CONNECTEDNESS AND SOME CONCEPTS RELATED TO

CONNECTEDNESS OF A TOPOLOGICAL SPACE

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THESIS

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PREFACE

The purpose of this thesis is to investigate the idea of topological "connectedness" by presenting some of the basic ideas concerning connectedness along with several related concepts. There are three chapters in the thesis. In Chapter I, the idea of "connectedness" in general will be examined, while Chapter II will deal with the idea of "local connectedness" and the related ideas of "connectedness im kleinen," "property S," and "uniform local connectedness." In Chapter III, the concept of "path-connectedness" will be investigated. All of the elementary properties of topological spaces will be freely used without statement or proof. The notation used is elementary set notation as discussed in Elementary General Topology, by Theral O. Moore.

TABLE OF CONTENTS

CHAPTER I

CONNECTEDNESS

1.1. Definition A topological space Y is connected if it is not the union of two nonempty, disjoint open sets. A subset $B \subset Y$ is connected if it is connected as a subspace of Y.

1.2. Definition Two subsets A and B of a topological space Y are said to be separated if $A \neq \emptyset$, $B \neq \emptyset$, and $A \cap \overline{B} = \emptyset = \overline{A} \cap B$.

1.3. Definition A subset A of a set B is called a proper subset of B if and only if $A \neq \emptyset$ and $A \neq B$.

1.4. Theorem Let A and B be nonempty, disjoint subsets of a topological space Y. Then, A and B are separated if and only if both A and B are open in $A \cup B$.

Proof:

Part 1 — Let A and B be separated. Thus, $\overline{A} \cap B = \emptyset$ which implies that A is closed in $A \cup B$. Consequently, $(A \cup B) - A = B$ is open in $A \cup B$. Similarly, A is open in $A \cup B$.

Part 2 — Let both A and B be open in $A \cup B$. There is an open set $V \subset Y$ such that $V \cap (A \cup B) = B$. Now, suppose that $V \cap A \neq \emptyset$. Then, there is a point $p \in V \cap A$. Thus, $p \in (V \cap A) \cup (V \cap B) = V \cap (A \cup B) = B$. But this implies that $p \in A \cap B$, a contradiction. Hence, $V \cap A = \emptyset$, and, since V is open, $V \cap \overline{A} = \emptyset$. Therefore, $B \cap \overline{A} = \emptyset$. Similarly, $A \cap \overline{B} = \emptyset$. Thus, (by 1.2) A and B are separated.

1.5. Theorem Let Y be a topological space. The following five properties are equivalent:

(1) Y is connected.

(2) Y is not the union of two separated sets.

(3) Y is not the union of two nonempty, disjoint closed sets.

(4) Y contains no proper subset which is both open and closed.

(5) No continuous mapping $f: Y \rightarrow 2$ is surjective, where 2 is the space consisting of the two points $\{0,1\}$ with the discrete topology.

Proof:

Show that (1) implies (2).

This follows directly from 1.1 and 1.4.

Show that (2) implies (3).

Assume that $Y = A \cup B$ where A and B are nonempty, disjoint closed sets. Then, $Y-A = B$ and $Y-B = A$ are both open; and (by 1.4) A and B are separated, a contradiction.

Show that (3) implies (4).

Assume that Y contains a proper subset A, which is both open and closed. Thus, Y-A is nonempty and closed. Since

$Y = (Y-A) \cup A$, then Y is the union of two nonempty, disjoint closed sets, namely A and Y-A, a contradiction.

Show that (4) implies (5).

Assume that there is a continuous $f: Y \longrightarrow 2$ which is surjective. Thus, $f^{-1}(0) \neq Y$ and $f^{-1}(0) \neq \emptyset$; consequently, $f^{-1}(0)$ is a proper subset of Y. Now, $\{0\}$ is both open and closed in 2, and, since f is continuous, $f^{-1}(0)$ is both open and closed in Y, a contradiction.

Show that (5) implies (1).

Assume that Y is not connected. Then, (By 1.1) $Y = A \cup B$ where A and B are nonempty, disjoint open sets.

Define $C_A: Y \longrightarrow 2$ by $C_A(x) = \{\begin{smallmatrix} 0 \text{ if } x \notin A \\ 1 \text{ if } x \in A \end{smallmatrix}\}$. Since A is nonempty and $B = Y-A$ is nonempty, then $C_A: Y \longrightarrow 2$ is surjective.

Now, the set $\{0,1\}$ is open in 2, and $C_A^{-1}(\{0,1\}) = A \cup B = Y$, which is open in Y. The set $\{1\}$ is open in 2, and $C_A^{-1}(1) = A$, which is open in Y. The set $\{0\}$ is open in 2, and $C_A^{-1}(0) = B$, which is open in Y.

Thus, the inverse image of each set open in 2 is also open in Y. Thus, C_A is continuous, a contradiction.

1.6. Lemma Let A and B be separated subsets of a topological space X. If C and D are nonempty sets such that $C \subset A$ and $D \subset B$, then C and D are separated.

Proof: Now, $C \subset A$ implies that $\overline{C} \subset \overline{A}$. Since A and B are separated, $\overline{A} \cap B = \emptyset$. Thus, since $D \subset B$, then $\overline{A} \cap D = \emptyset$, and, since $\overline{C} \subset \overline{A}$, then $C \cap D = \emptyset$. Likewise, $C \cap \overline{D} = \emptyset$. Therefore (by 1.2), C and D are separated.

<u>1.7. Theorem</u> Let A and B be separated subsets of a topological space X. If C is a connected subset of $A \cup B$, then $C \subset A$ or $C \subset B$.

<u>Proof</u>: Assume that $C \not\subset A$ and $C \not\subset B$. Thus, C contains points in both A and B; so $C = P \cup Q$, where $P = C \cap A$ and $Q = C \cap B$. Since A and B are separated, then (by 1.6) P and Q are separated, which implies (by 1.5) that C is not connected, a contradiction.

<u>1.8. Theorem</u> Let C be a family of connected subsets of a topological space. If no two members of C are separated, then $\cup C$ is connected.

<u>Proof</u>: Assume that $\cup C$ is not connected. Then (by 1.5) $\cup C = P \cup Q$, where P and Q are separated sets. Let $C_1 \varepsilon C$. Then (by 1.7) $C_1 \subset P$ or $C_1 \subset Q$. Suppose that the lettering is chosen such that $C_1 \subset P$. Since (by 1.2) Q is nonempty, there is an element $C_2 \varepsilon C$ such that $C_2 \cap Q \neq \emptyset$, and (by 1.7) $C_2 \subset Q$. However (by 1.6), C_1 and C_2 are separated, a contradiction.

<u>1.9. Corollary</u> If C is a family of connected subsets of a topological space which have at least one point in common, then $\cup C$ is connected.

Proof: Since each two members of C have a point in common, (by 1.2) no two members of C are separated. Thus (by 1.8), $\cup C$ is connected.

1.10. Remark Let A and B be subsets of a topological space X such that $A \subset B$. Then, A is connected in X if and only if A is connected in B.

1.11. Theorem The continuous image of a connected set is connected. That is, if X and Y are topological spaces, if A is a connected subset of X, and if $f:X \to Y$ is continuous, then f(A) is connected.

Proof: Assume that f(A) is not connected. Then (by 1.5), there is a proper subset P of f(A) such that P is both open and closed in f(A). Now, since $f:X \to Y$ is continuous, $f|A:A \to Y$ is continuous. Thus, it follows that $f^{-1}(P)$ is a proper subset of A which is both open and closed in A. Therefore (by 1.5), A is not connected, a contradiction.

1.12. Theorem Let $\{A_i : i \in Z^+ = \{1,2,3,\ldots\}\}$ be a family of connected sets of a topological space Y, with $A_i \cap A_{i+1} \neq \emptyset$ for each $i \in Z^+$. Then, $\cup\{A_i : i \in Z^+\}$ is connected.

Proof: Mathematical induction will be used. Let $n \in Z^+$ and let P(n) represent the statement "$\cup\{A_i : i \in Z_n^+ = \{1,2,\ldots,n\}\}$ is connected."

(1) P(1) is true since A_1 is connected.

(2) Assume that P(k) is true for some $k \in Z^+$. That is, assume that $\bigcup_{i=1}^{k} A_i$ is connected.

(3) Show that P(k+1) is true.

Since $A_k \cap A_{k+1} \neq \emptyset$, then $(A\cup \ldots \cup A_k) \cap A_{k+1} \neq \emptyset$. By (2), $\bigcup_{i=1}^{k} A_i$ is connected, and, by the hypothesis, A_{k+1} is connected, so (by 1.9) $(\bigcup_{i=1}^{k} A_i) \cup A_{k+1} = \bigcup_{i=1}^{k+1} A_i$ is connected.

Therefore, $\bigcup\{A_i : i \in Z^+\}$ is connected.

<u>1.13. Theorem</u> If $\{A_\alpha : \alpha \in T\}$ is a family of connected subsets of a topological space Y such that there exists a connected set A with $A \cap A_\alpha \neq \emptyset$ for each $\alpha \in T$, then $A\cup (\bigcup_{\alpha \in T} A_\alpha)$ is connected.

<u>Proof</u>: Consider the set $\{A\cup A_\alpha : \alpha \in T\}$. Since for each $\alpha \in T$, A is connected, A_α is connected, and $A\cap A_\alpha \neq \emptyset$, then (by 1.9) $A\cup A_\alpha$ is connected. Also, $A \subset \bigcap\{A\cup A_\alpha : \alpha \in T\}$. Thus (by 1.9), $\bigcup\{A\cup A_\alpha : \alpha \in T\} = A\cup(\bigcup_{\alpha \in T} A_\alpha)$ is connected.

<u>1.14. Theorem</u> If $\{A_\alpha : \alpha \in T\}$ is a family of connected sets such that any two of them have nonempty intersection, then $\bigcup_{\alpha \in T} A_\alpha$ is connected.

