=i}^{i_0-1} b_j.$$

Then $b_j = 1$ for all $i+1 \leq j \leq i_0 - 1$. Since $a_{i_0} = 2$ and $b_{i_0-1} = 1$, we have $(a_{\alpha_{i_0}-1,\alpha_{i_0}}, a_{\alpha_{i_0}-1,\alpha_{i_0}+1}) = (1,1)$. Thus $\alpha_{i_0} - \alpha_i - 1$ is even. This gives a contradiction because $\alpha_{i_0} - \alpha_i$ is even. $\square$

Now, we claim the following inequality which we asked.

Claim. $\sum_{j=1}^{t} a_j \leq \sum_{j=1}^{t} b_j + 2.$

Proof of Claim. If t is equal to 1, then $a_1 \leq b_1 + 2$. Also if $t = 2$, it is not fifficult to show that

$$a_1 + a_2 \leq b_1 + b_2 + 1$$

by considering all cases. From now we assume that $t \geq 3$.

If $a_j \leq 2$ for all $1 \leq j \leq t$, then we are done by Sublemma. Therefore, we assume that there exists j such that $a_j \geq 3$.

Suppose that i is the largest number such that

$$\sum_{j=1}^{i} a_j \leq \sum_{j=1}^{i} b_j + 1.$$

We want to show that i is equal to t. Suppose that i is less than t.

If there is no $j \geq i$ such that $a_j = 1$, then we have

$$\sum_{j=i}^{t} a_j \leq \sum_{j=i}^{t} b_j$$

by applying Lemma 3.2.4 or Lemma 3.2.5 sucessfully, which gives a contradiction by the choice of i. Therefore, there exists a smallest number k such that $k \geq i + 1$ and $a_k = 1$.

First, if $a_i, \ldots, a_{k-1}$ satisfy the conditions in Lemma 3.2.4, then $a_{k-1} \leq b_{k-1}$ and the string $(a_{\alpha_k,j})_{\alpha_k \leq j}$ in the α_k^{th} row is $(000 \ldots 1 \ldots)$ because a_{k-1} is a power of 2. Thus $b_k \geq 2$, which gives a contradiction by the choice of i.

Next, suppose that $a_i, \ldots, a_{k-1}$ satisfy the conditions in Lemma 3.2.5. Note that $b_{k-1} \geq a_{k-1}$ by Lemma 3.2.5. If there is some $k_0 > k$ such that $a_{k_0} \geq 2$ then

$$\sum_{l=i}^{s} a_l \leq \sum_{l=i}^{s} b_l$$

for some $s \geq j_0$ by Sublemma, which gives a contradiction by the choice of i. If $a_j = 1$ for all $k \leq j \leq t$, then i is equal to t, which gives a contradiction.

Finally, if $a_l \leq 2$ for all $j \leq l \leq t$, then by applying Sublemma, we have a contradiction by the choice of i.

By considering all cases, we prove the claim. $\square$

By the claim, we prove theorem. $\blacksquare$ The proof of Theorem shows that if $t \geq 2$ then the order of any induced cycle C can not achieve the upper bound. Therefore, we get the following:

Corollary 3.2.7 *Let G be a Steinhaus graph with n vertices and C be a simple induced cycle in G. If the order of C is $\lfloor \frac{1}{2}(n+3) \rfloor$ then C is either $\{1, 2, \ldots, \lfloor \frac{1}{2}(n+1) \rfloor, n\}$ or $\{1, n - \lfloor \frac{1}{2}(n+1) \rfloor, \ldots, n\}$.*

Now, we give an example of simple induced cycle which achieves the bound in the theorem. Let G be the Steinhaus graph with generating string $(a_{1,j})_{1 \leq j \leq n}$ given by

$$a_{1,j} = \begin{cases} 0 & \text{if } j = 1, 3, \ldots, \lceil \frac{n}{2} \rceil; \\ 1 & \text{otherwise.} \end{cases}$$

Then G has the induced cycle $\{1, 2, \ldots, \lceil \frac{n}{2} \rceil, n\}$ of order $\lfloor \frac{1}{2}(n+3) \rfloor$.

We close by mentioning the size of maximal induced cycles in Steinhaus graphs. The question is that "Does the order of any induced cycles in Steinhaus graphs have a reasonable bound like in Theorem 3.2.6?". But if $n \leq 30$, it is not difficult to show that the maximum size of an induced cycle in Steinhaus graphs with n vertices is $\frac{n+3}{2}$. Thus we give the following conjecture.

Conjecture. The size of any induced cycles in a Steinhaus graph with n vertices is at most $\lfloor \frac{n+3}{2} \rfloor$.

CHAPTER 4

CLIQUES IN STEINHAUS GRAPHS

4.1 Introduction

In this chaper, we investigate the classification of all Steinhaus graphs which contain cliques of large size. In section 4.2, we give some results (see [Ha], [BDD]) in Steinhaus graphs. Based on these results, we give definitions to classify several types of cliques in Steinhaus graphs and their classification. Also, we discuss maximal types of cliques in Steinhaus graphs. In section 4.3, we discuss the classification of Steinhaus graphs with n vertices which contain cliques of size $w(n) = \lceil \frac{n+3}{3} \rceil$ for $n \geq 27$. Also we investigate the number of Steinhaus graphs which contain cliques of size $w(n)$. In section 4.4, we generalize the results in section 3 by counting the number of Steinhaus graphs which contain a clique of size near $w(n)$. Finally, in section 4.5, we investigate similar results as in section 4.3 and section 4.4 on the complement of Steinhaus graphs.

4.2 Preliminaries

It is natural to expect that if a Steinhaus graph contains a clique of large size, then the Steinhaus graph must have many edges. Conversely, if a Steinhaus graph has many edges then the Steinhaus graph may have a clique of large size. Harborth investigated the upper bound for the number of edges in Steinhaus graphs, where he showed that the upper bound is given by the following theorem.

Theorem 4.2.1 ([Ha]) *The largest number of edges in Steinhaus graphs with n vertices is*

$$\lfloor \frac{n(n-1)+1}{3} \rfloor.$$

From this theorem one may expect that the order of any cliques in Steinhaus not so large. In [BDD], Brigham, Deo and Dutton find the maximum size of cliques in Steinhaus graphs by the following method.

Lemma 4.2.2 ([BDD]) *No clique in any Steinhaus graph contains two pairs of consecutively numbered vertices.*

Lemma 4.2.3 ([BDD]) *No clique in any Steinhaus graph contains two pairs of vertices such that the vertices in each pair are numbered with a difference 2.*

Here, we use the notations C, T and CT in [BDD]. It is possible for a clique to contain one pair of consecutively numbered vertices. If there is such a pair, we call it a configuration of type C. Similarly, there may be one pair of vertices whose numbers differ by two, and if such a pair exists, we call it a configuration of type T. Both configurations may occur in the same clique. If they do, they must either be separated in labelings by at least two nodes or they must overlap by occurring as nodes either i, $i + 1$ and $i + 3$ or i, $i + 2$ and $i + 3$. This latter case is designated a configuration of type CT, and if it occurs, there can be no other C or T. A vertex configuration is a vertex which occurs in a clique which is not in any of the three types C, T or CT. Consideration of the above two lemmas leads to the conclusion that in any clique of a Steinhaus graph the labelings for any two configurations must be separated by at least two vertices which are not in the clique. We include the proof of the following theorem which can be found in [BDD], because it is an essential tool to get for later results.

Theorem 4.2.4 ([BDD]) *The size of a largest clique in any Steinhaus graph is given by*

$$w(n) = \lceil (n + 3)/3 \rceil$$

for $n \geq 2$.

Proof. First, a bound will be placed on the number of C, T, CT and vertex configurations which can appear in a clique of an order n Steinhaus graph. Each configuration is assumed to occupy a single vertex position, and n is reduced by the number of vertices in the configuration less one. If n' represents this reduced value of n, then the total number of configurations in a clique is at most $\lceil n' \rceil$. The number of vertices in the clique is then the number of configurations plus the number of clique vertices in the configurations which exceeds one, called the clique exess. Results are summarized in the following two tables. The first lists the excess numbers for each of the four possible configurations.

unit	nodes in configuration	node exess	clique nodes in configuration	clique exess
node	1	0	1	0
C	2	1	2	2
T	3	2	2	2
CT	4	3	3	2

The maximum number of vertices possible in a clique depends on the types of configurations in the clique. In all events this number can be computed by the following formula where a total excess refers to the sum of the corresponding excesses for all configurations in the clique:

$$\text{number of vertices} \le \lceil (n - m)/3 \rceil + \text{total clique excess, where } m \text{ is}$$

the total vertex excess.

The following table shows this computation for the five possible combinations of configurations which can occue in a clique:

Combination	the number $w(n)$ of vertices in the clique
No C, T or CT	$w(n) \le \lceil n/3 \rceil + 0 = \lceil n/3 \rceil$
C, no T	$w(n) \le \lceil (n - 1)/3 \rceil + 1 = \lceil (n + 2)/3 \rceil$
T, no C	$w(n) \le \lceil (n - 2)/3 \rceil + 2 = \lceil (n + 1)/3 \rceil$
CT	$w(n) \le \lceil (n - 3)/3 \rceil + 2 = \lceil (n + 3)/3 \rceil$
C and T	$w(n) \le \lceil (n - 3)/3 \rceil + 2 = \lceil (n + 3)/3 \rceil$

This establishes the results as an upper bound. It is not difficult to see that this bound is achieved for the Steinhaus graphs generated by the sequence $0101101\cdots$. ∎

From the above Theorem 4.2.5, it is easy to get the following useful corollary.

Corollary 4.2.5 *Let G be a Steinhaus graph with n vertices. Let G have a clique $Q = \{y_1 < y_2 < y_3 < \ldots < y_{w(n)}\}$ of size $w(n) = \lceil (n+3)/3 \rceil$ in G. Then Q satisfies one of the followings:*

(1) $n = 3k + 1$ *for some $k \geq 2$.*

In this case $\max_{2 \leq i \leq w(n)}\{y_i - y_{i-1}\} = 3$. So Q contains either a CT configuration or C and T configurations.

(2) $n = 3k + 2$ *for some $k \geq 2$.*

In this case $\max_{2 \leq i \leq w(n)}\{y_i - y_{i-1}\}$ is either 3 or 4.

If $\max_{2 \leq i \leq w(n)}\{y_i - y_{i-1}\} = 3$ then Q contains a C configuration.

If $\max_{2 \leq i \leq w(n)}\{y_i - y_{i-1}\} = 4$ then Q contains either a CT configuration or C and T configurations.

(3) $n = 3k + 3$ *for some $k \geq 2$.*

In this case $\max_{2 \leq i \leq w(n)}\{y_i - y_{i-1}\}$ is either 3, 4 or 5.

If $\max_{2 \leq i \leq w(n)}\{y_i - y_{i-1}\} = 3$ then Q contains a T configuration.

If $\max_{2 \leq i \leq w(n)}\{y_i - y_{i-1}\} = 4$ then Q contains a C configuration.

If $\max_{2 \leq i \leq w(n)}\{y_i - y_{i-1}\} = 5$ then Q contains either a CT configuration or C and T configurations.

We will use Corollary 4.2.5 to classify all cliques of size $w(n)$ in G as the main tool in the following section.

4.3   Classification of maximal cliques

Let G be a Steinhaus graph with n vertices which has a clique of size $w(n)$. Then G may have several cliques of size $w(n)$. But G satisfies

$$\max_{2 \leq i \leq w(n)} \{y_i - y_{i-1}\} \leq 5$$

for any clique $Q = \{y_1 < y_2 < y_3 < \ldots < y_{w(n)}\}$ in G by Corollary 4.2.5. But we want to express all cliques $Q = \{y_1, y_2, y_3, \ldots, y_{w(n)}\}$ of size $w(n)$ such that

$$\max_{2 \leq i \leq w(n)} \{y_i - y_{i-1}\} \leq 3$$

because if Q satisfies the above inequalty then it will be easy to classify all cliques of size $w(n)$. Thus the first goal in this section is to show that at least one clique Q of size $w(n)$ in G satisfies

$$\max_{2 \leq i \leq w(n)} \{y_i - y_{i-1}\} \leq 3.$$

Let us define several types of cliques which may occur in Steinhaus graphs.

