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# THE VORTEX THEORY AND ITS SIGNIFICANCE IN AVIATION

## PART I. VORTEX THEORY

## PART II. WING THEORY

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# THE VORTEX THEORY AND ITS SIGNIFICANCE IN AVIATION.*

By A. Betz.

During the last decade considerable progress has been made in the theory of fluid motion, particularly in two directions. Prandtl's boundary-layer theory enabled us to determine, at least qualitatively, the origin of drag.**

By drag we mean the force experienced by a moving body in opposition to its motion. A theory of wing lift which enables the mathematical treatment of a vast number of practical problems with considerable reliability will be developed later.***

By lift we mean the force normal to the direction of motion. The vortex idea and a few other closely related conceptions play a very important part in both these problems. In spite of the fact that much theoretical work is being done to-day on the basis of the vortex idea, considerable confusion still prevails regarding the real significance of this theory.

*"Der Wirbelbegriff und seine Bedeutung für die Flugtechnik," from Unterrichtsblätter für Mathematik und Naturwissenschaften, Vol. 34 (1928), No. 12. (Lecture delivered in 1926 summer course at Göttingen."

**Prandtl, "Ueber Flussigkeitsbewegung bei sehr kleiner Reibung," Third International Congress of Mathematicians, p. 484. (For translation, see N.A.C.A. Technical Memorandum No. 452: "Motion of Fluids with Very Little Viscosity," (1928).

***Compare Prandtl's "Wing Theory," Parts I and II, Nachr. v. d. Kgl. Ges. d. Wissensch. zu Göttingen, Math.-Phys., Klasse 1918, p. 151; and 1919, p. 107; or the more popular presentation by Betz, "Einfuhrung in die Theorie der Flugzeug-Tragflugel," Die Naturwissenschaften 1918, pp. 557 and 573.

Aside from the difficulty of representation exhibited by fluid motion in general, the widely spread confusion about the vortex conception is also due to the fact that the term itself is not always used to indicate the same thing, so that misunderstandings easily arise. The ideas closely related to the vortex conception will be illustrated therefore by the simplest possible examples. In addition to these general considerations, I shall then devote my attention principally to showing the application of the vortex theory in connection with the wing theory.

## Part I: The Vortex Theory

Let us now observe a few typical forms of flow, so that we may learn from them the conception of rotation. In the simplest form of flow, all fluid particles move parallel to one another and at the same velocity (Fig. 1). The lines represent streamlines, while the length of the heavy arrows is proportional to the velocity. If a small stick s is placed in such a flow, it will be moved by and parallel to the latter, its orientation remaining unchanged, independently of its original position in the fluid.

Another simple flow is that in which a fluid rotates like a solid body (Fig. 2). The velocity of any point in this fluid is therefore proportional to its distance from the axis of rotation. If a small stick s is placed in this flow, at each rotation of the fluid body it will also revolve about the same axis; its orientation does not remain the same, however, but turns 360° with every revolution. In this process it is also immaterial as to what the original position of the stick is, the angular velocity not being affected thereby.

A third type of flow is represented in Figure 3. The streamlines are straight and parallel to each other as in the first type, but the velocity increases proportionally to the distance from a reference line of zero velocity. If we now bring our small stick parallel to the direction of the flow ($s_1$), it will experience no turning. If, on the contrary, we place it in a direction normal to the flow ($s_2$), it will then turn in the direction indicated by the arrow, with an angular velocity of

$$\omega_z = \frac{dv}{dy}.$$

The angular velocity in this case depends on the orientation of the small stick and varies between zero in the position $s_1$ and a maximum value in the position $s_2$.

As the last example, we will consider again a flow with concentric circular paths (Fig. 2). The velocity in this case decreases, however, inversely proportional to the distance from the middle point (Fig. 4). In this case a small stick placed parallel to the streamlines $s_1$ turns once in a clockwise direction for each revolution, while a small stick $s_2$ in the radial position will turn in the opposite direction, due to the given velocity distribution. If $v$ is the velocity at the point in question, r distant from the middle point, the time for a complete turn is then $\frac{2 r \pi}{v}$. Hence the angular velocity of the small stick $s_1$ is

$$\omega_1 = \frac{v}{r}.$$

Since v has been assumed to be inversely proportional to the radius, for any point in the fluid, we may write

$$v = \frac{v_0}{r},$$

in which $v_0$ is now a constant for every point. Therefore

$$\omega_1 = \frac{v_0}{r^2}.$$

The small stick $s_2$ has the angular velocity

$$\omega_2 = \frac{dv}{dr}.$$

However, since $v = \frac{v_0}{r}$, we get

$$\omega_2 = -\frac{v_0}{r^2},$$

that is, equal in magnitude but opposite in sign to $\omega_1$. In this example, as well as in the previous ones, we have therefore the case that the rotational speed depends on the position of the small stick.

