HOCHSCHILD COHOMOLOGY AND COMPLEX REFLECTION GROUPS

Briana A. Foster-Greenwood

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APPROVED:

Anne V. Shepler, Major Professor
Charles H. Conley, Committee Member
Neal Brand, Committee Member
Su Gao, Chair of the Department of Mathematics
Mark Wardell, Dean of the Toulouse Graduate School

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A concrete description of Hochschild cohomology is the first step toward exploring associative deformations of algebras. In this dissertation, deformation theory, geometry, combinatorics, invariant theory, representation theory, and homological algebra merge in an investigation of Hochschild cohomology of skew group algebras arising from complex reflection groups. Given a linear action of a finite group on a finite dimensional vector space, the skew group algebra under consideration is the semi-direct product of the group with a polynomial ring on the vector space.

Each representation of a group defines a different skew group algebra, which may have its own interesting deformations. In this work, we explicitly describe all graded Hecke algebras arising as deformations of the skew group algebra of any finite group acting by the regular representation. We then focus on rank two exceptional complex reflection groups acting by any irreducible representation. We consider in-depth the reflection representation and a nonfaithful rotation representation. Alongside our study of cohomology for the rotation representation, we develop techniques valid for arbitrary finite groups acting by a representation with a central kernel.

Additionally, we consider combinatorial questions about reflection length and codimension orderings on complex reflection groups. We give algorithms using character theory to compute reflection length, atoms, and poset relations. Using a mixture of theory, explicit examples, and calculations using the software GAP, we show that Coxeter groups and the infinite family G(m,1,n) are the only irreducible complex

reflection groups for which the reflection length and codimension orders coincide. We describe the atoms in the codimension order for the groups G(m,p,n). For arbitrary finite groups, we show that the codimension atoms are contained in the support of every generating set for cohomology, thus yielding information about the degrees of generators for cohomology.

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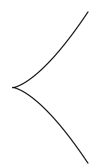
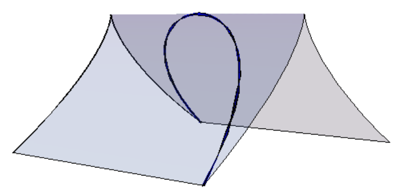
CHAPTER 1

INTRODUCTION

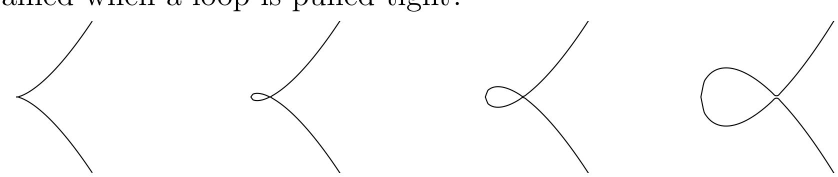
In this dissertation, we investigate cohomology governing deformations of algebras. To illuminate the idea of a deformation, we begin with an informal example that has both a geometric and algebraic interpretation. Consider the following curve in the plane:



Should we view the singularity (or sharp point) as an accident, or fundamental? We wonder: Is the curve the shadow of a smooth curve in another dimension?



Or the curve obtained when a loop is pulled tight?



$t = 0$ $\quad$ $t = \frac{1}{4}$ $\quad$ $t = \frac{1}{2}$ $\quad$ $t = 1$

The family of loops above is an example of a **deformation** of the curve $y^2 = x^3$. The family is parametrized—each loop has equation $y^2 = x^3 + tx^2$, for some real value t. As t varies, so do the curves, and the special case $t = 0$ gives the singular curve we are trying to deform. At the same time, we could consider the coordinate rings

$$\mathbb{R}[x, y, t]/\langle y^2 - x^3 - tx^2\rangle$$

of these curves, obtaining a parametrized family of algebras that may be regarded as a deformation of the algebra $\mathbb{R}[x, y]/\langle y^2 - x^3\rangle$ (see [8]).

Formal deformations of algebras (see Section 1.3) do not require us to have any geometric context in mind—they are defined for any associative algebra. Determining all possible associative deformations of an algebra invokes a study of Hochschild cohomology.

In this work, the algebras of interest are skew group algebras built from groups and their actions on space. Using invariant theory, representation theory, cohomology, combinatorics, and geometry, we explore the Hochschild cohomology and deformation theory of skew group algebras arising from different representations of reflection groups. Background definitions and motivation are contained in Sections 1.1−1.4, and the problems of investigation are outlined in Section 1.5.

1.1. Complex Reflection Groups

Let V be a real or complex n-dimensional vector space. An element of $GL(V)$ is a **reflection** if it has finite order and its fixed point space is a hyperplane. Thus the eigenvalues of a reflection are $1, \ldots, 1, \lambda$ for some root of unity $\lambda \neq 1$. A reflection is called real or complex according to whether V is a real or complex vector space. While a real reflection must have order two, a complex reflection can have any finite order greater than one. A finite subgroup of $GL(V)$ is a **reflection group** if it is generated by reflections. With respect to a suitable basis, a finite reflection group acts by isometries, i.e., preserves a Hermitian form and is represented by unitary matrices.

Examples of reflection groups include dihedral groups (acting on $\mathbb{R}^2$ as reflections and rotations of planar n-gons) and symmetric groups (generated by transpositions fixing the hyperplanes $x_i - x_j = 0$ in real or complex space). Finite irreducible complex reflection groups were classified in the 1950's by Shephard and Todd [21]. The classification consists of one infinite family $G(m, p, n)$ and 34 exceptional reflection groups, which are referred to as $G_4 - G_{37}$ in the literature.

Reflection groups are remarkable for several reasons. They include symmetry groups of the five Platonic solids in $\mathbb{R}^3$ as well as regular polytopes in complex space. The Shephard-Todd-Chevalley Theorem [4] of classical invariant theory establishes reflection groups as precisely those groups whose ring of polynomial invariants is a polynomial algebra (generated by algebraically independent invariant polynomials). In the realm of Lie theory, the combinatorics of Weyl groups is elegantly connected to the classification of simple complex Lie algebras. The action of a reflection group on the set of hyperplanes and also a set of root vectors allows one to study reflection groups using geometry and combinatorics.

1.2. Skew Group Algebras

Let G be a finite group acting on a $\mathbb{C}$-algebra A by algebra automorphisms (so, formally, we have a group homomorphism from G to $\mathrm{Aut}(A)$). The **skew group algebra** $A\#G$ is the semi-direct product of A with the group algebra $\mathbb{C}G$. As a vector space, $A\#G \cong A \otimes \mathbb{C}G$, so every element of $A\#G$ can be expressed uniquely as a sum $\sum_{g\in G} f_g \otimes g$ with $f_g \in A$. Multiplying two simple tensors requires use of the group action to move a group element past an element of A:

$$(a_1 \otimes g_1)(a_2 \otimes g_2) := a_1 \vec{g_1}(a_2) \otimes g_1 g_2,$$

where $\vec{g_1}(a_2)$ is the result in A of applying the action of g_1 to a_2. This definition extends bilinearly to an associative multiplication on all of $A\#G$. Note that $A\#G$ contains both A and $\mathbb{C}G$ as subalgebras. Furthermore, if G acts faithfully on a commutative algebra A, then the skew group algebra $A\#G$ contains as its center the ring $A^G = \{a \in A : \vec{g}(a) = a\}$ of G-invariants. For notational convenience, we often supress the tensor signs when working with a skew group algebra.

If G acts linearly on a vector space $V \cong \mathbb{C}^n$ with basis $v_1, \ldots, v_n$, then G also acts on the symmetric algebra $S(V) \cong \mathbb{C}[v_1, \ldots, v_n]$, and we can form the skew group algebra

$$S(V)\#G := S(V) \rtimes \mathbb{C}G.$$

Within this algebra, vectors commute ($vw - wv = 0$ for all v, w in V), group elements multiply as they do in the group, and when a group element moves from left to right past a vector, it acts on the vector. By changing the group action, we obtain a whole host of skew group algebras for a given group.

1.2.1. Example. Let G be the cyclic group generated by the permutation $(1\,2\,3)$. Then G acts on $V \cong \mathbb{C}^3$ by cyclically permuting basis vectors v_1, v_2, v_3. Every element of $S(V)\#G$ can be expressed uniquely in the form

$$f_1 + f_2(1\,2\,3) + f_3(3\,2\,1),$$

with each f_i a polynomial in $\mathbb{C}[v_1, v_2, v_3]$. To illustrate the multiplication in the skew group algebra, consider the product of $(1\,2\,3)$ and $v_1^2 v_2^7 v_3^4 (3\,2\,1)$:

$$(1\,2\,3)v_1^2 v_2^7 v_3^4 (3\,2\,1) = v_2^2 v_3^7 v_1^4 (1\,2\,3)(3\,2\,1) = v_1^4 v_2^2 v_3^7.$$

1.3. Deformation Theory and Hochschild Cohomology

In algebraic deformation theory (see [10], [13], and [8]), we seek to deform the multiplication in an algebra while retaining the same underlying vector space structure. Formally, we adjoin a central parameter t to an $\mathbb{F}$-algebra A and consider the vector space $A[t]$ with multiplication defined by

$$a * b = ab + \mu_1(a,b)t + \mu_2(a,b)t^2 + \cdots ,$$

where a, b are in A, and each $\mu_i : A \times A \to A$ is bilinear[1]. We obtain a family of algebras by specializing the parameter t to different values in the field (e.g. letting $t = 0$ recovers the algebra A).

Requiring an associative multiplication imposes conditions on the maps μ_i and leads to a study of Hochschild cohomology. Indeed, expanding $(a * b) * c$ and $a * (b * c)$ into power series and equating like terms produces a sequence of "associative deformation equations", the first of which provides the definition of a **Hochschild 2-cocycle**. In particular, the first multiplication map $\mu_1 : A \times A \to A$ of an associative deformation must satisfy the cocycle condition

$$a\mu_1(b,c) - \mu_1(ab,c) + \mu_1(a,bc) - \mu_1(a,b)c = 0.$$

[1]The multiplication is defined first on A and then extended $\mathbb{F}[t]$-bilinearly to all of $A[t]$. For each pair a, b in A we require $\mu_i(a,b) = 0$ for all but finitely many $i \geq 1$ so that $a * b$ is in $A[t]$ as opposed to $A[[t]]$.

Generally, for a $\mathbb{C}$-algebra A and an A-bimodule M, the Hochschild cohomology of A with coefficients in M is the space

$$\mathrm{HH}^\bullet(A, M) = \mathrm{Ext}^\bullet_{A\otimes A^{\mathrm{op}}}(A, M),$$

where we tensor over $\mathbb{C}$. When $M = A$, we simply write $\mathrm{HH}^\bullet(A)$.

In the setting of skew group algebras, Hochschild cohomology may be formulated in terms of **invariant theory**. Ştefan [27] finds cohomology of the skew group algebra $S(V)\#G$ as the space of G-invariants in a larger cohomology ring:

$$\mathrm{HH}^\bullet(S(V)\#G) \cong \mathrm{HH}^\bullet(S(V), S(V)\#G)^G,$$

and Farinati [7] and Ginzburg and Kaledin [11] describe the larger cohomology ring:

$$\mathrm{HH}^\bullet(S(V), S(V)\#G) \cong \bigoplus_{g\in G}\Big(S(V^g) \otimes \bigwedge^{\bullet-\mathrm{codim}(g)} (V^g)^* \otimes \bigwedge^{\mathrm{codim}(g)} \big((V^g)^*\big)^{\perp} \otimes \mathbb{C}g \Big).$$

Here, V^g denotes the fixed point space of g, and $\mathrm{codim}(g) := \mathrm{codim}\, V^g$. After switching the order of the operations "direct sum" and "take invariants", we compute cohomology by finding invariants under centralizer subgroups and taking a direct sum over a set of conjugacy class representatives.

1.4. Motivation

Let G be a finite group acting linearly on a vector space $V \cong \mathbb{C}^n$. Let $T(V)$ be the tensor algebra (over $\mathbb{C}$) of V. Any skew-symmetric bilinear map $\kappa : V \times V \to \mathbb{C}G$ defines a quotient algebra

$$\mathcal{H}_\kappa := T(V)\#G/I_\kappa, .$$

where I_κ is the two-sided ideal $I_\kappa = \langle v \otimes w - w \otimes v - \kappa(v, w) : v, w \in V\rangle$. We say $\mathcal{H}_\kappa$ is a **graded Hecke algebra** if it satisfies a Poincaré-Birkhoff-Witt property—namely, if the associated graded algebra $\mathrm{gr}(\mathcal{H}_\kappa)$ is isomorphic to the skew group algebra $S(V)\#G$. (This property is in analogy with the Poincaré-Birkhoff-Witt Theorem for universal enveloping algebras.) More concretely, $\mathcal{H}_\kappa$ is a graded Hecke algebra if the cosets $v_1^{e_1} \cdots v_n^{e_n} g + I_\kappa$ for e_i in $\mathbb{N}$ and g in G form a vector space basis of $\mathcal{H}_\kappa$, where $v_1, \ldots, v_n$ is any basis of V.

1.4.1. Example. As in Example 1.2.1, let G be the cyclic group generated by the permutation (1 2 3), and let G act on $V \cong \mathbb{C}^3$ by cyclically permuting basis vectors v_1, v_2, v_3. Define a skew-symmetric bilinear form $\kappa : V \times V \to \mathbb{C}G$ by

$$\kappa(v_1, v_2) = \kappa(v_2, v_3) = \kappa(v_3, v_1) = 1 + (1\,2\,3) + (3\,2\,1).$$

Then $\mathcal{H}_\kappa$ is a graded Hecke algebra, as we will show in Chapter 6.

Graded Hecke algebras appear in different guises under many different names. Though Drinfeld [5] was first to consider the algebras $\mathcal{H}_\kappa$ as defined above, the terminology "graded Hecke algebra" arises because Lusztig [16] independently explored the case when G is a Weyl group, defining the algebras as very different looking quotient algebras that gave a graded version of the affine Hecke algebra (see [18]). In the case of a doubled-up action of G on a vector space $V \oplus V^*$ with V a real or complex reflection representation, graded Hecke algebras are known as **rational Cherednik algebras** and play the key role in Gordon's proof [12] of a Weyl group analogue of the $n!$-conjecture. More generally, if G is a symplectic group acting on an even-dimensional vector space, then graded Hecke algebras are called **symplectic reflection algebras**—rediscovered by Etingof and Ginzburg [6] in the context of orbifold theory.

What does all this have to do with deformation theory? Every graded Hecke algebra is a formal deformation of some skew group algebra $S(V)\#G$. In fact, Witherspoon [28] shows that, up to isomorphism, graded Hecke algebras are precisely the deformations of $S(V)\#G$ such that the i^{th} multiplication map has degree $-2i$ for each i. However, Ram and Shepler [18] show that for most complex reflection groups *acting on V via the reflection representation*, the skew group algebra $S(V)\#G$ has no nontrivial graded Hecke algebras. This prompted the idea to consider skew group algebras arising from nonfaithful and nonreflection actions. For example, Shepler and Witherspoon [23] investigate a nonfaithful action of $G(m, p, n)$ and find more graded Hecke algebras.

What other interesting deformations of $S(V)\#G$ may play a role in representation theory? We turn to Hochschild cohomology to give a description of all possible deformations, not just the type discussed above.

1.5. Problems

In this dissertation, we explore two problems aimed towards understanding Hochschild cohomology and deformations of skew group algebras associated to reflection groups. In Chapter 2, we investigate a combinatorial problem about reflection groups and apply our results to obtain information about generators for cohomology rings. In Chapters 3−6, we turn our attention to determining Hochschild cohomology for skew group algebras arising from various nonreflection representations of reflection groups.

1.5.1. *Reflection Length and Codimension Posets for Complex Reflection Groups*

Given a reflection group, we consider two partial orders: a reflection length order (related to the word metric of geometric group theory) and a codimension order (capturing the geometry of the group action). Various reflection length orders are key tools in the theory of Coxeter groups, while codimension appears in the numerology of complex reflection groups and in recent work connecting combinatorics and Hochschild cohomology.

For Coxeter groups and the infinite family $G(m, 1, n)$ of monomial reflection groups, the reflection length and codimension orders are known to be identical (see Carter [2] and Shi [26]). Using a mixture of theory, explicit examples, and algorithms developed using the software GAP, my work examines the question of whether or not this pattern holds for other reflection groups. We complete the determination of the reflection groups for which reflection length coincides with codimension:

Theorem. *Let G be an irreducible complex reflection group. Absolute reflection length and codimension coincide if and only if G is a Coxeter group or $G = G(m, 1, n)$.*

This theorem has implications for generators of cohomology rings related to deformation theory of skew group algebras. The Hochschild cohomology ring of a skew group algebra may be identified with the G-invariant subalgebra of a larger cohomology ring, which Shepler and

Witherspoon [24] show is finitely generated (under cup-product) by elements corresponding to atoms of the codimension poset. Reflections are always atoms in the codimension poset, but are there any other atoms?

In the case of the infinite family $G(m, p, n)$, we obtain an explicit combinatorial description of the atoms in the codimension poset. For the exceptional reflection groups, we appeal to character theory to write code in the software GAP to compute absolute reflection length, atoms, and poset relations in the partial orders on the set of conjugacy classes of G.

1.5.2. *Determination of Hochschild Cohomology*

Working with the invariant theoretic formulation of cohomology requires an analysis of centralizer subgroups and computation of invariants in tensor product spaces. Using theoretical tools (such as character theory and Molien series) alongside computational algebra software (such as GAP and Mathematica) we determine Hochschild cohomology for the following cases:

- **Irreducible representations of $G_4 - G_{22}$:** Let G be one of the groups $G_4 - G_{22}$, and let V be any irreducible representation of G. To compute Hochschild cohomology of $S(V)\#G$, we need to understand centralizer subgroups and actions on fixed spaces (and their complements) before computing invariants in a tensor product. Fortunately, the centralizers in the groups $G_4 - G_{22}$ are very small. If z generates the center of the group, then the centralizer of a noncentral element g in G is generated by z and one more element (often g itself). Since these centralizers are abelian, much of the Hochschild cohomology can be expressed in terms of polynomials semi-invariant with respect to various linear characters. Using the software GAP, we implement code to compute the relevant linear characters.
- **Rotation representation of $G_4 - G_{22}$:** The groups $G_4 - G_{22}$ in the Shephard-Todd classification of irreducible complex reflection groups are called tetrahedral, octahedral, or icosahedral, according to whether the quotient by the center Z is isomorphic to the rotation group of the tetrahedron, octahedron, or icosahedron. Let $V \cong \mathbb{C}^3$.