Definition 4.3.1 *Let G be a Steinhaus graph with n vertices. A clique $Q(G) = \{y_1 < y_2 < \ldots < y_m\}$ in G is called a maximal clique type of G if $Q(G)$ satisfies the following conditions:*

(a) *$Q(G)$ is contained in a maximal clique in G.*

(b) *$y_i - y_{i-1} \leq 3$ for all $i \geq 2$.*

(c) *the size of $Q(G)$ is largest among all possible cliques in G satisfying (a) and (b).*

(d) *$y_m - y_1$ is also largest among all possible cliques in G satisfying (a), (b) and (c).*

Next, we will describe all maximal clique types precisely.

Definition 4.3.2 *Let G be a Steinhaus graph with n vertices. Let $Q(G)$ be a maximal clique type of G where $Q(G) = \{y_1, y_2, \ldots, y_m\}$.*

(a) $Q(G)$ is said to be of type I in G if $y_i = y_{i-1} + 3$ for all $i \geq 2$.

(b) $Q(G)$ is said to be of type II in G if either $y_2 = y_1 + 2$ and $y_i = y_{i-1} + 3$ for all $i \geq 3$ or $y_m = y_{m-1} + 2$ and $y_i = y_{i-1} + 3$ for all $2 \leq i \leq m - 1$.

(c) $Q(G)$ is said to be of type III in G if either $y_2 = y_1 + 1$ and $y_i = y_{i-1} + 3$ for all $i \geq 3$ or $y_m = y_{m-1} + 1$ and $y_i = y_{i-1} + 3$ for all $2 \leq i \leq m - 1$.

(d) $Q(G)$ is said to be of type IV in G if there is unique i such that either

$$y_j = \begin{cases} y_{j-1} + 1 & \text{if } j = i; \\ y_{j-1} + 2 & \text{if } j = i+1; \\ y_{j-1} + 3 & \text{otherwise} \end{cases}$$

or

$$y_j = \begin{cases} y_{j-1} + 2 & \text{if } j = i; \\ y_{j-1} + 1 & \text{if } j = i+1; \\ y_{j-1} + 3 & \text{otherwise.} \end{cases}$$

(e) $Q(G)$ is said to be of type V in G if either $y_2 = y_1 + 1$, $y_m = y_{m-1} + 2$ and $y_i = y_{i-1} + 3$ for all $3 \leq i \leq m-1$ or $y_2 = y_1 + 2$, $y_m = y_{m-1} + 1$ and $y_i = y_{i-1} + 3$ for all $3 \leq i \leq m - 1$.

Let G be a Steinhaus graph with n vertices. Then G seems to have several possible distinct maximal clique types Q of G. Note that if G has a maximal clique type then its partner $P(G)$ has a clique of same size and same type. Now we give several examples of maximal clique types of G.

Example 4.3.3 1. Let G be the Steinhaus graph which is generated by the string (0001101101). Then G has the clique $\{1, 4, 7, 10\}$ which is of type I in G.

2. Let G be the Steinhaus graph with generating string (011011011011). Then $\{1, 3, 6, 9, 12\}$ is a maximal clique of type II.

3. Let G be the Steinhaus graph with generating string (010010001001). Then $\{1, 2, 5, 8, 11\}$ is a maximal clique of type III.

4. Let G be the Steinhaus graph with generating string (0101101010101). Then $\{1,2,4,7,10,13\}$, $\{1,4,5,7,10,13\}$ and $\{1,4,7,8,10,13\}$ are maximal cliques of type IV.

5. Let G be the Steinhaus graph with generating string (0100100100101). Then $\{1,2,5,8,11,13\}$ is a maximal clique of type V.

Now, we want to classify a largest clique Q of size $w(n)$ in a Steinhaus graph with n vertices and show that Q must be one of the above maximal types for n sufficiently large.

Let G be a Steinhaus gragh with generating string $(a_{1,i})_{i=1}^{n}$. Through this chapter we assume that if G is a Steinhaus graph with n vertices then G has the generating string $(a_{1,i})_{i=1}^{n}$. Then the entry $a_{i,j}$ in the Steinhaus triangle is given by

$$a_{i,j} \equiv \sum_{k=0}^{i-l} \binom{i-l}{k} a_{l,j-k} \pmod 2$$

for all $1 \leq l \leq i-1$.

Thus we have two useful facts from the above identity.

Fact I. If $a_{i,j} \equiv a_{i,j+3} \pmod 2$ then $a_{i+3,j+3} \equiv a_{i,j+2} + a_{i,j+3} \pmod 2$.

Fact II. If $l \geq 0$ then $a_{i+2^l,j+2^l} \equiv a_{i,j} + a_{i,j+2^l} \pmod 2$ by Lucas Theorem ([Sta] p.53).

By using the above facts, we will decide the positions of configurations in the cliques by the following lemmas.

Lemma 4.3.4 *Let Q be a clique in a Steinhaus graph G. Then Q does not contain $\{i, i+3, i+4, i+7\}$ for any i.*

Proof. Assume that Q contains $\{i, i+3, i+4, i+7\}$. Then we have

$$a_{i,i+3} = a_{i,i+7} = a_{i+4,i+7} = 1.$$

This gives a contradiction by Fact II. Hence Q does not contain $\{i, i+3, i+4, i+7\}$.
∎

Lemma 4.3.5 *Let Q be a clique in a Steinhaus graph G. Then Q does not contain $\{i, i+3, i+8, i+11\}$ for any i.*

Proof. Assume that Q contains $\{i, i+3, i+8, i+11\}$. Then we have $a_{i,i+3} = a_{i,i+11} = a_{i+8,i+11} = 1$. This gives a contradiction by Fact II. Hence Q does not contain $\{i, i+3, i+8, i+11\}$. ∎

The next lemma is a generalization of Lemma 4.2.2 and Lemma 4.2.3.

Lemma 4.3.6 *Let Q be a clique in a Steinhaus graph G. Then Q does not contain $\{i, i+2^l, j, j+2^l\}$ for any $j \geq i+1$ and $l \geq 0$.*

Proof. Suppose that Q contains $\{i, i+2^l, j, j+2^l\}$ for some $j \geq i+2^l$. Then we have $a_{i,j} = 1, a_{i,j+2^l} = 1$ and $a_{i+2^l,j+2^l} = 1$. This gives a contradiction by Fact II. ∎

Clearly from the above lemma, we get Lemma 4.2.2 and Lemma 4.2.3 when $l = 0, 1$ respectively.

From the above lemmas, we will find nice classifications of cliques of size $w(n)$. More precisely, if a Steinhaus graph with $n \geq 27$ has a clique of size $w(n)$ then at least one of its cliques of size $w(n)$ in G is a maximal clique type of G.

Lemma 4.3.7 *Let G be a Steinhaus graph with $3k+1$ vertices for any $k \geq 8$. If G has a clique $Q = \{y_1, y_2, \ldots, y_{k+2}\}$ of size $w(3k+1) = k+2$ in G then Q is either of type IV or of type V in G.*

Proof. By Corollary 4.2.5, Q satisfies $\max_{2 \leq i \leq w(n)}\{y_i - y_{i-1}\} \leq 3$. Moreover, Q contains either a CT configuration or C and T configurations.

If Q contains a CT configuration, it is obvious that Q is of type IV from Corollary 4.2.5.

If Q contains configuration C and T, by Lemma 4.3.5, either $y_2 = y_1 + 1$ or $y_{k+2} = y_{k+1} + 1$. We want to show that either $y_2 = y_1 + 1$ and $y_{k+2} = y_{k+1} + 2$ or $y_2 = y_1 + 2$ and $y_{k+2} = y_{k+1} + 1$.

Assume that $y_2 = y_1 + 1$ and $y_{k+2} = y_{k+1} + 3$. Then there exists an i such that $y_i = y_{i-1} + 2$ for some $3 \leq i \leq k + 1$.

If $y_{i-1} \geq 8$, then we have $a_{y_{i-3},j} = 1$ for $j = y_{i-2}, y_{i+1}$ because Q is a clique. Since $y_{i+1} - y_{i-2} = 8$, by Fact II we have $a_{y_{i-3}+8,y_{i+1}} = 0$. This gives a contradiction because $y_{i-3} + 8 = y_i$ and $a_{y_i,y_{i+1}} = 1$.

If $y_{i-1} < 8$ then $i - 1$ must be 3, so $y_2 = 2, y_3 = 5, y_5 = 10$ and $y_6 = 13$. Then we have $a_{2,j} = 1$ for $j = 5, 13$. Thus $a_{10,13} = 0$ by Fact II. This gives a contradiction by $a_{y_5,y_6} = a_{10,13} = 1$.

Thus, we have $y_{k+2} = y_{k+1} + 2$, which gives that Q is of type V in G.

If we assume that $y_2 = y_1 + 3$ and $y_{k+2} = y_{k+1} + 1$ then we have a contradiction by considering its partner, $P(G)$, by the same arguement in the above. ∎

Lemma 4.3.8 *Let G be a Steinhaus graph with $3k + 2$ vertices for any $k \geq 8$. Let $Q = \{y_1, y_2, y_3, \ldots, y_{k+2}\}$ be a clique of size $w(3k+2) = k + 2$ in G. If $y_1 = 1$ and $y_{k+2} = 3k + 2$ then Q is of type III in G.*

Proof. By Corollary 4.2.5, Q satisfies one of the following.

(1) $\max_{2 \leq i \leq k+2} \{y_i - y_{i-1}\} = 3$.

(2) $\max_{2 \leq i \leq k+2} \{y_i - y_{i-1}\} = 4$.

Case 1 $\max_{2 \leq i \leq k+2} \{y_i - y_{i-1}\} = 3$.

By Corollary 4.2.5, Q contains a C configuration. Thus we have a pair $\{y_i, y_{i+1}\}$ in Q such that $y_{i+1} = y_i + 1$. By Lemma 4.3.5, i is either 1 or $k + 1$. Therefore Q is of type III in G.

Case 2 $\max_{2\leq i\leq k+2}\{y_i - y_{i-1}\} = 4$.

By Corollary 4.2.5, Q contains either a CT configuration or C and T configurations.

In both cases, we have three pairs $\{y_i, y_{i+1}\}$, $\{y_j, y_{j+1}\}$ and $\{y_l, y_{l+1}\}$ in Q such that $y_{i+1} = y_i + 1$, $y_{j+1} = y_j + 2$ and $y_{l+1} = y_l + 4$. Without loss of generality, we can assume that j is greater than i by considering its partner $P(G)$.

Suppose that Q contains a CT configuration. Then $y_j = y_{i+1} + 2$. By Lemma 4.3.7, Q satisfies either $i = 1$ and $l \leq 4$ or $l = i - 1$.

First, if $i = 1$ and $l = 3$ we have $a_{4,11} = a_{8,11} = 1$. Then we have $a_{4,7} = 0$ by Fact II. Since $a_{4,8} = 1$ we have $a_{5,8} = 1$. On the other hand, we have $a_{1,4} = a_{1,8} = 1$. Thus $a_{5,8} = 0$ by Fact II. This gives a contradiction.

If $i = 1$ and $l = 4$ then we have $a_{4,13} = 0$ by Fact I. Thus $a_{3,11} = 0$. On the other hand, we have $a_{3,7} = 0$ by Fact I again. Thus we have $a_{7,11} = 0$. But $a_{7,11} = 1$, which gives a contradiction.

Next, if $l = i - 1$ then $a_{y_{i-1},y_{i+2}} = a_{y_i,y_{i+2}} = 1$ since $j + 1 = i + 2$. Note that $y_i = y_{i-1} + 4$ and $y_{i-1} = y_{i-2} + 3$. By Fact II, we have $a_{y_{i-1},y_i-1} = 0$. Then $a_{y_{i-1}+1,y_i} = 1$, so $a_{y_{i-1}+1,y_i} = 1$. Thus we have $a_{y_{i-2},y_{i-1}} = 0$ by applying Fact II to a_{y_{i-2},y_i} and $a_{y_{i-1}+1,y_i}$. This gives a contradiction.

In both cases, Q does not satisfy (2).

If Q contains C and T configurations, then $y_j > y_i + 2$. By Lemma 4.3.7, $i = 1$ and $l = i + 1 = 2$. Then $j \leq l + 1$, otherwise $j \geq l + 2$ so Q contains either $\{y_1, y_4, y_{j-2}, y_{j+1}\}$ or $\{y_1, y_4, y_j, y_{j+3}\}$. But $y_4 - y_1 = y_{j+1} - y_{j-2} = 8$ and $y_{j+3} - y_j = 8$. This is impossible by Lemma 4.3.6.