These processes may appear clearer to the eye if we imagine a square formed of small sticks, as shown in Figures 1 to 4, and observe the deformation caused by the fluid motion. Figure 5 shows the variations in the four above-described typical flows. In the first case (uniform parallel flow), there is no other variation besides a lateral displacement. In the second case (rotation like a solid body), the figure rotates without changing its form. In the third and fourth cases the square is distorted into a rhombus. In the fourth case the dotted diagonals of the figure retain their original direction, since the angular velocity of the small sticks is exactly symmetrical to the diagonals. If we consider what forces produce this deformation of the fluid square, it is obvious that they must be symmetrical to the diagonals and therefore produce no rotational moment. A fluid particle is considered as self-rotating when a turning moment is required for the production of its state of motion. Fluid particles, the state of motion of which is produced without the intervention of turning moments, as in the above case 4, are classified as nonrotational. The fourth example represents therefore a nonrotational motion. In the third case we have the same deformation as in the last example, but also a rotation through which two sides of the parallelogram are brought back to their original direction, the angular velocity of rotation being

$$\omega = \frac{\omega_1 + \omega_2}{2} = \frac{1}{2}\frac{dv}{dy} .$$

That example 1 represents a nonrotational motion, and example 2, a rotational one, requires no further explanation. We may simply remark that the rotation is mathematically expressed by the vector "rot V." As you probably know, this is found by the formula

$$\text{rot } V = \frac{dv}{dy} = \frac{du}{dx} .$$

This is, however, equal to $\omega_1 + \omega_2$; consequently to double the angular velocity $\omega$ of the rotation.

We shall now undertake to answer the question as to how a rotational moment may be exerted on a fluid particle, thereby setting the latter in motion. We imagine a sphere separated from the rest of the fluid and let the same pressures act over the whole surface (Fig. 6). Since the pressures are normal to the surface, they are all directed toward the center and can therefore produce no turning moment. Symmetrical deformation, however, can be produced as, for example, when the horizontal forces are greater than the vertical ones. If we wish to set the spherical fluid particle in rotation, tangential shearing forces must be applied to the surface. These shearing stresses in a fluid particle are transferred to the adjacent medium by virtue of the fluid viscosity. In some deformations , such as those of examples 3 and 4, certain tangential forces appear on the peripheral surface which are proportional to the angular velocity $\omega_1 - \omega_2$ (Fig. 7). From these considerations, we learn that pure pressure forces can produce only nonrotational motion, while for the production of rotational motion, we must have forces produced by the viscosity of the fluid. We might perhaps infer that this consideration applies to a spherical particle, but not to any other shape. If we stop to think, however, that the internal portions of any fluid particle, in motion of any form whatever, possess rotation, we must then admit that in order to impart the motion to the whole particle, each portion within the latter must be set in motion.

In order to explain these considerations, I shall point out how motions similar to the four above-mentioned examples are produced.

The first form of flow (parallel) can be produced by letting the fluid from a large receptacle flow through a nozzle (Fig. 8). For this purpose, we need only take care that the pressure in the nozzle is lower than that prevailing in the receptacle A, so that the fluid may flow into the tube. At some distance from the tube inlet the streamlines will be parallel to the walls of the tube. The pressure at any section of the nozzle must be constant therefore, since each pressure drop normal to the direction of flow would deflect the flow from its path, and the streamlines would be no longer straight. Equality of pressure at any cross section, however, also requires equality of velocity, since these are only the result of the pressure difference between the nozzle and the large space A. It has been assumed that no force other than this pressure difference has any accelerating or retarding effect on the fluid. In particular, we must exclude even frictional effects.

The procedure is quite different if we wish to produce the second form of flow. The production of this is not possible by mere pressure difference for, in order to bring about the velocities of the outer portion, we must see that the pressure in this region is less than in the slower inner region. Due to the centrifugal force of the fluid particles, such a pressure distribution is not possible, however. These could, in fact, move in their circular path only if we had pressure difference directed toward the center in order to counteract the centrifugal force. A circular path requires therefore the pressure at the outer region to be greater than at the inner region. Such a pressure distribution, however, is also contrary to that needed for the production of the desired velocity distribution. This form of flow can be produced, however, by rotating a hollow cylinder filled with a liquid about its own axis. Due to viscosity (friction) the fluid is gradually carried along by the cylinder wall and eventually attains a uniform rotation. It is not necessary for the viscosity to be particularly strong. On the contrary, it suffices for it to be always present. The smaller the viscosity is, the longer it will take to impart the contemplated motion to the revolving fluid.

The third form of flow can be produced by letting the fluid flow between two smooth parallel walls, one of which stands still while the other moves along parallel to itself (Fig. 9). Due to its viscosity the fluid is then carried along by the moving wall and the desired flow is gradually acquired.