We compare cohomology of $S(V)\#G$ with the cohomology of $S(V)\#G/Z$, where the action of G on V is obtained by lifting the natural "rotation representation" of G/Z on V. Naively, one would hope to obtain the cohomology for $S(V)\#G$ by just taking multiple copies of the cohomology for $S(V)\#G/Z$, but the computation is more subtle because the centralizer of an element g in G need not project onto the full centralizer of the corresponding element $\overline{g}$ in G/Z.

Regular representation: In Chapter 6, we classify all graded Hecke algebras arising as deformations of $S(V)\#G$ when G is a finite group acting on $V \cong \mathbb{C}^{|G|}$ by the (left) regular representation. In the case of the regular representation, very few elements contribute to the cohomology $\mathrm{HH}^{\bullet}(S(V)\#G)$ in the low degrees that we are interested in for deformation theory. When $|G| > 4$, we identify the parameter space of graded Hecke algebras with the space of skew-symmetric, G-invariant, bilinear forms on $V \cong \mathbb{C}^{|G|}$. With respect to an appropriate basis, these forms may be described explicitly using matrices for the right regular representation.

CHAPTER 2

COMPARING REFLECTION LENGTH AND CODIMENSION

Reflection length and codimension of fixed point spaces induce partial orders on a complex reflection group. While these partial orders are of independent combinatorial interest, our investigation is motivated by a connection between the codimension order and the algebraic structure of cohomology governing deformations of skew group algebras. In this chapter, we compare the reflection length and codimension functions and discuss implications for cohomology of skew group algebras. We give algorithms using character theory for computing reflection length, atoms, and poset relations. Using a mixture of theory, explicit examples, and computer calculations in GAP, we show that Coxeter groups and the infinite family $G(m, 1, n)$ are the only irreducible complex reflection groups for which the reflection length and codimension orders coincide. We describe the atoms in the codimension order for the infinite family $G(m, p, n)$, which immediately yields an explicit description of generators for cohomology.

2.1. Reflection Length and Codimension Posets

Let V be an n-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$, and let $G \subset GL(V)$ be a finite group generated by reflections. We define two class functions on a reflection group G and use these functions to partially order the group. The first is a reflection length function, related to the word metric of geometric group theory:

2.1.1. Definition. The **(absolute) reflection length** of an element g of a reflection group G is the minimum number of factors needed to write g as a product of reflections:

$$\ell(g) = \min\{k \;:\; g = s_1 \cdots s_k \text{ for some reflections } s_1, \ldots, s_k \text{ in } G\}.$$

The identity is declared to have length zero.

Note that the absolute reflection length function gives length with respect to *all* reflections in the group, as opposed to a set of fundamental or simple reflections.

The second function of interest relates to the geometry of the group action. While each reflection in G fixes a hyperplane pointwise, each remaining element fixes an intersection of hyperplanes. The codimension function records codimensions of these fixed point spaces:

2.1.2. DEFINITION. The **codimension** of an element g in a group $G \subset GL(V)$ is the codimension of its fixed point space (or number of non-one eigenvalues):

$$\operatorname{codim}(g) = n - \dim\{v \in V \ : \ gv = v\}.$$

Note that the identity has codimension zero, and reflections have codimension one.

The reflection length and codimension functions satisfy the following properties:

- constant on conjugacy classes
- subadditive: $\ell(ab) \leq \ell(a) + \ell(b)$ and $\operatorname{codim}(ab) \leq \operatorname{codim}(a) + \operatorname{codim}(b)$
- $\operatorname{codim}(g) \leq \ell(g)$ for all g in G.

Now define the reflection length order on G by

$$a \leq_{\ell} c \qquad \Leftrightarrow \qquad \ell(a) + \ell(a^{-1}c) = \ell(c).$$

Analogously[1], define the codimension order on G by

$$a \leq_{\perp} c \qquad \Leftrightarrow \qquad \operatorname{codim}(a) + \operatorname{codim}(a^{-1}c) = \operatorname{codim}(c).$$

Since reflection length and codimension are constant on conjugacy classes, we get induced partial orders on the set of conjugacy classes of G. Define the reflection length (likewise codimension) of a conjugacy class to be the reflection length (likewise codimension) of the elements in the conjugacy class. If A and C are conjugacy classes of G, then set $A \leq_{\ell} C$ if there exists an element $a \in A$ and an element $c \in C$ with $a \leq_{\ell} c$. Analogously, define the

[1]Brady and Watt [1] prove $\leq_{\perp}$ is a partial order. Their proof is also valid when codimension is replaced by any function $\mu : G \to [0, \infty)$ satisfying $\mu(a) = 0$ iff $a = 1$ (positive definite) and $\mu(ab) \leq \mu(a) + \mu(b)$ for all a, b in G (subadditive).

codimension order on conjugacy classes. As an example, Figure 2.1 illustrates the reflection length and codimension orderings on the set of conjugacy classes of the dihedral group of order eight.

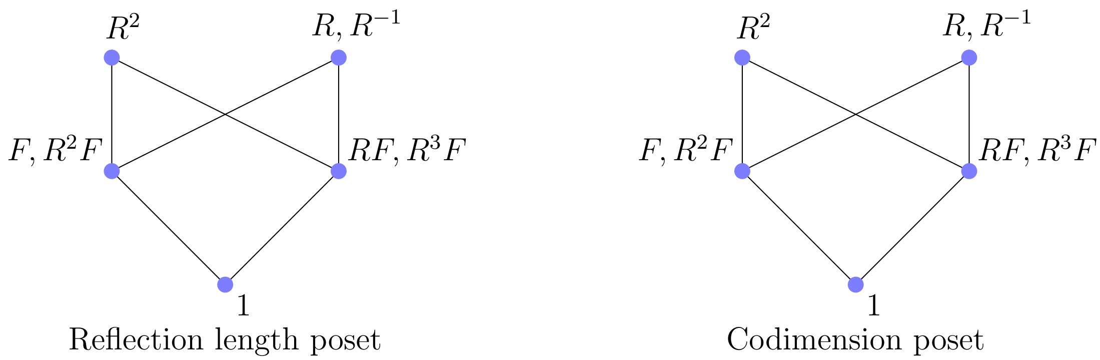


Figure 2.1. Reflection Length and Codimension Posets for the Dihedral Group D_4

In Section 2.5, we will appeal to character theory to deduce information about the partial orders on G by working with the (somewhat simpler) partial orders on the set of conjugacy classes of G.

In a poset $(P, \leq)$, we say b **covers** a if $b > a$ and the interval $\{x \in P : a < x < b\}$ is empty. The **atoms** of a poset are the covers of the minimum element (when it exists). The identity is the minimum element in the reflection length poset and in the codimension poset. For emphasis, we often refer to the atoms in the codimension poset as **codimension atoms**. Note that an element a in G is an atom in the poset on G if and only if its conjugacy class is an atom in the corresponding poset on the set of conjugacy classes of G.

Figure 2.2 illustrates the reflection length and codimension orderings on the set of conjugacy classes of the order 24 complex reflection group G_4. Note that the conjugacy classes labeled $3a$ and $3b$ consist of order three reflections and are the only atoms in the reflection length poset. However, there is an additional atom (conjugacy class $2a$) in the codimension poset.

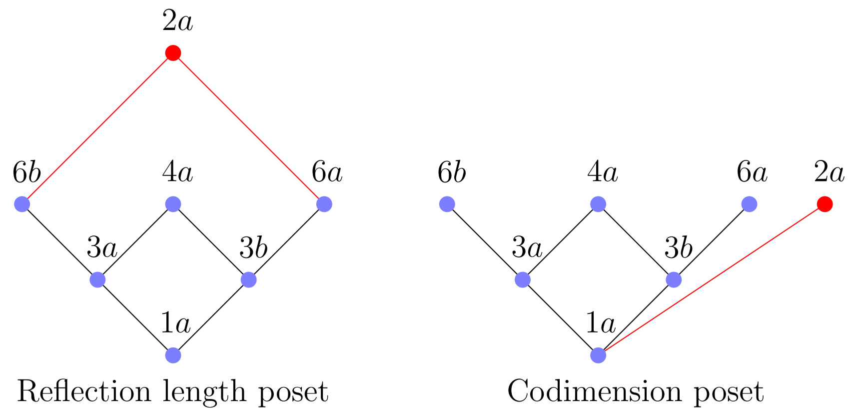


FIGURE 2.2. Reflection Length and Codimension Posets for Reflection Group G_4

2.2. Functions versus Posets

We now show that comparing the length and codimension functions is (in a sense) equivalent to comparing the set of atoms in each poset.

2.2.1. DEFINITION. We say $g = g_1 \cdots g_k$ is a **factorization of g with codimensions adding** if $\operatorname{codim}(g) = \operatorname{codim}(g_1) + \cdots + \operatorname{codim}(g_k)$.

Note that if $g = g_1 \cdots g_k$ is a factorization with codimensions adding, then, using the fact that codimension is subadditive and constant on conjugacy classes, we also have $g_1, \ldots, g_k \leq_\perp g$. Furthermore, since V is finite dimensional, we can work recursively to factor any nonidentity element of G into a product of codimension atoms with codimensions adding:

2.2.2. OBSERVATION. *Given a nonidentity group element g, there exist codimension atoms $a_1, \ldots, a_k \leq_\perp g$ such that $g = a_1 \cdots a_k$ and $\operatorname{codim}(g) = \operatorname{codim}(a_1) + \cdots + \operatorname{codim}(a_k)$.*

The next two lemmas follow from repeated use of subadditivity of length and codimension and the fact that codimension is bounded above by reflection length.

2.2.3. LEMMA. *Fix g in G. If $\ell(a) = \operatorname{codim}(a)$ for every codimension atom $a \leq_\perp g$, then $\ell(g) = \operatorname{codim}(g)$.*

PROOF. The statement certainly holds for the identity. Now let $g = a_1 \cdots a_k$ be a factorization of $g \neq 1$ into atoms with codimensions adding. (Note that necessarily $a_1, \ldots, a_k \leq_\perp g$.)

Then

$$\ell(g) \leq \ell(a_1) + \cdots + \ell(a_k) = \operatorname{codim}(a_1) + \cdots + \operatorname{codim}(a_k) = \operatorname{codim}(g) \leq \ell(g),$$

with equality throughout. □

2.2.4. Lemma. *Let $g \in G$ with $\ell(g) = \operatorname{codim}(g)$. If $h \leq_\ell g$, then $h \leq_\perp g$.*

Proof. Subadditivity gives $\operatorname{codim}(g) \leq \operatorname{codim}(h) + \operatorname{codim}(h^{-1}g)$. If $\ell(g) = \operatorname{codim}(g)$ and $h \leq_\ell g$, we also have the reverse inequality:

$$\operatorname{codim}(h) + \operatorname{codim}(h^{-1}g) \leq \ell(h) + \ell(h^{-1}g) = \ell(g) = \operatorname{codim}(g).$$

□

Lemma 2.2.3 and Lemma 2.2.4 combine to reveal that the reflection length and codimension functions coincide on all of G if and only if every codimension atom is a reflection.

2.2.5. Proposition. *The following are equivalent:*

(i) $\ell(g) = \operatorname{codim}(g)$ *for every g in G.*

(ii) $\ell(g) = \operatorname{codim}(g)$ *for every codimension atom g in G.*

(iii) *Every codimension atom is a reflection.*

(iv) *For every $g \neq 1$, there exists a reflection s in G such that* $\operatorname{codim}(gs) < \operatorname{codim}(g)$.

Proof. The implication (1) ⇒ (2) is immediate. Application of Lemma 2.2.4 with g a codimension atom and h a reflection shows (2) ⇒ (3). Lastly, if every codimension atom is a reflection, then the hypothesis of Lemma 2.2.3 holds for each g in G, and hence (3) ⇒ (1). It is straightforward to work (4) into the loop via (1) ⇒ (4) and (4) ⇒ (3). □

2.3. The Infinite Family $G(m, p, n)$

The group $G(m, 1, n) \cong (\mathbb{Z}/m\mathbb{Z})^n \rtimes \mathfrak{S}_n$ consists of all $n \times n$ monomial matrices having m^{th} roots of unity for the nonzero entries. For p dividing m, the group $G(m, p, n)$ is the subgroup of $G(m, 1, n)$ consisting of those elements whose nonzero entries multiply to an $(\frac{m}{p})^{th}$ root of unity. Throughout this section let $\zeta_m = e^{2\pi i/m}$. Each group $G(m, p, n)$ contains the order two *transposition type reflections* of the form $\delta\sigma$, where σ is a transposition swapping the i^{th} and j^{th} basis vectors, and $\delta = \mathrm{diag}(1, \ldots, \zeta_m^a, \ldots, \zeta_m^{-a}, \ldots, 1)$ scales rows i and j of σ ($\delta = 1$ is a possibility). When p properly divides m, the group $G(m, p, n)$ also contains the *diagonal reflections* $\mathrm{diag}(1, \ldots, \zeta_m^a, \ldots, 1)$ where $0 < a < m$ and p divides a. The $G(m, p, n)$ family includes the following Coxeter groups ($n \geq 2$):

- symmetric group: $G(1, 1, n)$ (not irreducible)
- Weyl groups of type B_n and C_n: $G(2, 1, n)$
- Weyl groups of type D_n: $G(2, 2, n)$
- dihedral groups: $G(m, m, 2)$

In this section, we describe the atoms in the codimension poset for an arbitrary group $G(m, p, n)$. In the groups for which the reflection length and codimension functions do not coincide, we give explicit examples of elements with length exceeding codimension.

2.3.1. Definition. Let $V = V_1 \oplus \cdots \oplus V_n$ be a decomposition of $V \cong \mathbb{C}^n$ into one-dimensional subspaces permuted by $G(m, p, n)$. Let g be in $G(m, p, n)$, and partition $\{V_1, \ldots, V_n\}$ into g-orbits, say $\mathcal{O}_1, \ldots, \mathcal{O}_r$. The action of g on $\bigoplus_{V_j \in \mathcal{O}_i} V_j$ can be expressed as $\delta_i \sigma_i$, where δ_i is diagonal and σ_i is a cyclic permutation. (Thus, up to conjugation by a permutation matrix, g is block diagonal with i^{th} block $\delta_i \sigma_i$.) The **cycle-sum** of g corresponding to orbit $\mathcal{O}_i$ is the exponent c_i (well-defined modulo m) such that $\det(\delta_i) = \zeta_m^{c_i}$.

Cycle-sums allow us to quickly read off codimension of an element:

$$\mathrm{codim}(g) = n - \#\{i \,:\, c_i \equiv 0 \pmod{m}\}.$$

Note that for a reflection t and any group element g, the relation $t \leq_{\perp} g$ is equivalent to $\mathrm{codim}(t^{-1}g) = \mathrm{codim}(g) - 1$. Letting $s = t^{-1}$ and noting that the conjugate elements sg and gs have the same codimension, we obtain the following convenient observation:

2.3.2. Observation. *An element $g \neq 1$ is comparable with a reflection in the codimension poset if and only if there exists a reflection s such that* $\mathrm{codim}(gs) < \mathrm{codim}(g)$.

We recall from Shi [26] (see Corollary 1.8 and the proof of Theorem 2.1) the three possibilities for how the cycle-sums change upon multiplying by a reflection:

2.3.3. LEMMA (Shi [26]). *Let $g \in G(m,p,n)$ with cycle-sums $c_1, \ldots, c_r$ corresponding to g-orbits $\mathcal{O}_1, \ldots, \mathcal{O}_r$. If s is a transposition type reflection interchanging V_i and V_j, then the cycle-sums of g split or merge into the cycle-sums of gs:*

(i) *If V_i and V_j are in the same g-orbit, say $\mathcal{O}_k$, then gs has cycle sums $c_1, \ldots, \widehat{c_k}, \ldots, c_r, d, c_k - d$ for some integer d.*

(ii) *If V_i and V_j are in different g-orbits, say $\mathcal{O}_k$ and $\mathcal{O}_l$, then gs has cycle sums $c_1, \ldots, \widehat{c_k}, \ldots, \widehat{c_l}, \ldots, c_r, c_k + c_l$.*

Let s be a diagonal reflection scaling V_i by non-1 eigenvalue ζ_m^a (where p divides a).

(iii) *If V_i is in the g-orbit $\mathcal{O}_k$, then gs has cycle sums $c_1, \ldots, c_k + a, \ldots, c_r$.*

Note that if g is nondiagonal, then by choosing a suitable transposition type reflection s, we can arrange for the cycle-sum d of gs in part (1) of Lemma 2.3.3 to be any of $0, \ldots, m-1$. In particular, we can choose s so that $d = 0$, thereby increasing the number of zero cycle-sums and decreasing codimension. Hence *every nonreflection atom in the codimension poset must be diagonal*. The converse is false, but we come closer to the set of nonreflection atoms by considering only p-connected diagonal elements.

2.3.4. DEFINITION. A diagonal matrix $g \neq 1$ whose non-1 eigenvalues are $\zeta_m^{c_1}, \ldots, \zeta_m^{c_k}$ (listed with multiplicities) is **p-connected** if p divides $c_1 + \cdots + c_k$ but p does not divide $\sum_{i \in I} c_i$ for $I \subsetneq \{1, \ldots, k\}$. (Note that g is in $G(m,p,n)$ iff p divides $c_1 + \cdots + c_k$.)