Finally, if $j = l + 1$ then $a_{2,y_k+1} = a_{6,y_k+1} = 1$. Thus $a_{2,y_k-1} = 0$ by Fact II, which gives a contradiction.

In any case, Q does not satisfy (2) also.

This proves the lemma. ∎

Lemma 4.3.9 *Let $Q = \{y_1, y_2, y_3, \ldots, y_{k+2}\}$ be a clique of size $w(3k + 3) = k + 2$ in a Steinhaus graph G with $3k + 3$ vertices. If $y_1 = 1$ and $y_{k+2} = 3k + 3$, then G has either a clique of type II, a clique of type III or a clique type IV in G of size $k + 2$.*

Proof. By Corollary 4.2.5, Q satisfies one of the followings.

(1) $\max_{2 \leq i \leq w(3k+3)} \{y_i - y_{i-1}\} = 3$.

(2) $\max_{2 \leq i \leq w(3k+3)} \{y_i - y_{i-1}\} = 4$.

(3) $\max_{2 \leq i \leq w(3k+3)} \{y_i - y_{i-1}\} = 5$.

Case 1 $\max_{2 \leq i \leq w(3k+3)} \{y_i - y_{i-1}\} = 3$.

Q contains configuration T by Corollary 4.2.5. Then there is a pair $\{y_i, y_{i+1}\}$ in Q such that $y_{i+1} = y_i + 2$ for some $1 \leq i \leq k + 1$. We want to show that either $i = 1$ or $i = k + 1$.

Suppose that i is neither equal to 1 nor $k + 1$ i.e. $2 \leq i \leq k$. Without loss of generality, we can assume that $y_{i+1} \geq 3(k + 1)/2$ by considering its partner $P(G)$. Then Q contains $\{y_i - 6, y_i - 3, y_i + 2, y_i + 5\}$ because $y_{i+1} = y_i + 2$ and $y_{i+2} = y_i + 5$. This gives a contradiction by Lemma 4.3.7. Hence either $i = 1$ or $i = k + 1$. Thus Q is of type II in G.

Case 2 $\max_{2 \leq i \leq w(3k+3)} \{y_i - y_{i-1}\} = 4$.

By Corollary 4.2.5, Q contains either a C configuration, a CT configuration or C and T configurations.

Subcase 2.1 Q contains a C configuration.

In this case, Q has two pairs $\{y_i, y_{i+1}\}, \{y_j, y_{j+1}\}$ such that

$$y_{i+1} = y_i + 1$$

and

$$y_{j+1} = y_j + 4.$$

We can assume that i is less than j by condering its partner $P(G)$. First by Lemma 4.3.5, i is either 1 or $j - 2$. Then by Lemma 4.3.7, $i = 1$ and $j = 3$. By applying Facts I and II, G has the generating string $01\alpha(001)^k$ where $\alpha = 0, 1$ and $(abc)^k$ means the sequence $abcabcabc\ldots$ that abc is repeated k times. If $\alpha = 0$, then $\{2, 3, 6, \cdots, 3k + 3\}$ is a clique of size $k + 2$, which is of type III in G.

If $\alpha = 1$, then $\{1, 3, 6, \cdots, 3k + 3\}$ is a clique of size $k + 2$, which is of type II in G.

In this case, G has either a clique of type II, size $k + 2$ in G or a clique of type III, size $k + 2$ in G which is different from Q.

Subcase 2.2 Q contains either a CT configuration or C and T configurations.

In both cases, Q has two pairs $\{y_i, y_{i+1}\}$, $\{y_j, y_{j+1}\}$ such that

$$y_{i+1} = y_i + 4$$

and

$$y_{j+1} = y_j + 4.$$

Thus Q contains $\{y_i, y_i + 4, y_j, y_j + 4\}$, which gives a contradiction by Lemma 4.3.8.

Case 3 $\max_{2 \leq i \leq w(3k+3)}\{y_i - y_{i-1}\} = 5$.

By Corollary 4.2.5, Q contains either a CT configuration or C and T configurations. In both cases, Q has three pairs $\{y_i, y_{i+1}\}$, $\{y_j, y_{j+1}\}$ and $\{y_l, y_{l+1}\}$ such that

$$y_{i+1} = y_i + 1,$$

$$y_{j+1} = y_j + 2$$

and

$$y_{l+1} = y_l + 5.$$

Without loss of generality, we assume that i is less than or equal to $j - 2$ by considering its partner $P(G)$.

Subcase 3.1 Q has a CT configuration.

Since Q contains a CT configuration, j is equal to $i + 2$. Moreover, l is equal to either 1 or $k + 1$ otherwise Q contains $\{y_{l-1}, y_l, y_{l+1}, y_{l+2}\}$ and $y_l = y_{l-1} + 3, y_{l+1} = y_l + 5$ and $y_{l+2} = y_{l+1} + 3$. This is impossible by Lemma 4.3.6.

First, if $l = k + 1$ then Q satisfies that y_l is equal to either y_{i+2} or y_{i+3} otherwise Q contains $\{y_{i+1}, y_{i+4}, y_{l-1}, y_{l+1}\}$, which is impossible by Lemma 4.3.8. In any case, $(a_{1,j'})_{j'=1}^{3k-2} = 0(101)^{k-1}$ by $y_{k+1} = 3k - 2$. For example, if $l = i + 2$ then $a_{3r+1, y_{i+1}+1} = 0$ for $0 \le r \le k - 2$ by Fact I. Then $(a_{3r+1,s})_{y_i \le s \le y_{i+2}} = (1101)$ for all $0 \le r \le k - 2$. From these entries, it is not difficult to show that $(a_{1,j'})_{1 \le j' \le 3k-2} = 0(101)^{k-1}$.

Now, we want to find the entries $a_{1,3k-1}$, $a_{1,3k}$ and $a_{1,3k+1}$ in the Steinhaus triangle of G. By using the entries $a_{i',3k+3}$ where $i'=1,4,7,10,13,16,19$ and part of $(3k - 5)^{th}$ column which is $(a_{i',3k-5})_{i'=1}^{3k-5} = 0(101)^{k-2}$ in the Steinhaus triangle of G and by applying Fact II several times to the above facts, we have $(a_{i',3k+3})_{i'=7}^{13} = (1, 0, 0, 1, 0, 0, 1)$. By those entries we can find the generating string of G such that

$$a_{1,3k-1} = 1,$$

$$a_{1,3k} = 0,$$

$$a_{1,3k+1} = 1,$$

$$a_{1,3k+2} = 1$$

and

$$a_{1,3k+3} = 1.$$

Therefore, $a_{y_{i+1},3k+1} = a_{y_{i+1},3k+3} = 1$. By Fact II, we have $a_{y_{i+2},3k+3} = a_{y_{i+2},y_{i+1}} = 0$, which is impossible.

If $l = 1$, we have the same result by the same argument in the above.

Subcase 3.2 Q has C and T configurations.

Let us remind some facts from Lemma 4.3.5, Lemma 4.3.6 and Lemma 4.3.7;

(a) Q does not contain $\{x, x+3, x+4, x+7\}$ by Lemma 4.3.5.

(b) Q does not contain $\{x, x+3, x+8, x+11\}$ by Lemma 4.3.6.

(c) Q does not contain $\{x, x+8, y, y+8\}$ by Lemma 4.3.7.

Suppose that y_l is equal to either y_1 or y_{k+2}.

Then either the subgraph $G - \{1,2,3,4,5\}$ has the maximal clique of type V $\{y_2, y_3, \ldots, y_{3k+2}\}$ with size $k+1$ or $G - \{3k-1, 3k, \ldots, 3k+3\}$ has the maximal clique of type V $\{y_1, y_2, \ldots, y_{k+1}\}$ with with size $k+1$ by Lemma 4.3.8. If $y_l = y_{k+2}$, then $\{y_1, y_2, \ldots, y_{k+1}\} = \{1, 2, 5, \ldots, 3k-2\}$. Therefore $(a_{1,j'})_{1 \leq j' \leq 3k-2} = (0(100)^{k-2}101)$ by Steinhaus property. Note that $a_{5,3k-5}$ is 0, $a_{9,3k-5}$ is 1, $a_{12,3k-5}$ is 1 and $a_{8,3k-2}$ is 1. We want to show that $a_{8,3k+2}$ and $a_{12,3k+2}$ are equal to 1. First, since $a_{17,3k+3}$ is 1, $a_{9,3k+3}$ is 0 by applying Fact II to $a_{9,3k-5}$. Since $a_{8,3k+3}$ is 1, $a_{8,3k+2}$ is 1. Next, since $a_{20,3k+3}$ is 1, $a_{12,3k+3}$ is 0 by applying Fact II to $a_{9,3k-5}$. Also, by applying Fact II to $a_{5,3k-5}$ and $a_{5,3k+3}$, $a_{13,3k+3}$ is 1. Therefore, $a_{12,3k+2}$ is 1. By applying Fact II to $a_{8,3k+2}$ and $a_{12,3k+2}$, $a_{8,3k-2}$ is 0, which is impossible. If $y_l = y_1$, this gives a contradiction by the similar argument in the above.

Suppose that $y_2 \leq y_l \leq y_k$.

If $y_i = y_1$ then either $y_l = y_{k-1}$ and $y_j = y_k$ or $y_l = y_k and y_j = y_{k+1}$ by (b) and (c). When $y_l = y_{k-1}$ and $y_j = y_k$, we want to show that

$a_{7,y_{k-1}} = 1$, $a_{7,y_k-1} = a_{11,y_k-1} = 0$, which gives a contradiction by Fact II. Since $a_{11,y_k} = a_{11,y_{k+1}} = 1$, $a_{13,y_{k+1}} = 0$ by Fact II. Since $a_{5,y_{k+1}} = 1$, $a_{5,y_{k-1}-1} = 1$ by fact II. Since $a_{5,y_{k-2}} = a_{5,y_{k-1}} = 1$, $a_{5,y_{k-2}+1} = 0$, therefore, $a_{7,y_{k-1}} = 1$ by Fact II. Since $a_{1,y_k} = a_{1,y_{k+1}} = a_{2,y_k} = a_{2,y_{k+1}} = a_{5,y_k} = a_{5,y_{k+1}} = 1$, $a_{1,y_k-1} = a_{2,y_k-1} = 0$ by Steinhaus property. Since $a_{1,y_k-1} = a_{2,y_k-1} = 1$, $a_{5,y_k-1} = a_{6,y_k-1} = 1$, by Fact II. Thus $a_{7,y_k} = 1$. So $a_{7,y_k-1} = 0$ since $a_{8,y_k} = 1$. Finally, from above it is easy to get that $a_{4,y_k-1} = 0$ and $a_{5,y_k-1} = a_{6,y_k-1} = 1$. Since $a_{4,y_k-1} = a_{5,y_k-1} = 1$ and $a_{6,y_k-1} = 0$, $a_{8,y_k-1} = 1$, $a_{9,y_k-1} = 0$ and $a_{10,y_k-1} = 1$ by Fact II. Therefore by Fact I and Steinhaus property, $a_{11,y_k-1} = 0$. Similarly, we will get a contradiction by the same process in the above when $y_l = y_k$ and $y_j = y_{k+1}$.

If $y_i \geq y_2$ and $y_i \geq y_l$ then $y_l = y_{k-2}$, $y_i = y_{k-1}$ and $y_j = y_{k+1}$ by (2). Since $a_{3r+1,y_{k-3}} = a_{3r+1,y_k} = 1$, $a_{3r+9,y_k} = 0$ by Fact II for $r = 0, 1, \ldots$. Since $a_{3r+10,y_{k+1}} = 1$ for $r = 0, 1, \ldots$, G has the generating string $0(011)^k(11)$ by the y_k^{th} column and applying Fact I several times. Thus G has the clique $\{1, 3, 4, 7, \ldots, 3k+1\}$ of size $k+2$ which is of type IV.