The flow given in the fourth example can be produced, at least approximately, by the method illustrated in Figure 10. The fluid flows out of the large container A through a semicircular channel. By virtue of the centrifugal force, the pressure on the outer wall of this curved path will be greater than that on the inner wall, and consequently the velocity near the outer wall will be smaller than that near the inner wall. Deviations from the contemplated form of flow occur, however, on account of the proximity of the straight portion of the channel. These deviations become smaller as the channel becomes narrower with respect to the radius of curvature. The existence of a velocity distribution equal to $v = \frac{v_o}{r}$ in such a flow can be mathematically proved.

Let $p_o$ be the pressure in the large container A and p the pressure at a point in the bend. According to Bernoulli's equation, the relation between pressure and velocity in a frictionless flow is

$$\frac{\rho}{2} v^2 = p_o - p.$$

The radial pressure increase, due to the centrifugal force, is

$$\frac{\partial p}{\partial r} = \frac{\rho v^2}{r}.$$

Differentiating Bernoulli's equation, we obtain

$$\rho\, v \frac{\partial v}{\partial r} = - \frac{\partial p}{\partial r} = - \rho \frac{v^2}{r},$$

from which

$$\frac{\partial v}{\partial r} = - \frac{v}{r}$$

or

$$v = \frac{v_o}{r}$$

in which $v_o$ is the integration constant.

Thus we see that a rotational fluid motion can result only from frictional forces which are in turn attributable to the viscosity of the fluid. Such frictional forces, however, do not necessarily produce a rotational motion. If we have a nonrotational motion, such as that in our fourth example, the shearing forces will also be symmetrical and will therefore exert no turning moment on a fluid particle and will not set it in rotation. We cannot easily convince ourselves of this fact if we consider the shearing forces acting on the four sides of the distorted square (Fig. 7), which are perfectly symmetrical. The same is true when the rotation in the vicinity of the particle is constant, as in the second and third examples.

If a fluid, originally at rest, is set in motion by a pressure difference, similarly to the case of the fluid flowing from a vessel (Fig. 8), or even when a body is set in motion in a fluid, no shearing forces are present at the beginning of the motion, since the velocity and also the velocity gradients necessary for the shearing forces are everywhere zero. The fluid particles are therefore nearly all accelerated by pressure forces, and the resulting weak flow is consequently nonrotational. Consequently, as we have just considered, the gradually appearing shearing forces likewise produce no turning moment and the motion therefore remains nonrotational.

From this we obtain the remarkable result that, when a viscous fluid is set in motion by pressure forces, a nonrotational motion develops internally. The question instinctively arises as to how rotation takes place in spite of all this, since experience shows that in every fluid motion, regions of more or less rotational flow develop. The solution of this difficulty is found by observing the phenomena at the fluid boundary. We have already seen that the shearing forces in a nonrotational motion are so distributed over the four sides of the square (Figs. 4-7) that they produce no rotational moment. The assumption was therefore that the motion was surrounded by a nonrotational flow on all four sides. At the boundaries of the fluid, however, this is not the case. If, for example, we place the square in Figures 4 or 7 with one side coinciding with the surface of the fluid, the shearing force on that side will disappear and the shearing forces on the other sides will produce a turning moment. Still more powerful is the effect if the particle comes in contact with a rigid surface. In this case the velocities, at first unstable, fall to zero and produce an infinitely large shearing force, which immediately disturbs the symmetry of the shearing forces.

As soon as the outermost strata are set in rotation, other particles adhere, thus causing further asymmetry of the shearing forces, so that the latter particles likewise begin to rotate. The rotation of the boundary layer is thus gradually increased from the outside inward, quite similarly to the temperature changes in a body heated from the outside. The rate of propagation of the rotation depends on the viscosity of the fluid. In most technical processes the viscosity is very small, so that in the motion of a body in general, only a very thin layer of fluid particles adheres to its surface. This layer is commonly known as the "boundary layer." How thick this boundary layer is with respect to the proportions of the body depends on the Reynolds Number which, as you have heard in Prandtl's lecture, expresses the relation between the inertia and viscosity forces.

As you may likewise have heard in Prandtl's lecture, the boundary layer separates from the surface of the body under certain circumstances, and some of the rotating fluid particles get into the steady flow, thereby causing most of the important disturbances in the main flow. Due to the thinness of the boundary layer, the rotating portions of the fluid which penetrate the free fluid are naturally very small, while much the greater portion of the flow remains practically free from rotation. This is very important for the theoretical treatment of the flow since, as you must know from vector analysis, a nonrotational vector field, like the velocity field of a nonrotational flow, can be expressed by the differential quotient of a potential function. For this reason, nonrotational flows are also called potential flows. The theory of this potential problem has, however, been mathematically very extensively worked out, so that nonrotational motions can be treated mathematically much easier than rotational motions.