It is easy to see that each nonidentity diagonal element of $G(m,p,n)$ factors in $G(m,p,n)$ into p-connected elements with codimensions adding. Thus *every nonreflection atom in the codimension poset must be p-connected*. We next check for poset relations among the reflections and p-connected elements.

2.3.5. Lemma. *The p-connected elements of $G(m,p,n)$ are pairwise incomparable in the codimension poset.*

Proof. Suppose a, b in $G(m,p,n)$ are diagonal elements such that ab is p-connected and $\mathrm{codim}(a) + \mathrm{codim}(b) = \mathrm{codim}(ab)$. Since codimensions add, it is not hard to show that the non-1 eigenvalues of ab are the non-1 eigenvalues $\zeta_m^{a_1}, \ldots, \zeta_m^{a_{\mathrm{codim}(a)}}$ of a together with the non-1 eigenvalues $\zeta_m^{b_1}, \ldots, \zeta_m^{b_{\mathrm{codim}(b)}}$ of b. If $a, b \neq 1$, we have a contradiction to p-connectedness of ab, as p divides $a_1 + \cdots + a_{\mathrm{codim}(a)}$ by virtue of a being in $G(m,p,n)$. □

2.3.6. Lemma. *Let g in $G(m,p,n)$ be p-connected and not a diagonal reflection. Then there exists a reflection $s \leq_{\perp} g$ if and only if g has codimension two and non-1 eigenvalues ζ_m^c and ζ_m^{-c} for some c.*

Proof. If g has codimension two and non-1 eigenvalues ζ_m^c and ζ_m^{-c}, then g factors into two reflections with codimensions adding. The factorization in dimension two illustrates the general case:

$$\begin{pmatrix} \zeta_m^c & \\ & \zeta_m^{-c} \end{pmatrix} = \begin{pmatrix} & \zeta_m^c \\ \zeta_m^{-c} & \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}.$$

Conversely, if g is comparable with a reflection, then there must be a reflection s with $\mathrm{codim}(gs) < \mathrm{codim}(g)$. If s is a diagonal type reflection, then we get a contradiction to p-connectedness of g. If s is a transposition type reflection, then, since g is diagonal, we use Lemma 2.3.3 (2) to see that g must have nonzero cycle-sums c_k and c_l such that $c_k + c_l \equiv 0 \pmod{m}$. By p-connectedness, c_k and c_l must be the only nonzero cycle-sums of g, and hence the only non-1 eigenvalues of g are $\zeta_m^{c_k}$ and $\zeta_m^{c_l} = \zeta_m^{-c_k}$. □

Since every element of $G(m,p,n)$ must be above *some* atom, we now have the collection of codimension atoms:

2.3.7. Proposition. *The codimension atoms for $G(m,p,n)$ are the reflections together with the p-connected elements except for those with codimension two and determinant one.*

It is known that length and codimension coincide for Coxeter groups and the family $G(m,1,n)$, which, incidentally, includes the rank one groups $G(m,p,1) = G(\frac{m}{p},1,1)$. For the remaining groups in the family $G(m, p, n)$, we give explicit examples of codimension atoms with reflection length exceeding codimension.

2.3.8. Corollary. *The reflection length and codimension functions do not coincide in the following groups:*

- $G(m, p, n)$ *with* $1 < p < m$ *and* $n \geq 2$
- $G(m, m, n)$ *with* $m \geq 3$ *and* $n \geq 3$.

Proof. Let I_k be the $k \times k$ identity matrix, and let M_2 and M_3 be the matrices

$$M_2 = \begin{pmatrix} \zeta_m & \\ & \zeta_m^{p-1} \end{pmatrix} \text{ and } M_3 = \begin{pmatrix} \zeta_m & & \\ & \zeta_m^{-2} & \\ & & \zeta_m \end{pmatrix}.$$

In $G(m, p, n)$ with $1 < p < m$ and $n \geq 2$, the direct sum matrix $M_2 \oplus I_{n-2}$ has reflection length three and codimension two. In $G(m, m, n)$ with $m \geq 3$ and $n \geq 3$, the direct sum matrix $M_3 \oplus I_{n-3}$ has reflection length four and codimension three. □

2.3.9. Remarks.

- A 1-connected element must be a diagonal reflection, so the set of codimension atoms in $G(m, 1, n)$ is simply the set of reflections. By Lemma 2.2.5, this recovers the result that length and codimension coincide in $G(m, 1, n)$.
- Shi [26] gives a formula for reflection length in $G(m, p, n)$ in terms of a maximum over certain partitions of cycle-sums. He also uses existence of a certain partition of the cycle-sums as a necessary and sufficient condition for an element to have reflection length equal to codimension.

2.4. Rank Two Exceptional Reflection Groups

The complex reflection groups $G_4 - G_{22}$ act irreducibly on $V \cong \mathbb{C}^2$. Each has at least one conjugacy class of elements for which length and codimension differ. In all except for G_8 and G_{12}, an argument comparing the order of the reflections with the order of the center of the group demonstrates the existence of a central element with length greater than codimension.

2.4.1. Lemma. *Let G be an irreducible complex reflection group acting on $V \cong \mathbb{C}^2$. If $z \in G$ is central and G does not contain any reflections with the same order as z, then $\ell(z) > \mathrm{codim}(z)$.*

Proof. Since G acts irreducibly on $V \cong \mathbb{C}^2$, each central element $z \neq 1$ is represented by a scalar matrix of codimension two. Note that if $z = st$ is a product of two reflections, then s and t are actually *commuting* reflections. Then, working with s, t, and z simultaneously in diagonal form, it is easy to deduce that the reflections s and t must have the same order as z. Thus if G does not contain any reflections of the same order as z, we have $\ell(z) > 2$. ☐

Note that if g is a group element with $\mathrm{codim}(g) = 2$, then $\ell(g) = \mathrm{codim}(g)$ if and only if g can be expressed as a product of two reflections. Thus, we describe the codimension atoms for a rank two reflection group:

2.4.2. Lemma. *The codimension atoms in a rank two complex reflection group are the reflections together with the elements g such that $\ell(g) > \mathrm{codim}(g)$.*

2.4.3. Proposition. *Reflection length and codimension do not coincide in the rank two exceptional complex reflection groups $G_4 - G_{22}$.*

Proof. Inspection of Tables I, II, and III in Shephard-Todd [21] and an application of Lemma 2.4.1 shows that each rank two group G_i with $i \neq 8$ or 12 has a central element z such that $\ell(z) > \mathrm{codim}(z)$.

The group G_8 can be generated by the order four reflections

$$r_1 = \begin{pmatrix} i & \\ & 1 \end{pmatrix} \text{ and } r_2 = \frac{1}{2}\begin{pmatrix} 1+i & 1+i \\ -1-i & 1+i \end{pmatrix}.$$

The element

$$g = r_1(r_1 r_2^2 r_1^{-1})r_2 = \frac{1}{2}\begin{pmatrix} -1+i & 1-i \\ -1-i & -1-i \end{pmatrix}$$

has length three and codimension two. Note that if g were the product of two reflections, then gs would be a reflection for some reflection s in G_8. However, computation shows $\operatorname{codim}(gs) = 2$ for all reflections s in G_8.

For G_{12}, let S and T be the generators given in Shephard-Todd [21]. Although S is a reflection, the element T has codimension two. We express T as the product of two reflections (each a conjugate of S):

$$T = (STST^{-1}S^{-1})(T^{-1}ST^{-1}STS^{-1}T).$$

The element ST has length three and codimension two. (We verify the length by noting that all reflections in G_{12} have determinant -1, so ST also has determinant -1 and must have odd length.)

□

2.4.4. Remarks. Carter [2] proves length equals codimension in Weyl groups. Although Carter's proof applies equally well to any Coxeter group, we indicate two places where the proof can break down for a general complex reflection group.

- Carter's proof shows that in a real reflection group, if g has maximum codimension, i.e., $\operatorname{codim}(g) = n$, then $\operatorname{codim}(gs) < n$ for all reflections s in the group. This may fail in a general complex reflection group, as illustrated by the element g in G_8 given above in the proof of Proposition 2.4.3.
- Though G_{12} only has order two reflections, Carter's proof fails for G_{12} because a complex inner product is not symmetric.

2.5. Exceptional Reflection Groups $G_{23} - G_{37}$

For the exceptional reflection groups, we work with the partial orders on the set $\mathscr{C}_G$ of conjugacy classes of G. With the aid of the software GAP [19], we compute reflection length, atoms, and poset relations, appealing to character theory to speed up the computations. Some of the exceptional reflection groups are Coxeter groups, for which reflection length and codimension are known to agree. For the remaining groups, our computations show reflection length and codimension do not coincide.

We first recall class algebra constants, which we use to aid our computations. Let X, Y, and C be conjugacy classes of G, and let c be a fixed representative of C. The class algebra constant $\text{ClassAlgConst}(X, Y, C)$ counts the number of pairs (x, y) in $X \times Y$ such that $xy = c$. These are the structure constants for the center of the group algebra and have a formula in terms of the irreducible characters of G (details can be found in James and Liebeck [14], for example).

Using class algebra constants, we can inductively find the elements of each reflection length without having to multiply individual group elements. Let $L(k)$ denote the set of conjugacy classes whose elements have reflection length k. Suppose conjugacy class C is not in $L(0) \cup \cdots \cup L(k)$ so that $\ell(C)$ is at least $k+1$. Then C is in $L(k+1)$ if and only if $\text{ClassAlgConst}(X, Y, C)$ is nonzero for some X in $L(1)$ and Y in $L(k)$. Since the class algebra constants are nonnegative, we have $\ell(C) = k+1$ if and only if

$$\sum \{\text{ClassAlgConst}(X, Y, C) \;:\; X \in L(1) \text{ and } Y \in L(k)\} \neq 0.$$

Using the same idea, we can easily compute all relations in the reflection length and codimension posets on the set of conjugacy classes of G. For example, in the codimension poset, we have

$$A \leq_{\perp} C \qquad \Leftrightarrow \qquad \sum_{\substack{X \in \mathscr{C}_G \\ \text{codim}(A) + \text{codim}(X) = \text{codim}(C)}} \text{ClassAlgConst}(A, X, C) \neq 0.$$

In particular,

$$C \text{ is an atom in } (\mathscr{C}_G, \leq_{\perp}) \qquad \Leftrightarrow \qquad \sum_{\substack{X, Y \in \mathscr{C}_G \setminus \{\{1\}, C\} \\ \text{codim}(X) + \text{codim}(Y) = \text{codim}(C)}} \text{ClassAlgConst}(X, Y, C) = 0.$$

Table 2.1 summarizes the data collected for the groups $G_{23} - G_{37}$. (The Coxeter groups are included for contrast.) The middle columns compare the number of conjugacy classes of nonreflection atoms with the number of conjugacy classes C such that $\ell(C) \neq \text{codim}(C)$. The final columns compare maximum reflection length with the dimension n of the vector

group	# conj classes	# length≠codim	# nonref atoms	$\dim V$	max ref length
23	10	0	0	3	3
24	12	2	2	3	4
25	24	3	1	3	4
26	48	9	5	3	4
27	34	12	12	3	5
28	25	0	0	4	4
29	37	10	4	4	6
30	34	0	0	4	4
31	59	27	5	4	6
32	102	27	6	4	6
33	40	12	6	5	7
34	169	78	14	6	10
35	25	0	0	6	6
36	60	0	0	7	7
37	112	0	0	8	8

TABLE 2.1. Atom Count in $(\mathscr{C}_G, \leq_\perp)$ for Exceptional Reflection Groups $G_{23} - G_{37}$

space on which the group acts. Note that in each case the maximum reflection length is at most $2n-1$, usually less. In Appendix B, we record, for each exceptional reflection group, a two-variable polynomial with the coefficient of the $x^l y^d$ term indicating the number of group elements with absolute reflection length l and codimension d.

2.6. When Does Reflection Length Equal Codimension?

Combining the existing results for Coxeter groups and $G(m,1,n)$ with our computations for the remaining irreducible complex reflection groups, we complete the determination of which reflection groups have length equal to codimension.

2.6.1. Theorem. *Let G be an irreducible complex reflection group. The reflection length and codimension functions coincide if and only if G is a Coxeter group or $G = G(m,1,n)$.*

Proof. Carter's proof [2] that reflection length coincides with codimension in Weyl groups works just as well for Coxeter groups, and Shi [26] proves reflection length coincides with codimension in the infinite family $G(m,1,n)$ (also see Shepler and Witherspoon [24] for a more linear algebraic proof). For the converse, Corollary 2.3.8 gives counterexamples for the remaining groups in the family $G(m,p,n)$, while Proposition 2.4.3 and Table 2.1 show reflection length and codimension do not coincide in the non-Coxeter exceptional complex reflection groups. □

2.7. Applications to Hochschild Cohomology of Skew Group Algebras

The codimension poset has applications to Hochschild cohomology and deformation theory of skew group algebras $S(V)\#G$ for a finite group G acting linearly on V. Deformations of skew group algebras include graded Hecke algebras and symplectic reflection algebras. Hochschild cohomology detects potential deformations. For a $\mathbb{C}$-algebra A and an A-bimodule M, the Hochschild cohomology of A with coefficients in M is the space

$$\mathrm{HH}^\bullet(A, M) = \mathrm{Ext}^\bullet_{A\otimes A^{\mathrm{op}}}(A, M),$$

where we tensor over $\mathbb{C}$. When $M = A$, we simply write $\mathrm{HH}^\bullet(A)$. We refer the reader to Gerstenhaber and Schack [10] for more on algebraic deformation theory and Hochschild cohomology.

In the setting of skew group algebras, Hochschild cohomology may be formulated in terms of invariant theory. Ştefan [27] finds cohomology of the skew group algebra $S(V)\#G$ as the space of G-invariants in a larger cohomology ring:

$$\mathrm{HH}^\bullet(S(V)\#G) \cong \mathrm{HH}^\bullet(S(V), S(V)\#G)^G,$$

and Farinati [7] and Ginzburg and Kaledin [11] describe the larger cohomology ring:

$$\mathrm{HH}^\bullet(S(V), S(V)\#G) \cong \bigoplus_{g\in G}\Big(S(V^g) \otimes \bigwedge^{\bullet-\mathrm{codim}(g)} (V^g)^* \otimes \bigwedge^{\mathrm{codim}(g)} \big((V^g)^*\big)^\perp \otimes \mathbb{C}g \Big),$$

which we identify with a subspace of $S(V) \otimes \bigwedge^\bullet V^* \otimes \mathbb{C}G$. Here, $V^g = \{v \in V \,:\, gv = v\}$ denotes the fixed point space of g.

Shepler and Witherspoon [24] further show that the cohomology $\mathrm{HH}^\bullet(S(V), S(V)\#G)$ is generated as an algebra under cup product by $\mathrm{HH}^\bullet(S(V))$ together with derivation forms corresponding to atoms in the codimension poset. More specifically, for each g in G, fix a choice of volume form $\mathrm{vol}_g^\perp$ in the one-dimensional space $\bigwedge^{\mathrm{codim}(g)}\big((V^g)^*\big)^\perp$. (If s is a reflection, we may take $\mathrm{vol}_s^\perp$ in V^* to be a linear form defining the hyperplane about which s reflects.) Then [24, Corollary 9.4] asserts that the cohomology ring $\mathrm{HH}^\bullet(S(V), S(V)\#G)$ is generated by $\mathrm{HH}^\bullet(S(V)) \cong S(V) \otimes \bigwedge^\bullet V^* \otimes 1_G$ and the set of volume forms tagged by codimension atoms:

$$\{1 \otimes \mathrm{vol}_g^\perp \otimes g \,:\, g \text{ is an atom in the codimension poset for } G\}.$$

2.7.1. Example. Consider the group $G = G(m,p,n)$ acting on $V \cong \mathbb{C}^n$ by its standard reflection representation. Let $v_1, \ldots, v_n$ denote the standard basis of V and $v_1^*, \ldots, v_n^*$ the dual basis of V^*. As in Section 2.3, let $\zeta_m = e^{2\pi i/m}$.

Proposition 2.3.7 describes the codimension atoms in $G(m,p,n)$, and we can easily find the corresponding volume forms $\mathrm{vol}_g^\perp$. The cohomology $\mathrm{HH}^\bullet(S(V), S(V)\#G(m,p,n))$ is thus generated as a ring under cup product by $\mathrm{HH}^\bullet(S(V))$ and the elements

- $1 \otimes (v_i^* - \zeta_m^c v_j^*) \otimes s$, where s is a reflection about the hyperplane $v_i^* - \zeta_m^c v_j^* = 0$, and
- $1 \otimes v_{i_1}^* \wedge \cdots \wedge v_{i_{\mathrm{codim}(g)}}^* \otimes g$, where g is p-connected and $v_{i_1}, \ldots, v_{i_{\mathrm{codim}(g)}}$ form a basis of $(V^g)^\perp$.

(Note that we have included the elements $1 \otimes (v_{i_1}^* \wedge v_{i_2}^*) \otimes g$ with $\det(g) = 1$, but these do not arise from codimension atoms and are superfluous generators.)

Shepler and Witherspoon [24, Corollary 10.6] show that if G is a Coxeter group or $G = G(m,1,n)$, then, in analogy with the Hochschild-Kostant-Rosenberg Theorem, the cohomology $\mathrm{HH}^\bullet(S(V), S(V)\#G)$ is generated in cohomological degrees 0 and 1. We use our comparison of the reflection length and codimension posets to show this analogue fails for the other irreducible complex reflection groups.