If $y_i \geq y_2$ and $y_i \leq y_l$, then y_j is either y_k or y_{k+1} by (a), (b) and (c). Say, $y_j = y_{k+1}$. Then either $y_i = y_{k-2}, y_l = y_{k-1}$ or $y_i = y_{k-1}, y_l = y_k$. First, if $y_i = y_{k-2}$ and $y_l = y_{k-1}$, we want to show that $a_{y_{k-3}+1,y_{k-1}} = a_{y_{k-3}+1,y_{k+1}} = 0$, which gives a contradiction since $a_{y_{k-1},y_{k+1}} = a_{y_{k-3}+1,y_{k-2}} + a_{y_{k-3}+1,y_{k+1}} = 0$. Since $a_{y_{k-3},y_{k-2}} = a_{y_{k-3},y_{k-1}} = 1$, $a_{y_{k-3}+1,y_{k-1}} = 0$. Since $a_{y_{k-3},y_{k+2}} = a_{y_{k-1},y_{k+2}} = 1$, $a_{y_{k-3},y_k+1} = 0$. So, $a_{y_{k-3}+1,y_{k+1}} = 0$ since $a_{y_{k-2},y_{k+1}} = 1$ and apply to get $a_{y_k,y_{k+2}} = 1$. Next, if $y_i = y_{k-1}$ and $y_l = y_k$, we want to show that $a_{r,y_{k-4}} = 1$ for $y_{k-6} \leq r \leq y_{k-5}$, which gives $a_{y_{k-6},y_{k-5}} = 0$. It is impossible. Note that $a_{y_{k-1},y_{k+2}} = a_{y_k,y_{k+2}} = 1$ and $a_{y_{k+1},y_{k+2}} = a_{y_{k+2},y_{k+2}} = 0$. Since $a_{y_{k-6},y_{k+2}} = 1$ and $a_{y_{k-6}+2,y_{k+2}} = 0$, $a_{y_{k-6},y_{k-4}+1} = a_{y_{k-6}+2,y_{k-4}+1} = 0$. By Fact I, we get the claim.

Finally, if $y_j = y_k$, we have the same conclusion by the above argument.

By combining all cases, we prove lemma. ∎

Theorem 4.3.10 *If $n \geq 24$, then at least one of the largest cliques of size $w(n)$ in a Steinhaus graph G with n vertices must be of*

$$\begin{cases} \text{type IV or type V} & \text{if } n = 3k+1; \\ \text{type III, type IV or type V} & \text{if } n = 3k+2; \\ \text{type II, type III, type IV or type V} & \text{if } n = 3k+3. \end{cases}$$

Proof. Let $Q = \{y_1, y_2, y_3, \cdots, y_{w(n)}\}$ be a clique of size $w(n)$ in G.

Case 1 $n = 3k+1$.

It follows from Lemma 4.3.8 that Q is either type IV or type V in G.

Case 2 $n = 3k+2$.

If $y_1 = 1$ and $y_{w(n)} = n$, then Q is of type III in G by Lemma 4.3.9.

If Q does not contain either 1 or $3k+2$, then consider the induced subgraph of G which is either $G - \{1\}$ or $G - \{n\}$ having a clique of size $w(3k+1)$. By Case 1, Q is either of type IV or of type V in G.

Case 3 $n = 3k+3$.

If $y_1 = 1$ and $y_{w(n)} = n$, then G has either a clique of type II, a clique of type III or a clique of type IV in G by Lemma 4.3.10.

Otherwise, consider the induced subgraph of G which is either $G-\{1\}$ or $G-\{n\}$ having a clique of size $w(3k+2)$. Then by Case 2, Q is either of type III, of type IV or type V in G.

By combining three cases, we prove theorem. ∎

Now, we are able to get the number of the largest cliques in the Steinhaus graph with n vertices from Theorem 4.3.11.

Before to get the number, we introduce two notations for our convenience. The following notations give simple expressions in the generating string of G with n vertices if we know the generating string of the induced subgraph of G which is either $G - \{1\}$ or $G - \{n\}$.

(1) $(\alpha * \beta) \equiv \alpha + \beta \pmod{2}$ where $\alpha, \beta = 0, 1$.

(2) $\alpha * (a_{1,i})_1^n$ means the generating string $(b_{1,i})_1^{n+1}$ such that $b_{2,i} = a_{1,i}$ for $2 \le i \le n$, $b_{1,n+1} = \alpha$ where $\alpha = 0, 1$.

(3) $(a_{1,i})_1^n * \alpha$ means the generating string $(b_{1,i})_1^{n+1}$ such that $b_{1,i} = a_{1,i}$ for $2 \le i \le n$, $b_{1,n+1} = \alpha$ where $\alpha = 0, 1$.

As examples of the above notations, we give the following fact.

Fact III.

(1) $0 * 0(101)^k = 0(011)^k * 0$ for any k.

(2) $1 * 0(101)^k = 01(001)^k$ for any k.

(3) $1 * 0(011)^k = 01(101)^k$ for any k.

(4) $0 * 0(100)^{k-1}(101) = \begin{cases} 00(111000)^{\frac{k-1}{2}}(110) & \text{if } k \text{ is odd;} \\ 01000(111000)^{\frac{k-2}{2}}(110) & \text{if } k \text{ is even.} \end{cases}$

(5) $0 * 0(010)^{k-1}(011) = \begin{cases} 000(111000)^{\frac{k-1}{2}}(10) & \text{if } k \text{ is odd;} \\ 011000(111000)^{\frac{k-2}{2}}(10) & \text{if } k \text{ is even.} \end{cases}$

(6) $1 * 0(100)^{k-1}(101) = \begin{cases} 01(000111)^{\frac{k-1}{2}}(001) & \text{if } k \text{ is odd;} \\ 00111(000111)^{\frac{k-2}{2}}(001) & \text{if } k \text{ is even.} \end{cases}$

(7) $1 * 0(010)^{k-1}(011) = \begin{cases} 011(000111)^{\frac{k-1}{2}}(01) & \text{if } k \text{ is odd;} \\ 000111(000111)^{\frac{k-2}{2}}(01) & \text{if } k \text{ is even.} \end{cases}$

(8) $\alpha * (R * \beta) = [(\alpha * \beta) * R] * \alpha$ for any generating string R.

Lemma 4.3.11 *Let Q be a clique in a Steinhaus graph. If Q contains $\{i, i+3, i+6, i+9\}$, then $(a_{i,j})_{j=i+3}^{i+9}$ is either $1(101)^2$ or $1(011)^2$.*

Proof. First, by Fact I, we have the following identities,

$$a_{i,i+4} + a_{i,i+5} \equiv 1 \pmod 2$$

and

$$a_{i,i+7} + a_{i,i+8} \equiv 1 \pmod 2.$$

Suppose that $(a_{i,i+4}, a_{i,i+5}) \neq (a_{i,i+7}, a_{i,i+8})$.

If $(a_{i,i+4}, a_{i,i+5}) = (1,0)$ and $(a_{i,i+7}, a_{i,i+8}) = (0,1)$, then $(a_{i+3,i+7}, a_{i+3,i+8}) = (0,0)$. Thus we have $a_{i+6,i+9} = 0$, by Fact II. This gives a contradiction.

If $(a_{i,i+4}, a_{i,i+5}) = (0,1)$ and $(a_{i,i+7}, a_{i,i+8}) = (1,0)$, then $(a_{i+3,i+7}, a_{i+3,i+8}) = (1,1)$, which gives a contradiction because $a_{i+6,i+9} = 0$, by Fact II again. ∎

Before we prove the next theorem, we want to observe two facts. Let G be a Steinhaus graph with n vertices which has a maximal clique type of G, $Q = \{y_1, y_2, \ldots, y_{w(n)}\}$.

First, G may have several different cliques of size $w(n)$ but they are of same type in G. If they are not of type IV, it is not difficult to show that G has only one maximal clique type of size $w(n)$ by Lemma 4.3.7. If they are of type IV, there are two pairs $\{y_i, y_{i+1}\}$ and $\{y_j, y_{j+1}\}$ such that if $j = i + 1$ then

$$y_{i+1} = y_i + 1, \quad y_{j+1} = y_i + 3$$

and if $i = j + 1$ then

$$y_{j+1} = y_j + 1, \quad y_i = y_j + 3.$$

Both are not cliques at same time in G.

Next, to find the number of all Steinhaus graphs which have a clique of certain type, first we will choose one of maximal clique types of and find all Steinhaus graphs which have the chosen maximal clique type. Then by adding their partners, we will get the number of all Steinhaus graphs which have a clique of the given type.

Theorem 4.3.12 *The number of all Steinhaus graphs with $n \geq 25$ vertices which have a largest clique of size $w(n)$ is*

$$\begin{cases} 4 & \text{if } n = 3k+1; \\ 19 & \text{if } n = 3k+2; \\ 60 & \text{if } n = 3k+3, \ k \text{ odd}; \\ 63 & \text{if } n = 3k+3, \ k \text{ even}. \end{cases}$$

Proof. Let Q be a clique of maximal type, size $w(n)$, in a Steinhaus graph G with n vertices.

Case 1 $n = 3k+1$.

By Theorem 4.3.11, Q is either of type IV or of type V in G.

If Q is of type IV in G, say $Q = \{1, 4, 7, \cdots, 3i+1, 3i+2, 3i+4, \cdots, 3k+1\}$ for some $0 \leq i \leq k-1$, then by Lemma 4.4.12, G has the generating string $0(101)^k$. So its partner, $P(G)$, has the generating string $0(011)^k$.

If Q is of type V in G, say $Q = \{1, 2, 5, \cdots, 3k-4, 3k-1, 3k+1\}$, then by Lemma 4.4.13 again, G has the generating string $0(100)^{k-1}(101)$. So its partner, $P(G)$, has the generating string $0(010)^{k-1}(011)$.

By counting both types, we have 4 distinct Steinhaus graphs which have a clique either of tpye IV or of type V with size $w(3k+1)$.

Case 2 $n = 3k+2$.

By Theorem 4.4.11, Q is either of type III, of type IV or of type V in G.

Subcase 2.1 Q is of type III in G.

Since Q is of type III in G, it is clear that Q is either $\{1, 2, 5, 8, \cdots, 3k+2\}$ or $\{1, 4, 7, \cdots, 3k+1, 3k+2\}$. If Q is $\{1, 2, 5, 8, \cdots, 3k+2\}$, then G has the generating string either $(01\alpha01)(001)^{k-1}$ or $(01\alpha01)(101)^{k-1}$ by applying Lemma 4.3.12 to $G - \{1\}$ where $\alpha = 0, 1$.

If Q is $\{1, 4, 7, \ldots, 3k + 1, 3k + 2\}$, then G is one of the partners of the above Steinhaus graphs. In fact, the generating string of G is given by

$$\begin{cases} \text{either } 0\alpha(011)^k \text{ or } 00\alpha(101)^{k-1}11 & \text{if } k \text{ is even} \\ \text{either } 0\alpha\beta11(011)^{k-1} \text{ or } 0\alpha\beta(101)^{k-1}11 & \text{if } k \text{ is odd} \end{cases}$$

where $\alpha, \beta = 0, 1$ and $\alpha + \beta \equiv 1 \pmod 2$.

Since none of them are doubly symmetric and they have different partners, we have 8 distinct Steinhaus graphs G which have a clique of type III with size $w(3k + 2)$ in G.

Subcase 2.2 Q is of type IV in G.

Since Q is of type IV in G, Q is contained in either $\{1, 2, 3, \ldots, 3k + 1\}$ or $\{2, 3, 4, \ldots, 3k + 2\}$.

If Q is contained in $\{2, 3, 4, \ldots, 3k + 2\}$ then by applying Case 1 to $G - \{1\}$ and Q, G has the generating string which is either $\alpha * 0(101)^k$ or $\alpha * 0(011)^k$ where $\alpha = 0, 1$. By Fact III. (2) and (3), $1 * 0(101)^k$ and $1 * 0(011)^k$ are of type III in G. Thus we have 2 distinct Steinhaus graphs G which have a clique Q of type IV with size $w(3k + 2)$ in G where Q is contained in $\{2, 3, 4, \ldots, 3k + 2\}$. By adding their partners $P(G)$ which have a clique of type IV where the clique is contained in $\{1, 2, 3, \ldots, 3k + 1\}$, we may have 4 Steinhaus graphs G which have a clique of type IV, size $w(3k + 2)$ in G. But by Fact III. (1), we have only 3 distinct Steinhaus graphs G which have a clique of type IV with size $w(3k + 2)$ in G.