After these general remarks on rotational and nonrotational motions, and their occurrence in ordinary processes, we shall once more turn our attention to the fourth typical flow (Fig. 4) which represents a circular and at the same time nonrotational flow. If we take the small stick, which we employed to investigate the rotation, and lay it so that it coincides with the center of the circle, it will indicate the same rotation for whatever direction we give it. The shorter the stick is, the faster it will revolve about the center, since the velocity of flow increases indefinitely as we approach the center of the circle. While at all other points in the flow we found nonrotational conditions, at this special point we find a rotation of infinite strength. If we wish to inquire into the physical significance of this result, we must first remember that a flow of this type is only possible if we separate the middle point and a small region in its immediate surroundings, since it is impossible to produce velocities of infinite magnitude. Nevertheless, this fourth type forms an extraordinarily important type of flow, whereby we must realize that this flow occurs only beyond a certain distance from the center, the flow prevailing within this distance being of a different nature.

The flow represented by Figure 11 may be described as follows. The fluid flows in concentric circles. Outside a certain circle a nonrotational flow of type 4 (Fig. 4) prevails. Inside of this circle (hatched area) there is a rotational flow of type 2 (Fig. 2). Such a type of motion we shall call a single vortex.* The region of revolving fluid is called the vortex core, and the nonrotational flow outside of the core is called the vortex field. It is not necessary, as in the above example, for the rotation in the vicinity of the middle point (that is, in the vortex core) to be of the same magnitude everywhere. A point of importance is that there must be a distinct region of revolving fluid particles separated from a field of nonrotational motion, or at least from a field of small rotation. Very different types of rotational motion can be conceived for the vortex core of a given vortex field. Thus, for example (Fig. 12), the revolving fluid can be confined in a ring (hatched area). Outside of this ring the flow is nonrotational; inside of it there is a region of zero velocity. Since in ordinary fluids of low viscosity, according to what we have said, the rotating portion of the fluid is very limited, the vortex core is generally very small and the phenomena which interest us are dispersed to a great measure in the vortex field. In most cases it is therefore unimportant as to how each individual vortex core appears so long as it is adapted

*Very often the term "vortex" is applied to any motion not free from rotation.

to the vortex field. In most cases it is therefore unimportant as to how each individual vortex core appears so long as it is adapted to the vortex field. However, since it is immaterial as to what the phenomena are in the vortex core, for simplicity we can assume the cross section of the core to have the form of a point and return to the flow of Figure 4 (fourth type). We must understand, however, that this means only that the actual flow outside of a certain small region agrees with that represented in Figure 4, while inside of this region other velocities may prevail, which individually do not interest us. This ideal form of a vortex, in which the size of the core is infinitely small, is frequently called a potential vortex.

In order to describe the entire process of flow outside of the vortex core in all its characteristics, it suffices, in addition to the assumption that the motion is nonrotational, to know the center and a single number which, for example, represents the velocity at a unit distance from the center. Instead of the latter number, we generally take a number $2\pi$ times as large, which is called the circulation. This is the line integral of the velocity along a closed curve surrounding the vortex core. As we know from the theory of functions, this is independent of the path as long as the path is in the nonrotational field. In the present case the circulation is obtained as the product of the circumference of any circular streamline and the velocity along this line. This product is the same for every circle outside of the vortex core, since the velocity decreases in the same proportion as the circumference increases. This circulation is the characteristic value for the vortex core.

It appears at first surprising that the whole fluid motion extending to infinity should be so definitely determined from so few data (location of center and magnitude of circulation). The explanation of this is the assumption that the flow is nonrotational as far as the small region of the vortex core. The vast number of conceivable flow processes are all eliminated by this method of determination except this particular one, since all the other processes are not free from rotation.* One of the already mentioned, extraordinary simplifications is introduced by the assumption that the flow is free from rotation. And, as we have pointed out above, this assumption is often fulfilled for the greatest portion of the fluid.

We have considered only the nonrotational motion of four different types and their relation to a vortex core. We may in this manner generalize the argument, and turn to other nonrotational motions by superposition of two or more forms of flow. If we have a motion, a simple vortex field with the center $A_1$ (Fig. 13), we shall then have a certain velocity $v_1$, at the point P. If we take another motion, a vortex field with its center at $A_2$, the velocity at the point P will be $v_2$. By

*Strictly speaking, we should also introduce the condition that the flow extends to infinity and that it exhibits no restrictions due to the presence of other bodies. The vortex core and the point at infinity then form the boundaries of the fluid. The velocity at infinity is zero.