We recall from [24, Section 8] the volume algebra $A_{\mathrm{vol}} := \mathrm{Span}_{\mathbb{C}}\{1 \otimes \mathrm{vol}_g^\perp \otimes g \,:\, g \in G\}$, isomorphic to a (generalized) twisted group algebra with multiplication

$$(1 \otimes \mathrm{vol}_g^\perp \otimes g) \smile (1 \otimes \mathrm{vol}_h^\perp \otimes h) = \theta(g,h)(1 \otimes \mathrm{vol}_{gh}^\perp \otimes gh)$$

for some cocycle $\theta : G \times G \to \mathbb{C}$. The cocycle θ is generalized in that its values may include zero; in fact, the **twisting constant** $\theta(g,h)$ is nonzero if and only if $g \leq_\perp gh$. Iterating the product formula, we find

$$(1 \otimes \mathrm{vol}_{g_1}^\perp \otimes g_1) \smile \cdots \smile (1 \otimes \mathrm{vol}_{g_k}^\perp \otimes g_k) = \lambda(1 \otimes \mathrm{vol}_{g_1\cdots g_k}^\perp \otimes g_1 \cdots g_k),$$

where $\lambda = \theta(g_1, g_2)\theta(g_1g_2, g_3)\cdots\theta(g_1 \cdots g_{k-1}, g_k)$. The twisting constant λ is nonzero if and only if $g_1 \leq_\perp g_1g_2 \leq_\perp \cdots \leq_\perp g_1 \cdots g_k$. We make use of this fact in the proof of Lemma 2.7.2 below.

Once a choice of volume forms $\mathrm{vol}_g^\perp$ has been made, then given an element α in the cohomology ring $\mathrm{HH}^\bullet(S(V), S(V)\#G)$, there exist unique elements α_g in $S(V^g) \otimes \bigwedge^\bullet(V^g)^*$ such that

$$\alpha = \sum_{g\in G} \alpha_g \otimes \mathrm{vol}_g^\perp \otimes g.$$

Let the **support** of α be $\mathrm{supp}(\alpha) = \{g \in G \,:\, \alpha_g \neq 0\}$. For a set $B \subset \mathrm{HH}^\bullet(S(V), S(V)\#G)$, let $\mathrm{supp}(B) = \bigcup_{\beta\in B} \mathrm{supp}(\beta)$. In the next lemma, we relate the support of a subring of $\mathrm{HH}^\bullet(S(V), S(V)\#G)$ to the support of a set of generators for the subring.

2.7.2. Lemma. *Let B be a subring of $\mathrm{HH}^\bullet(S(V), S(V)\#G)$, and let $\mathcal{G}(B)$ be a set of generators for B as a ring under cup product. If g is in $\mathrm{supp}(B)$, then there exist group elements $g_1, \ldots, g_k$ in $\mathrm{supp}(\mathcal{G}(B))$ such that $g_1 \leq_\perp g_1g_2 \leq_\perp \cdots \leq_\perp g_1 \cdots g_k = g$.*

Proof. First consider the support of a finite cup product $\beta_1 \smile \cdots \smile \beta_k$ of generators β_i from $\mathcal{G}(B)$. Using the cup product formula [24, Equation (7.4)][2], we find that a typical

[2]Note that the factor $dv_g \wedge dv_h$ in Equation (7.4) may not a priori be an element of $\bigwedge^\bullet(V^{gh})^*$. To interpret the equation correctly, we must apply to the wedge product $dv_g \wedge dv_h$ the projection $\bigwedge^\bullet V^* \to \bigwedge^\bullet(V^{gh})^*$ induced by the orthogonal projection $V^* \to (V^{gh})^*$. After the last iteration of the cup product

summand of $\beta_1 \smile \cdots \smile \beta_k$ has the form

$$\omega \otimes \theta(g_1, g_2)\theta(g_1g_2, g_3)\cdots\theta(g_1\cdots g_{k-1}, g_k)\,\mathrm{vol}_g^{\perp} \otimes g,$$

where each g_i is in $\mathrm{supp}(\beta_i)$, $g = g_1 \cdots g_k$, and ω is a (possibly zero) derivation form in $S(V^g) \otimes \bigwedge^{\bullet}(V^g)^*$. The scalar

$$\theta(g_1, g_2)\theta(g_1g_2, g_3)\cdots\theta(g_1\cdots g_{k-1}, g_k)$$

is a twisting constant from the volume algebra and, as noted above, is nonzero if and only if $g_1 \leq_{\perp} g_1g_2 \leq_{\perp} \cdots \leq_{\perp} g_1\cdots g_k$. Thus

$$\mathrm{supp}(\beta_1 \smile \cdots \smile \beta_k) \subseteq \{g_1 \cdots g_k : g_i \in \mathrm{supp}(\beta_i) \text{ and } g_1 \leq_{\perp} g_1g_2 \leq_{\perp} \cdots \leq_{\perp} g_1 \cdots g_k\}.$$

Now note that for arbitrary elements $\alpha_1, \ldots, \alpha_k$ in B, we have

$$\mathrm{supp}(\alpha_1 + \cdots + \alpha_k) \subseteq \mathrm{supp}(\alpha_1) \cup \cdots \cup \mathrm{supp}(\alpha_k).$$

This proves the lemma since every element of B is a sum of finite cup products of elements of $\mathcal{G}(B)$. □

2.7.3. Corollary. *The set of codimension atoms for G is contained in the support of every generating set for* $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$.

Proof. Let $\mathcal{G}$ be a set of generators for $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$. Applying Lemma 2.7.2, we have that for each $g \neq 1$ in G there exist nonidentity group elements $g_1, \ldots, g_k$ in $\mathrm{supp}(\mathcal{G})$ such that $g_1 \leq_{\perp} g_1g_2 \leq_{\perp} \cdots \leq_{\perp} g_1 \cdots g_k = g$. In particular, $g_1 \leq_{\perp} g$. If g is a codimension atom, then since $g_1 \neq 1$ we must have $g_1 = g$, and hence g lies in $\mathrm{supp}(\mathcal{G})$. □

2.7.4. Remark. The exterior products in the description of cohomology force a homogeneous generator supported on a group element g to have cohomological degree at least $\mathrm{codim}(g)$ (and no more than $\dim V = n$). In light of Corollary 2.7.3, a set of homogeneous generators for $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ may well require elements of maximum cohomological degree n.

formula, we also apply the projections $S(V) \to S(V)/I((V^g)^{\perp}) \cong S(V^g)$ to the polynomial parts to obtain a representative in $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$.

For instance, in the group $G(n,n,n)$ for $n \geq 3$, the element $g = \mathrm{diag}(e^{2\pi i/n}, \ldots, e^{2\pi i/n})$ is a codimension atom, and a homogeneous generator supported on g must have cohomological degree $\mathrm{codim}(g) = n$. Thus, using Corollary 2.7.3, we see that every set of homogeneous generators for $\mathrm{HH}^\bullet(S(V), S(V)\#G(n,n,n))$ includes an element of cohomological degree n.

2.7.5. Corollary. *Let G be an irreducible complex reflection group. Then the cohomology ring $\mathrm{HH}^\bullet(S(V), S(V)\#G)$ is generated in cohomological degrees 0 and 1 if and only if G is a Coxeter group or a monomial reflection group $G(m,1,n)$.*

Proof. By Corollary 2.7.3, the support in G of a set of generators for $\mathrm{HH}^\bullet(S(V), S(V)\#G)$ must contain the set of codimension atoms. It follows that any set of generators contains elements of cohomological degree at least as great as the codimensions of the atoms in the codimension poset. If G is not a Coxeter group and not a monomial reflection group $G(m,1,n)$, then there are nonreflection atoms in the codimension poset, so a generating set for $\mathrm{HH}^\bullet(S(V), S(V)\#G)$ will necessarily include elements of cohomological degree greater than one.

Conversely, Shepler and Witherspoon show in [24, Corollary 10.6] that if G is a Coxeter group or a monomial reflection group $G(m,1,n)$, then $\mathrm{HH}^\bullet(S(V), S(V)\#G)$ can in fact be generated in degrees 0 and 1. $\square$

CHAPTER 3

TECHNIQUES FOR COMPUTING HOCHSCHILD COHOMOLOGY OF SKEW GROUP ALGEBRAS

In this chapter, we outline various tools and observations that simplify Hochschild cohomology computations in the case of a skew group algebra.

3.1. Invariant Theory

In the setting of skew group algebras, Hochschild cohomology may be formulated in terms of invariant theory. Let G be a finite group acting linearly on a complex vector space $V \cong \mathbb{C}^n$. Without loss of generality, G preserves the inner product on V and acts by unitary matrices. Ştefan [27] finds cohomology of the skew group algebra $S(V)\#G$ as the space of G-invariants in a larger cohomology ring:

$$\mathrm{HH}^\bullet(S(V)\#G) \cong \mathrm{HH}^\bullet(S(V), S(V)\#G)^G,$$

and Farinati [7] and Ginzburg and Kaledin [11] describe the larger cohomology ring:

$$\mathrm{HH}^\bullet(S(V), S(V)\#G) \cong \bigoplus_{g\in G} \mathrm{H}_g^\bullet \otimes g,$$

where

$$\mathrm{H}_g^\bullet = S(V^g) \otimes \bigwedge^{\bullet-\mathrm{codim}(g)} (V^g)^* \otimes \bigwedge^{\mathrm{codim}(g)} ((V^g)^\perp)^*.$$

Here, $V^g = \{v \in V : \vec{g}(v) = v\}$ denotes the fixed point space of g, and $\mathrm{codim}(g) := \mathrm{codim}\, V^g$. The group G acts diagonally on the tensor product, and the action on the tensor factor g is by conjugation in the group. Note that the centralizer $Z(g)$ of an element g in G preserves the fixed point space V^g and its orthogonal complement $(V^g)^\perp$, so the g-summand $\mathrm{H}_g^\bullet \otimes g$ is a $Z(g)$-module. After switching the order of the operations "direct sum" and "take invariants", one can compute cohomology by finding invariants under centralizer subgroups and taking a direct sum over a set of conjugacy class representatives (see Shepler-Witherspoon [23]). If $\mathscr{C}(G)$ is any set of conjugacy class representatives of G, then

$$\mathrm{HH}^{\bullet}(S(V)\#G) \cong \left(\bigoplus_{g\in G} \mathrm{H}_g^{\bullet} \otimes g\right)^G \cong \bigoplus_{g\in\mathscr{C}(G)} \left(\mathrm{H}_g^{\bullet} \otimes g\right)^{Z(g)}.$$

We refer to the summand corresponding to the conjugacy class of g as the **g-component of Hochschild cohomology** and denote it by $\mathrm{HH}^{\bullet}(g)$. Note that

$$\text{(1)} \qquad \mathrm{HH}^{\bullet}(g) \cong (\mathrm{H}_g^{\bullet})^{Z(g)} = \left(S(V^g) \otimes \bigwedge^{\bullet-\mathrm{codim}(g)} (V^g)^* \otimes \bigwedge^{\mathrm{codim}(g)} ((V^g)^{\perp})^*\right)^{Z(g)},$$

where the tensor factor g has been omitted since it carries a trivial $Z(g)$-action.

3.2. Actions of Centralizers on Fixed Point Spaces

Let χ be the character for a representation $V \cong \mathbb{C}^n$ of a finite group G. For a group element g in G, let $\chi \downarrow Z(g)$ be the restriction of χ to the centralizer subgroup $Z(g)$, and let

$$\chi \downarrow Z(g) = m_1\chi_1 + \cdots + m_r\chi_r$$

be the unique decomposition of $\chi \downarrow Z(g)$ into a combination of the irreducible characters $\chi_1, \ldots, \chi_r$ of $Z(g)$. Because g is in the center of $Z(g)$, Schur's Lemma reveals that g acts as a scalar multiple of the identity in each irreducible representation of $Z(g)$. Thus, we can pick out the constituents where g acts as the identity to get the character

$$\chi^g = \sum_{\substack{i \\ \chi_i(g)=\chi_i(1)}} m_i\chi_i$$

describing the action of $Z(g)$ on the fixed point space V^g, while the remaining constituents give the character for the action of $Z(g)$ on the orthogonal complement $(V^g)^{\perp}$.

3.3. Poincaré Series

Poincaré series (also called Hilbert or Molien series) are a tool for keeping track of the dimensions of the pieces of a graded vector space. A graded vector space $A = \bigoplus_{d=0}^{\infty} A_d$, where each A_d is a finite dimensional complex vector space, has Poincaré series

$$P_x(A) = \sum_{d=0}^{\infty} (\dim_{\mathbb{C}} A_d) x^d.$$

3.3.1. Example. The Poincaré series for the polynomial ring $\mathbb{C}[t]$ is

$$P_x(\mathbb{C}[t]) = 1 + x + x^2 + \cdots = \frac{1}{1-x}.$$

More generally, the Poincaré series for a polynomial ring with generators $t_{d_1}, \ldots, t_{d_r}$ of homogeneous degrees $d_1, \ldots, d_r$ has Poincaré series

$$P_x(\mathbb{C}[t_{d_1}, \ldots, t_{d_r}]) = \frac{1}{(1-x^{d_1}) \cdots (1-x^{d_r})}.$$

Hochschild cohomology of a skew group algebra is bigraded by polynomial degree and cohomological degree, so we seek a two-variable series with the coefficient of $x^i y^j$ recording the vector space dimension of the piece with polynomial degree i and cohomological degree j. A generalization of Molien's Theorem from invariant theory gives a generating function for the two-variable Poincaré series of the vector space $(S(V) \otimes \bigwedge^{\bullet} V^*)^G$ of invariant differential forms (see, for example, Kane [15, Chapters 17 and 22]). Since the g-component of Hochschild cohomology (Equation 1) may be expressed in terms of invariants in a tensor product, a slight variation yields the Poincaré series

$$P_{x,y}(\mathrm{HH}^{\bullet}(g)) = \frac{1}{|Z(g)|} \sum_{h \in Z(g)} \frac{\det^{-1}(h^{\perp}) \det(1 + h^* y)}{\det(1 - hx)}\, y^{\operatorname{codim}(g)},$$

where $h^{\perp}$ is the action of h on $(V^g)^{\perp}$, h^* in the numerator acts on $(V^g)^*$, and h in the denominator acts on V^g. In turn, the Poincaré series for $S(V)\#G$ is

$$P_{x,y}\left(\mathrm{HH}^{\bullet}(S(V)\#G)\right) = \sum_{g \in \mathscr{C}(G)} P_{x,y}(\mathrm{HH}^{\bullet}(g)).$$

In Section 5.3.1, we use Poincaré series in an example to verify that we have found a complete set of generators for the identity component of Hochschild cohomology.

3.4. Necessary Conditions for Nonzero Cohomology

For convenience, we record some necessary conditions in order to have $\mathrm{HH}^k(g) \neq 0$.

3.4.1. Observation. *Let G be any finite group acting linearly on $V \cong \mathbb{C}^n$. If the cohomology component* $\mathrm{HH}^k(g)$ *is nonzero, then* $\det(g) = 1$ *and* $\operatorname{codim}(g) \leq k \leq n$.

Proof. Note that g acts trivially on the tensor product $S(V^g) \otimes \bigwedge^{k-\operatorname{codim}(g)}(V^g)^*$ and scales the one-dimensional vector space $\bigwedge^{\operatorname{codim}(g)}((V^g)^{\perp})^*$ by $\det^{-1}(g)$. So necessarily $\det(g) = 1$ in order to have any nonzero $Z(g)$-invariants in $\mathrm{H}_g^{\bullet}$. For the second criteria, note that the exterior factor $\bigwedge^{k-\operatorname{codim}(g)}(V^g)^*$ is zero unless $0 \leq k - \operatorname{codim}(g) \leq n - \operatorname{codim}(g)$. □

Based on the above observation, the following table illustrates, for $n = 4$, which components of cohomology may be potentially nonzero.

codim(g)	0	1	2	3	4
$\mathrm{HH}^0(g)$	*	0	0	0	0
$\mathrm{HH}^1(g)$	*	0	0	0	0
$\mathrm{HH}^2(g)$	*	0	*	0	0
$\mathrm{HH}^3(g)$	*	0	*	*	0
$\mathrm{HH}^4(g)$	*	0	*	*	*

Note that elements of codimension one (reflections) never have determinant one, so based on the codimension restrictions, $\mathrm{HH}^2(g)$ can be nontrivial only for elements with codimension zero or two.

3.5. Cohomology Component for an Element with an Abelian Centralizer

If the centralizer of g in G is abelian, then the g-component of Hochschild cohomology can be described in terms of polynomials semi-invariant under various linear characters.

We first establish some notation. Let G be a finite group acting linearly on $V \cong \mathbb{C}^n$, and suppose the centralizer of g in G is abelian. Then $Z(g)$ acts diagonally on V with respect to some basis, say $v_1, \ldots, v_n$. We may assume the vectors are ordered so that $\vec{g}(v_i) = v_i$ if and only if $1 \leq i \leq \dim V^g$. Let k be a cohomological degree with $\operatorname{codim}(g) \leq k \leq n$, and let $p = k - \operatorname{codim}(g)$. (Note that $0 \leq p \leq \dim V^g$.) Define an eigenvector basis of

$\bigwedge^{k-\operatorname{codim}(g)}(V^g)^* \otimes \bigwedge^{\operatorname{codim}(g)}((V^g)^\perp)^*$ as follows: For each tuple $I = (i_1, \ldots, i_p)$ in the set

$$\mathcal{I}_k = \{(i_1, \ldots, i_p) : 1 \leq i_1 < \cdots < i_p \leq \dim V^g\},$$

define the element v_I^* in $\bigwedge^{k-\operatorname{codim}(g)}(V^g)^* \otimes \bigwedge^{\operatorname{codim}(g)}((V^g)^\perp)^*$ by

$$v_I^* = v_{i_1}^* \wedge \cdots \wedge v_{i_p}^* \otimes \operatorname{vol}_g^\perp,$$

where $\operatorname{vol}_g^\perp = 1$ if $\dim V^g = n$ and $\operatorname{vol}_g^\perp = v_{\dim V^g+1}^* \wedge \cdots \wedge v_n^*$ if $\dim V^g < n$. By convention, $\mathcal{I}_{\operatorname{codim}(g)} = \{()\}$ and $v_{()}^* = 1 \otimes \operatorname{vol}_g^\perp$. Then $\{v_I^*\}_{I \in \mathcal{I}_k}$ is a $Z(g)$-eigenvector basis for $\bigwedge^{k-\operatorname{codim}(g)}(V^g)^* \otimes \bigwedge^{\operatorname{codim}(g)}((V^g)^\perp)^*$, and for each tuple I in $\mathcal{I}_k$, there exists a linear character $\chi_I : Z(g) \to \mathbb{C}^\times$ such that

$$\vec{h}(v_I^*) = \overline{\chi_I(h)} v_I^*$$

for every h in the centralizer $Z(g)$.