Subcase 2.3 Q is of type V in G.

By the same arguement in Subcase 2.2, if Q is contained in $\{2, 3, 4, \ldots, 3k + 3\}$ then G has the generating string either $\alpha * 0(100)^{k-1}(101)$ or $\alpha * 0(010)^{k-1}(011)$ where $\alpha = 0, 1$. By Fact III. (4) - (7), they all are of type V. Therefore we have 4 distinct Steinhaus graphs G which have a clique Q of type V with size $w(3k + 2)$ in G where Q is contained in

$\{2, 3, 4, \ldots, 3k + 2\}$. Since none of them are doubly symmetric, by adding their partners we have 8 distinct Steinhaus graphs.

Thus by combining three cases, we have 19 distinct Steinhaus graphs G which have a clique of type V with size $w(3k + 2)$ in G.

Case 3 $n = 3k + 3$.

Again, by Theorem 4.3.11, we can assume that Q is either of type II, type III, type IV or type V in G.

Subcase 3.1 Q is of type II in G.

Since Q is of type II in G, Q is either $\{1, 3, 6, \ldots, 3k+3\}$ or $\{1, 4, 7, \ldots, 3k+1, 3k + 3\}$. If Q is $\{1, 3, 6, \ldots, 3k + 3\}$, then by Lemma 4.3.12, $(a_{3,j})_{j=6}^{3k+3}$ is either $1(101)^{k-1}$ or $1(011)^{k-1}$. By applying Fact II several times, G has a generating string which is either $0\alpha1(011)^k$ or $0\alpha1(001)^k$ where $\alpha = 0, 1$. Since none of them are doubly symmetric, by adding their partners we have 8 distinct Steinhaus graphs G which have a clique of type II with size $w(3k + 3)$ in G.

Subcase 3.2 Q is of type III in G.

Since Q is of type III, Q is contained in either $\{1, 2, 3, \ldots, 3k + 2\}$ or $\{2, 3, 4, \ldots, 3k + 3\}$. Suppose that Q is $\{1, 2, 5, \ldots, 3k + 2\}$.

If k is an odd number, the generating string of G is either $01\alpha01(001)^{k-1}\beta$, $01\alpha01(101)^{k-1}\beta$, $0\alpha\gamma11(011)^{k-1}\beta$ or $0\alpha\gamma(101)^{k-1}\beta$ by Subcase 2.1 where $\alpha, \beta, \gamma = 0, 1$ and $\alpha + \gamma \equiv 1 \pmod 2$. First, when the generating string of G is $01\alpha01(001)^{k-1}\beta$, its partner, $P(G)$, has the generating string $0(\beta*1)(\alpha*\beta*1)\beta(\beta*1)[(\beta*1)\beta(\beta*1)]^{k-1}\beta$ which has a clique of type III, size $w(3k + 3)$ in $P(G)$. Since none of them are doubly symmetric, there are 8 distinct Steinhaus graphs which have a clique of type III. When the generating string of G is $01\alpha01(101)^{k-1}\beta$, its partner, $P(G)$, has the generating

string $0(\beta*1)(\alpha*\beta*1)[\beta(\beta*1)\beta]^k$. But when $\alpha, \beta = 1$, the generating string is $(011)^{k+1}$ which has a clique of type II by Subcase 3.1. Also when $\beta = 0$ and either $\alpha = 0$ or $\alpha = 1$, the generating string is either $010(011)^{k-1}010$ or $(011)^k 010$ respectively which is the same generating string as given above. Since the remaining generating strings are not doubly symmetric, there are 2 more Steinhaus graphs which have a clique of type III. When the generating string of G is $0\alpha(\alpha*1)11(011)^{k-1}\beta$, its partner, $P(G)$, has the generating string $0(\alpha*\beta)(\alpha*\beta*1)[(\beta*1)(\beta*1)\beta]^k$. But when $\alpha, \beta = 1$, the generating string is $010(110)^{k-1}111$ which has a clique of type II by Subcase 3.1. When $\beta = 0$ and either $\alpha = 0$ or $\alpha = 1$, the generating string is either $001(110)^k$ or $010(110)^k$ respectively each of which is doubly symmetric. Since the remaining generating string is not doubly symmetric, there are 4 more Steinhaus graphs which have a clique of type III. Finally, when the generating string of G is $0\alpha(\alpha*1)(101)^{k-1}11\beta$, its partner, $P(G)$, has the generating string $0(\alpha*\beta)(\alpha*\beta*1)(\beta*1)(\beta*1)[\beta\beta\beta(\beta*1)(\beta*1)(\beta*1)]^{(k-1)/2}\beta$. Since none of them are doubly symmetric, there are 8 Steinhaus graphs which have a clique of type III. By combining all cases, we have 22 distinct Steinhaus graphs which have a clique of type III with size $w(3k+3)$ if k is odd.

If k is an even number, the generating string of G is either $01\alpha01(001)^{k-1}\beta$, $01\alpha01(101)^{k-1}\beta$, $01\alpha11(011)^{k-1}\beta$ or $00\alpha(101)^{k-1}11\beta$ by Subcase 2.1 where $\alpha, \beta = 0, 1$. When the generating string of G is $01\alpha01(001)^{k-1}\beta$, its partner, $P(G)$, has the generating string $0(\alpha*\beta*1)^2\beta(\beta*1)[(\beta*1)\beta(\beta*1)]^{k-1}\beta$. Since none of them is doubly symmetric, we have 8 Steinhaus graphs which have a clique of type III. When the generating string of G is $01\alpha01(101)^{k-1}\beta$, its partner, $P(G)$, has the generating string $0(\alpha*\beta)(\alpha*\beta*1)[\beta(\beta*1)\beta]^k$. But when $\alpha, \beta = 1$, the generating string of G is $(011)^{k+1}$, which is of type II by subcase 3.1. When $\alpha = 1$ and $\beta = 0$, the gener-

ating string of G is $(011)^k010$, which is the same generating string in the above. Since the remaining generating strings are not doubly symmetric, there are 4 more Steinhaus graphs which have a clique of type III. When the generating string of G is $01\alpha11(011)^{k-1}\beta$, its partner, $P(G)$, has the generating string $0(\alpha*\beta*1)(\alpha*\beta)(\beta*1)(\beta*1)[\beta(\beta*1)(\beta*1)]^{k-1}\beta$. But when $\alpha=0$ and $\beta=1$, the generating string of G is $010(110)^{k-1}111$, which is of type II by Subcase 3.1. When $\alpha=0$ and $\beta=0$, the generating string of G is $010(110)^k$, which is doubly symmetric. Since the remaining generating strings are not doubly symmetric, there are 5 more Steinhaus graphs which have a clique of type III. When the generating of G is $00\alpha(101)^{k-1}(11\beta)$, its partner, $P(G)$, has the generating string $0(\alpha*\beta)(\alpha*\beta*1)\beta\beta[(\beta*1)(\beta*1)(\beta*1)\beta\beta\beta]^{(k-2)/2}(\beta*1)(\beta*1)(\beta*1)\beta$. Since none of them are doubly symmetric, there are 8 more Steinhaus graphs which have a clique of type III. By combining all cases, we have 25 distint Steinhaus graphs which have a clique of type III with size $w(3k+3)$ if k is even.

Subcase 3.3 Q is of type IV in G.

Since Q is of type IV in G, Q is contained either in $\{1,2,3,\ldots,3k+1\}$, $\{2,3,4,\ldots,3k+2\}$ or $\{3,4,5,\ldots,3k+3\}$.

First, suppose that Q is contained in $\{1,2,3,\ldots,3k+1\}$. Then by Case 1, the generating string of G is either $0(101)^k*\alpha*\beta$ or $0(011)^k*\alpha*\beta$ where $\alpha,\beta=0,1$. Assume that the generating string of G is $0(101)^k*\alpha*\beta$. When $\alpha=1$ and $\beta=0$, the generating string of G is $010(110)^k$ which is of type III by Subcase 3.2. When $\alpha,\beta=1$, the generating string of G is $010(110)^{k-1}111$ which is of type II by Subcase 3.1. Assume that the generating string of G is $0(011)^k*\alpha*\beta$. When $\alpha=0$ and $\beta=1$, the generating string of G is $001(101)^k$ which is of type II by Subcase 3.1. When $\alpha=1$, the generating string of G is $0(011)^k1*\beta$ which is of type III

by Subcase 3.2. By considering their partners, we have 6 distinct Steinhaus graphs which have a clique of type IV with size $w(3k + 3)$.

Next, suppose that Q is contained in $\{2, 3, 4, \ldots, 3k + 2\}$. By Case 1 and Fact III. (8), the generating string of G is either $\alpha * [0(101)^k * \beta] = 0[(\alpha * \beta)(\alpha * \beta * 1)(\alpha * \beta * 1)]^k * (\alpha * \beta) * \alpha$ or $\alpha * [0(011)^k * \beta] = 0[(\alpha * \beta)(\alpha * \beta)(\alpha * \beta * 1)]^k * (\alpha * \beta) * \alpha$. If α is not equal to β, then the generating string of G is either $01(001)^k * \alpha$ or $01(101)^k * \alpha$, which is of type III by Subcase 3.2. If α is equal to β, then the generating string of G is either $0(011)^k * 0 * \alpha$ or $0(001)^k * 0 * \alpha$ where $\alpha = 0, 1$. When α is equal to 1, $0(011)^k 01$ is of type II by Subcase 3.1 and $0(001)^k 01$ is of type II because its partner has generating string $001(101)^k$ and apply Subcase 3.2. When α is equal to 0, $0(011)^k 00$ is of type IV which appeared in the above and $0(001)^k 00$ is of type IV which appeared in the above because its partner has the generating string $0(011)^k 00$. Thus we do not have any new Steinhaus graphs which have a clique of type IV in this case.

By combining both cases, we have 6 distinct Steinhaus graphs which have a clique of type IV with size $w(3k + 3)$.

Subcase 3.4 Q is of type V in G.

Since Q is of type V in G, Q is contained in either $\{1, 2, 3, \ldots, 3k + 1\}$, $\{2, 3, 4, \ldots, 3k + 2\}$ or $\{3, 4, 5, \ldots, 3k + 3\}$.

First, suppose that Q is contained in $\{1, 2, 3, \ldots, 3k + 1\}$. Then by Case 1, the generating string of G is either $0(100)^{k-1}(101) * \alpha * \beta$ or $0(010)^{k-1}(011) * \alpha * \beta$ where $\alpha, \beta = 0, 1$. By considering their partners, we have 16 distinct Steinhaus graphs which have a clique of type IV with size $w(3k + 3)$.

Next, suppose that Q is contained $\{2, 3, 4, \cdots, 3k + 2\}$. By Case 1 and Fact III. (8), the generating string of G is either $\alpha * [0(100)^{k-1}(101) * \beta]$ or $\alpha * [0(010)^{k-1}(011) * \beta]$ where $\alpha, \beta = 0, 1$. Note that if G has a

generating string $\alpha * [0(100)^{k-1}(101) * \beta]$ then the partner of G, $P(G)$, has the generating string $\alpha * [0(010)^{k-1} * (\alpha * \beta)]$. Thus in this case we have 8 distinct Steinhaus graphs which have a clique of type IV with size $w(3k + 3)$.

By combining both cases, we have 24 distinct Steinhaus graphs which have a clique of type IV with size $w(3k + 3)$.

From the above Subcases, the number of all Steinhaus graphs which have a clique of size $w(3k + 3)$ is

$$\begin{cases} 60 & \text{if } k \text{ is odd;} \\ 63 & \text{if } k \text{ is even.} \end{cases}$$

By combining all cases, we prove theorem. $\blacksquare$

Before we close this section, we list all generating strings of Steinhaus graphs of order $n \geq 27$ which have a clique of size $w(n)$.

(I) $n = 3k + 1$.

    type IV: $0(101)^k$ $0(011)^k$

    type V: $0(100)^{k-1}(101)$ $0(010)^{k-1}(011)$

(II) $n = 3k + 2$.