adding together the velocities $v_1$ and $v_2$, and combining them by a parallelogram of forces, we obtain the new velocity v. If we perform the same operation for each point in the space, we obtain a new velocity for each point. These new velocities give a new form of flow, and we say that the resultant flow is produced by the superposition or addition of both original fields of flow. It can be shown now that the rotation exhibited at any particular point by the flow resulting from the superposition is equal to the sum of the rotations which the original flow components had at the same point. In particular, if nonrotational flows are superposed, a nonrotational motion is obtained again, and if a nonrotational field of motion is superposed over a rotational field, the rotation then remains unchanged. This law can be easily understood by observing that the velocities which bring about the motiom of our test-sticks add themselves by superposition. The same thing is true of the angular velocities of the little stick and, also of the rotations, since these are the result of the angular velocities. By means of such superpositions, we can now construct the most different kinds of nonrotational forms of flow, for the explicit description of which suffice the location of the vortex cores and the respective circulations. It is not necessary therefore to make a separate analysis of the vortex cores. These can also mingle uninterruptedly with one another. Instead of adding the velocities of the individual vortex fields, an integration then takes place. The vortex core often consists of a thin layer of rotating fluid. We can then generally disregard the thickness of this layer and obtain a surface of unsteady motion (Fig. 14). We can visualize this as composed of potential vortices with their centers located very close together (a vortex surface), and compute the field by superposition of these potential vortices (Fig. 15).

If the origin and motion of the rotating fluid particles in any flow whatever can be followed, so that their distribution and rotational magnitude are known, the behavior of the remaining nonrotational flow will be known. One often wonders over this fact and finds it incomprehensible that any spatially restricted figure, such as a vortex core, can have any decisive effect on the remaining flow. The vortex core is therefore erroneously looked upon as the mechanical cause of the vortex field. In reality the nonrotational flow is produced by the pressure forces and, at the place where the influence of friction is felt in a noticeable measure, rotating fluid regions develop, which in turn are somehow distributed through space by the other fluid motion. One could therefore much better say that the whole fluid motion causes the production and space distribution of the vortex cores. In any case, the magnitude and distribution of the vortex cores are greatly affected by the motion of the remaining fluid, so that the corresponding flow can be computed backward from the core distribution in the same way that causes can be determined from their effects.

Due to the small effect of viscosity on the motion, a vortex, once generated, continues for a long time. In the presence of several cores, however, their relative positions change and consequently the entire field of flow changes, but the partial fields corresponding to the individual vortex cores remain the same. They overlap one another only in a different way. In the course of a long time, however, the viscosity makes itself felt and alters the individual cores. The fluid particles near the core are gradually set in rotation also, and form the components of the core itself. In time, therefore, the cores assume a surface extension. However, the circulation around the core is not altered thereby, since it can be measured at any distance from the core and hence at so great a distance that the effect of the viscosity has not made itself felt there. If, however, the vortex surface is enlarged, while the circulation remains unchanged, the mean rotation of the fluid in the core must become smaller, because it has been distributed over a greater area. Figure 16 represents the velocity distribution of a vortex at two different points of time. The longer arrows give the original velocities; the shorter arrows, the later velocities. As long as the fluid remains nonrotational, the velocities remain unchanged. Inside the core they have become smaller. The rule that the circulation remains constant in this enlargement of the core holds good only so long as the different cores do not grow into one another. If two cores of opposite rotation flow into each other by reason of this growth, their rotations are entirely or partially arrested. If the circulations around these cores were originally of the same magnitude but opposite in direction, both vortices would disappear entirely in the course of sufficient time. The case is similar, if the fluid is confined within walls, in which event the circulation diminishes as soon as the cores reach the solid boundary or the boundary layer which possesses a rotation opposite to that of the cores in question. Such conditions are, however, always present. Either solid walls are in the vicinity of a vortex or, if the flow is very wide, more vortices are constantly appearing, the rotation of which is partly in one direction and partly in the other, so that the sum of the circulations around the individual cores is zero. In a flow extending to infinity the kinetic energy of a single vortex would be infinitely great. It is not possible, therefore, to generate such a vortex, but there must always be many vortices rotating in opposite directions with finite energy, if the sum of their circulations is zero. By virtue of this circumstance all vortices would gradually disappear in a fluid, if left to itself for a long time without being acted on by external forces.

Our discussion has been thus far limited to uniplanar flow phenomena, namely, those in which each particle remains in one plane through the whole course of its motion, and the same flow form is found in all planes parallel to it. Naturally these are more obvious and point out the essential features of the process.