<u>Proof</u>: Let $A_\beta \in \{A_\alpha : \alpha \in T\}$. For any $A_\alpha \in \{A_\alpha : \alpha \in T-\{\beta\}\}$, $A_\beta \cap A_\alpha \neq \emptyset$. Thus (by 1.13), $A_\beta \cup(\bigcup\{A_\alpha : \alpha \in T-\{\beta\}\}) = A_\beta \cup(\bigcup\{A_\alpha : \alpha \in T\}-A_\beta) = \bigcup_{\alpha \in T} A_\alpha$ is connected.

<u>1.15. Definition</u> Given two nonempty sets U_α and U_β of a topological space, a collection of sets $U_1,\ldots,U_n$ is a <u>chain</u> from U_α to U_β, provided that $U_\alpha \cap U_1 \neq \emptyset$, $U_\beta \cap U_n \neq \emptyset$, and $U_i \cap U_{i+1} \neq \emptyset$ for $i=1,\ldots,n-1$.

<u>1.16. Theorem</u> A topological space Y is connected if and only if every open covering $\{U_\alpha : \alpha \varepsilon T\}$ of Y has the following property: for each pair U_β, $U_\varphi \varepsilon \{U_\alpha : \alpha \varepsilon T\}$, there is a subcollection of $\{U_\alpha : \alpha \varepsilon T\}$ which forms a chain from U_β to U_φ.

<u>Proof</u>:

Part 1 — Suppose that the given property holds, and assume that Y is not connected. Thus $Y = A \cup B$ where A and B are nonempty, disjoint open sets. Therefore, $\{A,B\}$ is an open covering for Y, and, by the hypothesis, $A \cap B \neq \emptyset$, which is a contradiction.

Part 2 — Suppose that Y is connected. Let $\{U_\alpha : \alpha \varepsilon T\}$ be an open covering of Y. Let $U_\beta \varepsilon \{U_\alpha : \alpha \varepsilon T\}$, and let C be the collection of sets consisting of U_β together with all sets $U_\delta \varepsilon \{U_\alpha : \alpha \varepsilon T\}$ such that there is a chain consisting of elements of $\{U_\alpha : \alpha \varepsilon T\}$ from U_β to U_δ. C is nonempty, since $U_\beta \varepsilon C$. Therefore, $\cup C$ is nonempty, and, since C is a collection of open sets, then $\cup C$ is open.

To show that $\cup C$ is closed, let $p \varepsilon (\cup C)'$. Then, $p \varepsilon Y = \cup \{U_\alpha : \alpha \varepsilon T\}$. This implies that $p \varepsilon U$ for some $U \varepsilon \{U_\alpha : \alpha \varepsilon T\}$. Thus, U is an open set containing p, and, since $p \in (\cup C)'$, then $U \cap (\cup C) \neq \emptyset$, which implies that there is some $U_\lambda \in C$ such that $U \cap U_\lambda \neq \emptyset$. Since $U_\lambda \in C$, there is a chain $U_1, \ldots U_n$ from U_β to U_λ which consists of elements of $\{U_\alpha : \alpha \in T\}$. But, since $U \cap U_\lambda \neq \emptyset$, the collection $U_1, \ldots, U_n, U_\lambda$ is a chain from U_β to U. Hence, $U \in C$. Thus, $p \in U \subset \cup C$ which implies that $\cup C$ is closed.

Thus, $\cup C$ is a nonempty set which is both open and closed in Y, and, since Y is connected, (by 1.5) $\cup C = Y$.

Let $U_\varphi \in \{U_\alpha : \alpha \in T\}$. Then $U_\varphi \subset \cup C$, which implies that there is some $U_k \in C$ such that $U_\varphi \cap U_k \neq \emptyset$. Since $U_k \in C$, there is a chain $U_1, \ldots, U_n$ from U_β to U_k consisting of elements of $\{U_\alpha : \alpha \in T\}$. Since $U_\varphi \cap U_k \neq \emptyset$, the collection $U_1, \ldots, U_n, U_k$ is a chain from U_β to U_φ.

<u>1.17. Theorem</u> Let A be a connected subset of the topological space Y. Then, any set B satisfying $A \subset B \subset \overline{A}$ is also connected. In particular, the closure of a connected set is connected.

<u>Proof</u>: Assume that B is not connected. Then (by 1.5), $B = P \cup Q$ where P and Q are separated. Since $A \subset B$ and A is connected, then (by 1.7) either $A \subset P$ or $A \subset Q$. Suppose that the labeling is chosen so that $A \subset P$. Thus, $Q \cap A = \emptyset$. Since $Q \subset B \subset \overline{A}$, $Q \cap A = \emptyset$, and Q is nonempty, then Q contains a limit point of A. But, since $A \subset P$, then Q contains a limit point of P, which implies that P and Q are not separated, a contradiction. Thus, B is connected.

In particular, if C is any connected set, then, since $C \subset \overline{C} \subset \overline{C}$, C is connected.

<u>1.18.Theorem</u> Let A and B be subsets of a topological space X. If A and B are closed in X, and $A \cup B$ and $A \cap B$ are connected, then A and B are connected.

<u>Proof</u>: The conclusion is immediate if $A \subset B$ or $B \subset A$. So, suppose that $A-B \neq \emptyset$ and $B-A \neq \emptyset$.

Assume that A is not connected. Then $A = P \cup Q$ where P and Q are nonempty, disjoint open sets in A. Thus (by 1.4), P and Q are separated, and, since $A \cap B \subset A \cup B$ and $A \cap B$ is connected, then (by 1.7) either $A \cap B \subset P$ or $A \cap B \subset Q$. Suppose the labeling is chosen so that $A \cap B \subset P$. Then, $A-B = A-(A \cap B) \supset A-P = Q$, and, since $A \supset A-B$ and Q is open in A, then Q is open in A-B. Also, since B is closed in $A \cup B$, then $A \cup B - B = A-B$ is open in $A \cup B$. Thus, Q open in A-B and A-B open in $A \cup B$ imply that Q is open in $A \cup B$. Also, since P is open in A, then $A-P = P \cup Q - P = Q$ is closed in A, and, since A is closed in $A \cup B$, then Q is closed in $A \cup B$. Thus, Q is a proper subset of $A \cup B$ which is both open and closed in $A \cup B$. Therefore (by 1.5), $A \cup B$ is not connected, a contradiction. Thus, A is connected, and, similarly, B is connected.

<u>1.19.Theorem</u> Let A be a connected subset of a connected topological space X. If B is a subset of X-A which is both open and closed in X-A, then $A \cup B$ is connected.

Proof: The proof is immediate if either $B = X-A$, $B = \emptyset$, or $A = \emptyset$. So, suppose that $B \neq X-A$, $B \neq \emptyset$, and $A \neq \emptyset$.

Let $H = (X-A)-B$. Since B is a proper subset of X-A which is both open and closed in X-A, then H is nonempty and open in X-A. Thus (by 1.4), H and B are separated.

Assume that $A \cup B$ is not connected. Then (by 1.5), $A \cup B = R \cup S$, where R and S are separated. But, since A is connected, (by 1.7) either $A \subset R$ or $A \subset S$. Suppose that the labeling is chosen such that $A \subset R$. Now, $S \subset B$ for, if not, $S \cap A \neq \emptyset$ which implies that $S \cap R \neq \emptyset$, a contradiction. Thus, since H and B are separated, (by 1.6) H and S are separated. Therefore, $X = H \cup (A \cup B) = H \cup (R \cup S) = (H \cup R) \cup S$ and since H and S are separated and R and S are separated, then $H \cup R$ and S are separated. This implies (by 1.5) that X is not connected, a contradiction.

1.20.Definition Let A be a subset of the topological space X. The boundary of A, written Fr(A), is $\overline{A} \cap \overline{X-A}$.

1.21.Theorem Let A be a subset of the space X. If p is a point in Fr(A), then each open set containing p contains at least one point in A and at least one point not in A.

Proof: Let $p \in Fr(A)$ and let U be an open set containing p. Since (by 1.20) $Fr(A) = \overline{A} \cap \overline{X-A}$, then $p \in \overline{A} \cap \overline{X-A}$. Now, $p \in A$ or $P \in X-A$. Suppose that $p \in A$.

Then p $\not\in$ X-A, and, since p ε $\overline{X-A}$, then p ε (X-A)'. Thus, U contains a point in X-A, and, since p ε A, then U contains a point in A. Similarly, if p ε X-A, then U contains points in A and points not in A.

1.22. Definition Let A be a subset of the topological space X. The interior of A, written Int(A), is the largest open set contained in A.

The following properties will be assumed without proof.

1.23. Theorem Let A be a subset of the topological space S. Then:

(1) Fr (A) = $\overline{A}$ - Int(A)

(2) Fr (A) $\cap$ Int(A) = $\emptyset$

(3) $\overline{A}$ = Int(A) $\cup$ Fr(A)

(4) X = Int(A) $\cup$ Fr(A) $\cup$ Int(X-A) is a pairwise disjoint union.

1.24. Theorem Let A be a subset of a topological space Y. If C is a connected subset of Y which contains points of A and points not in A, then C must contain points of the boundary of A.

Proof: The set C contains points of A and points not in A, so A$\cap$C $\neq \emptyset$ and C-A $\neq \emptyset$. But, C = (A$\cap$C)$\cup$(C-A) and since C is connected (by 1.5) (A$\cap$C)$\cap$(C-A)' $\neq \emptyset$ or (A$\cap$C)'$\cap$(C-A) $\neq \emptyset$.