3.5.1. Observation. *Let G be a finite group acting linearly on a vector space $V \cong \mathbb{C}^n$, and suppose the centralizer of g in G is abelian. Then, using the notation above, the g-component of Hochschild cohomology of the skew group algebra $S(V)\#G$ is given by*

$$\operatorname{HH}^k(g) \cong \begin{cases} \displaystyle\bigoplus_{I \in \mathcal{I}_k} S(V^g)^{\chi_I} \otimes v_I^* & \text{if } \det(g) = 1 \text{ and } \operatorname{codim}(g) \leq k \leq n \\ 0 & \text{otherwise,} \end{cases}$$

where $S(V^g)^{\chi_I} = \{f \in S(V^g) : \vec{h}(f) = \chi_I(h) f \text{ for all } h \text{ in } Z(g)\}$.

3.5.2. Observation. *Let G be a finite group with a cyclic center generated by an element z. Suppose G acts irreducibly on a vector space $V \cong \mathbb{C}^n$ so that z scales vectors in V by some primitive r^{th} root of unity ζ_r. If the centralizer of g in G is $Z(g) = \langle g, z \rangle$, then the g-component of Hochschild cohomology of $S(V)\#G$ is given by*

$$\operatorname{HH}^k(g) \cong \begin{cases} \displaystyle\bigoplus_{\substack{I \in \mathcal{I}_k, \\ d \equiv k \pmod{r}}} S^d(V^g) \otimes v_I^* & \text{if } \det(g) = 1 \text{ and } \operatorname{codim}(g) \leq k \leq n \\ 0 & \text{otherwise,} \end{cases}$$

where $S^d(V^g)$ is the subspace of $S(V^g)$ spanned by the degree d monomials.

PROOF. Suppose $\det(g) = 1$ and $\operatorname{codim}(g) \le k \le n$ (otherwise $\mathrm{HH}^k(g) = 0$ by Observation 3.4.1). The g-component of Hochschild cohomology is

$$\begin{aligned}\mathrm{HH}^k(g) &\cong \left(S(V^g) \otimes \bigwedge^{k-\operatorname{codim}(g)} (V^g)^* \otimes \bigwedge^{\operatorname{codim}(g)} ((V^g)^\perp)^* \right)^{Z(g)} \\ &\cong \left(\bigoplus_{\substack{I \in \mathcal{I}_k \\ d \ge 0}} S^d(V^g) \otimes \mathbb{C} v_I^* \right)^{Z(g)} .\end{aligned}$$

Since $Z(g)$ preserves the summands $S^d(V^g) \otimes \mathbb{C} v_I^*$, it suffices to consider the action of $Z(g)$ on an element $f \otimes v_I^*$, where f is a homogeneous polynomial in $S(V^g)$. Since z scales vectors in V by ζ_r and vectors in V^* by ζ_r^{-1}, we have

$$\vec{g}(f \otimes v_I^*) = f \otimes \det{}^{-1}(g) v_I^* = f \otimes v_I^* \qquad \text{and} \qquad \vec{z}(f \otimes v_I^*) = \zeta_r^{\deg(f)} f \otimes \zeta_r^{-k} v_I^*.$$

Thus, $f \otimes v_I^*$ is $Z(g)$-invariant if and only if $\deg(f) - k \equiv 0 \mod r$. ☐

3.6. Cohomology Component for an Element with a Trivial Fixed Point Space

If g is a group element with maximum codimension $\operatorname{codim}(g) = n$, then we can describe the g-component of Hochschild cohomology explicitly just by knowing the determinants of the elements in the centralizer of g.

3.6.1. LEMMA. *Let G be a finite group acting linearly on a vector space $V \cong \mathbb{C}^n$ with basis $v_1, \ldots, v_n$. Suppose g is a group element with maximum codimension $\operatorname{codim}(g) = n$. Then $\mathrm{HH}^k(g) = 0$ for $k \ne n$, and*

$$\mathrm{HH}^n(g) \cong \begin{cases} \mathbb{C} v_1^* \wedge \cdots \wedge v_n^* & \text{if } \det(h) = 1 \text{ for all } h \text{ in } Z(g) \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Let g be an element of G with $\operatorname{codim}(g) = n$. Then $\mathrm{HH}^k(g) = 0$ when $k \ne n$ by Observation 3.4.1. Turning to the case $k = n$, we have

$$\mathrm{HH}^n(g) \cong \left(S(V^g) \otimes \bigwedge^{n-\operatorname{codim}(g)} (V^g)^* \otimes \bigwedge^{\operatorname{codim}(g)} ((V^g)^\perp)^* \right)^{Z(g)} = \left(\mathbb{C} \otimes \mathbb{C} \otimes \bigwedge^n V^* \right)^{Z(g)} .$$

The volume form $\mathrm{vol}_g^\perp = v_1^* \wedge \cdots \wedge v_n^*$ is a basis for $\bigwedge^n V^*$, and

$$\vec{h}(1 \otimes 1 \otimes \mathrm{vol}_g^\perp) = \det{}^{-1}(h)(1 \otimes 1 \otimes \mathrm{vol}_g^\perp)$$

for every h in $Z(g)$. Thus $1 \otimes 1 \otimes \mathrm{vol}_g^\perp$ is $Z(g)$-invariant if and only if $\det(h) = 1$ for all h in $Z(g)$. □

3.7. Galois Conjugate Representations

If the action of a finite group G on a finite-dimensional vector space V is represented by matrices with entries in the field $\mathbb{Q}(\zeta)$ for some root of unity ζ, then applying a field automorphism γ to the entries of the matrices gives another representation V_γ of G, not necessarily equivalent to the representation V. However, since the Poincaré series for Hochschild cohomology of a skew group algebra is given in terms of characteristic polynomials and determinants of matrices (see Section 3.3), we have

$$P_{x,y}(\mathrm{HH}^\bullet(S(V_\gamma)\#G)) = \gamma\left(P_{x,y}(\mathrm{HH}^\bullet(S(V)\#G))\right).$$

Since the coefficients in the Poincaré series are integers, applying the field automorphism γ leaves the series unchanged, and $P_{x,y}(\mathrm{HH}^\bullet(S(V_\gamma)\#G)) = P_{x,y}(\mathrm{HH}^\bullet(S(V)\#G))$. In particular, if the g-component of one Hochschild cohomology ring is zero, then the g-component of the other must also be zero. This is a convenient observation when deciding which skew group algebras to investigate.

3.8. Representations Differing by a Group Automorphism

It is possible for inequivalent representations to give rise to isomorphic skew group algebras. For example, if two representations of a group differ by a group automorphism, then the skew group algebras are isomorphic, and so is the resulting cohomology.

Let G be a finite group acting on $V \cong \mathbb{C}^n$, say by representation $\rho : G \to GL(V)$. For any group automorphism ϕ of G, composing with ρ gives another representation $\rho \circ \phi$ of G. We write V_ϕ to indicate when the action of G on V is the composition $\rho \circ \phi$. An isomorphism between the skew group algebras $S(V)\#G$ and $S(V_\phi)\#G$ is given by $v \mapsto v$ and $g \mapsto \phi^{-1}(g)$, and an explicit conversion between their Hochschild cohomology rings is accomplished by merely changing group element tags.

CHAPTER 4

RANK TWO EXCEPTIONAL REFLECTION GROUPS

Since the centralizers of the rank two complex reflection groups $G_4 - G_{22}$ are easy to describe, we can get a fairly good idea of what the Hochschild cohomology looks like. In this chapter, we describe the centralizers of elements in each rank two complex reflection group and then give methods for using GAP (Groups, Algorithms, and Programming) to compute Hochschild cohomology.

4.1. Centralizer Subgroups of Reflection Groups $G_4 - G_{22}$

In any group G, the smallest possible centralizer of an element g is the subgroup generated by g and the center of G, while the largest possible centralizer is the whole group. We say an element g has a **medium centralizer** if

$$\langle g, Z(G)\rangle \subsetneq Z(g) \subsetneq G.$$

In the tetrahedral and icosahedral complex reflection groups, there are no elements with medium centralizers:

4.1.1. Lemma. *Let G be a tetrahedral complex reflection group (one of $G_4 - G_7$ in the Shephard-Todd table) or an icosahedral group (one of $G_{16} - G_{22}$ in the Shephard-Todd table). Let z be a generator of the (cyclic) center of G. If g is a noncentral element of G, then $Z(g) = \langle g, z\rangle$.*

Proof. This is easily verified by using GAP to compare `Size(Centralizer(G,g))` with `Size(G)` and `Size(Closure(Centre(G),g))`. □

In an octahedral reflection group, there are elements with medium centralizers. Some elements with medium centralizers are squares, and upon computing the centralizer of a particular square element h^2 to be $Z(h^2) = \langle h, z\rangle$, we notice that every element with a medium centralizer is conjugate to an element of the h^2Z coset, leading to the following description of all medium centralizers:

4.1.2. LEMMA. *Let G be an octahedral complex reflection group (one of $G_8 - G_{15}$ in the Shephard-Todd table). Let z be a generator of the (cyclic) center Z of G. If g in G is an element with a medium centralizer, i.e., $\langle g, z\rangle \subsetneq Z(g) \subsetneq G$, then there exists an element h in G such that g is in the coset h^2Z and $Z(g) = Z(h) = \langle h, z\rangle$.*

PROOF. Let g be an element with a medium centralizer. Suppose h is an element with centralizer $Z(h) = \langle h, z\rangle$ and $|Z(h)| = |Z(g)|$. If g is in the coset h^2Z, then $Z(g) \supseteq Z(h)$, and, by comparing sizes, $Z(g) = Z(h)$. In fact, it suffices to know that g is *conjugate* to an element of the coset h^2Z. For, if $g = xh^2z^mx^{-1}$ for some power m and some x in G, then the hypotheses of the first argument hold with h replaced by xhx^{-1}.

Using the software GAP [19] with the CHEVIE [9] package for complex reflection groups, one can verify the hypothesis: for each g with a medium centralizer, there exists an element h such that $Z(h) = \langle h, z\rangle$, $|Z(h)| = |Z(g)|$, and g is conjugate to an element of the coset h^2Z. □

4.2. Cohomology for Irreducible Representations

If G is any of the groups $G_4 - G_{22}$ acting irreducibly on $V \cong \mathbb{C}^n$, then, since the noncentral elements in G have small abelian centralizers, we can compute all nonidentity components of Hochschild cohomology of $S(V)\#G$ using the techniques in Chapter 3. This can be automated using the software GAP [19], where we have access to character tables, conjugacy classes, centralizers, and eigenvalues. The CHEVIE [9] package is used to load complex reflection groups.

Let G be any of the groups $G_4 - G_{22}$. Given an irreducible character of G, a conjugacy class representative g, and a cohomological degree k, we automate the following steps to determine $\mathrm{HH}^k(g)$:

(i) Compute $\det(g)$ and $\mathrm{codim}(g)$ with the help of the `Eigenvalues` command.

(ii) Apply Observation 3.4.1. If $\det(g) = 1$ and $\operatorname{codim}(g) \leq k \leq n$, then continue to the next step. Otherwise, record $\mathrm{HH}^k(g) = 0$.

(iii) If $\operatorname{codim}(g) < n$, then continue to the next step. Otherwise, record $\mathrm{HH}^k(g)$ using Lemma 3.6.1.

(iv) Compare `Size(Centralizer(G,g))` with `Size(Closure(Centre(G),g))` and with `Size(G)` to determine the centralizer type.

(v) If $Z(g) = \langle g, z\rangle$, apply Observation 3.5.2.

(vi) If g has a medium centralizer, then find an element z that generates the center of G, and find the element h guaranteed by Lemma 4.1.2. Then use the idea in Section 3.2 to compute the linear characters needed to apply Observation 3.5.1.

(vii) If $Z(g) = G$, then, since the representation of G is irreducible and we have already determined $\operatorname{codim}(g) < n$, the element g must act as the identity. Now,

$$\mathrm{HH}^\bullet(g) \cong \mathrm{HH}^\bullet(1) \cong \left(S(V) \otimes \bigwedge^{\bullet} V^*\right)^G,$$

which our code does not describe explicitly. (See Section 5.3.1 for an example using standard invariant theory techniques to determine the identity component by hand.)

4.2.1. Example. To illustrate, a summary of the Hochschild cohomology of $S(V)\#G$ where $G = G_6$ acts irreducibly on $V \cong \mathbb{C}^n$ (with $n > 1$) is included in Appendix C. The group G_6 has fourteen irreducible representations with characters labeled X.1, ... ,X.14. We omit the cohomology tables for the six one-dimensional representations since all components of Hochschild 2-cohomology are zero, implying that there are no nontrivial deformations of the corresponding skew group algebras (see Gerstenhaber and Schack [10]).

In each of the cohomology tables, an entry $*$ denotes a cohomology component

$$\mathrm{HH}^k(g) \cong \left(S(V) \otimes \bigwedge^{k} V^*\right)^G,$$

isomorphic to the identity component of degree k cohomology.

The zeros in the cohomology tables for all of the two-dimensional representations may be deduced from Observation 3.4.1 and Lemma 3.6.1. Only the identity component is nontrivial. Representations X.7, X.8, X.11, and X.12 are reflection representations.

The zeros in the cohomology tables for X.13 and X.14 follow from the determinant and codimension restrictions given in Observation 3.4.1. An entry of the form $k\mathrm{m}r$ denotes a cohomology component

$$\mathrm{HH}^k(g) \cong \bigoplus_{\substack{I \in \mathcal{I}_k, \\ d \equiv k \mod r}} S^d(V^g) \otimes \mathbb{C}v_I^*$$

given by Observation 3.5.2. The congruences are unsimplified to emphasize that they can be read off from the cohomological degree and the action of the center.

Hochschild 2-cohomology contains all possible first multiplication maps of an associative deformation of an algebra. Looking at the cohomology tables in the G_6 example, a few representations stand out: X.7, X.8, X.11, and X.12 (reflection representations) have a scarcity of nontrivial Hochschild 2-cohomology, while X.13 (a "rotation representation") has an abundance of nontrivial Hochschild 2-cohomology. The analogous representations are present for each of the groups $G_4 - G_{22}$. In Section 4.3, we compute the Hochschild cohomology for the reflection representation of each of the rank two exceptional reflection groups. In Chapter 5, we define a "rotation representation" for each of the groups $G_4 - G_{22}$ and compute the corresponding cohomology in the case of the tetrahedral groups $G_4 - G_7$.

4.3. Cohomology for the Reflection Representation

In this section, we describe the Hochschild cohomology for the skew group algebra $S(V)\#G$, where G is any of the groups $G_4 - G_{22}$ acting on $V \cong \mathbb{C}^2$ by its reflection representation.

4.3.1. Proposition. *Let G_i with $4 \leq i \leq 22$ be a rank two exceptional complex reflection group acting on $V \cong \mathbb{C}^2$ by its standard irreducible reflection representation. Then*

$$\mathrm{HH}^\bullet(S(V)\#G_i) \cong \begin{cases} \mathrm{HH}^\bullet(1) & \textit{if } i \neq 4, 12 \\ \mathrm{HH}^\bullet(1) \oplus \mathrm{HH}^\bullet(g_4) & \textit{if } i = 4 \\ \mathrm{HH}^\bullet(1) \oplus \mathrm{HH}^\bullet(g_6) \oplus \mathrm{HH}^\bullet(g_6^2) & \textit{if } i = 12, \end{cases}$$

where

- g_4 *is a representative of the conjugacy class of order four elements in* G_4*, and*
- g_6 *is a representative of the conjugacy class of order six elements in* G_{12}*.*

Descriptions of the cohomology components are indicated in the proof.

Proof. Let g be an element of one of the rank two reflection groups G_i.

If $\mathrm{codim}(g) = 0$, then $g = 1$, and $\mathrm{HH}^k(1) \cong \left(S(V) \otimes \bigwedge^k V^*\right)^G$, which is well-studied for the reflection representation (see Orlik and Terao [17]).

If $\mathrm{codim}(g) = 1$, then $\mathrm{HH}^k(g) = 0$ for all k by Observation 3.4.1.

If $\mathrm{codim}(g) = 2$, then g has a trivial fixed point space. By Lemma 3.6.1, $\mathrm{HH}^k(g) = 0$ for $k \neq 2$, and

$$\mathrm{HH}^2(g) \cong \begin{cases} \mathbb{C}v_1^* \wedge v_2^* & \text{if } \det(h) = 1 \text{ for all } h \in Z(g) \\ 0 & \text{otherwise.} \end{cases}$$

To determine whether $\mathrm{HH}^2(g) \neq 0$ for some elements, one considers the groups G_i individually in an argument that parallels Section $2C$ of Ram and Shepler [18]. In fact, Ram and Shepler were investigating the polynomial degree zero part of Hochschild 2-cohomology, and the formula above shows there is no cohomology with higher polynomial degree, so their computation of $\mathrm{HH}^2_0(g)$ completely describes $\mathrm{HH}^2(g)$ when $\mathrm{codim}(g) = 2$. □

CHAPTER 5

COHOMOLOGY FOR LIFTS OF QUOTIENT GROUP REPRESENTATIONS

In this chapter, we consider the deformation theory of skew group algebras $S(V)\#G$ and $S(V)\#G/K$, where the action of the group G on the vector space V is lifted from an action of a quotient group G/K on V. After observing some general simplifications, we apply them to the specific example of tetrahedral complex reflection groups acting on $V \cong \mathbb{C}^3$ by a nonfaithful "rotation representation" lifted from a representation of the quotient group $G/Z \cong \mathrm{Alt}_4$.