At this point we shall briefly indicate a few peculiarities brought about by the multiplicity of the three-dimensional processes. For the flow to be uniplanar, the vortex cores must form cylinders normal to the plane of flow in question. In the case of an infinite flow they must extend to infinity, and when the flow is confined between parallel walls, parallel to the flow, they terminate at these walls. In the general case of three-dimensional flow the vortex cores do not follow a straight line, but follow some sort of curved vortex tube around which the flow takes place. The field of such a vortex core is likewise definitely determined, as soon as the location of the core and the circulation about the same are known. The velocity computation is, to be sure, somewhat less easy than in the two-dimensional flow, though still quite simple. The velocities pertaining to the same vortex core are governed by the same law, which determines the strength of a magnetic field about an electric conductor. The velocity corresponds to the strength of the field, the vortex core to the conductor, and the circulation to the current strength. Just as the strength of the current is the same in every cross section of an electric conductor, the circulation around a vortex core cannot change along its course. The vortex core therefore cannot end anywhere in the fluid. It must either form a ring-shaped self-contained body, or extend to infinity, or terminate at the boundaries of the fluid. The possibility may also arise that a vortex core splits into several branches. The sum of the circulations in the different branches then remains the same, exactly like the current in a branched system of electric conductors. Such a core can be visualized as a bundle of several cores located close together so that they appear as a single core. This important theorem on the spatial constancy of circulation follows entirely from geometrical considerations which, however, are not easily understood. It can be shown, in fact, that when different circulations prevail at two neighboring sections of a vortex core, the difference in the velocities $v_1$ and $v_2$ (Fig. 17) imparts a rotation to the fluid. This is, however, contrary to the assumption that the flow outside of the core is nonrotational. A well-known example of a vortex core which is not rectilinear is the vortex ring often visible in smoke.

## Part II: Wing Theory

As a wing, one prefers a body which, when in motion, produces a small drag but a relatively large lift (Fig. 18). The drag W, that is, the force component opposing the motion has, in flight, a detrimental effect to overcome, which requires energy that must be provided by an engine. The lift A, or the force normal to the motion, is the useful force. It requires no expenditure of energy, since it is perpendicular to the direction of motion. Since the drag is small in comparison with the lift, it can be neglected in many cases, but of course, only when one is not interested in the energy required for flying, which depends exclusively on the drag.

When lift is to be produced by a wing, the pressure on its upper side must obviously be smaller than that on the under side. Since the region in which viscosity is felt is restricted to the immediate vicinity of the wing, and to a small region behind the wing, Bernoulli's equation $p + \frac{\rho}{2} v^2 = \text{constant}$ applies to the rest of the flow. Therefore, when the pressure on the upper side is smaller than that on the under side, the velocity on the upper side $(v_1)$ of Figure 19, must also be greater than $(v_2)$ on the under side. Consequently, by plotting the line integral $\oint v \, ds$ for a line circumscribing the wing, the contributions of the upper and lower surfaces are not destroyed, and we obtain the circulation $\Gamma$. The lift is therefore obviously connected with a circulation around the wing. A quantitative determination of this relation between the lift $A$ and the circulation $\Gamma$ leads to the equation

$$d A = \rho \, v_0 \, \Gamma \, d b$$

wherein $\rho$ is the air density, $v_0$ the airfoil velocity, and $db$ the width of the wing strip on which the lift $dA$ acts. This equation was discovered by the German scientist Kutta and the Russian scientist Joukowski, independently of each other, and is now known as the Kutta-Joukowski equation.

On account of the circulation, the flow around a wing has a great similarity to the flow in a vortex field. The wing itself represents the vortex core. The essential difference consists in the fact that an ordinary vortex core is carried along by the flow, while the wing moves relatively to the flow. An ordinary vortex core, in spite of its circulation, produces no lift, since the velocity $v_0$ in the Kutta-Joukowski equation is zero.

It was stated in Part I of this paper that a single vortex cannot be produced, since the energy of its field would be infinite. The same is also true regarding the circulation around an airfoil. In practice, the process at the beginning of the motion is such that, since the flow cannot follow around this sharp edge, a vortex of reversed circulation develops at the trailing edge of the wing (Fig. 20). It grows, and then gradually recedes, but produces a smooth uniform flow at the trailing edge. The circulation around the airfoil develops in the same proportion as the vortex and, when the vortex has traveled far away, the circulation remains constant. Due to its great distance from the wing the vortex itself does not generally need to be considered. The magnitude of the circulation therefore depends on the smoothness of the flow at the trailing edge.

Let us now turn our attention to the lateral ends of the wing. Positive pressure prevails on the lower side of the wing and negative pressure on the upper side. This pressure difference cannot exist at the wing tips, since all pressure differences are there immediately equalized. Consequently, both positive and negative pressures diminish toward the wing tips, with the result that the fluid particles which flow above the wing near the tips are deflected laterally (Fig. 21). When they meet again behind the wing, their velocities no longer coincide. The particles which flowed above the wing have acquired a velocity component away from the edge, while those which flowed below the wing have a velocity component toward the edge.