Case 1 — Suppose $(A \cap C)' \cap (C-A) \neq \emptyset$. Thus, there is a point x such that $x \in (A \cap C)'$ and $x \in (C-A)$. But since $A \cap C \subset A$, $x \in (A \cap C)'$ implies that $x \in A' \subset \overline{A}$. Also, since $x \in C-A \subset Y-A$, then $x \in \overline{Y-A}$. Thus, $x \in \overline{A} \cap \overline{Y-A} = Fr(A)$, which implies that C contains points in Fr(A).

Case 2 — Suppose $(A \cap C) \cap (C-A)' \neq \emptyset$. Thus, there is a point x such that $x \in (A \cap C)$ and $x \in (C-A)'$. But, $x \in (A \cap C)$ implies that $x \in A \subset \overline{A}$. Also, since $x \in (C-A)'$ and $C-A \subset Y-A$, then $x \in (Y-A)' \subset (\overline{Y-A})$. Thus, $x \in \overline{A} \cap (\overline{Y-A}) =$ Fr (A), which implies that C contains points in Fr(A).

<u>1.25. Theorem</u> Let A and B be subsets of a topological space X, each of which is closed in $A \cup B$. If $A \cup B$ is connected and $A \cap B$ contains at most two points, then A is connected or B is connected.

<u>Proof</u>: Assume that both A and B are not connected. Then $A = P_1 \cup P_2$ where P_1 and P_2 are nonempty, disjoint open sets in A. Likewise, $B = P_3 \cup P_4$, where P_3 and P_4 are nonempty, disjoint open sets in B. Thus, $A-P_1 = P_2$ and $A-P_2 = P_1$ are closed in A, and, likewise, $B-P_3 = P_4$ and $B-P_4 = P_3$ are closed in B. Since A and B are closed in $A \cup B$, then P_1, P_2, P_3, and P_4 are closed in $A \cup B$.

Case 1 — Suppose that $(A \cap B) \cap P_i = \emptyset$ for some $i \in \{1,2,3,4\}$. Thus, $P_i \subset A-B$ or $P_i \subset B-A$. If $P_i \subset A-B$, then $P_i \subset A$, and, since $A-B \subset A$ and P_i is open in A, then P_i is open in $A-B$. But, since B is closed in $A \cup B$, $A-B$ is open in $A \cup B$. Thus, P_i is open in $A \cup B$, and, since P_i is also a proper subset of $A \cup B$ which is closed in $A \cup B$, then (by 1.5) $A \cup B$ is not connected, a contradiction. Similarly, if $P_i \subset B-A$, a contradiction is obtained.

Case 2 — Suppose that $(A \cap B) \cap P_i \neq \emptyset$ for all $i \in \{1,2,3,4\}$. Thus, $A \cap B \neq \emptyset$. Suppose that there is a point $p \in X$ such that $A \cap B = \{p\}$. Then, $(A \cap B) \cap P_1 = \{p\} \cap P_1 = \{p\}$, and $(A \cap B) \cap P_2 = \{p\} \cap P_2 = \{p\}$. Thus, $p \in P_1$ and $p \in P_2$, which implies that P_1 and P_2 are not disjoint, a contradiction. Thus, $A \cap B = \{p,q\}$ where $p,q \in X$ and $p \neq q$. This implies that P_1 or P_2 intersects P_3 or P_4. Let the labeling be chosen such that $P_1 \cap P_3 \neq \emptyset$. Suppose that $P_1 \cap P_3 = \{p,q\}$. Since $\{p,q\} \cap P_2 = (A \cap B) \cap P_2 \neq \emptyset$, then P_2 contains either p or q, implying that $P_1 \cap P_2 \neq \emptyset$, a contradiction. Thus, $P_1 \cap P_3$ contains only one point, say p. Consequently, $P_2 \cap P_4 = \{q\}$. Thus, $P_1 \cup P_3$ and $P_2 \cup P_4$ are disjoint, and, since each is closed in $A \cup B$, then each is open in $A \cup B$. Finally, since $A \cup B = (P_1 \cup P_2) \cup (P_3 \cup P_4) = (P_1 \cup P_3) \cup (P_2 \cup P_4)$, then (by 1.1) $A \cup B$ is not connected, a contradiction.

<u>1.26. Definition</u> A subset C of a topological space Y is called a <u>component</u> of Y if C is a maximal connected set in Y; that is, there is no connected subset of Y that properly contains C.

1.27. Theorem Let X be a topological space and $p \varepsilon X$. Then the component C of X containing p is the union of all connected subsets of X that contain p.

Proof: Let $\{A_\alpha : \alpha \varepsilon T\}$ be the family of all connected subsets of X that contain p. Then (by 1.9) $\cup\{A_\alpha : \alpha \varepsilon T\}$ is connected. But, since C is connected and contains p, then $C \varepsilon \{A_\alpha : \alpha \varepsilon T\}$, and, thus, $C \subset \cup\{A_\alpha : \alpha \varepsilon T\}$. However (by 1.26), C is a maximal connected set; so $\cup\{A_\alpha : \alpha \varepsilon T\} \subset C$. Thus $C = \cup\{A_\alpha : \alpha \varepsilon T\}$.

1.28. Definition If $\{A_\alpha : \alpha \varepsilon T\}$ is a covering of a topological space Y, and, if $A_\alpha \cap A_\beta = \emptyset$ whenever $\alpha, \beta \varepsilon T$ and $\alpha \neq \beta$, then the family $\{A_\alpha : \alpha \varepsilon T\}$ is called a partition of Y.

1.29. Theorem The set of all distinct components of a topological space Y forms a partition of Y.

Proof: For each point $y \varepsilon Y$, there is a component containing y. Thus, if S is the set of all distinct components of Y, then S is a covering for Y. Let C_1, $C_2 \varepsilon S$ such that $C_1 \neq C_2$, and suppose that $C_1 \cap C_2 \neq \emptyset$. Then (by 1.9), $C_1 \cup C_2$ is connected. Since $C_1 \neq C_2$, then C_1 is properly contained in $C_1 \cup C_2$, thus implying that C_1 is not a component. This is a contradiction, and, therefore, $C_1 \cap C_2 = \emptyset$. Hence, S is a partition of Y.

1.30. Theorem Each component C of a topological space Y is closed.

<u>Proof</u>: Since C is a component, C is connected, and, thus (by 1.17) $\overline{C}$ is connected. Also, since C is a maximal connected set, then $\overline{C} \subset C$. However, $C \subset \overline{C}$, and, thus, $\overline{C} = C$, which implies that C is closed.

<u>1.31. Theorem</u> If X and Y are topological spaces and $f:X \to Y$ is continuous, then the image of each component of X must lie in a component of Y.

<u>Proof</u>: Let C be a component of X, and let $x \in C$. Since f is continuous and C is connected, (by 1.11) f(C) is connected. Also, since $x \in C$, then $f(x) \in f(C)$. Thus, if D is a component of Y containing f(x) then $f(C) \subset D$, since D is a maximal connected set in Y containing f(x).

<u>1.32. Theorem</u> Let B be a connected subset of a topological space Y. If B is both open and closed, then B is a component of Y.

<u>Proof</u>: Assume that B is not a component of Y. Then there is a connected subset C of Y which properly contains B. But, since B is both open and closed in Y, B is both open and closed in C. Thus (by 1.5), C is not connected, a contradiction.

<u>1.33. Theorem</u> Let A be a subset of a topological space Y, where both A and Y are connected. If C is a component of Y-A, then Y-C is connected.

<u>Proof</u>: Assume that Y-C is not connected. Then $Y-C = P \cup Q$, where P and Q are nonempty, disjoint sets each of which

is open in Y-C. Thus, (Y-C) - P = Q and (Y-C) - Q = P are both closed in Y-C. Also, since C is connected in Y-A $\subset$ Y, then C is connected in Y. Thus (by 1.19), C∪Q and C∪P are both connected.

Now, since C $\subset$ Y-A, then A $\subset$ Y-C. However (by 1.4), P and Q are separated so (by 1.6) A $\subset$ P or A $\subset$ Q. Suppose A $\subset$ P. Now, P∩Q = ∅ implies that A∩Q = ∅. Thus, Q $\subset$ Y-A, and, since C $\subset$ Y-A, then C∪Q $\subset$ Y-A. But Q $\subset$ Y-C, so Q∩C = ∅, and, since Q is nonempty, then C∪Q is a connected subset of Y-A which properly contains C, a contradiction of the fact that C is a component of Y-A. Similarly, if A $\subset$ Q, a contradiction is reached. Therefore, Y-C is connected.

<u>1.34.Corollary</u> If Y is a connected topological space of at least two points, then there exist two connected subsets M and N, of Y, which are distinct from Y and such that M∪N = Y and M∩N = ∅.

<u>Proof</u>: Since any set consisting of a single point is connected, then Y contains a connected subset A such that A is distinct from Y. Let M be a component of Y-A and let N = Y-M. Thus, M∪N = Y, M∩N = ∅, M is connected, and (by 1.33) N is connected.

<u>1.35.Definition</u> Let X be a topological space and x ε X. Then the <u>quasicomponent</u> of X containing x is the set consisting of x together with all points y of X such that X is not the union of two disjoint open sets, one of which contains x, and the other y.

<u>1.36. Lemma</u> Let X be a topological space, and let Q be a quasicomponent of X. If $X = M \cup N$, where M and N are nonempty, disjoint open sets, then Q is a subset of either M or N.

<u>Proof</u>: Assume that $Q \not\subset M$ and $Q \not\subset N$. Then $Q \cap M \neq \emptyset$ and $Q \cap N \neq \emptyset$, which implies that there is a point $p \in Q \cap M$ and a point $q \in Q \cap N$. Thus, $p, q \in Q$ and $p \in M$ and $q \in N$, which implies (by 1.35) that Q is not a quasicomponent of X, a contradiction.

<u>1.37. Theorem</u> If Q is a quasicomponent of a topological space X, then Q is closed.