    (1) k is odd.

    type III: $01(001)^k$ $01101(001)^{k-1}$ $01(101)^k$ $01001(101)^{k-1}$

             $00111(011)^{k-1}$ $01(011)^k$ $001(101)^{k-1}(11)$ $010(101)^{k-1}(11)$

    type IV: $00(110)^k$ $0(001)^k 0$ $0(101)^k 0$

    type V: $0(100)^{k-1}(1010)$ $0(100)^{k-1}(1011)$ $0(010)^{k-1}(0110)$

             $0(010)^{k-1}(0111)$ $00(111000)^{\frac{k-1}{2}}(110)$ $000(111000)^{\frac{k-1}{2}}(10)$

             $01(000111)^{\frac{k-1}{2}}(001)$ $011(000111)^{\frac{k-1}{2}}(01)$

(2) k is even.

type III: $\quad 01(001)^k \quad 01101(001)^{k-1} \quad 01(101)^k \quad 01001(101)^{k-1}$

$\qquad 00(011)^k \quad 01(011)^k \quad 000(101)^{k-1} \quad 001(101)^{k-1}11$

type IV: $\quad 00(110)^k \quad 0(001)^k0 \quad 0(101)^k0$

type V: $\quad 0(100)^{k-1}(1010) \quad 0(100)^{k-1}(1011) \quad 0(010)^{k-1}(0110$

$\qquad 0(010)^{k-1}(1011) \quad 01000(111000)^{\frac{k-2}{2}}(110) \quad 011000(111000)^{\frac{k-2}{2}}(10)$

$\qquad 00111(000111)^{\frac{k-2}{2}}(001) \quad (000111)^{\frac{k}{2}}(01)$

III $\quad n = 3k + 3.$

(1) k is odd.

type II: $\quad 001(011)^k \quad (011)^{k+1} \quad (001)^{k+1} \quad 011(001)^k$

$\qquad 00(011)^k1 \quad 0(101)^k(11) \quad 0(011)^k(01) \quad 01(110)^k1$

type III: $\quad 01(001)^k0 \quad 01(101)^k0 \quad 01(001)^k1 \quad 00(010)^k1$

$\qquad 011(010)^k \quad 010(011)^{k-1}(010) \quad 0110(100)^{k-1}(11) \quad 001(100)^{k-1}(101)$

$\qquad 010(011)^k \quad 000(101)^k \quad 001(110)^k \quad 010(110)^k$

$\qquad 001(110)^{k-1}(111) \quad 010(001)^k \quad 001(101)^{k-1}(110) \quad 001(101)^{k-1}(110)$

$\qquad 00111(000111)^{\frac{k-1}{2}}0 \quad 01000(111000)^{\frac{k-1}{2}}1 \quad 010(101)^{k-1}(110)$

$\qquad 01011(000111)^{\frac{k-1}{2}}0 \quad 00100(111000)^{\frac{k-1}{2}}1 \quad 010(101)^{k-1}(111)$

type IV: $\quad 0(101)^k(00) \quad 0(101)^k(01) \quad 0(011)^k(00) \quad 000(100)^k$

$\qquad 0000(111000)^{\frac{k-1}{2}}(11) \quad 0111(000111)^{\frac{k-1}{2}}(00)$

type V: $0(100)^{k-1}(10100)$ $0(100)^{k-1}(10101)$ $0(100)^{k-1}(10110)$

$0(100)^{k-1}(10111)$ $0(010)^{k-1}(01100)$ $0(010)^{k-1}(01101)$

$0(010)^{k-1}(01110)$ $0(010)^{k-1}(01111)$ $00(111000)^{\frac{k-1}{2}}(1100)$

$01(000111)^{\frac{k-1}{2}}(0011)$ $01(000111)^{\frac{k-1}{2}}(0010)$

$00(111000)^{\frac{k-1}{2}}(1101)$ $000(111000)^{\frac{k-1}{2}}(100)$

$011(000111)^{\frac{k-1}{2}}(011)$ $011(000111)^{\frac{k-1}{2}}(010)$ $000(111000)^{\frac{k-1}{2}}(101)$

$$\begin{cases} 0001(011110100001)^l(00) & \text{if } k = 4l+1; \\ 0110100001(011110100001)^l(00) & \text{if } k = 4l+3. \end{cases}$$

$$\begin{cases} 0011110100001(011110100001)^{l-1}(11011) & \text{if } k = 4l+1; \\ 0100001(011110100001)^l(11011) & \text{if } k = 4l+3. \end{cases}$$

$$\begin{cases} 0(011110100001)^l(01110) & \text{if } k = 4l+1; \\ (000010101111)^l(000010101110) & \text{if } k = 4l+3. \end{cases}$$

$$\begin{cases} 010100001(011110100001)^{l-1}(01111011) & \text{if } k = 4l+1; \\ 00001(011110100001)^l(01111011) & \text{if } k = 4l+3. \end{cases}$$

$$\begin{cases} 0111(010000101111)^l(00) & \text{if } k = 4l+1; \\ (010000101111)^{l+1} & \text{if } k = 4l+3. \end{cases}$$

$$\begin{cases} 00001(011110100001)^l 1 & \text{if } k = 4l+1; \\ 01110100001(011110100001)^l & \text{if } k = 4l+3. \end{cases}$$

$$\begin{cases} 01011(110100001011)^l 0 & \text{if } k = 4l+1; \\ 01100001011(110100001011)^l 0 & \text{if } k = 4l+3. \end{cases}$$

$$\begin{cases} 001(000010111101)^l(001) & \text{if } k = 4l+1; \\ 010111101(000010111101)^l(001) & \text{if } k = 4l+3. \end{cases}$$

(2) k is even.

type II: $001(011)^k$ $(011)^{k+1}$ $(001)^{k+1}$ $011(001)^k$

$00(011)^k 1$ $0(101)^k(11)$ $0(011)^k(01)$ $01(110)^k 1$

type III: $(010)^{k+1}$ $(011)^k(010)$ $01(001)^k 1$ $00(010)^k 1$

$011(010)^k$ $00001(101)^{k-1}1$ $01101(001)^{k-1}1$ $01110(010)^{k-1}1$

$010(011)^{k-1}9010$ $001(010)^k$ $010(011)^k$ $010(101)^k$

$010(110)^k$ $011(110)^k$ $001(110)^k$ $011(110)^{k-1}(111)$

$010(001)^k$ $000(101)^{k-1}(110)$ $000(101)^{k-1}(111)$ $001(101)^{k-1}(110)$

$001(101)^{k-1}(111)$ $00100(111000)^{\frac{k-2}{2}}(1110)$ $01011(000111)^{\frac{k-2}{2}}(0001)$

$01000(111000)^{\frac{k-2}{2}}(1110)$ $00111(000111)^{\frac{k-2}{2}}(0001)$

type IV: $0(101)^k(00)$ $0(101)^k(01)$ $0(011)^k(00)$ $000(100)^k$

         $0(000111)^{\frac{k}{2}}(00)$ $0(111000)^{\frac{k}{2}}(11)$

type V: $0(100)^{k-1}(10100)$ $0(100)^{k-1}(10101)$ $0(100)^{k-1}(10110)$

         $0(100)^{k-1}(10111)$ $0(010)^{k-1}(01100)$ $0(010)^{k-1}(01101)$

         $0(010)^{k-1}(01110)$ $0(010)^{k-1}(01111)$ $011000(111000)^{\frac{k-2}{2}}(100)$

         $(000111)^{\frac{k}{2}}(100)$ $(000111)^{\frac{k}{2}}(010)$ $011000(111000)^{\frac{k-2}{2}}(010)$

         $01000(111000)^{\frac{k-2}{2}}(1100)$ $00111(000111)^{\frac{k-2}{2}}(0011)$

         $00111(000111)^{\frac{k-2}{2}}(0010)$ $01000(111000)^{\frac{k-2}{2}}(1101)$

$$\begin{cases} 0(011110100001)^l(00) & \text{if } k = 4l; \\ 0100001(011110100001)^l(00) & \text{if } k = 4l+3. \end{cases}$$

$$\begin{cases} 0110100001(011110100001)^{l-1}(11011) & \text{if } k = 4l; \\ 0001(011110100001)^l(11011) & \text{if } k = 4l+3. \end{cases}$$

$$\begin{cases} 0110100001(011110100001)^{l-1}(01110) & \text{if } k = 4l; \\ 0011110100001(011110100001)^{l-1}(01110) & \text{if } k = 4l+3. \end{cases}$$

$$\begin{cases} 0100001(011110100001)^{l-1}(01111011) & \text{if } k = 4l; \\ 0(011110100001)^l(01111011) & \text{if } k = 4l+3. \end{cases}$$

$$\begin{cases} 0(010000101111)^l(00) & \text{if } k = 4l; \\ 0101111(010000101111)^l(00) & \text{if } k = 4l+3. \end{cases}$$

$$\begin{cases} 01(011110100001)^l1 & \text{if } k = 4l; \\ 00100001(011110100001)^l1 & \text{if } k = 4l+3. \end{cases}$$

$$\begin{cases} 01(110100001011)^l0 & \text{if } k = 4l; \\ 00001(011110100001)^l(0110) & \text{if } k = 4l+3. \end{cases}$$

$$\begin{cases} (000010111101)^l(001) & \text{if } k = 4l; \\ 0111(010000101111)^l(01001) & \text{if } k = 4l+3. \end{cases}$$

4.4   The number of Steinhaus graphs which have a clique of large size

In Theorem 4.3.13, we showed that the number of all Steinhaus graphs with n vertice which have a clique of size $w(n)$ has period 6. In other word, $W(n,0) = W(n+6,0)$, where l is a nonnegative integer and $W(n,l)$ is the number of all Steinhaus graphs with n vertices which have a largest clique of size $w(n) - l$. So, it is natural to ask the following question:

$$W(n,l) \overset{?}{=} W(n+6,l)$$

for a fixed l and large enough n.

A series of lemmas will give the answer of this question.

Lemma 4.4.1 *Let G be a Steinhaus graph with generating string $(a_{1,j})_{1 \leq j \leq n}$. Let $Q(G) = \{y_1, y_2, y_3, \ldots, y_m\}$ be a clique of type V in G which is contained in a largest clique Q in G. Let $j \geq 3$ be a positive integer. Let k be the smallest integer such that $2^k \geq j$. If $y_m \geq 2^{k+1} + j + 8$ then $y_m + j \notin Q$.*

Proof. Without loss of generality, we can assume that $y_1 = 1$ by considering the induced subgraph $G - \{1, 2, 3, \ldots, y_1 - 1\}$ of G.

Suppose that $y_m + j \in Q$. Then by Fact II, we are able to find all entries $a_{i, y_m + j}$ in Steinhaus triangle where $2^k + 1 \leq i \leq y_m - 2^k - 1$. Using these entries, we are able to find a_{i, y_m} where $2^k + 1 \leq i \leq 2^k + 7$. Moreover, there exists an i such that

$$a_{i, y_m} = a_{i+1, y_m} = 1$$

for some $1 \leq i \leq 7$.

On the other hand, since $Q(G)$ is of type V in G, the first row in the induced Steinhaus triangle on the subgraph $G - \{y_m + 1, \ldots, n\}$ of G is either $0(010)^{(y_m - 4)/3}(011)$ or its partner $0(100)^{(y_m - 4)/3}(101)$. This gives a contradiction by the same argument in the above. ∎

Lemma 4.4.2 *Let G be a Steinhaus graph with n vertices.*

Let $Q(G) = \{y_1, y_2, y_3, \ldots, y_m\}$ be a maximal clique type of G which is not of type V in G. Under the same conditions on j, k, y_m and Q in Lemma 4.4.1, we have one of the followings:

(1) $y_m + j \notin Q$.

(2) $Q \cup \{y_m + 3\}$ *is a clique.*

Proof. Suppose that $y_m + 3 \in Q$. Then by the same argument in Lemma 4.4.1, we conclude that $Q \cup \{y_m + 3\}$ is a clique. ∎

Now, we are ready to prove the following theorem.

Theorem 4.4.3 *Let l be a fixed nonnegative integer. Let G be a Steinhaus graph with n vertices which has a largest clique of size $w(n) - l$. Then at least one of largest cliques in G must be one of maximal clique type of G if n is large enough.*

Proof. Let $Q(G) = \{y_1, y_2, y_3, \dots, y_m\}$ be a maximal clique type of G. Suppose that a clique Q contains $Q(G)$ where $Q = \{x_1, x_2, x_3, \dots, x_{w(n)-l}\}$ is of size $w(n) - l$ in G.

We want to show that $Q = Q(G)$.