These lateral disturbing velocities (or induced velocities) form a potential motion, since they are produced by pressure differences. At the point beyond the wing where they meet again, they form an unstable motion which corresponds to a surface vortex. When the vortex distribution in this surface is known, the whole disturbing field can then be computed from it. Since the streamline deflection on the wing depends on the lateral pressure drop, and since this in turn represents a lift drop or a circulation drop toward the wing tips, it is clear that the vortex in the surface of discontinuity corresponds to the circulation drop. We can easily understand this relation quantitatively, if we consider the theorem mentioned in Part I of this paper, namely, that a vortex core or, more generally, the core of a flow with circulation cannot terminate. If, therefore, the circulation around a wing becomes smaller near the tips, then vortices of corresponding circulation must pass off from the wing. These are indeed the vortices in the surface of discontinuity. The previously mentioned relation between the circulation around the wing and the vortices behind the wing means, in other words, that the circulation along line I of Figure 22 (circulation around the wing) is equal to the circulation along line II, (circulation around the vortex sheet extending from the point in question to the wing tip). Therefore, if the lift distribution along the wing span is known, the vortex distribution in the surface of discontinuity and hence the whole flow can be calculated.

In order to produce the transverse motion, work must be done. This is manifested by the fact that the wing experiences a drag which depends on the lift distribution along the span. This drag could be computed from the energy of the vortex field. It is simpler, however, to follow the development of the drag on the wing. The vortex field is accompanied, on the wing, by a downward velocity $w$, which generally varies from point to point. If the circulation around a wing is at a distance $x$ from the middle $\Gamma$, there is then developed, together with the downward velocity $w$, according to the Kutta-Joukowski theorem, a force $dW = \rho\, w\, \Gamma\, dx$, opposing the wing motion (normal to $w$) for a piece of wing $dx$ long.

By integrating over the whole span, we obtain the entire drag produced by the lateral edges, which is called induced or rim drag:

$$W_i = \rho \int_{-b/2}^{+b/2} w\,\Gamma dx.$$

The induced velocity $w$ can be determined very easily from the distribution of $\Gamma$ over the span $\Gamma(x)$.

The question now arises as to how a given lift must be distributed over a wing of a given span, in order that the induced drag shall be a minimum. A simple minimum calculation shows that this can be accomplished if the induced velocity $w$ is constant along the entire span. This takes place when the lift distribution over the span is elliptical, that is, when

$$\Gamma = \Gamma_m \sqrt{1 - \left(\frac{2x}{b}\right)^2}$$

in which $\Gamma_m$ is the circulation at midspan, $\Gamma$ that at the distance $x$ from the middle, and $b$, the wing span. The induced drag for this most favorable lift distribution is

$$W_i = \frac{A^2}{\pi\, b^2\, \frac{\rho}{2}\, v_o^2}.$$

It is obvious that, for wing forms occurring in practice, the induced drag does not vary materially from this minimum value, so that this simple formula can be safely used for all calculations of induced drag. Naturally, we must add to this the resistance of the profile itself, the so-called "profile drag," even when the wing is of infinite length. In practical wings, however, the profile drag is considerably smaller than the computable induced drag.

In the above example, I have illustrated one of the best known aerotechnical applications of the vortex theory. Another series of problems connected with the wing theory can be treated in a similar way.

All problems connected with the resistance of fluids, however, lead us back to the vortex theory. Naturally, the practical solution of these problems requires experience. The limited time at your disposal and the many other branches of science make it impossible for you to acquire this experience. In my lecture I have therefore endeavored to explain the principal phenomena. If the fundamental theories you have probably already encountered in other lectures in a more mathematical form have been made clearer, the main purpose of this lecture will have been accomplished.

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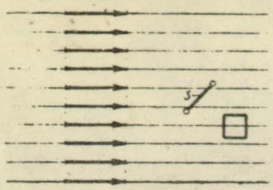



Fig.1 Parallel flow at constant velocity.

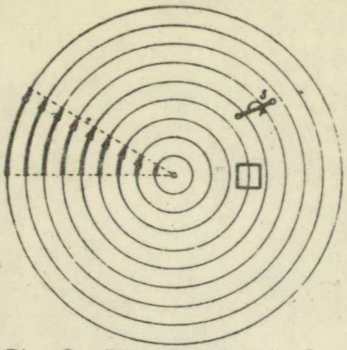


Fig.2 Fluid rotating like a solid body.

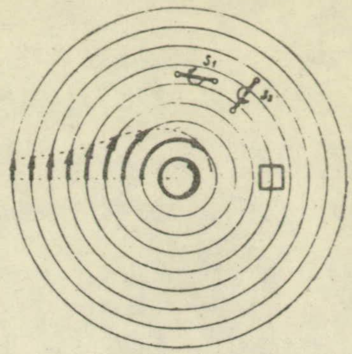



Fig.4 Nonrotational flow with concentric streamlines (potential vortex)

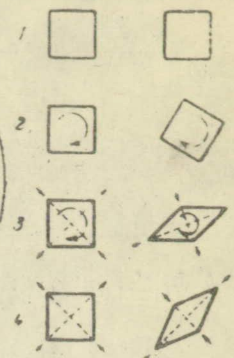



Fig.5 From variation of squares shown in Figs.1 to 4 due to the fluid flow.