5.1. The Lazy Mathematician's Dream Formula

Before investigating the tetrahedral example, we consider the general setup. Let K be a normal subgroup of a finite group G and $\pi : G \twoheadrightarrow G/K$ the canonical quotient homomorphism. Suppose G/K acts on a vector space $V \cong \mathbb{C}^n$, say by a representation $\rho : G/K \to GL(V)$. Then the composition $G \xrightarrow{\pi} G/K \xrightarrow{\rho} GL(V)$ is a representation of G, and we can compare the deformation theory of the skew group algebras $S(V)\#G$ and $S(V)\#G/K$. Keeping in mind that a conjugacy class in the quotient group G/K lifts to a union of conjugacy classes in G, the lazy mathematician's dream formula would be

$$\mathrm{HH}^\bullet(S(V)\#G) \cong \bigoplus_{gK \in \mathscr{C}(G/K)} \mathrm{HH}^\bullet(gK)^{\oplus m_g},$$

where $\mathscr{C}(G/K)$ is a set of conjugacy class representatives for the quotient group G/K, and the multiplicity m_g is the number of conjugacy classes of G that intersect the coset gK nontrivially. This formula, however, breaks down in the following two ways:

(i) The conjugacy class of g in G and the conjugacy class of gK in G/K need not give isomorphic contributions to $\mathrm{HH}^\bullet(S(V)\#G)$ and $\mathrm{HH}^\bullet(S(V)\#G/K)$, respectively.

(ii) Conjugacy classes of elements in the same coset of K need not give isomorphic contributions to the cohomology of $S(V)\#G$.

In this section we highlight sufficient conditions for (i) and (ii) to hold, giving us techniques for efficient computations in the tetrahedral example in Section 5.4.

Recall from Section 3.1 that the g-component of Hochschild cohomology of the skew group algebra $S(V)\#G$ is $\mathrm{HH}^\bullet(g) \cong \left(\mathrm{H}_g^\bullet\right)^{Z(g)}$, where

$$\mathrm{H}_g^\bullet = S(V^g) \otimes \bigwedge^{\bullet-\mathrm{codim}(g)} (V^g)^* \otimes \bigwedge^{\mathrm{codim}(g)} ((V^g)^\perp)^*.$$

Since G acts on V by $\rho \circ \pi$, and G/K acts on V by ρ, we have

$$V^g = \{v \in V : ((\rho \circ \pi)(g))(v) = v\} = \{v \in V : \rho(\pi(g))(v) = v\} = V^{\pi(g)}$$

and $\mathrm{H}_g^\bullet = \mathrm{H}_{\pi(g)}^\bullet$. The g-component of $\mathrm{HH}^\bullet(S(V)\#G)$ is

$$\mathrm{HH}^\bullet(g) \cong (\mathrm{H}_g^\bullet)^{Z_G(g)} = (\mathrm{H}_{\pi(g)}^\bullet)^{\pi(Z_G(g))},$$

while the $\pi(g)$-component of $\mathrm{HH}^\bullet(S(V)\#G/K)$ is

$$\mathrm{HH}^\bullet(\pi(g)) \cong (\mathrm{H}_{\pi(g)}^\bullet)^{Z_{\pi(G)}(\pi(g))}.$$

When the centralizers $\pi(Z_G(g))$ and $Z_{\pi(G)}(\pi(g))$ are equal, we have $\mathrm{HH}^\bullet(g) \cong \mathrm{HH}^\bullet(\pi(g))$:

5.1.1. Proposition. *Let K be a normal subgroup of the finite group G. If the image of the centralizer $Z_G(g)$ under the canonical quotient homomorphism $G \to G/K$ is all of the centralizer $Z_{G/K}(gK)$, then the contribution $\mathrm{HH}^\bullet(g)$ to the cohomology $\mathrm{HH}^\bullet(S(V)\#G)$ is isomorphic to the contribution $\mathrm{HH}^\bullet(gK)$ to the cohomology $\mathrm{HH}^\bullet(S(V)\#G/K)$.*

Unfortunately, $\pi(Z_G(g))$ is often a *proper* subgroup of $Z_{\pi(G)}(\pi(g))$. In this case the space of $\pi(Z_G(g))$-invariants is typically larger than the space of $Z_{\pi(G)}(\pi(g))$-invariants (fewer group elements, easier to be invariant), and we cannot directly describe the contribution $\mathrm{HH}^\bullet(g)$ in terms of the contribution $\mathrm{HH}^\bullet(\pi(g))$.

5.1.2. Example. The order 48 reflection group G_6 has an order four center Z, and the quotient group G_6/Z is isomorphic to the alternating group Alt_4. The diagram here shows how the fourteen conjugacy classes of G_6 project down onto the four conjugacy classes of Alt_4.

```
         Z

  G6    1a     4a   4a   4a     12c  12c  12c  12c     12b  12b  12b  12b
        4b     2a   2a   2a      6b   6b   6b   6b      6a   6a   6a   6a
        2b     4a   4a   4a     12d  12d  12d  12d     12a  12a  12a  12a
        4c     2a   2a   2a      3b   3b   3b   3b      3a   3a   3a   3a
Alt4    __     __   __   __     ___  ___  ___  ___     ___  ___  ___  ___
        id       (12)(34)            (123)                  (321)
```

The conjugacy classes of G_6 are denoted by $1a$, $4a$, etc. (The number in a label indicates the order of each element in the conjugacy class.) Each column represents a coset of the center, and the labels in that column record the conjugacy classes of the elements in the coset.

Suppose G_6 acts on a vector space V by a representation with kernel Z, the center of the group. A priori,

$$\mathrm{HH}^\bullet(S(V)\#G_6) \cong \bigoplus_{g\in\mathscr{C}(G_6)} \mathrm{HH}^\bullet(g),$$

a sum over the fourteen conjugacy classes of G_6. However, elements of the same coset of the kernel Z act the same on V, so they have the same fixed point space. Furthermore, elements in the same coset of the center always have the same centralizer. Thus, $\mathrm{HH}^\bullet(g) \cong \mathrm{HH}^\bullet(g')$ if g and g' are in the same coset of the center. Now, we can shorten our work by writing

$$\mathrm{HH}^\bullet(S(V)\#G_6) \cong \mathrm{HH}^\bullet(1a)^{\oplus 4} \oplus \mathrm{HH}^\bullet(4a)^{\oplus 2} \oplus \mathrm{HH}^\bullet(12c)^{\oplus 4} \oplus \mathrm{HH}^\bullet(12b)^{\oplus 4},$$

a sum of four components with multiplicities.

5.1.3. Proposition. *Let G be a finite group acting linearly on a vectorspace $V \cong \mathbb{C}^n$, and suppose the kernel $K = \{g \in G : \vec{g}(v) = v \text{ for all } v \text{ in } V\}$ of the action is a subset of the center of the group G. Let $\mathscr{C}(G/K)$ be a set of conjugacy class representatives of the quotient group G/K. Then the Hochschild cohomology of the skew group algebra $S(V)\#G$ is*

$$\mathrm{HH}^\bullet(S(V)\#G) \cong \bigoplus_{gK\in\mathscr{C}(G/K)} \mathrm{HH}^\bullet(g)^{\oplus m_g},$$

where the multiplicity $m_g = \dfrac{|Z_G(g)|}{|Z_{G/K}(gK)|}$ is the number of conjugacy classes of G that intersect the coset gK nontrivially.

Proof. Suppose g and g' are in the same coset of the kernel K. Then they act the same on V, so their fixed point spaces are the same. Furthermore, since K is a subset of the center, g and g' have the same centralizer. Thus, $\mathrm{HH}^\bullet(g) \cong \mathrm{HH}^\bullet(g')$. It follows that

$$\mathrm{HH}^\bullet(S(V)\#G) \cong \bigoplus_{gK \in \mathscr{C}(G/K)} \mathrm{HH}^\bullet(g)^{\oplus m_g},$$

where the multiplicity m_g is the number of conjugacy classes that intersect the coset gK nontrivially.

We compute the multiplicities in terms of centralizer sizes. Let g be an element of G. Recall from group theory that the conjugacy class $\mathrm{Cl}_{G/K}(gK) = \{gK, g_1K, \ldots, g_rK\}$ lifts to a union of conjugacy classes in G. Furthermore, the elements of the conjugacy class $\mathrm{Cl}_G(g)$ of g in G distribute equally across the cosets $gK, g_1K, \ldots, g_rK$, so

$$|\mathrm{Cl}_G(g) \cap gK| = \frac{|\mathrm{Cl}_G(g)|}{|\mathrm{Cl}_{G/K}(gK)|}.$$

If g and g' are in the same coset of the central kernel K, then $|\mathrm{Cl}_G(g) \cap gK| = |\mathrm{Cl}_G(g') \cap gK|$, as g and g' have the same centralizer (and hence the same conjugacy class sizes). Now the number of conjugacy classes intersecting the coset gK nontrivially is

$$m_g = \frac{|K|}{|\mathrm{Cl}_G(g) \cap gK|} = |K| \cdot \frac{|\mathrm{Cl}_{G/K}(gK)|}{|\mathrm{Cl}_G(g)|} = \frac{|K||G|}{|K||Z_{G/K}(gK)|} \cdot \frac{|Z_G(g)|}{|G|} = \frac{|Z_G(g)|}{|Z_{G/K}(gK)|},$$

as required. □

The following example illustrates that for an arbitrary nonfaithful action, elements of the same coset of the kernel of the action may give nonisomorphic contributions to cohomology.

5.1.4. Example. Consider the dihedral group D_4 generated by an order four rotation r and a reflection f. Define a nonfaithful action of D_4 on $V \cong \mathbb{C}^2$ by letting elements of the rotation subgroup $\{1, r, r^2, r^3\}$ act as the identity and elements of the coset $\{f, rf, r^2f, r^3f\}$ act by the matrix $\mathrm{diag}(-1, 1)$ with respect to a basis v_1, v_2 of V. Then

$$\mathrm{HH}^\bullet(r) \cong \left(S(V) \otimes \bigwedge^\bullet V^*\right)^{\langle r\rangle} = S(V) \otimes \bigwedge^\bullet V^*,$$

but

$$\mathrm{HH}^\bullet(1) \cong \left(S(V) \otimes \bigwedge^\bullet V^*\right)^{D_4} \subsetneq S(V) \otimes \bigwedge^\bullet V^*.$$

The last inclusion is proper since, for example, $\mathrm{HH}^2(1) \cong \{f \otimes v_1^* \wedge v_2^* : f \text{ is in } v_1\mathbb{C}[v_1^2, v_2]\}$.

5.2. The Rotation Representation

Complex reflection groups G_4–G_{22} in the Shephard-Todd classification share the property that their central quotient is the rotational symmetry group of a Platonic solid. Write G/Z for the quotient of a group by its center. For groups $G_4 - G_7$, the quotient G/Z is isomorphic to Alt_4, the rotational symmetry group of the tetrahedron. For groups $G_8 - G_{15}$, the quotient G/Z is isomorphic to Sym_4, the rotational symmetry group of the octahedron. For groups $G_{16} - G_{22}$, the quotient G/Z is isomorphic to Alt_5, the rotational symmetry group of the icosahedron.

Let G be any of the groups $G_4 - G_{22}$, and write Z for the center of G. Let $V \cong \mathbb{C}^3$, and let $\rho : G/Z \to GL(V)$ be the complexification of the natural rotation action of G/Z on $\mathbb{R}^3$. We call the composition $G \xrightarrow{\pi} G/Z \xrightarrow{\rho} GL(V)$ the **rotation representation** of G.

The remainder of this chapter is devoted to cohomology for the skew group algebras of the alternating group on four letters and the tetrahedral reflection groups $G_4 - G_7$, each acting on $V \cong \mathbb{C}^3$ by the rotation representation. Although we only include full details for the tetrahedral groups, a similar analysis could be carried out for the octahedral and icosahedral groups acting by the rotation representation.

5.3. Cohomology for the Alternating Group on Four Letters

Let Alt_4 denote the alternating group on four letters. The symmetric group Sym_4 acts on the vector space $V \cong \mathbb{C}^4$ by permuting the standard basis vectors e_1, e_2, e_3, e_4. The subspace $L = \mathrm{Span}_{\mathbb{C}}\{e_1+e_2+e_3+e_4\}$ is an invariant line, and an orthonormal basis for the orthogonal complement $L^\perp$ is

$$\begin{aligned} v_1 &= \frac{1}{2}(e_1 - e_2 + e_3 - e_4) \\ v_2 &= \frac{1}{2}(e_1 - e_2 - e_3 + e_4) \end{aligned}$$

$$v_3 = \frac{1}{2}(e_1 + e_2 - e_3 - e_4).$$

The group Alt_4 acts irreducibly on $V := L^\perp$, and with respect to the basis v_1, v_2, v_3 we have generating matrices

$$(3\ 4\ 2) \mapsto \begin{pmatrix} & & 1 \\ 1 & 0 & \\ 0 & 1 & \end{pmatrix} \quad \text{and} \quad (3\ 2\ 1) \mapsto \begin{pmatrix} & & -1 \\ -1 & 0 & \\ 0 & 1 & \end{pmatrix}.$$

The alternating group Alt_4 has four conjugacy classes, so there are four components of Hochschild cohomology to compute:

$$\mathrm{HH}^\bullet(S(V)\#\,\mathrm{Alt}_4) \cong \mathrm{HH}^\bullet(1) \oplus \mathrm{HH}^\bullet((12)(34)) \oplus \mathrm{HH}^\bullet(123) \oplus \mathrm{HH}^\bullet(321).$$

Subsection 5.3.1 is devoted to the identity component; Subsection 5.3.2 describes the component for (12)(34); and Subsection 5.3.3 describes the components for the three-cycles.

5.3.1. *Identity Component*

The cohomology contribution from the identity is $\mathrm{HH}^\bullet(1) \cong \left(S(V) \otimes \bigwedge^\bullet V^*\right)^{\mathrm{Alt}_4}$. One standard approach to computing Alt_4-invariants is to observe that if W is a finite dimensional Sym_4-module, then

$$W^{\mathrm{Alt}_4} = W^{\mathrm{Sym}_4} \oplus W^{\mathrm{Sym}_4}_{\mathrm{sgn}},$$

where $W^{\mathrm{Sym}_4}_{\mathrm{sgn}} = \{w \in W : \vec{g}(w) = \mathrm{sgn}(g)w \text{ for all } g \text{ in } \mathrm{Sym}_4\}$. (This decomposition follows from the fact that Alt_4 is an index two subgroup of Sym_4.) We choose instead a direct (and perhaps longer) approach, as it nicely highlights a wide range of techniques from invariant theory and is instructive to the nonexpert.

Let $G = \mathrm{Alt}_4$. We first compute the Poincaré series for $\mathrm{HH}^\bullet(1)$, one for each cohomological degree:

$$P_x((S(V) \otimes \bigwedge^0 V^*)^G) = \frac{1 + x^6}{(1-x^2)(1-x^3)(1-x^4)}$$

$$P_x((S(V) \otimes \bigwedge^1 V^*)^G) = \frac{x + x^2 + 2x^3 + x^4 + x^5}{(1-x^2)(1-x^3)(1-x^4)}$$

$$P_x((S(V)\otimes\bigwedge^2 V^*)^G) = \frac{x+x^2+2x^3+x^4+x^5}{(1-x^2)(1-x^3)(1-x^4)}$$

$$P_x((S(V)\otimes\bigwedge^3 V^*)^G) = \frac{1+x^6}{(1-x^2)(1-x^3)(1-x^4)}$$

The denominator for each Poincaré series is $(1-x^2)(1-x^3)(1-x^4)$, suggesting that $(S(V)\otimes\bigwedge^\bullet V^*)^G$ is a finitely generated free $\mathbb{C}[f_2,f_3,f_4]$-module for some algebraically independent invariant polynomials f_2, f_3, and f_4 of degrees 2, 3, and 4, respectively. Since the group action is given by monomial matrices, it is easy to guess invariant polynomials with these degrees:

$$f_2 = v_1^2+v_2^2+v_3^2$$

$$f_3 = v_1v_2v_3$$

$$f_4 = v_1^4+v_2^4+v_3^4.$$

The Jacobian determinant

$$\det\left(\frac{\partial f_i}{\partial v_j}\right) = \det\begin{pmatrix} 2v_1 & 2v_2 & 2v_3 \\ v_2v_3 & v_1v_3 & v_1v_2 \\ 4v_1^3 & 4v_2^3 & 4v_3^3 \end{pmatrix} \doteq v_1^4(v_2^2-v_3^2)+v_2^4(v_3^2-v_1^2)+v_3^4(v_1^2-v_2^2)$$

is a nonzero polynomial, so f_2, f_3, and f_4 are algebraically independent.

The numerators of the Poincaré series predict the polynomial degrees of the secondary generators of $(S(V)\otimes\bigwedge^k V^*)^G$. (The identity component of cohomology will be the free $\mathbb{C}[f_2,f_3,f_4]$-span of the secondary generators.) In cohomological degrees zero and three, the numerator $(1+x^6)$ predicts generators of polynomial degrees 0 and 6, and in cohomological degrees one and two, the numerator $(x+x^2+2x^3+x^4+x^5)$ predicts generators of polynomial degrees $1,2,3,3,4,5$.

Before constructing secondary generators, we establish some notation. Define ordered bases $\mathcal{B}_k$ of $\bigwedge^k V^*$ as follows:

$$\mathcal{B}_0 = \{1\}$$

$$
\begin{aligned}
\mathcal{B}_1 &= \{v_1^*,\ v_2^*,\ v_3^*\} \\
\mathcal{B}_2 &= \{v_2^* \wedge v_3^*,\ v_3^* \wedge v_1^*,\ v_1^* \wedge v_2^*\} \\
\mathcal{B}_3 &= \{v_1^* \wedge v_2^* \wedge v_3^*\}.
\end{aligned}
$$

Note that $\bigwedge^0 V^*$ and $\bigwedge^3 V^*$ both carry the trivial representation of Alt_4, while $\bigwedge^1 V^*$ and $\bigwedge^2 V^*$ are isomorphic to V as Alt_4-modules. In fact, an easy check reveals that the matrices for the action of Alt_4 on V with respect to the basis $\{v_1, v_2, v_3\}$ also serve as matrices for the action of Alt_4 on the spaces $\bigwedge^1 V^*$ and $\bigwedge^2 V^*$ with respect to the bases $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively.