Suppose that $Q \neq Q(G)$. Since the size of Q is $w(n) - l$, we have the inequality

$$\max_{2 \le i \le w(n)-l} \{x_i - x_{i-1} - 1\} \le 3l.$$

Let x be the smallest integer in Q which is greater than y_m. Then $4 \le x - y_m \le 3l + 1$. Let j be the integer $x - y_m$. Since n is large enough, y_m is also large enough to satisfy the conditions in the lemmas.

Case 1 $Q(G)$ is of type V.

By Lemma 4.4.1, $y_m + j = x \notin Q$. This gives a contradiction.

Case 2 $Q(G)$ is not of type V.

Since $y_m + j \in Q$, $Q(G) \cup \{y_m + 3\}$ is not a clique by Lemma 4.4.2. On the other hand, we can show that $Q \cup \{y_m + 3\} - \{y_m + 4\}$ is a clique of size at least $w(n) - l$. Thus $Q(G) \cup \{y_m + 3\}$ is a maximal clique type of G which contains $Q(G)$. This gives a contradiction by the choice of $Q(G)$.

By combining two cases, we prove theorem. ∎

To prove that the question in this section is true, we need the following lemmas.

Lemma 4.4.4 *Let G be a Steinhaus graph wuth n vertices which have a largest clique of size $w(n) - l$. Then there exists exactly one maximal clique type of G,size $w(n) - l$ where l is a fixed nonnegative integer and n is large enough.*

Proof. Suppose that G has two maximal clique types of G. then by combining two maximal cliques, G has another maximal clique which is larger than two maximal cliques by applying Fact I and II, which is impossible. $\blacksquare$

Lemma 4.4.5 *Let $Q(G) = \{y_1, y_2, y_3,, \ldots, y_{w(n)}\}$ be a type I of G, size $w(n)$. Then the two subtriangles generated by $(a_{i,k})_{k=i}^{i+5}$ and $(a_{j,k})_{k=j}^{j+5}$ in the Steinhaus triangle of G are same if $y_1 \leq i \leq j \leq y_{w(n)}$ and $i \equiv j \pmod 6$.*

Proof. By Lemma 4.3.13, the y_1^{th} row in the Steinhaus triangle of G is either $0\alpha\beta(101)^{(w(n)-4)/3}$ or $0\alpha\beta(011)^{(w(n)-4)/3}$ where $\alpha, \beta = 0, 1$. Then the $(y_1 + 6)^{th}$ row is either $0\alpha\beta(101)^{(w(n)-10)/3}$ or $0\alpha\beta(011)^{(w(n)-10)/3}$ respectively. $\blacksquare$

Now, we are ready to prove the following theorem.

Theorem 4.4.6 *Let l be a fixed nonnegative integer. Then we have the following:*

$$W(n,l) = W(n + 6, l)$$

if n is large enough.

Proof. Let $\Gamma(n,l)$ be the collection of all Steinhaus graphs G with n vertices which have a maximal clique type of G, size $w(n) - l$. To prove the theorem, we want to find a bijection function from $\Gamma(n,l)$ to $\Gamma(n+6,l)$. Here, we observe some fact about Steinhaus triangle. In fact, there are several ways to generate the same Steinhaus triangle. For example, $1 \leq j \leq n$, we may denote $(a_{1,j}, a_{2,j}, \ldots, a_{j,j})(a_{j,j+1}, \ldots, a_{j,n})$ as a generating string of G with the Steinhaus triangle $(a_{i,j})$. In partucular, when $j = 1$, we have the standard generating string.

Let G be a Steinhaus graph with n vertices which has the unique maximal type clique $Q(G) = \{y_1, y_2, y_3, \ldots, y_{w(n)-l}\}$ of size $w(n) - l$. Consider the Steinhaus triangle $(a_{i,j})$ of G and the generating string $(a_{1,y_1}, \ldots, a_{y_1,y_1})(a_{y_1,y_1+1}, \ldots, a_{y_1,n})$ of G. From this generating string, we will construct a Steinhaus graph G' with $n + 6$ vertices. It suffices to construct a string which is the generating string of G'. Define the string $(b_{1,y_1}, \ldots, b_{y_1,y_1})(b_{y_1,y_1+1}, \ldots, b_{y_1,n+6})$ by

$$b_{i,j} = \begin{cases} a_{i,j} & \text{if } 1 \leq i \leq y_5 \text{ and } y_1 \leq j \leq y_5 \\ a_{i,j-6} & \text{otherwise} \end{cases}.$$

It is clear that the Steinhaus graph G' with the above generating string has a largest clique of size $w(n + 6) - l$ which is the same type of G' as $Q(G)$ of G by Lemma 4.4.5. Moerover, it is not difficult to show that the above contruction gives a one to one correspondence between $\Gamma(n, l)$ and $\Gamma(n + 6, l)$. ∎

4.5 Maximal clique in the complements of Steinhaus graphs

In [Dy2], Dymacek study several properties of complements of Steinhaus graphs. In this section, we investigate the results in previous sections on the complements of Steinhaus graphs.

For each positive integer n, it is obious that the largest size of cliques in the set of all the complements of Steinhaus graphs is n, which can be obtained by the empty graph E^n with the generating string consisting of zeros.

Let l be a fixed nonnegative integer. Let $\overline{W}(n, l)$ be the number of all steinhaus graphs with n vertices such that their complements have a maximal clique of size $n - l$. Now we give a nice expression of clique of large size in the complement of Steinhaus graph.

Lemma 4.5.1 *Let l be a fixed positive integer. Let G be a Steinhaus graph with n vertices which has the Steinhaus triangle $(a_{i,j})$. Let $\overline{Q}$ be a clique in the complement of G. Then the size of $\overline{Q}$ is $n - l$ if and only if $Q = \{i, i+1, i+2, \ldots, n-l+i-1\}$*

and $(a_{1,i}, a_{2,i}, a_{3,i}, \ldots, a_{i-2,i}, 1)0^{n-l}(1, a_{i,n-l+i+1}, \ldots, a_{i,n})$ *is the generating string of* G *for some* $1 \leq i \leq l+1$ *if* n *is large enough.*

Proof. Suppose that the size of $\overline{Q}$ is $n - l$. Since n is large enough, $\overline{Q}$ contains a set A of size at least n/l which consists of consecutive vertices. First, assume that j is the largest vetex in A and k is a vertex in $\overline{Q}$ which is greater than j. Since $k - j$ is far less than n/l, all entries in the Steinhaus triangle of the induced subgraph $A \cup \{j+1, j+2, j+3, \ldots, k\}$ are zero. By applying this arguement continuously, we conclude that $\overline{Q}$ consists of consecutive vertices. If i is the smallest vertex in $\overline{Q}$, G has the generating string in the above.

Conversely, suppose that G has the generating string in the above and $\{i, i+1, i+2, \ldots, n-l+i-1\}$ is a clique of size $n - l$ in the complement of G. Since n is large enough, this set is maximal clique in the complement of G. ∎

Theorem 4.5.2 *Let l be fixed.*

$$\overline{W}(n, l) = \overline{W}(n+1, l)$$

if n is large enough. Futhermore, $\overline{W}(n, l)$ is

$$\overline{W}(n, l) = \begin{cases} 1 & \text{if } l = 0; \\ 2 & \text{if } l = 1; \\ 2^l + (l-1)2^{l-2} & \text{otherwise.} \end{cases}$$

Proof. Let $\overline{\Gamma}(n, l)$ be the collection of all Steinhaus graphs G with n vertices which have a maximal clique of size $n - l$ in the complement of G. It is sufficient to find a bijection fuction between $\overline{\Gamma}(n, l)$ and $\overline{\Gamma}(n+1, l)$.

Let G be a Steinhaus graph whose complement has a maximal clique of size $n - l$. G has the generating string $(a_{1,i}, \ldots, a_{i-2,i}, 1)0^{n-l}(1, a_{i,n-l+i+1}, \ldots, a_{i,n})$ for some $1 \leq i \leq l+1$ by Lemma 5.5.1. So, $\{i, i+1, i+2, \ldots, n-l+i-1\}$ is the maximal clique in the complement of G. From this string, we construct the string $(a_{1,i}, \ldots, a_{i-2,i}, 1)0^{n-l+1}(1, a_{i,n-l+i+1}, \ldots, a_{i,n})$ of lenghth $n + 1$. Clearly, the

Steinhaus graph with the generating string in the above has the maximal clique $\{i, i+1, i+2, \ldots, n-l+i\}$ of size $n-l+1$. It is not difficult to show that this construction gives a bijection.

Next, we will decide the number $\overline{W}(n,l)$. Clearly, $\overline{W}(n,0) = 1$ and $\overline{W}(n,1) = 2$.

Suppose that $l \geq 2$. For each $1 \leq i \leq l+1$, let $\overline{W}_i$ be the number of the Steinhaus graphs G in $\overline{\Gamma}(n,l)$ such that the smallest vertex in the maximal clique in the complement of G is i. If i is either 1 or $l+1$, then $\overline{W}_i = 2^{l-1}$. If $2 \leq i \leq l$, then $\overline{W}_i = 2^{l-2}$ by Lemma 4.5.1.

By combining both cases, $\overline{W}(n,l) = 2^l + (l-1)2^{l-2}$ if $l \geq 2$. ∎

CHAPTER 5

HAMILTON CYCLES IN RANDOM STEINHAUS GRAPHS

5.1 Introduction

Since its introduction by Erdős and Rényi ([ErRe1], [ErRe2]), the theory of random graphs has been greatly developed and many properties of a random graph have been studied in detail [Bo4], [Bo5], [Ma] etc. One of the important questions Erdős and Rényi [ErRe2] raised in their fundamental paper on the evolution of random graphs is "is almost every graph Hamiltonian?" A breakthrough was achieved by Pósa [Po2] and Korshunov [Kor]. They prove that for some constant c almost every labelled graph with n vertices and at least $cn \log n$ edges is Hamiltonian.

On the other hand, it would be useful to have a criterion by which to decide whether a specific graph behaves like a random graph, that is, has the property (of almost every graph) that interests us. Such a criterion gives the concepts of pseudo-random graphs and quasi-random graphs which is a special type of pseudo-random graphs. In [Th2], Thomason shows that a (p, α)-jumbled graph behaves like a random graph with edge probability p.

The properties of random Steinhaus graphs and random generalized Steinhaus graphs have been investigated by Brand and other authers. The first paper to address a question of this nature is [Br1] in which Brand answered in the affirmative Brigham and Dutton's [BrDu] conjecture that almost all Steinhaus graphs have diameter two where $P[a_{1,j} = 1] = 1/2$. In [BCDJ], Brand, Curran, Das and Jacob generalize this result to the case where $0 < P[a_{1,j} = 1] < 1$. A much more general result is obtained in [BrJa] in which Brand and Jackson show that the theory of random Steinhaus graphs is first order complete and identitical with the first order theory of random graphs. Thus a first order statement is true for almost all graphs if and only if it is

true for almost all Steinhaus graphs. Moreover in [BrMo], Brand and Morton show that almost all generalized Steinhaus graphs are quasi-random graphs.

In section 5.2, we investigate if almost all Steinhaus graphs are Hamiltonian.

5.2   Hamilton cycles

Many theorems on Hamilton graphs require a degree condition (see [Be], [Bol]). But not many graphs satisfy the degree conditions. For example, in [Pa] we see that almost all graphs G do not satisfy the condition that for every pair of nonadjacent vertices u and v, $d(u) + d(v) \geq n$. Also some theorems on Hamilton graphs require an edge condition (see [Be], [Ore]). But not many graphs satisfy the edges conditions. For example, it is clear that almost all graphs G do not satisfy the condition that the number of edges of G is at least $\lfloor \frac{(n-1)(n-2)}{2} + 3 \rfloor$. Therefore it is natural to think that if we combine an edge condition (sometimes, called a global condition) with a degree condition then a given graph satisfying both conditions may be Hamiltonian. From the definition of quasi-random graph G, with n vertices ([CGW]) we find the following property P_4 which we use through this Chapter 5 because of its good global condition:

P_4.   For each subset S of G, $e(S) = \frac{1}{4}|S|^2 + o(n^2)$

where $o(n^2)$ means $\frac{o(n^2)}{n^2} \to 0$ as $n \to \infty$.