<u>Proof</u>: Let $x \in Q$ and assume that Q is not closed. Then Q has a limit point p such that $p \notin Q$. Thus (by 1.35), there are two disjoint open sets M and N such that $X = M \cup N$ and $x \in M$ and $p \in N$. But, since $p \in Q'$, then N contains a point $q \in Q$. Now (by 1.36), $Q \subset M$ or $Q \subset N$. However, since $x \in Q$ and $x \in M$, then $Q \subset M$. Thus, $q \in M$ and $q \in N$ which implies that M and N are not disjoint, a contradiction.

<u>1.38. Theorem</u> Let X be a topological space and $x \in X$. If Q is a quasicomponent containing x, then Q is the intersection of all sets which are both open and closed and contain x.

<u>Proof</u>: Let $\{A_\alpha : \alpha \in T\}$ be the family of all sets which are both open and closed and contain x. The set $\{A_\alpha : \alpha \in T\}$ is nonempty for $X \in \{A_\alpha : \alpha \in T\}$.

Part 1 — Let $p \varepsilon Q$. Assume that $p \not\varepsilon \cap\{A_\alpha : \alpha \varepsilon T\}$. Thus, for some $A \varepsilon \{A_\alpha : \alpha \varepsilon T\}$, $p \not\varepsilon A$. This implies that $p \varepsilon X-A$, and, since A is closed, X-A is open. Thus, $X = A\cup(X-A)$, where A and X-A are disjoint open sets such that $x \varepsilon A$ and $p \varepsilon X-A$. But this implies that Q is not a quasicomponent, a contradiction. Thus $p \varepsilon \cap\{A_\alpha: \alpha \varepsilon T\}$.

Part 2 — Let $p \varepsilon \cap\{A_\alpha : \alpha \varepsilon T\}$. Assume that $p \not\varepsilon Q$. Thus, $X = M\cup N$, where M and N are disjoint open sets such that $x \varepsilon M$ and $p \varepsilon N$. Now, since N is open, $X-N = M$ is closed. Thus, since $x \varepsilon M$ and M is both open and closed, then $M \varepsilon \{A_\alpha : \alpha \varepsilon T\}$. But since $p \varepsilon N$, and M and N are disjoint, then $p \not\varepsilon M$. Thus, $p \not\varepsilon \cap\{A_\alpha : \alpha \varepsilon T\}$, which is a contradiction. Therefore, $p\varepsilon Q$. It follows that $Q=\cap\{A_\alpha:\alpha\varepsilon T\}$.

<u>1.39.Theorem</u> Each component C of a topological space X is a subset of some quasicomponent.

<u>Proof</u>: Let $x \varepsilon C$, and let Q be a quasicomponent containing x. If $C = \{x\}$, then immediately $C \subset Q$. Suppose that $y \varepsilon C$, where $y \neq x$. Assume that $y \not\varepsilon Q$. Thus (by 1.35), there are two disjoint open sets A and B such that $X = A\cup B$ and $x \varepsilon A$ and $y \varepsilon B$. Since A is open, $X-A = A\cup B-A = B$ is closed. Now, $y \varepsilon B\cap C$, and, since B is both open and closed in X, then $B\cap C$ is both open and closed in C. Also, since $x \not\varepsilon B$, then $x \not\varepsilon B\cap C$, which implies that $B\cap C \neq C$. Thus, $B\cap C$ is a proper subset of C, which implies (by 1.5) that C is not connected. This is a contradiction, since C is a component. Thus, $y \varepsilon Q$. It follows that $C \subset Q$.

The proof of the next theorem will depend upon the maximal principle, which will be stated for reference, and also upon a lemma which will follow the statement of the maximal principle.

Maximal Principle Let A be a set partially-ordered by a relation $<$. Let B be a subset of A and assume that B is simply-ordered by $<$. Then there is a subset M of A that is simply-ordered by $<$, contains B, and is not a proper subset of any other subset of A with these properties.

1.40. Lemma Let a and b be points of a compact Hausdorff space X, and let $\{H_\alpha : \alpha \in T\}$ be a collection of closed sets, and suppose that $\{H_\alpha : \alpha \in T\}$ is simply-ordered by inclusion. If each H_α contains both a and b and is not the union of two separated sets, one containing a and the other containing b, then the intersection $\cap\{H_\alpha : \alpha \in T\}$ also has this property.

Proof: Let $H = \cap\{H_\alpha : \alpha \in T\}$ and assume that $H = A \cup B$ where $a \in A$, $b \in B$ and A and B are separated. Thus, A and B are closed in H, and, since H is closed, then A and B are closed. Therefore, since X is compact, A and B are compact. This implies that since $A \cap B = \emptyset$, there are two disjoint open sets U and V such that $A \subset U$ and $B \subset V$. Since $a \in A$ and $b \in B$, then $a \in U$ and $b \in V$. Thus, for each $\alpha \in T$, $a \in H_\alpha \cap U$ and $b \in H_\alpha \cap V$.

Now, let $R_\alpha = H_\alpha \cap [X-(U \cup V)]$ for each $\alpha \in T$. Since H_α is closed and $[X-(U \cup V)]$ is closed, then R_α is closed.

Assume that $R_\theta = \emptyset$ for some $\theta \in T$. Then, $H_\theta = H_\theta \cap (U \cup V) = (H_\theta \cap U) \cup (H_\theta \cap V)$, and, since U and V are disjoint open sets, then $(H_\theta \cap U)$ and $(H_\theta \cap V)$ are disjoint and open in H_θ. Thus (by 1.4), $(H_\theta \cap V)$ and $(H_\theta \cap U)$ are separated, a contradiction. Therefore, $R_\alpha \neq \emptyset$ for all $\alpha \in T$.

Consider any two distinct sets R_β, $R_\varphi \in \{R_\alpha : \alpha \in T\}$. Then, $R_\beta = H_\beta \cap [X-(U \cup V)]$, and $R_\varphi = H_\varphi \cap [X-(U \cup V)]$. Since $\{H_\alpha : \alpha \in T\}$ is simply-ordered by inclusion, given H_β and H_φ, one is a subset of the other. Supposing that H_β is a subset of H_φ, then $H_\beta \cap [X-(U \cup V)]$ is a subset of $H_\varphi \cap [X-(U \cup V)]$ implying that $\{R_\alpha : \alpha \in T\}$ is simply-ordered by inclusion. Therefore, the intersection of any finite number of elements of $\{R_\alpha : \alpha \in T\}$ is an element of $\{R_\alpha : \alpha \in T\}$ and, consequently, nonempty. Thus, $\{R_\alpha : \alpha \in T\}$ satisfies the finite intersection hypothesis, and, since X is compact, $\cap\{R_\alpha : \alpha \in T\} \neq \emptyset$. But, $\cap\{R_\alpha : \alpha \in T\} = \cap\{H_\alpha \cap [X-(U \cup V)] : \alpha \in T\} = \cap\{H_\alpha : \alpha \in T\} \cap [X-(U \cup V)] = H \cap [X-(U \cup V)]$. Thus, $H \cap [X-(U \cup V)] \neq \emptyset$, which implies that $H \not\subset (U \cup V)$, a contradiction.

<u>1.41.Theorem</u> In a compact Hausdorff space X, every quasicomponent is a component.

<u>Proof</u>: Let Q be a quasicomponent of X, let $q \in Q$, and let C be a component of X containing q. Assume that $Q \neq C$. Since (by 1.39) $C \subset Q$, then there must be a point $x \in Q$ such that $x \notin C$. Now, let $\{A_\alpha : \alpha \in T\}$ be the

collection of all closed subsets of X, each of which contains both q and x but none of which is the union of two separated sets, one containing q and the other containing x. Now, since $q, x \varepsilon Q$, X cannot be the union of two separated sets, one containing q and the other containing x. Thus, $X \varepsilon \{A_\alpha : \alpha \varepsilon T\}$. Let $\{A_\alpha : \alpha \varepsilon T\}$ be partially-ordered by inclusion. By the maximal principle, there is a maximal, simply-ordered subcollection $\{B_\beta : \beta \varepsilon S\}$ of $\{A_\alpha : \alpha \varepsilon T\}$. Thus, $K = \cap\{B_\beta : \beta \varepsilon S\}$ is closed, and (by 1.40) $K \varepsilon \{A_\alpha : \alpha \varepsilon T\}$. Assume that K is not connected. Then $K = K_1 \cup K_2$ where K_1 and K_2 are separated sets. Since $K \varepsilon \{A_\alpha : \alpha \varepsilon T\}$, then either K_1 or K_2 must contain both q and x. Suppose $q, x \varepsilon K_1$. Now, K_1 is closed in K and since K is closed in X, then K_1 is closed in X. Also, q and x cannot be separated in K_1 because, if so, they could be separated in K. Thus, $K_1 \varepsilon \{A_\alpha : \alpha \varepsilon T\}$ and K_1 is a proper subset of K, implying that $\{B_\beta : \beta \varepsilon S\}$ is not maximal, a contradiction. Hence, K must be connected. But, since $q \varepsilon C$ and $q \varepsilon K$, then (by 1.9) $C \cup K$ is connected. Also, since $x \varepsilon C \cup K$ and $x \not\varepsilon C$, then $C \cup K$ properly contains C, implying that C is not a maximal connected set, a contradiction. Thus, Q = C.

CHAPTER II

LOCAL CONNECTEDNESS

2.1. Definition A topological space Y is locally connected if for each point $p \in Y$ and each neighborhood U of p there is a connected neighborhood V of p such that $V \subset U$. A subset A of Y is locally connected if it is locally connected as a subspace of Y.

2.2. Definition Let Y be a topological space and let B be a collection of open sets in Y. Then B is a basis for Y if for each open set U and each point $x \in U$ there is a set $V \in B$ such that $x \in V \subset U$.

2.3. Theorem Let Y be a topological space. The following three properties are equivalent:

(1) Y is locally connected

(2) The components of each open set in Y are open sets.