For $0 \le k \le 3$, let $n_k = \binom{3}{k}$ denote the size of the basis $\mathcal{B}_k = \{b_1, \dots, b_{n_k}\}$ of $\bigwedge^k V^*$. Given an n_k-tuple of polynomials p_i, let $(p_1, \dots, p_{n_k})_{\mathcal{B}_k}$ denote the element $\sum p_i \otimes b_i$ of $S(V) \otimes \bigwedge^k V^*$.

The following standard key facts motivate the construction of Alt_4-invariant forms:

(i) Wedging two G-invariant forms produces another G-invariant form.

(ii) Suppose W and W^* are dual irreducible representations of a group G. If the sets $\mathcal{B} = \{w_1, \dots, w_n\}$ and $\mathcal{B}^* = \{w_1^*, \dots, w_n^*\}$ are bases of W and W^* such that $[g]_{\mathcal{B}^*} = \overline{[g]}_{\mathcal{B}}$, then $(W \otimes W^*)^G$ is spanned by $\sum_{i=1}^n w_i \otimes w_i^*$.

When a representation is self-dual, the gradient of an invariant polynomial determines an invariant derivation. Using the invariant polynomials f_2, f_3, f_4 from above, construct invariant derivations by scaling $(\frac{\partial f_i}{\partial v_1}, \frac{\partial f_i}{\partial v_2}, \frac{\partial f_i}{\partial v_3})$:

$$
\begin{aligned}
\alpha_{1,1} &= (v_1, v_2, v_3)_{\mathcal{B}_1} \\
\alpha_{2,1} &= (v_2v_3, v_3v_1, v_1v_2)_{\mathcal{B}_1} \\
\alpha_{3,1} &= (v_1^3, v_2^3, v_3^3)_{\mathcal{B}_1}.
\end{aligned}
$$

Wedge them together two at a time to get invariant 2-forms of polynomial degrees $3, 4, 5$:

$$
\begin{aligned}
\beta_{3,2} := \alpha_{1,1}\alpha_{2,1} &= (v_1(v_2^2 - v_3^2), v_2(v_3^2 - v_1^2), v_3(v_1^2 - v_2^2))_{\mathcal{B}_2} \\
\beta_{4,2} := \alpha_{3,1}\alpha_{1,1} &= (v_2v_3(v_2^2 - v_3^2), v_1v_3(v_3^2 - v_1^2), v_1v_2(v_1^2 - v_2^2))_{\mathcal{B}_2}
\end{aligned}
$$

$$\beta_{5,2} := \alpha_{2,1}\alpha_{3,1} \quad = \quad (v_1(v_2^4 - v_3^4), v_2(v_3^4 - v_1^4), v_3(v_1^4 - v_2^4))_{\mathcal{B}_2}.$$

Wedge them together three at a time to get an invariant 3-form of polynomial degree 6:

$$f_{6,3} := \alpha_{1,1}\alpha_{2,1}\alpha_{3,1} = (v_1^4(v_2^2 - v_3^2) + v_2^4(v_3^2 - v_1^2) + v_3^4(v_1^2 - v_2^2))_{\mathcal{B}_3}.$$

(Up to a scalar, this is the Jacobian determinant from earlier.) The indices in the above notation indicate the polynomial and cohomological degrees of the elements. For example, $\beta_{4,2}$ has polynomial degree 4 and cohomological degree 2.

So far, we have only half of the required generators. But, since there is an Alt_4-module isomorphism $V \cong V^* \cong \bigwedge^2 V^*$, we are able to use the polynomials from above and simply change the bases to obtain the remaining generators. For $i = 1, 2, 3$, define the 2-form $\alpha_{i,2}$ by switching the basis on $\alpha_{i,1}$ from $\mathcal{B}_1$ to $\mathcal{B}_2$. So, for example, $\alpha_{1,2} = (v_1, v_2, v_3)_{\mathcal{B}_2}$. Similarly, for $i = 3, 4, 5$, define the 1-form $\beta_{i,1}$ by switching the basis on $\beta_{i,2}$ from $\mathcal{B}_2$ to $\mathcal{B}_1$. Finally, identify $(S(V) \otimes \bigwedge^0 V^*)$ with $S(V)$, and let $f_6 = f_{6,0}$ be the polynomial part of $f_{6,3}$.

We claim all of the elements $\alpha_{i,k}$ and $\beta_{i,k}$ are Alt_4-invariant. To illustrate, consider $\alpha_{2,1}$ and $\alpha_{2,2}$. The set $\mathcal{P} = \{v_2v_3, v_3v_1, v_1v_2\}$ of polynomial parts is a basis for an Alt_4-stable subspace of $S(V)$. It is easily verified that

$$[g]_{\mathcal{P}} = [g]_{\mathcal{B}_1} = [g]_{\mathcal{B}_2} = \overline{[g]}_{\mathcal{B}_1} = \overline{[g]}_{\mathcal{B}_2},$$

so, by the second key fact, both $\alpha_{2,1}$ and $\alpha_{2,2}$ are Alt_4-invariant. Similarly, one can show that all of the forms $\alpha_{i,k}$ and $\beta_{i,k}$ are Alt_4-invariant. Note that f_6 is an Alt_4-invariant polynomial.

5.3.1.1. Claim. *Let* Alt_4 *act on* $V \cong \mathbb{C}^3$ *by the rotation representation. Let* $R = \mathbb{C}[f_2, f_3, f_4]$*. Then the identity component of* $\mathrm{HH}^\bullet(S(V)\# \mathrm{Alt}_4)$ *is given by*

$$\begin{aligned}
\mathrm{HH}^0(1) &\cong R \oplus Rf_6 \\
\mathrm{HH}^1(1) &\cong R\alpha_{1,1} \oplus R\alpha_{2,1} \oplus R\alpha_{3,1} \oplus R\beta_{3,1} \oplus R\beta_{4,1} \oplus R\alpha_{5,1} \\
\mathrm{HH}^2(1) &\cong R\alpha_{1,2} \oplus R\alpha_{2,2} \oplus R\alpha_{3,2} \oplus R\beta_{3,2} \oplus R\beta_{4,2} \oplus R\alpha_{5,2} \\
\mathrm{HH}^3(1) &\cong Rv_1^* \wedge v_2^* \wedge v_3^* \oplus Rf_{6,3}.
\end{aligned}$$

In each cohomological degree k, we have already verified that the proposed secondary generators are Alt_4-invariant, so the R-span of the generators on the right-hand side is a subset of $\mathrm{HH}^k(1)$. By showing the sums on the right are *direct*, i.e. the generators are R-linearly independent, we will have the same Poincaré series for each side, and since one side is already a subset of the other, we will have equality.

The next step is to verify that in each cohomological degree the proposed secondary generators are $\mathbb{C}[f_2, f_3, f_4]$-linearly independent. The trick for doing this is to extend the action of the alternating group Alt_4 on V to an action of the symmetric group Sym_4 on V. Recalling that the basis vectors v_i were initially defined in terms of basis vectors e_i naturally permuted by the symmetric group, we can generate Sym_4 with the additional matrix

$$(1\ 2) \mapsto \begin{pmatrix} 0 & -1 & \\ -1 & 0 & \\ & & 1 \end{pmatrix}.$$

Let $\sigma = (1\ 2)$. Note that the polynomials f_2, f_3, f_4 are invariant under σ (and hence Sym_4), but f_6 is not: $\vec{\sigma}(f_6) = -f_6$. We first show 1 and f_6 are $\mathbb{C}[f_2, f_3, f_4]$-linearly independent. Suppose $p + qf_6 = 0$ for some polynomials p, q in $\mathbb{C}[f_2, f_3, f_4]$. Applying σ to both sides of the equation yields $p - qf_6 = 0$. Now $qf_6 = -qf_6$ forcing $q = 0$ and, in turn, $p = 0$. Thus 1 and f_6 are $\mathbb{C}[f_2, f_3, f_4]$-linearly independent, and $R + Rf_6 \subseteq \mathrm{HH}^0(1)$ is a *direct* sum with the same Poincaré series as $\mathrm{HH}^0(1)$, so $\mathrm{HH}^0(1) \cong R \oplus Rf_6$.

The same idea shows that the $\alpha_{i,1}$'s and $\beta_{i,1}$'s are R-linearly independent. First note that the $\alpha_{i,1}$'s are invariant under σ, but the $\beta_{i,1}$'s are only semi-invariant:

$$\vec{\sigma}(\alpha_{i,1}) = \alpha_{i,1} \qquad \text{and} \qquad \vec{\sigma}(\beta_{i,1}) = -\beta_{i,1}.$$

If

$$p_1\alpha_{1,1} + p_2\alpha_{2,1} + p_3\alpha_{3,1} + q_3\beta_{3,1} + q_4\beta_{4,1} + q_5\beta_{5,1} = 0$$

for some invariant polynomials p_i and q_i in $\mathbb{C}[f_2, f_3, f_4]$, then applying σ to the equation gives us also that

$$p_1\alpha_{1,1} + p_2\alpha_{2,1} + p_3\alpha_{3,1} - q_3\beta_{3,1} - q_4\beta_{4,1} - q_5\beta_{5,1} = 0.$$

Then $q_3\beta_{3,1} + q_4\beta_{4,1} + q_5\beta_{5,1} = 0$, and in turn $p_1\alpha_{1,1} + p_2\alpha_{2,1} + p_3\alpha_{3,1} = 0$. Expanding out what this means gives us two matrix equations:

$$\begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix} \begin{pmatrix} v_1 & v_2 & v_3 \\ v_2v_3 & v_3v_1 & v_1v_2 \\ v_1^3 & v_2^3 & v_3^3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} q_3 & q_4 & q_5 \end{pmatrix} \begin{pmatrix} v_1(v_2^2 - v_3^2) & v_2(v_3^2 - v_1^2) & v_3(v_1^2 - v_2^2) \\ v_2v_3(v_2^2 - v_3^2) & v_1v_3(v_3^2 - v_1^2) & v_1v_2(v_1^2 - v_2^2) \\ v_1(v_2^4 - v_3^4) & v_2(v_3^4 - v_1^4) & v_3(v_1^4 - v_2^4) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}.$$

But the determinants of the 3×3 matrices are nonzero polynomials, so we must have $p_1 = p_2 = p_3 = 0$ and $q_3 = q_4 = q_5 = 0$. Then

$$R\alpha_{1,1} + R\alpha_{2,1} + R\alpha_{3,1} + R\beta_{3,1} + R\beta_{4,1} + R\beta_{5,1} \subseteq \mathrm{HH}^1(1)$$

is actually a *direct* sum with the same Poincaré series as $\mathrm{HH}^1(1)$, so

$$\mathrm{HH}^1(1) \cong R\alpha_{1,1} \oplus R\alpha_{2,1} \oplus R\alpha_{3,1} \oplus R\beta_{3,1} \oplus R\beta_{4,1} \oplus R\alpha_{5,1}.$$

Analogous arguments establish Claim 5.3.1.1 for cohomological degrees two and three. Note that there is a role reversal in degree two: the $\beta_{i,2}$'s are invariant under σ, while the $\alpha_{i,2}$'s are only semi-invariant.

5.3.2. *Double-flip Component*

In this subsection, we compute the Hochschild cohomology component for the conjugacy class of double-flips. Let $g = (13)(24)$. The centralizer of g in Alt_4 is the copy of the Klein four group generated by the matrices

$$(13)(24) \mapsto \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \quad \text{and} \quad (14)(23) \mapsto \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}.$$

By Observation 3.4.1, $\mathrm{HH}^k((13)(24)) = 0$ for $k \neq 2, 3$. The two-form $\mathrm{vol}_g^\perp = v_2^* \wedge v_3^*$ is a basis for $\bigwedge^2((V^g)^\perp)^*$. In the following table, we record the information needed to calculate the invariants describing the Hochschild cohomology contribution from the conjugacy class of g:

generators of $Z(g)$	(13)(24)	(14)(23)
action on $V^g = \mathbb{C}v_1$	1	−1
action on $\bigwedge^2((V^g)^\perp)^* = \mathbb{C}\,\mathrm{vol}_g^\perp$	1	−1

From this information, we find that

$$\begin{aligned} \mathrm{HH}^2((13)(24)) &\cong \left(S(V^g) \otimes \mathbb{C} \otimes \textstyle\bigwedge^2 \left((V^g)^\perp\right)^* \right)^{Z((13)(24))} \\ &= \bigoplus_{d \overset{2}{\equiv} 1} \left(S^d(V^g) \otimes \mathbb{C} \otimes \textstyle\bigwedge^2 \left((V^g)^\perp\right)^* \right). \\ &\cong v_1\, \mathbb{C}[v_1^2] \otimes \mathbb{C} \otimes \mathbb{C}\,\mathrm{vol}_g^\perp \end{aligned}$$

and

$$\begin{aligned} \mathrm{HH}^3((13)(24)) &\cong \left(S(V^g) \otimes (V^g)^* \otimes \textstyle\bigwedge^2 \left((V^g)^\perp\right)^* \right)^{Z((13)(24))} \\ &= \bigoplus_{d \overset{2}{\equiv} 0} \left(S^d(V^g) \otimes (V^g)^* \otimes \textstyle\bigwedge^2 \left((V^g)^\perp\right)^* \right). \\ &\cong \mathbb{C}[v_1^2] \otimes \mathbb{C}v_1^* \otimes \mathbb{C}\,\mathrm{vol}_g^\perp \end{aligned}$$

Thus, $\mathrm{HH}^2(g)$ is the $\mathbb{C}[v_1^2]$-span of $v_1 \otimes 1 \otimes \mathrm{vol}_g^\perp$, and $\mathrm{HH}^3(g)$ is the $\mathbb{C}[v_1^2]$-span of $1 \otimes v_1^* \otimes \mathrm{vol}_g^\perp$.

5.3.3. *Three-cycle Components*

In this subsection, we compute the Hochschild cohomology components for the conjugacy classes of $g = (243)$ and $g^{-1} = (342)$. The choice of representative does not matter; we simply chose these because of the simplicity of the basis for V^g:

$$V^g = \mathrm{Span}_{\mathbb{C}}\{v_1 + v_2 + v_3\} \quad \text{and} \quad (V^g)^\perp = \mathrm{Span}_{\mathbb{C}}\{v_1 + \omega v_2 + \omega^2 v_3,\ v_1 + \omega^2 v_2 + \omega v_3\},$$

where $\omega = e^{2\pi i/3}$. The two-form $\mathrm{vol}_g^\perp = v_1^* \wedge v_2^* + v_2^* \wedge v_3^* + v_3^* \wedge v_1^*$ is a basis for the one dimensional space $\bigwedge^2((V^g)^\perp)^*$.

Because g has determinant one and generates its own centralizer, we have

$$\begin{aligned} \mathrm{HH}^k(g) &\cong \left(S(V^g) \otimes \bigwedge^{k-\mathrm{codim}(g)} (V^g)^* \otimes \bigwedge^{\mathrm{codim}(g)} ((V^g)^\perp)^*\right)^{Z(g)} \\ &= S(V^g) \otimes \bigwedge^{k-\mathrm{codim}(g)} (V^g)^* \otimes \bigwedge^{\mathrm{codim}(g)} ((V^g)^\perp)^*. \end{aligned}$$

By Observation 3.4.1, $\mathrm{HH}^k(g) = 0$ for $k \neq 2, 3$. For $k = 2, 3$, we describe $\mathrm{HH}^k(g)$ with respect to an explicit basis. $\mathrm{HH}^2(g)$ is isomorphic to the $\mathbb{C}[v_1+v_2+v_3]$-span of $1 \otimes 1 \otimes \mathrm{vol}_g^\perp$, and $\mathrm{HH}^3(g)$ is isomorphic to the $\mathbb{C}[v_1 + v_2 + v_3]$-span of $1 \otimes (v_1^* + v_2^* + v_3^*) \otimes \mathrm{vol}_g^\perp$.

5.4. Cohomology for Tetrahedral Groups Acting by the Rotation Representation

We now apply the techniques of Section 5.1 and, whenever possible, use components from the cohomology of $S(V)\#\,\mathrm{Alt}_4$ (see Section 5.3) to determine Hochschild cohomology for any tetrahedral group $G_4 - G_7$ acting on $V \cong \mathbb{C}^3$ by the rotation representation defined in Section 5.2.

We first use the software GAP [19] to record the centralizer sizes necessary to apply Propositions 5.1.1 and 5.1.3. Let Z denote the center of G, and let $\pi : G \to G/Z \cong \mathrm{Alt}_4$ be the canonical quotient homomorphism.

Centralizer sizes for tetrahedral groups $G_4 - G_7$				
conjugacy class of $\pi(g)$	1	(13)(24)	(243)	(342)
size of $Z_G(g)$	$12\|Z\|$	$2\|Z\|$	$3\|Z\|$	$3\|Z\|$
size of $\pi(Z_G(g))$	12	2	3	3
size of $Z_{G/Z}(\pi(g))$	12	4	3	3

The sizes of the centers of the groups G_4, G_5, G_6, and G_7 are 2, 6, 4, and 12, respectively.

5.4.1. Proposition. *Let G be any of the tetrahedral groups $G_4 - G_7$ acting on $V \cong \mathbb{C}^3$ by the rotation representation, and let Z denote the center of G. Then*

$$\begin{aligned}\mathrm{HH}^\bullet(S(V)\#G) &\cong \mathrm{HH}^\bullet(g_1)^{\oplus|Z|} \oplus \mathrm{HH}^\bullet(g_2)^{\oplus|Z|/2} \oplus \mathrm{HH}^\bullet(g_3)^{\oplus|Z|} \oplus \mathrm{HH}^\bullet(g_3^{-1})^{\oplus|Z|} \\ &\cong \mathrm{HH}^\bullet(1)^{\oplus|Z|} \oplus \mathrm{HH}^\bullet(g_2)^{\oplus|Z|/2} \oplus \mathrm{HH}^\bullet(243)^{\oplus|Z|} \oplus \mathrm{HH}^\bullet(342)^{\oplus|Z|},\end{aligned}$$

where $g_1 = 1$, and g_2 and g_3 in G are chosen so that $\pi(g_2) = (13)(24)$ and $\pi(g_3) = (243)$. Explicit descriptions of the components are indicated in the proof.