In [CGW], they showed that this property P_4 gives the following property P_0':

P_0'.   All but $o(n)$ vertices of G have degree $\frac{1}{2}(1 + o(1))n$

where $o(1)$ means $o(1) \to 0$ as $n \to \infty$.

In this case we say that G is *almost-regular*. Note that the property P_0' does not imply the property P_4 ([CGW]). Thus we conclude that a quasi-random graph has a good global condition but does not have a good degree condition. From now we

assume that the probability of an edge is $\frac{1}{2}$. In [BrMo] and [Pa], we can find that almost every Steinhaus graph satisfies the desired global and degree conditions.

Theorem 5.2.1 ([BrMo]) *Almost all Steinhaus graphs are quasi-random.*

Theorem 5.2.2 ([Si]) *Let $\varepsilon > 0$. Then almost all Steinhaus graphs satisfy*

$$\frac{1}{2}(1 - \varepsilon)n < d(v) < \frac{1}{2}(1 + \varepsilon)n$$

for all of their vertices v.

In this section, we present two proofs that almost all Steinhaus graphs are Hamiltonian. The first proof follows from a result in [Th2] and the second proof follows the standard method in the theory of random graphs (see [Bo2], [Po2]) with the above theorems. Let us give the first proof.

Let G be a quasi-random and Steinhaus graph with n vertices. Let S be a subset of G. Then we have

$$e(S) = \frac{1}{4}|S|^2 + o(n^2).$$

Thus

$$e(S) - \frac{1}{2}\binom{|S|}{2} = \frac{1}{4}|S| + o(n^2) = o(n^2).$$

If $|S| \leq o(n)$ then

$$\left| e(S) - \frac{1}{2}\binom{|S|}{2} \right| \leq \frac{1}{4}|S|^2 \leq o(n)|S|$$

and if $|S| \geq o(n)$ then

$$\left| e(S) - \frac{1}{2}\binom{|S|}{2} \right| = o(n^2) \leq o(n)|S|.$$

By combining both cases, we show that the graph G is $(\frac{1}{2}, o(n))$-jumbled. Thus almost all Steinhaus graphs are $(\frac{1}{2}, o(n))$-jumbled. Since almost all the Steinhaus graphs G satisfy the degree condition in Theorem 5.2.2, $\delta(G) \geq (1 + o(1))\frac{n}{2}$ where $\delta(G)$ is the minimum degree of G. This gives the first proof by the following theorem.

Theorem 5.2.3 ([Th2]) *Let G be a (p,α)-jumbled graph, and P be a path in G of length $l \geq 0$ and $\delta(G)$ be the minimum degree of G. If $\delta(G) \geq 6\alpha p^{-1} + l$, then G has a Hamilton cycle containing P.*

Now we give the second proof.

Let $G = (V, E)$ be a Steinhaus graph with n vertices. Also, assume that G is a quasi-random graph with the degree condition in Theorem 5.2.2. Let x_0 be a vertex in G. Let S be a longest x_0-path in G, that is a path beginning at x_0: $S = x_0 x_1 \ldots x_k$. Then the neighbor of x_k, $\Gamma(x_k)$, is contained in $\{x_0, x_1, \ldots, x_{k-1}\}$ since otherwise S could be continued to a longer path. If x_k is adjacent to x_j, $0 \leq j < k - 1$, then $S' = x_0 x_1 \ldots x_j x_k x_{k-1} \ldots x_{j+1}$ is another longest x_0-path. We call S' a simple transform of S.

Let L be the set of end vertices (different from x_0) of transforms of S and put $N = \{x_j \in S : x_{j-1} \in L \text{ or } x_{j+1} \in L\}$ and $R = V - N \cup L$. We are now ready to state Pósa's lemma.

Theorem 5.2.4 ([Bo2], [Po1]) *The graph G has no L-R edges.*

Corollary 5.2.5 *If $|L| \leq n/3$ then there are disjoint sets of size $|L|$ and $n - 3|L| + 1$, that are joined by no edges of G.*

Proof. Consider L and R in Theorem 5.2.4. Then we have

$$|R| = n - |N \cup L| \geq n - 2|l| \geq n - 3|l| + 1.$$

Choose any subset W of R such that the size of W is $n - 3|L| + 1$. ∎

Let U and W be two subsets of G. Then from Theorem 5.2.1 and Theorem 5.2.2, we get the following corollaries.

Corollary 5.2.6 *Let k be the number of edges between U and $G - W$. Then k is given by*

$$k = \frac{1}{2}|U||W - U| + o(n^2).$$

Proof. Apply the property P_4 to the subsets U, W, $U \cup W$, $U - W$ and $W - U$. ∎

Corollary 5.2.7 $|U \cup \Gamma(U)| \geq \frac{1}{2}(1 + o(n))n.$

Now we give a simple lemma in the vein of Theorem 5.2.10.

Lemma 5.2.8 *Let $0 < \gamma < 1/3$ be a constant. Then almost all Steinhaus graphs G are such that if U is a subset of G and $|U| \leq \gamma n$ then*

$$|U \cup \Gamma(U)| \geq 3|U|.$$

Proof. Suppose that there is a quasi-random Steinhaus graph G with n vertices such that

$$|U \cup \Gamma(U)| < 3|U|$$

for some γ and some subset U of G and $|U| \leq \gamma n$.

Let W be the complement of $U \cup \Gamma(U)$. Denote a, b and c by the size of subsets U, $\Gamma(U) - U$ and W respectively. By Corollary 5.2.6, $(1 + o(1))\frac{n}{6} < a$. Also, $\frac{ac}{2} = o(n^2)$ by Corollary 5.2.5.

Since $a \leq \gamma n$ and $a + b < 3a$, we have $c \geq (1 - 3\gamma)n$. Thus we have

$$o(n^2) = \frac{ac}{2} \geq \frac{n^2}{12}(1 + o(1))(1 - 3\gamma).$$

This gives a contradiction for all n large enough. ∎

Let $G = (V, E)$ be a Steinhaus graph with n vertices which is quasi-random. Denote D_t by the number of pairs (X, Y) of disjoint subsets of U such that $|X| = t$, $|Y| = n - 3t$ and G has no $X - Y$ edges. In fact, Corollary 5.2.5 provides an example of D_t. Lemma 5.2.8 gives the following corollary.

Corollary 5.2.9 *Let $D = \{G : D_t = 0 \text{ for every } t, 1 \leq t \leq \gamma n\}$ where $0 < \gamma < \frac{1}{3}$ is a constant. Then we have*

$$P(\bar{D}) = o(1)$$

where $\bar{D}$ is the complement of D in G.

Now we give the second proof.

Theorem 5.2.10 *Almost all Steinhaus graphs contain a Hamiltonian path. More precisely, if x and y are arbitrary distinct vertices, then almost every Steinhaus graph contains a Hamilton path from x to y.*

Proof. Since almost all Steinhaus graphs are quasi-random, we can assume that the Steinhaus graphs in this proof are quasi-random. Let us introduce the following notation for certain events whose general element is denoted by G.

- Let D be the collection of all Steinhaus graphs such that $D_t = 0$ for every t where $1 \le t \le \gamma n$ and $0 < \gamma < \frac{1}{3}$.

- Let $E(W, x)$ be the collection of all Steinhaus graphs G such that the induced subgraph $G[W]$ of G has a path of maximal length whose end vertex is joined to x.

- Let $E(W, x|w)$ be the collection of all Steinhaus graphs G such that the induced subgraph $G[W]$ has a w-path of maximal length among the w-paths whose end vertex is joined to x.

- Let $F(x)$ be the collection of all Steinhaus graphs G such that every path of maximal length in G contains x.

- Let $H(W)$ be the collection of all Steinhaus graphs G such that the induced subgraph $G[W]$ of G has a Hamilton path.

- Let $H(x, y)$ be the collection of all Steinhaus graphs G such that G has a Hamilton x-y path.

- The complement of an event A is $\overline{A}$.

Note that by Corollary 5.2.9 we have

$$P(\overline{D}) = 1 - P(D) = o(1).$$

Let $|W| = n - 2$ or $n - 1$ and let us estimate the probability of the event $D \cap \overline{E}(W, x)$ and $P(D \cap \overline{E}(W, x))$ where x is not in W. Let $G \in D \cap \overline{E}(W, x)$ and consider a path $S = x_0 x_1 \ldots x_k$ of maximal length in $G[W]$. (By introducing an ordering in W, we can easily achieve that S is determined by $G[W]$.) Let $L = L(G[W])$ be the set of end vertices of the transforms of the x_0-path S and let R be as in Theorem 5.2.4 (applied to $G[W]$). Recall that $|R| \geq |W| + 1 - 3|L|$ and there is no L-R edge, so no L-$R \cup \{x\}$ edge either. Since $G \in D$ and $|R \cup \{x\}| \geq n - 3|L|$, we find that $|L| \geq \gamma n$. As L is independently of the edges incident with x, we have

$$P(D \cap \overline{E}(W, x)) \leq P(G \in D \text{ and } x \text{ is not joined to } L(G[W])) \leq (\tfrac{1}{2})^{\lfloor \gamma n \rfloor}.$$

Exactly the same proof implies that

$$P(D \cap \overline{E}(W, x | w) \leq (\tfrac{1}{2})^{\lfloor \gamma n \rfloor}$$

provided $|W| = n - 2$ or $n - 1$, $x \in W$ and $x \notin W$.

Note now that $\overline{F}(x) \subset \overline{E}(V - \{x\}, \{x\})$, so

$$
\begin{aligned}
P(\overline{H}(V)) &= P(\cup_{x \in V} \overline{F}(x)) \\
&\leq P(D \cap \cup_{x \in V} \overline{F}(x)) + P(\overline{D}) \\
&\leq \sum_{x \in V} P(D \cap \overline{F}(x)) + P(\overline{D}) \\
&\leq n P(D \cap \overline{E}(V - \{x\}, x)) + P(\overline{D}) \\
&\leq n (\tfrac{1}{2})^{\lfloor \gamma n \rfloor} + o(1).
\end{aligned}
$$

This proves that almost every Steinhaus graph has a Hamilton path.

Now let x and y be distinct vertices and put $W = V - \{x, y\}$. By the first part

$$P(\overline{H}(W)) \leq 2n(\tfrac{1}{2})^{\lfloor \gamma n \rfloor} + o(1).$$

Since $H(x, y) \supset H(W) \cap E(W, y) \cap E(W, x | y)$ we have

$$
\begin{aligned}
P(\overline{H}(x, y)) &\leq P(\overline{H}(W)) + P(D \cap \overline{E}(W, y)) + P(D \cap \overline{E}(W, x | y)) + P(\overline{D}) \\
&\leq 2n(\tfrac{1}{2})^{\lfloor \gamma n \rfloor} + 2(\tfrac{1}{2})^{\lfloor \gamma n \rfloor} + o(1).
\end{aligned}
$$

Thus almost every Steinhaus graph contains a Hamilton path from x to y. ∎

Corollary 5.2.11 *Almost all Steinhaus graphs are Hamiltonian.*

Proof. Let $\varepsilon > 0$ be given. Choose k such that $(\frac{1}{2})^k < \frac{\varepsilon}{2}$. Let $H([n,i])$ be the collection of all Steinhaus graphs with n vertices which have a Hamilton path from the vertex 1 to the vertex i and $A([n,i])$ be the collection of all Steinhaus graphs such that the vertex 1 is adjacent to the vertex i for $2 \leq i \leq n$. Then by Theorem 5.2.10 there exists $n_0 > k$ such that

$$P(H([n,i])) > 1 - \frac{\varepsilon}{2^k}$$

for all $n \leq n_0$ and $2 \leq i \leq k$.

Therefore, we have

$$
\begin{aligned}
P(\bigcap_{i=2}^{k} H([n,i]) \cap \bigcap_{i=2}^{k} A([n,i])) \ &> \ 1 - \sum_{i=2}^{k} \frac{\varepsilon}{2^i} - \frac{1}{2^k} \\
&> \ 1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} \\
&= \ 1 - \varepsilon
\end{aligned}
$$

for all $n \geq n_0$.

This shows that almost all Steinhaus graphs are Hamiltonian. ∎

We close by mentioning Hamiltonian connected property on Steinhaus graphs. While almost all graphs are Hamiltonian connected ([Bo2]), it is still not known that almost all Steinhaus graphs are Hamiltonian connected.

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