(3) Y has a basis consisting of connected sets.

Proof:

Show that (1) implies (2).

Let U be open in Y, C be a component of U, and $y \in C$. Thus, $y \in U$, and, since Y is locally connected (by 2.1) there is a connected neighborhood V of y such that $V \subset U$.

However, since C is a maximal connected set in U which contains y, then $V \subset C$. Thus, C is open.

Show that (2) implies (3).

Let B be the family of all components of all open sets in Y. Let U be open in Y and $x \in U$. If C is a component of U containing x, then C is open and $C \in B$. Thus (by 2.2), is a basis for Y, and B consists of connected sets.

Show that (3) implies (1).

Let B be a basis for Y, where B consists of connected sets. Let $x \in Y$ and U be a neighborhood of x. Then (by 2.2) there is a $V \in B$ such that $x \in V \subset U$. This implies that Y is locally connected.

<u>2.4.Theorem</u> Let X be a locally connected topological space. If A is an open subset of X, then A is locally connected.

<u>Proof</u>: Let $p \in A$, and let U be a neighborhood of p in A. Since U is open in A and A is open in X, then U is open in X. Since X is locally connected, (by 2.1) there is a connected neighborhood V of p such that $V \subset U$. Thus, V is also connected and open in A. This implies that A is locally connected.

<u>2.5.Theorem</u> Local connectedness is a topological invariant.

Proof: Let X and Y be topological spaces where X is locally connected, and let $f:X \to Y$ be a homeomorphism. Let $y \in Y$, and let U be a neighborhood of y in Y. Then $f^{-1}(y) \in f^{-1}(U)$, and, since $f:X \to Y$ is a homeomorphism, then $f^{-1}(U)$ is open in X. Since X is locally connected, there is a connected neighborhood $V \subset X$ such that $f^{-1}(y) \in V \subset f^{-1}(U)$. Thus, $f(f^{-1}(y)) \in f(V) \subset f(f^{-1}(U))$, which implies that $y \in f(V) \subset U$. But (by 1.11), f(V) is connected. Also, since $f:X \to Y$ is a homeomorphism and V is open in X, then f(V) is open in Y. Thus (by 2.1), Y is locally connected.

2.6. Theorem If X is a locally connected Hausdorff space, then every quasicomponent is a component.

Proof: Let Q be a quasicomponent of X, let $q \in Q$, and let C be a component containing q. Thus (by 1.39), $C \subset Q$. Also, since C is a component, (by 1.30) C is closed. Thus, X-C is open. Since X is locally connected, (by 2.3) C is open. Hence, $X = C \cup (X-C)$ where C and X-C are disjoint, open sets. Therefore (by 1.36), $Q \subset C$ or $Q \subset X-C$. Since $Q \cap C \neq \emptyset$, then $Q \subset C$. Thus, Q = C.

2.7. Theorem Let Y be a locally connected topological space. If U is a component of the open set $G \subset Y$, then $G \cap Fr(U) = \emptyset$.

Proof: Assume that $G \cap Fr(U) \neq \emptyset$. Then, there is a point $x \in G \cap Fr(U) = G \cap (\overline{U} \cap \overline{Y-U})$.

Suppose $x \varepsilon U$. Thus, $x \not\varepsilon Y-U$, and, since $x \varepsilon \overline{Y-U}$, then $x \varepsilon (Y-U)'$. But, since Y is locally connected, (by 2.3) U is open and, thus, U contains a point of Y-U, a contradiction.

Suppose $x \not\varepsilon U$. Since $x \varepsilon \overline{U}$, then $x \varepsilon U'$. Now, G is a neighborhood of x, and, since Y is locally connected, (by 2.1) there is a connected neighborhood V of x such that $V \subset G$. Thus, since $x \varepsilon U'$, V contains a point of U different from x, and (by 1.9) $V \cup U$ is connected. But, since $V \cup U$ properly contains U, then U cannot be a component of G, a contradiction. Therefore, $G \cap Fr(U) = \emptyset$.

<u>2.8. Theorem</u> Let Y be a locally connected topological space, let $A \subset Y$, and let C be a component of A. Then the following properties hold:

(1) $Int(C) = C \cap Int(A)$

(2) $Fr(C) \subset Fr(A)$

(3) If A is closed, then $Fr(C) = C \cap Fr(A)$.

<u>Proof of (1)</u>:

Since $C \subset A$, then $Int(C) \subset Int(A)$. Thus, since $Int(C) \subset C$, then $Int(C) \subset (C \cap Int(A))$. Now, to show that $C \cap Int(A) \subset Int(C)$, let $y \varepsilon C \cap Int(A)$. Since Int(A) is open in Y and Y is locally connected, there is a connected neighborhood U of y such that $U \subset Int(A)$. This implies that $y \varepsilon U \subset A$, and, since $y \varepsilon C$ and C is a component of A , then $U \subset C$. But U is open, so $U \subset Int(C)$. Thus, $y \varepsilon Int(C)$, implying that $C \cap Int(A) \subset Int(C)$.

Proof of (2):

Assume that $Fr(C) \not\subset Fr(A)$. Then there is a point $x \in Fr(C)$ such that $x \notin Fr(A)$. This implies (by 1.21) that there is a neighborhood U of x such that $U \cap A = \emptyset$, or $U \cap (Y-A) = \emptyset$. Now, $x \in Fr(C)$; so $U \cap C \neq \emptyset$, and, since $C \subset A$, then $U \cap A \neq \emptyset$. Thus, $U \cap (Y-A) = \emptyset$. Since Y is locally connected, there is a connected neighborhood V of x such that $V \subset U$. Thus, $V \cap (Y-A) = \emptyset$, which implies that $V \subset A$. Now, since $x \in Fr(C)$ and $x \in V$, then V contains points in C and points not in C. Thus (by 1.9), $V \cup C$ is connected. Also, $V \cup C \subset A$ and $V \cup C$ properly contains C, which implies that C is not a component of A, a contradiction.

Proof of (3):

Part i — Let $x \in Fr(C)$. By part (2), $Fr(C) \subset Fr(A)$, implying that $x \in Fr(A)$. Now, $x \in Fr(C) = \overline{C} \cap (\overline{Y-C})$, which implies that $x \in \overline{C}$. But, since C is a component of A, (by 1.30) C is closed in A, and, since A is closed in Y, then C is closed in Y, implying that $\overline{C} = C$. Thus, $x \in C \cap Fr(A)$.

Part ii — Let $x \in C \cap Fr(A)$, and let U be a neighborhood of x. Since $x \in Fr(A)$ (by 1.21), U contains a point $p \in A$ and a point $q \notin A$. Since $C \subset A$, then $q \notin C$. Thus, $x \in C$, $q \notin C$, and $x, q \in U$, which implies that $x \in Fr(C)$. Thus, it follows that $Fr(C) = C \cap Fr(A)$.

2.9. Theorem Let Y be a locally connected topological space and $A \subset Y$. If $S \subset A$ is connected and open in A, then $S = A \cap C$, where C is connected and open in Y.

Proof: Since $A \subset Y$ and S is open in A, then there is an open set U such that $S = A \cap U$. Let $p \in S$ and C be the component of U containing p. Since $p \in S \subset U$ and S is connected, then $S \subset C \subset U$, which implies that since $S = A \cap U$, then $S = A \cap C$. Finally, since Y is locally connected, (by 2.3) C is open.

2.10. Theorem Let Y be a locally connected topological space which is not connected. Then, a decomposition of Y into two nonempty, disjoint open sets can always be accomplished by taking any component as one of the sets, and all the rest as the other set.

Proof: Let C be the collection of all components of Y. Let $A \in C$, and let $K = C - \{A\}$. The set K is nonempty since Y is not connected, and $Y = A \cup (\cup K)$. Since Y is locally connected and Y is open, (by 2.3) each component in Y is open. Thus, A is open, and $\cup K$ is open.

Now, for each $B \in K$, $A \cap B = \emptyset$, since A and B are maximal connected sets. Thus, $A \cap (\cup K) = \emptyset$. Hence, A and $\cup K$ are nonempty, disjoint open sets that decompose Y.

2.11. Theorem Let X be a connected, locally connected topological space. If A is a nonempty, closed subset of X, then the closure of each component of $X-A$ meets A.

Proof: Assume that there is a component C of $X-A$ such that $\overline{C} \cap A = \emptyset$. If $X-A = \emptyset$, then the proof is trivial. Suppose that $X-A \neq \emptyset$. Then $C \neq \emptyset$, and, since A is nonempty and $C \subset X-A$, then $C \neq X$. Thus, C is a proper subset of X.

Now, since A is closed, X-A is open. Thus, since X is locally connected, (by 2.3) C is open. Also, since C is a component of X-A, then (by 1.30) C is closed in X-A. Thus, $\overline{C} \cap (X-A-C) = \emptyset$, and, since $\overline{C} \cap A = \emptyset$, then $\emptyset = \overline{C} \cap \left[(X-A-C) \cup A\right] = \overline{C} \cap (X-C)$. This implies that C is closed in X. Therefore, C is a proper subset of X which is both open and closed in X, implying (by 1.5) that X is not connected, a contradiction.

2.12. Theorem Let X be a connected, locally connected topological space. If A and B are two disjoint, closed subsets of X, then $X-(A \cup B)$ has a component whose closure meets both A and B.

Proof: If $X-(A \cup B) = \emptyset$, the proof is trivial. So, suppose that $X-(A \cup B) \neq \emptyset$. Since A and B are closed, $A \cup B$ is closed, and, thus, (by 2.11) the closure of each component of $X-(A \cup B)$ meets $A \cup B$. Assume that if C is a component of $X-(A \cup B)$, then either $\overline{C} \cap A = \emptyset$ or $\overline{C} \cap B = \emptyset$. Let J and K be the sets of all components of $X-(A \cup B)$ whose closures meet A and B respectively. Let $J^* = \cup J$ and $K^* = \cup K$. Thus, $X-(A \cup B) = J^* \cup K^*$, and, since $X-(A \cup B) \neq \emptyset$, either $J^* \neq \emptyset$ or $K^* \neq \emptyset$. Let the labeling be chosen such that $J^* \neq \emptyset$.