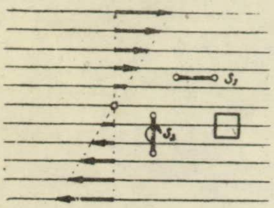



Fig.3 Shearing flow in parallel streamlines.

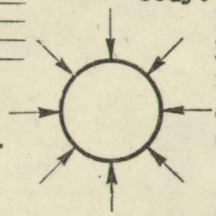


Fig.6 Pressure forces on a fluid sphere.

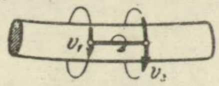
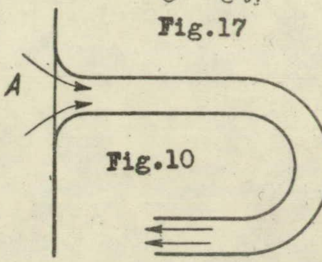



Fig.17




Formation of a nonrotational flow with curved streamlines like Fig.4.

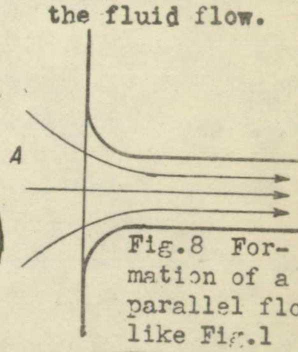



Fig.8 Formation of a parallel flow like Fig.1

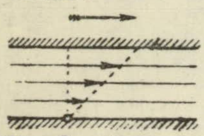
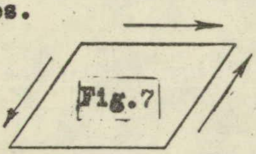


Fig.9 Formation of a shearing flow like Fig.3

Fig.15 Surface of discontinuity formed from individual vortices.




Shearing forces changing the form of a fluid particle.

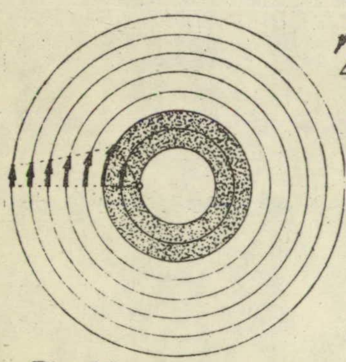


Fig.12 Vortex with annular core

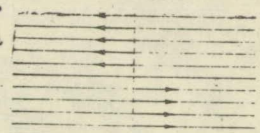


Fig.14 Surface of discontinuity.

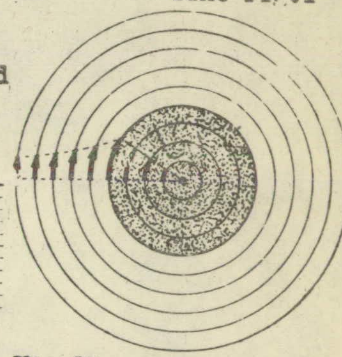


Fig.11 Field and core of a vortex.

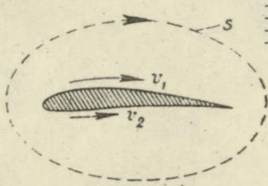



Fig.19 Circulation around a wing.

Fig.20 Vortex formation on trailing edge at beginining of motion.

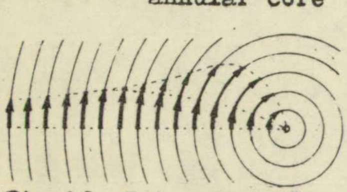


Fig.16 Extension of a vortex core.

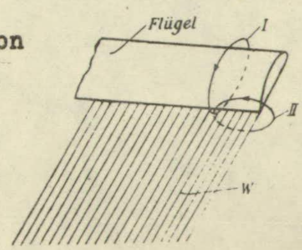



Fig.22 Relation between circulation around wing (line integ.I) and that around vortex sheet (line integ. II.)

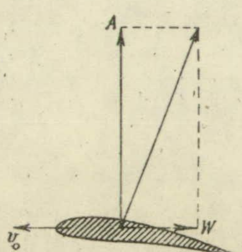



Fig.18 Lift and drag of a wing.

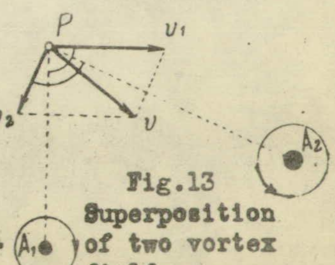



Fig.13 Superposition of two vortex fields.

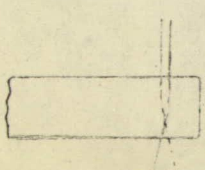


Fig.21 Lateral bending of steamlines near wing tips.