Now, assume that $B \cap (J^*)' \neq \emptyset$. Thus, there is a point $p \in B \cap (J^*)'$. Since $A \cap B = \emptyset$ and A and B are closed, then $p \notin A'$. Thus, there is a neighborhood U of p such that $U \cap A = \emptyset$. Let V be a component of U which contains p. Since

X is locally connected, (by 2.3) V is open. Since $p \varepsilon (J^*)'$, V contains a point $q \varepsilon J^*$. Thus, q is a point in some element C of J, and (by 1.9) $V \cup C$ is connected. Also, since both V and C are open, then $V \cup C$ is open. Therefore, since X is locally connected, (by 2.4) $V \cup C$ is locally connected. Also, since B is closed, then $B \cap (V \cup C)$ is closed in $V \cup C$.

Now, since C is a component of $X-(A \cup B)$ and $C \subset (V \cup C) -B \subset X-(A \cup B)$, then C is a component of $(V \cup C)-B$. Thus (by 2.11), $\overline{C} \cap \left[B \cap (V \cup C)\right] \neq \emptyset$, implying that $\overline{C} \cap B \neq \emptyset$, a contradiction, since $C \varepsilon J$ implies that $\overline{C} \cap A \neq \emptyset$. Thus, $B \cap (J^*)' = \emptyset$. Likewise, if $K^* \neq \emptyset$, then $A \cap (K^*)' = \emptyset$.

Now, if $K^* = \emptyset$, then $X-(A \cup B) = J^*$, and $X = (A \cup J^*) \cup B$. Since A and B are disjoint and closed, then A and B are separated. Also, since $B \cap (J^*)' = \emptyset$, $B \cap (J^*) = \emptyset$, and $B = \overline{B}$, then $B \cap (\overline{J^*}) = \emptyset = \overline{B} \cap (J^*)$, implying that B and J^* are separated. Thus, $A \cup J^*$ and B are separated, which implies (by 1.5) that X is not connected, a contradiction. Likewise, if $J^* = \emptyset$, a contradiction is reached.

Suppose that $J^* \neq \emptyset$ and $K^* \neq \emptyset$. Then, $X = (A \cup J^*) \cup (B \cup K^*)$. Assume that J^* and K^* are not separated. Then either $\overline{J^*} \cap K^* \neq \emptyset$ or $J^* \cap \overline{K^*} \neq \emptyset$. Let the labeling be chosen such that $\overline{J^*} \cap K^* \neq \emptyset$. Since $J^* \cap K^* = \emptyset$, then $(J^*)' \cap K^* \neq \emptyset$. Let $p \varepsilon (J^*)' \cap K^*$. Thus, p belongs to some $C \varepsilon K$. Since C is open and $p \varepsilon (J^*)'$, then C contains a point $q \varepsilon J^*$. Thus, q belongs to some $D \varepsilon J$. Therefore, (by 1.9) $C \cup D$ is connected, and, since $C \neq D$, then $C \subset C \cup D$, contradicting the maximality of C. Thus, J^* and K^* are separated. Consequently, $(A \cup J^*)$ and $(B \cup J^*)$ are separated, implying (by 1.5) that X is not connected, a contradiction.

Hence, $X-(A \cup B)$ has a component whose closure meets both A and B.

Closely related to local connectedness is the idea of "connected im kleinen."

<u>2.13.Definition</u> A topological space X is <u>connected im kleinen at a point x</u> provided that for each open set U containing x there is an open set V containing x such that $V \subset U$ and, if y is any point in V, then there is a connected subset of U containing x and y.

<u>2.14.Theorem</u> If X is a topological space which is locally connected at a point x, then X is connected im kleinen at x.

<u>Proof</u>: Let U be a neighborhood of x. Since X is locally connected at x, there is a connected neighborhood V of x such that $V \subset U$. Thus, if $y \in V$, then V is a connected subset of U which contains x and y. This implies that X is connected im kleinen at x.

<u>2.15.Theorem</u> If X is a topological space which is connected im kleinen at each point, then X is locally connected.

<u>Proof</u>: Let U be an open set in X, let C be a component of U, and let $x \in C$. Thus (by 2.13), there is

an open set $V \subset U$ containing x such that if $p \in V$, then there is a connected subset D_p of U, which contains p and x. Thus, $V \subset \cup\{D_p : p \in V\}$, and since (by 1.9) $\cup\{D_p : p \in V\}$ is connected, then $V \subset \cup\{D_p : p \in V\} \subset C$. This implies that C is open, and (by 2.3) X is locally connected.

<u>2.16.Theorem</u> Let Y be a topological space such that $Y = A\cup B$, where A and B are closed. If Y is locally connected and $A\cap B$ is locally connected, then both A and B are locally connected.

<u>Proof</u>: If either $A\cap B = \emptyset$, $A \subset B$, or $B \subset A$, then the theorem is trivial. Therefore, suppose that $A\cap B \neq \emptyset$, $A \not\subset B$, and $B \not\subset A$. Let $x \in A$ and let U be an open set containing x. Since A is closed, $A = \overline{A} = \mathrm{Int}(A) \cup \mathrm{Fr}(A)$. Thus, $x \in \mathrm{Int}(A)$ or $x \in \mathrm{Fr}(A)$.

Suppose that $x \in \mathrm{Int}(A)$. Then $x \in U\cap\mathrm{Int}(A)$. Since $U \cap \mathrm{Int}(A)$ is open and Y is locally connected, there is a connected neighborhood V of x such that $V \subset U \cap \mathrm{Int}(A)$. Thus, $V \subset U\cap A$, and, if $y \in V$ then $x,y \in V \subset U\cap A$ where V is connected.

Suppose that $x \in \mathrm{Fr}(A)$. Now, $\mathrm{Fr}(A) \subset A\cap B$, for, if not, there is a point $p \in \mathrm{Fr}(A)$ such that $p \notin A\cap B$. Thus, either $p \in A-B$ or $p \in B-A$. If $p \in A-B$, then $p \in \mathrm{Int}(A)$, a contradiction. If $p \in B-A$, then, since $p \in \mathrm{Fr}(A)$, (by 1.21) $(B-A) \cap A \neq \emptyset$, a contradiction. Thus, $x \in A\cap B$. Since $A\cap B$ is locally connected, (by 2.14) $A\cap B$ is connected im kleinen at x. Therefore, there is an open set $W \subset U$

containing x such that if $y \in W \cap (A \cap B)$, then there is a connected set $M(x,y) \subset U \cap (A \cap B)$ which contains x and y.

Since Y is locally connected (by 2.14) Y is connected im kleinen at x. Thus, there is an open set $V \subset W$ containing x such that if $y \in V$, then there is a connected set $N(x,y) \subset W$ which contains x and y.

Now, consider the set $V \cap A = V \cap [\text{Int}(A) \cup \text{Fr}(A)] = [V \cap \text{Int}(A)] \cup [V \cap \text{Fr}(A)]$. The set $V \cap A$ is nonempty, since $x \in V \cap A$. Let $t \in V \cap A$. Then, either $t \in V \cap \text{Int}(A)$ or $t \in V \cap \text{Fr}(A)$.

Suppose that $t \in V \cap \text{Fr}(A)$. Thus, $t \in W \cap \text{Fr}(A)$. Since $\text{Fr}(A) \subset A \cap B$, then $t \in W \cap (A \cap B)$. Thus, there is a connected set $M(x,t) \subset U \cap (A \cap B) \subset U \cap A$ which contains x and t.

Suppose that $t \in V \cap \text{Int}(A)$. Since $t \in V$, there is a connected set $N(x,t) \subset W$ which contains x and t. Let C be a component of $W \cap \text{Int}(A)$, which contains t. Since $W \cap \text{Int}(A)$ is open and Y is locally connected, then (by 2.3) C is open.

Let $K = N(x,t) \cap C$ and let $H = [N(x,t) \cap \text{Int}(A)] - K$. Now, it will be shown that $\overline{C} \cap [\text{Fr}(A) \cap N(x,t)] \neq \emptyset$. Assume that $\overline{C} \cap [\text{Fr}(A) \cap N(x,t)] = \emptyset$. Clearly, $N(x,t) = K \cup H \cup [N(x,t) \cap \text{Fr}(A)] \cup [N(x,t) \cap (B-A)]$.

Now $C \subset W \cap \text{Int}(A)$. Since $N(x,t) \subset W$, then $N(x,t) \cap \text{Int}(A) \subset W \cap \text{Int}(A)$. Thus, $H \subset W \cap \text{Int}(A)$, and $C \cup H \subset W \cap \text{Int}(A)$. Now, C is open in $W \cap \text{Int}(A)$, and, since C is a component of W ∩ Int(A), then (by 1.30) C is closed in W ∩ Int(A). Thus, C is both open and closed in C∪H, which implies (by 1.4) that C and H are separated. Since K ⊂ C, then (by 1.6) K and H are separated.

Now, $\overline{C}\cap[Fr(A) \cap N(x,t)] = \emptyset$, and, since C is open, $C\cap[\overline{Fr(A) \cap N(x,t)}] = \emptyset$. Thus, C and [Fr(A) ∩ N(x,t)] are separated. Since K ⊂ C, then (by 1.6) K and [Fr(A) ∩ N(x, t)] are separated.

Now, C ⊂ W ∩ Int(A) ⊂ Int(A) ⊂ A which implies that C and B-A are disjoint. Also, C and B-A are both open. Thus (by 1.4), C and B-A are separated. Since K ⊂ C, then (by 1.6) K and B-A are separated. Thus, K and [(B-A) ∩ N(x,t)] are separated.

From the above three paragraphs, it is concluded that K and H∪[N(x,t) ∩ Fr(A)] ∪ [N(x,t) ∩ (B-A)] are separated. Thus, N(x,t) is not connected, a contradiction. Therefore, $\overline{C}\cap[Fr(A) \cap N(x,t)] \neq \emptyset$. Let $p \in \overline{C}\cap[Fr(A) \cap N(x,t)]$. Since N(x,t) ⊂ W, then p ε [W ∩ Fr(A)] ⊂ [W ∩ (A∩B)]. Then, there is a connected set M(x,p) containing x and p such that M(x,p) ⊂ U∩(A∩B). Now, C ⊂ W ∩ Int(A) ⊂ U ∩ Int(A) ⊂ U∩A. Thus, C∪M(x,p) ⊂ U∩A. Since $p \in \overline{C}$, p ε M(x,p), and C and M(x,p) are connected, then (by 1.8) C∪M(x,p) is connected. Also, C∪M(x,p) contains both x and t. Therefore, A is connected im kleinen at x, and (by 2.15) A is locally connected. Similarly, B is locally connected.

2.17 Theorem Let Y be a locally connected topological space, and let A be a subset of Y. If Fr(A) is locally connected, then $\overline{A}$ is locally connected.

Proof: The space $Y = \overline{A} \cup \overline{Y-A}$, and both $\overline{A}$ and $\overline{Y-A}$ are closed. Also, $A \cap Y-A = Fr(A)$, which is locally connected. Thus (by 2.16) $\overline{A}$ is locally connected.

2.18. Theorem A metric space (X,d) is connected im kleinen at a point x if and only if, given $e > 0$, there is a number $\delta > 0$ such that if $d(x,y) < \delta$, then x and y lie in a connected set of diameter less than e.

Proof:

Part 1 — Let $x \in X$. Suppose that, given $e > 0$, there is a number $\delta > 0$ such that if $d(x,y) < \delta$, then x and y lie in a connected set of diameter less than e. Let U be an open set containing x. Then, there is an open set $W = B(x,e_1)$ such that $W \subset U$. Since $e_1 > 0$, there is a number $\delta_1 > 0$ such that if $d(x,y) < \delta_1$, then x and y lie in a connected set of diameter less than e_1.

Let $V = B(x,\delta_1)$, and let $p \in V$. Thus, $d(x,p) < \delta_1$, and $x,p \in C_p$, where C_p is a connected set of diameter less than e_1. Thus, $C_p \subset W \subset U$.

Assume that $V \not\subset W$. Then, there is a point $q \in V$ such that $q \notin W$. This implies that $d(x,q) < \delta_1$ and $d(x,q) \geq e_1$. Since $d(x,q) < \delta_1$, then $x,q \in C_q$ where C_q is a connected set of diameter less than e_1. Thus, $d(x,q) < e_1$, a contradiction. Hence, $V \subset W \subset U$, and (by 2.13) X is connected im kleinen at x.

Part 2 — Suppose that X is connected im kleinen at x. Let $e > 0$, and let $U = B(x, \frac{e}{3})$. Then (by 2.13), there is an open set V containing x such that $V \subset U$ and, if $y \in V$, then there is a connected subset of U containing x and y.

Now, there is a $\delta > 0$ such that the open set $W = B(x,\delta)$ is a subset of V. Let $p \in W$. Then, $p \in V$, and $d(x,p) \leq \delta$. Also, there is a connected subset C of U which contains x and p. Since the diameter of $U = 2(\frac{e}{3}) < e$ and $C \subset U$, then the diameter of $C < e$.

Thus, for $e > 0$, there is a number $\delta > 0$ such that if $d(x,p) < \delta$, then x and p lie in a connected set C where the diameter of $C < e$.

Another concept which is related to local connectedness but used only in metric spaces is "property S."

<u>2.19.Definition</u> A metric space M has <u>property S</u> if for every $e > 0$, M is the union of a finite number of connected sets, each of diameter less than e.

<u>2.20.Theorem</u> If (X,d) is a metric space having property S, then X is connected im kleinen at each of its points and, hence, is locally connected.

<u>Proof</u>: Let $x \in X$ and let U be an open set containing x. There is an open set $G = B(x,e)$ such that $G \subset U$. Since X has property S, $X = \cup\{C_i : i = 1, \ldots, n\}$ where $\{C_i : i = 1, \ldots, n\}$ is a collection of connected sets each of diameter

less than $\frac{e}{3}$. Let C be the collection of all elements of $\{C_i : i = 1,\ldots,n\}$ whose closure contains x. Now, $x \in C_a$ where $C_a \in C$. Thus, if $C_b \in C$, then $x \in C_a \cap \overline{C_b}$, which implies (by 1.8) that $C_a \cup C_b$ is connected. Hence, $\cup C$ is connected. Now, to show that $\cup C \subset U$, let $y \in \cup C$. Then, $y \in$ some $C_k \in C$, which implies that $x \in C_k$. Thus, $d(x,y) \leq \frac{e}{3} < e$ implying that $\cup C \subset G \subset U$.

Now, consider the collection $D = \{C_i : i = 1,\ldots,n\} - C$. Thus, if $C_j \in D$, then $x \notin \overline{C_j}$. It follows that $x \notin \cup\{\overline{C_j} : C_j \in D\}$. Hence, $x \in X - \cup\{\overline{C_j} : C_j \in D\}$, which is open, since $\cup\{\overline{C_j} : C_j \in D\}$ is closed.

Next, it will be shown that $X - \cup\{\overline{C_j} : C_j \in D\} \subset \cup C$. Let $p \in X - \cup\{\overline{C_j} : C_j \in D\}$. Thus, $p \notin \cup\{\overline{C_j} : C_j \in D\} \supset \cup\{C_j : C_j \in D\} = \cup D$. Since $p \in X = \cup\{C_i : i = 1,\ldots,n\} = (\cup C) \cup (\cup D)$ and $p \notin \cup D$, then $p \in \cup C$. Hence, $X - \cup\{\overline{C_j} : C_j \in D\}$ $\cup C$.

In summary, $X - \cup\{\overline{C_j} : C_j \in D\}$ is an open set containing x, and $X - \cup\{\overline{C_j} : C_j \in D\} \subset \cup C \subset U$ where $\cup C$ is connected. Thus X is connected im kleinen at x and hence (by 2.3) is locally connected.

<u>2.21.Theorem</u> If X is a compact, locally connected metric space, then X has property S.

<u>Proof</u>: Let $e > 0$, let $x \in X$, and let $U_x = B(x,\frac{e}{3})$. Since X is locally connected, there is a connected neighborhood V_x of x such that $V_x \subset U_x$. Thus, the diameter of V_x is less than or equal to the diameter of U_x, which is less than or equal to $\frac{2}{3}e$. The collection $\{V_p : p \in X\}$

of all such connected neighborhoods forms an open cover for X, and, since X is compact, $\{V_p : p \in X\}$ has a finite subcover for X. Thus, X has property S.

__2.22. Theorem__ Let (X,d) be a metric space and let M be a subset of X such that M has property S. If N is a subset of X such that $M \subset N \subset \overline{M}$, then N has property S.

__Proof__: Let $e > 0$. Then $M = \cup\{C_i : i = 1,\ldots,n\}$, where $\{C_i : i=1,\ldots,n\}$ is a collection of connected sets, each of diameter less than e. Consider the set $\cup\{N \cap \overline{C_i} : i=1, \ldots,n\}$. Clearly, $\cup\{N \cap \overline{C_i} : i = 1,\ldots,n\} \subset N$. Now, if $p \in N$, then, since $N \subset \overline{M} = \cup\{\overline{C_i} : i = 1,\ldots,n\}$, $p \in \overline{C_k}$ for some $1 \leq k \leq n$. Thus, $p \in \cup\{N \cap \overline{C_i} : i = 1,\ldots,n\}$, which implies that $N \subset \cup\{N \cap \overline{C_i} : i = 1,\ldots,n\}$. Therefore, $N = \cup\{N \cap \overline{C_i} : i = 1,\ldots,n\}$. Next, it will be shown that $C_j \subset \overline{C_j} \cap N \subset \overline{C_j}$ for each $1 \leq j \leq n$. To show this, let $p \in C_j$. Thus, $p \in \overline{C_j}$. Also, $p \in M$, since $M = \cup\{C_i : i = 1,\ldots,n\}$, and this implies that $p \in N$, since $M \subset N$. Thus, $p \in \overline{C_j} \cap N$, implying that $C_j \subset \overline{C_j} \cap N$, and, clearly, $\overline{C_j} \cap N \subset \overline{C_j}$. Thus (by 1.17), $\overline{C_j} \cap N$ is connected. Also, since the diameter of $C_j < e$, then the diameter of $\overline{C_j} < e$, which implies that the diameter of $\overline{C_j} \cap N < e$. Thus, N has property S.

Another concept relating to local connectedness which applies only to metric spaces is "uniform local connectedness."

__2.23. Definition__ A metric space (X,d) is __uniformly locally connected__ provided that, given $e > 0$, there is a number $\delta > 0$, independent of position, such that any two points x and y with $d(x,y) < \delta$, lie in a connected set of diameter less than e.

2.24. Theorem If (X,d) is a compact, locally connected metric space, then (X,d) is uniformly locally connected.

Proof: Let $e > 0$. Since X is compact and locally connected, (by 2.21) X has property S. Thus, $X = \bigcup\{C_i : i = 1,\ldots,n\}$, where $\{C_i : i = 1,\ldots,n\}$ is a finite collection of connected sets, each of diameter less than $\frac{e}{3}$. If for each pair (C_k,C_l), where $C_k,C_l \in \{C_i : i = 1,\ldots,n\}$, $\overline{C_k} \cap \overline{C_l} \neq \emptyset$, then the proof is immediate. Therefore, assume that the collection $D = \left[(C_k,C_l) : C_k,C_l \in \{C_i : i=1,\ldots,n\} \text{ and } \overline{C_k} \cap \overline{C_l} = \emptyset\right]$ is nonempty. Thus, if $\delta_{k,l} = d(C_k,C_l)$ for all $(C_k,C_l) \in D$, then $\delta_{k,l} > 0$. Since D is finite, the collection of all such $\delta_{k,l}$ is finite. Let δ be one half the minimum of this collection. Thus, $\delta > 0$. Now, let x,y be any two points in X such that $d(x,y) < \delta$. Since $X = \bigcup\{C_i : i=1,\ldots,n\}$, then there are a $C_a,C_b \in \{C_i : i=1,\ldots,n\}$ such that $x \in C_a$ and $y \in C_b$. If $C_a = C_b$, then, clearly, x and y lie in a connected set of diameter less than e. Suppose that $C_a \neq C_b$. Since $d(x,y) < \delta$, then $d(C_a,C_b) < \delta$. This implies that $(C_a,C_b) \notin D$, which, in turn, implies that $\overline{C_a} \cap \overline{C_b} \neq \emptyset$. Since C_a and C_b are connected, (by 1.17) $\overline{C_a}$ and $\overline{C_b}$ are connected. Thus (by 1.9), $\overline{C_a} \cup \overline{C_b}$ is connected. Also, since the diameters of C_a and $C_b < \frac{e}{3}$, the diameters of $\overline{C}_a$ and $\overline{C}_b \leq \frac{e}{3}$. Thus, the diameter of $\overline{C}_a \cup \overline{C}_b \leq \frac{e}{3} + \frac{e}{3} = \frac{2e}{3} < e$. Hence, X is uniformly locally connected.

CHAPTER III

PATH-CONNECTEDNESS

3.1. Definition A path in a topological space Y is a continuous mapping $f:I \to Y$, where I is the unit interval. The point $f(0) \in Y$ is called the initial (or starting) point, and $f(1) \in Y$, the terminal (or end) point of the path f, and f is said to run from $f(0)$ to $f(1)$, or join $f(0)$ to $f(1)$.

It will be noted that if f is a path running from $f(0)$ to $f(1)$, then the mapping $g:I \to Y$ defined by $g(t) = f(1-t)$, where $t \in I$, is a path running from $f(1)$ to $f(0)$.

3.2. Definition A topological space Y is path-connected if each pair of its points can be joined by a path.

3.3. Theorem Let Y be a topological space, and let $y_0 \in Y$. Y is path-connected if and only if each $y \in Y$ can be joined to y_0 by a path.

Proof:

Part 1 — Let Y be path-connected. Thus, each $y \in Y$ can be joined to y_0 by a path.

Part 2 — Suppose that each $y \in Y$ can be joined to y_0 by a path. Let $a, a' \in Y$. Then, there is a path $f:I \to Y$ joining a to y_0. Thus, $f(0) = a$, and $f(1) = y_0$. Also, there is a path joining a' to y_o, which implies there is a path $g:I \longrightarrow Y$ joining y_o to a'. Thus, $g(0) = y_o$, and $g(1) = a'$.

Now, let $A = \{t: 0 \leq t \leq 1/2\}$, $B = \{t: 1/2 \leq t \leq 1\}$ and $h(t) = \{\begin{matrix} f(2t) & ,t \in A \\ g(2t-1) & ,t \in B \end{matrix}\}$. Thus, h maps I into Y, $h(0) = f(0) = a$, and $h(1) = g(1) = a'$.

Next, it will be shown that h is continuous. Let $q \in h(I)$, and let U be a closed set containing q. Since $(h|A)(t) = f(2t)$ for all $t \in A$ and $(h|B)(t) = g(2t-1)$ for all $t \in B$, then $h|A$ and $h|B$ are continuous on A and B respectively. Now, $h^{-1}(U) \cap A \neq \emptyset$ or $h^{-1}(U) \cap B \neq \emptyset$. Let the labeling be chosen such that $h^{-1}(U) \cap A \neq \emptyset$. Clearly, $h^{-1}(U) \cap A = (h|A)^{-1}(U) \cap A = (h|A)^{-1}\left[U \cap (h|A)(A)\right]$. Since $(h|A)(A) \subset Y$ and U is closed in Y, then $U \cap (h|A)(A)$ is closed in $(h|A)(A)$. Since $h|A$ is continuous, then $(h|A)^{-1}\left[U \cap (h|A)(A)\right]$ is closed in A. But, since A is closed in I, then $(h|A)^{-1}\left[U \cap (h|A)(A)\right]$ is closed in I. Thus, $h^{-1}(U) \cap A$ is closed in I.

Suppose that $h^{-1}(U) \cap B = \emptyset$. Then, $h^{-1}(U) = h^{-1}(U) \cap A$, which is closed in I.

Suppose that $h^{-1}(U) \cap B \neq \emptyset$. Then, $h^{-1}(U) \cap B = (h|B)^{-1}\left[U \cap (h|B)(B)\right]$, which is closed in I. Thus, $h^{-1}(U) = \left[h^{-1}(U) \cap A\right] \cup \left[h^{-1}(U) \cap B\right]$, implying that $h^{-1}(U)$ is closed, since $h^{-1}(U) \cap A$ and $h^{-1}(U) \cap B$ are both closed. Consequently, h is continuous.

Therefore, h is a path joining a to a', implying that Y is path-connected.

3.4.Theorem The union of any family of path-connected spaces having a point in common is path-connected.

Proof: Let $\{Y_\alpha: \alpha \in T\}$ be a family of path-connected spaces so that for each $\alpha \in T$, $y_o \in Y_\alpha$. Thus, $y_o \in \cup\{Y_\alpha: \alpha \in T\}$. Now, let $y \in \cup\{Y_\alpha: \alpha \in T\}$. Thus, for some $\beta \in T$, $y \in Y_\beta$, and, since $y_o \in Y_\beta$ and Y_β is path-connected, there is a path joining y to y_o. Therefore (by 3.3), $\cup\{Y_\alpha:\alpha \in T\}$ is path-connected.

3.5.Definition A subset C of a topological space Y is called a path component of Y if C is a maximal path-connected set in Y.

3.6.Theorem Each path-connected topological space Y is connected.

Proof: Let $y_o \in Y$. Now, if $y \in Y$, then, since Y is path-connected, y can be joined to y_o by a path $f_y:I \to Y$. Thus, $y_o, y \in f_y(I)$. Also, since $f_y:I \to Y$ is continuous and I is connected, then (by 1.11) $f_y(I)$ is connected. Thus, $y_o \in f_x(I)$ for all $x \in Y$, and (by 1.9) $\cup\{f_x(I):x \in Y\} = Y$ is connected.

3.7.Corollary Each path component is connected.

3.8.Theorem The following two properties of a topological space Y are equivalent:

(1) Each path component is open (and, therefore, also closed).

(2) Each point of Y has a path-connected neighborhood.

Proof:

Part 1 — Suppose that each path component is open. Then, if $y \in Y$ and C is a path component containing y, then C is a path-connected neighborhood of y.

Part 2 — Suppose that each point of Y has a path-connected neighborhood. Let C be a path component in Y, and let $x \in C$. Thus, x has a path-connected neighborhood U, and, since C is a maximal path-connected set containing x, then $x \in U \subset C$, which implies that C is open.

Since X-C is the union of the remaining path components in Y, each of which is open, then X-C is open implying that C is closed.

3.9. Theorem If each path component of a topological space Y is open and closed, then the path components of Y coincide with the components of Y.

Proof:

Part 1 — Let C be a path component of Y. Thus, C is open and closed, and (by 3.7) C is connected. Then (by 1.4) there is no connected set which properly contains C, implying that C is a component of Y.

Part 2 — Let C be a component of Y, let $y \in C$, and let D be a path component of Y containing y. Since D is path-connected, (by 3.6) D is connected, and, thus, $D \subset C$ since C is a maximal connected set containing y. But D is both open and closed, so (by 1.4) $D = C$.

<u>3.10.Theorem</u> A topological space Y is path-connected if and only if it is connected, and each $y \varepsilon Y$ has a path-connected neighborhood.

<u>Proof</u>:

Part 1 — Suppose that Y is path-connected. Then (by 3.6) Y is connected. Also, if $y \varepsilon Y$, then Y is a path-connected neighborhood of y.

Part 2 — Suppose that Y is connected and each $y \varepsilon Y$ has a path-connected neighborhood. Then (by 3.8), each path component C is both open and closed. If $C \neq Y$, then C is a proper subset of Y, which implies (by 1.4) that Y is not connected, a contradiction. Thus, $Y = C$, implying that Y is path-connected.

<u>3.11.Theorem</u> The continuous image of a path-connected topological space is path-connected.

<u>Proof</u>: Let X and Y be spaces where X is path-connected, and let $f:X \rightarrow Y$ be continuous. Thus, $f:X \rightarrow f(X)$ is continuous. Let $y_0, y_1 \varepsilon f(X)$. Thus, $f^{-1}(y_0)$ and $f^{-1}(y_1)$ are nonempty subsets of X. Let $p \varepsilon f^{-1}(y_0)$ and $q \varepsilon f^{-1}(y_1)$. Since X is path-connected, there is a path $g:I \rightarrow X$ joining p to q. Therefore, g is continuous, $g(0) = p$ and $g(1) = q$, which implies that $(f \circ g)(0) = f(g(0)) = f(p) = y_0$ and $(f \circ g)(1) = f(g(1)) = f(q) = y_1$. Also, since $g:I \rightarrow X$ and $f:X \rightarrow f(X)$ are continuous, then $(f \circ g):I \rightarrow f(X)$ is continuous, and, therefore, is a path which joins y_0 to y_1. Thus, f(X) is path-connected.

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