Proof. Note that the kernel of the rotation representation for G is the center of the group. Thus Proposition 5.1.3 applies, and we read the multiplicities $m_g = \dfrac{|Z_G(g)|}{|Z_{G/Z}(gZ)|}$ off of the table of centralizer sizes. Secondly, when $|\pi(Z_G(g))| = |Z_{\mathrm{Alt}_4}(\pi(g))|$, then we may apply Proposition 5.1.1 and use components from the cohomology of $S(V)\#\,\mathrm{Alt}_4$.

The component $\mathrm{HH}^\bullet(1)$ is given in Subsection 5.3.1, and the components $\mathrm{HH}^\bullet(243)$ and $\mathrm{HH}^\bullet(342)$ are given in Subsection 5.3.3. Since g_2 has determinant one in the rotation representation and generates its own centralizer in G, we have

$$\begin{aligned}\mathrm{HH}^k(g_2) &\cong \left(S(V^{g_2}) \otimes \bigwedge^{k-\mathrm{codim}(g_2)} (V^{g_2})^* \otimes \bigwedge^{\mathrm{codim}(g_2)} ((V^{g_2})^\perp)^*\right)^{Z_G(g_2)} \\ &= S(V^{g_2}) \otimes \bigwedge^{k-\mathrm{codim}(g_2)} (V^{g_2})^* \otimes \bigwedge^{\mathrm{codim}(g_2)} ((V^{g_2})^\perp)^*.\end{aligned}$$

If we choose the conjugacy class representative g_2 in G such that $\pi(g_2) = (13)(24)$, then, using the matrices from Section 5.3.1, $V^{g_2} = \mathbb{C}v_1$. By Observation 3.4.1, $\mathrm{HH}^k(g_2) = 0$ for $k \neq 2, 3$. For degrees two and three, we have that $\mathrm{HH}^2(g_2)$ is the $\mathbb{C}[v_1]$-span of $1 \otimes 1 \otimes \mathrm{vol}_{g_2}^\perp$, and $\mathrm{HH}^3(g_2)$ is the $\mathbb{C}[v_1]$-span of $1 \otimes v_1^* \otimes \mathrm{vol}_{g_2}^\perp$, where $\mathrm{vol}_{g_2}^\perp = v_2^* \wedge v_3^*$. □

Shepler and Witherspoon have shown in [23, Theorem 8.7] that the parameter space of graded Hecke algebras for a skew group algebra is isomorphic (as a $\mathbb{C}$-vector space) to the polynomial degree zero part of Hochschild 2-cohomology. We apply the theorem to the rotation representation of the tetrahedral groups:

5.4.2. Corollary. *Let G be any of the tetrahedral groups $G_4 - G_7$ acting on $V \cong \mathbb{C}^3$ by the rotation representation, and let Z denote the center of G. The parameter space of graded Hecke algebras for $S(V)\#G$ is $\frac{5|Z|}{2}$-dimensional (as a vector space over $\mathbb{C}$).*

5.4.3. Remark. While the parameter space of graded Hecke algebras for $S(V)\#\,\mathrm{Alt}_4$, where Alt_4 acts on $V \cong \mathbb{C}^3$ by the rotation representation, is only two-dimensional, there are four dimensions (as a $\mathbb{C}$-vector space) worth of cohomology in polynomial degree one, cohomological degree two. It would be interesting to investigate these in the context of the Drinfeld orbifold algebras discussed in Shepler-Witherspoon [22].

CHAPTER 6

REGULAR REPRESENTATION

In this chapter, we classify and give an explicit description of all graded Hecke algebras obtained as deformations of the skew group algebra $S(V)\#G$ where the finite group G acts on $V \cong \mathbb{C}^{|G|}$ by the regular representation. With respect to the basis $\{v_g : g \in G\}$ of $\mathbb{C}^{|G|}$, the left regular action of a group element g on a vector v_x is denoted by $\vec{g}(v_x) = v_{gx}$.

Shepler and Witherspoon showed in [23, Theorem 8.7] that the parameter space of graded Hecke algebras for a skew group algebra is isomorphic (as a vector space over $\mathbb{C}$) to the polynomial degree zero part of Hochschild 2-cohomology, which we denote by $\mathrm{HH}^2_0(S(V)\#G)$. Recall from Section 3.1 that the g-component of Hochschild cohomology of a skew group algebra $S(V)\#G$ is the space of centralizer invariants

$$\mathrm{HH}^{\bullet}(g) \cong \left(S(V^g) \otimes \bigwedge^{\bullet-\mathrm{codim}(g)} (V^g)^* \otimes \bigwedge^{\mathrm{codim}(g)} ((V^g)^{\perp})^* \right)^{Z(g)} .$$

The formula simplifies considerably when restricting to polynomial degree zero and cohomological degree two. For convenience, we record the simplifications in an observation, omitting any one-dimensional tensor factors on which the centralizer is guaranteed to act trivially.

6.0.4. Observation. *Let G be a finite group acting linearly on a vector space $V \cong \mathbb{C}^n$. Then the g-component of $\mathrm{HH}^2_0(S(V)\#G)$ is*

$$\mathrm{HH}^2_0(g) \cong \begin{cases} \left(\bigwedge^2 V^*\right)^{Z(g)} & \text{if } \mathrm{codim}(g) = 0 \\ \left(\bigwedge^2((V^g)^{\perp})^*\right)^{Z(g)} & \text{if } \mathrm{codim}(g) = 2 \\ 0 & \text{otherwise.} \end{cases}$$

In the formula above, the cases $\mathrm{codim}(g) \neq 0, 2$ follow from Observation 3.4.1.

6.1. G-invariant Two-forms

Consider the identity component in polynomial degree zero and cohomological degree two:

$$\mathrm{HH}_0^2(1) \cong \left(\bigwedge^2 V^*\right)^G.$$

We first use inner products of characters to compute the vector space dimension of $\mathrm{HH}_0^2(1)$. Let χ_{reg} be the character for the action of G on V (and also V^* since the regular representation is self-dual); let χ_2^{Alt} be the character for the action of G on $\bigwedge^2 V^*$; and let ι be the trivial character. Then, by character theory (see [20], for example), the dimension of $\mathrm{HH}_0^2(1)$ is

$$\begin{aligned}
\dim_{\mathbb{C}} \left(\bigwedge^2 V^*\right)^G &= \left\langle \chi_2^{Alt}, \iota \right\rangle \\
&= \frac{1}{|G|} \sum_{g \in G} \chi_2^{Alt}(g)\overline{\iota(g)} \\
&= \frac{1}{|G|} \sum_{g \in G} \chi_2^{Alt}(g) \\
&= \frac{1}{|G|} \sum_{g \in G} \frac{\chi_{reg}^2(g) - \chi_{reg}(g^2)}{2}.
\end{aligned}$$

Recalling that $\chi_{reg}(1) = |G|$ and $\chi_{reg}(g) = 0$ for $g \neq 1$, the summands are zero except when $g^2 = 1$. Thus

$$\begin{aligned}
\dim_{\mathbb{C}} \left(\bigwedge^2 V^*\right)^G &= \frac{1}{|G|} \sum_{\substack{g \in G \\ g^2 = 1}} \frac{\chi_{reg}^2(g) - \chi_{reg}(g^2)}{2} \\
&= \frac{1}{2|G|} \left(\sum_{\substack{g \in G \\ g^2 = 1}} \chi_{reg}^2(g) - \sum_{\substack{g \in G \\ g^2 = 1}} \chi_{reg}(g^2) \right) \\
&= \frac{1}{2|G|} \left(|G|^2 - |G| \cdot \#\{g \in G : g^2 = 1\} \right) \\
&= \frac{|G| - \#\{g \in G : g^2 = 1\}}{2}.
\end{aligned}$$

6.1.1. Lemma. *Let G be a finite group acting on $V \cong \mathbb{C}^{|G|}$ by the regular representation. Then the vector space dimension of the polynomial degree zero part of the identity component of* $\mathrm{HH}^2(S(V)\#G)$ *is*

$$\dim_{\mathbb{C}} \mathrm{HH}^2_0(1) = \frac{|G| - \#\{g \in G : g^2 = 1\}}{2}.$$

The next task is to explicitly describe a basis for $\left(\bigwedge^2 V^*\right)^G$. A standard technique for producing G-invariants is averaging over the group.

6.1.2. Lemma. *Let G act on $V \cong \mathbb{C}^{|G|}$ by the left regular representation. For each g in G, define a G-invariant two-form α_g in $\left(\bigwedge^2 V^*\right)^G$ by*

$$\alpha_g = \sum_{x \in G} \vec{x}(v_1^* \wedge v_g^*) = \sum_{x \in G} v_x^* \wedge v_{xg}^*.$$

Partition $G \backslash \{g : g^2 = 1\}$ into two-element sets $\{g, g^{-1}\}$, and let B be a subset of G containing exactly one representative from each block of the partition. Then

(i) $\alpha_{g^{-1}} = -\alpha_g$,

(ii) $\alpha_g = 0$ *if and only if* $g^2 = 1$, *and*

(iii) *the set* $\{\alpha_g : g \in B\}$ *is a basis of* $\left(\bigwedge^2 V^*\right)^G$.

Proof. (i) Note that for fixed g in G, $\{x : x \in G\} = \{xg^{-1} : x \in G\}$, so

$$\begin{aligned}
\alpha_g &= \sum_{x \in G} \vec{x}(v_1^* \wedge v_g^*) \\
&= \sum_{x \in G} \overrightarrow{xg^{-1}}(v_1^* \wedge v_g^*) \\
&= \sum_{x \in G} \vec{x}(v_{g^{-1}}^* \wedge v_1^*) \\
&= \sum_{x \in G} \vec{x}(-v_1^* \wedge v_{g^{-1}}^*) \\
&= -\alpha_{g^{-1}}.
\end{aligned}$$

(ii) If $\alpha_g = 0$, then there exist group elements x and y such that $yg = x$ and $y = xg$. But then $y = xg = (yg)g$, and $g^2 = 1$. Conversely, if $g^2 = 1$, then $g = g^{-1}$ and $\alpha_g = 0$ by part (i).

(iii) By (ii), $\alpha_g \neq 0$ for all g in B. To show $\{\alpha_g : g \in B\}$ is a linearly independent set, it suffices to show that for distinct group elements g and h in B, the sets $\{\pm v_x^* \wedge v_{xg}^* : x \in G\}$ and $\{\pm v_y^* \wedge v_{yh}^* : y \in G\}$ are disjoint.

Suppose the intersection $\{\pm v_x^* \wedge v_{xg}^* : x \in G\} \cap \{\pm v_y^* \wedge v_{yh}^* : y \in G\}$ is nonempty. Then there exist elements x and y such that $x = y$ and $xg = yh$, or there exist elements x and y such that $x = yh$ and $xg = y$. The first case leads to $g = h$, while the second case leads to $g = h^{-1}$, both contradictions if g and h are distinct elements of B. Now, since the size of the linearly independent set $\{\alpha_g : g \in B\}$ equals $\dim_{\mathbb{C}}(\bigwedge^2 V^*)^G$, we have our basis. □

6.2. Polynomial Degree Zero Hochschild 2-cohomology

In this section, we compute the components $\mathrm{HH}_0^2(g)$ for the nonidentity group elements g. Combined with the previous section, this yields a complete description of $\mathrm{HH}_0^2(S(V)\#G)$ and allows us to apply Theorem 8.7 of Shepler and Witherspoon [23] to determine the dimension of the parameter space of graded Hecke algebras for $S(V)\#G$ when G acts by the regular representation.

For most groups, $\mathrm{HH}_0^2(g)$ is zero for the nonidentity elements:

6.2.1. Proposition. *Let G act on $V \cong \mathbb{C}^{|G|}$ by the regular representation. If $|G| > 4$, then*

$$\mathrm{HH}_0^2(S(V)\#G) \cong \mathrm{HH}_0^2(1),$$

and the parameter space of graded Hecke algebras for $S(V)\#G$ has dimension

$$\frac{|G| - \{g \in G : g^2 = 1\}}{2}$$

as a vector space over $\mathbb{C}$. A basis of $\mathrm{HH}_0^2(1)$ is given in Lemma 6.1.2.

Proof. We show that $\mathrm{codim}(g) > 2$ for all nonidentity group elements g. Then, by Observation 3.4.1, $\mathrm{HH}^2(g) = 0$ for all $g \neq 1$.

Suppose $g \neq 1$, and let $|g|$ denote the order g. Then

$$\mathrm{codim}(g) = \dim V - \dim V^g = |G| - \frac{|G|}{|g|} = |G|\left(1 - \frac{1}{|g|}\right) \geq \frac{|G|}{2} > 2,$$

where we have used $|g| > 1$ and $|G| > 4$ to conclude the inequalities. □

There are four remaining groups to consider: the Klein four group and the cyclic groups of orders two, three, and four. We consider the cyclic group of order two and the Klein four group as special cases of an arbitrary finite direct product of $\mathbb{Z}/2\mathbb{Z}$.

6.2.2. PROPOSITION. *Let* $G \cong \underbrace{\mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z}}_{m}$ *act on* $V \cong \mathbb{C}^{2m}$ *by the regular representation. Then*

$$\mathrm{HH}_0^2(S(V)\#G) = 0,$$

and $S(V)\#G$ *has no nontrivial graded Hecke algebras.*

PROOF. Since every element of G squares to the identity, we have $\mathrm{HH}_0^2(1) = 0$. To show the nonidentity components are also zero, we consider the three cases $m = 1$, $m = 2$, and $m \geq 3$ separately.

If $m = 1$, then write $G = \{1, a\}$. The nonidentity element a has determinant -1 in the regular representation, so $\mathrm{HH}_0^2(a)$ is zero by Observation 3.4.1.

If $m = 2$, then write $G = \{1, a, b, ab\}$. Each element is in its own conjugacy class, so there are three nonidentity components to consider. We show details for the a-component, and the others are computed similarly. Note that

$$V^a = \mathrm{Span}_{\mathbb{C}}\{v_1 + v_a, v_b + v_{ab}\} \qquad \text{and} \qquad (V^a)^\perp = \mathrm{Span}_{\mathbb{C}}\{v_1 - v_a, v_b - v_{ab}\}.$$

Thus, $\mathrm{codim}(a) = 2$, and, by Observation 6.0.4,

$$\mathrm{HH}_0^2(a) \cong \left(\bigwedge^2((V^a)^\perp)^*\right)^{Z(a)}.$$

The two-form $\mathrm{vol}_a^\perp = (v_1^* - v_a^*) \wedge (v_b^* - v_{ab}^*)$ is a basis for $\bigwedge^2((V^a)^\perp)^*$. However, b centralizes a and does not fix $\mathrm{vol}_a^\perp$, so we have $\mathrm{HH}_0^2(a) = 0$. Similarly, one can argue $\mathrm{HH}_0^2(b) = 0$ and $\mathrm{HH}_0^2(ab) = 0$.

If $m \geq 3$, then $|G| > 4$ and, by Proposition 6.2.1, $\mathrm{HH}_0^2(g) = 0$ for all $g \neq 1$.

We have now shown $\mathrm{HH}_0^2(S(V)\#G) = 0$, so there are no nontrivial graded Hecke algebras for $S(V)\#G$.

$\square$

The cyclic groups of orders three and four have nonzero contribution from some nonidentity elements.

6.2.3. Proposition. *Let* $G \cong \mathbb{Z}/3\mathbb{Z}$ *act on* $V \cong \mathbb{C}^3$ *by the regular representation. Write* $G = \{1, g, g^2\}$. *Then*

$$\mathrm{HH}^2_0(S(V)\#G) \cong \mathrm{HH}^2_0(1) \oplus \mathrm{HH}^2_0(g) \oplus \mathrm{HH}^2_0(g^2),$$

where $\mathrm{HH}^2_0(1) \cong \mathrm{HH}^2_0(g) \cong \mathrm{HH}^2_0(g^2) \cong \mathbb{C}\alpha_g$ *with* $\alpha_g = v_1^* \wedge v_g^* + v_g^* \wedge v_{g^2}^* + v_{g^2}^* \wedge v_1^*$. *The parameter space of graded Hecke algebras for* $S(V)\#G$ *is 3-dimensional.*

Proof. Since each element of G is in its own conjugacy class, there are two nonidentity components to compute. However, since g and g^{-1} share the same fixed point space and centralizer, $\mathrm{HH}^\bullet(g) \cong \mathrm{HH}^\bullet(g^{-1})$. Note that

$$V^g = \mathrm{Span}_{\mathbb{C}}\{v_1 + v_g + v_{g^2}\} \quad \text{and} \quad (V^g)^\perp = \mathrm{Span}_{\mathbb{C}}\{v_1 + \omega v_g + \omega^2 v_{g^2}, v_1 + \omega^2 v_g + \omega v_{g^2}\},$$

where $\omega = e^{2\pi i/3}$. Thus, $\mathrm{codim}(g) = 2$, and, by Observation 6.0.4,

$$\mathrm{HH}^2_0(g) \cong \left(\bigwedge^2((V^g)^\perp)^*\right)^{Z(g)}.$$

The two-form $\alpha_g = v_1^* \wedge v_g^* + v_g^* \wedge v_{g^2}^* + v_{g^2}^* \wedge v_1^*$ is a basis of $\bigwedge^2((V^g)^\perp)^*$ and is invariant under $Z(g) = G$. Hence, $\mathrm{HH}^2_0(g) \cong \mathrm{HH}^2_0(g^{-1}) \cong \mathbb{C}\alpha_g$. By Lemma 6.1.2, α_g also spans $\mathrm{HH}^2_0(1)$. □

6.2.4. Proposition. *Let* $G \cong \mathbb{Z}/4\mathbb{Z}$ *act on* $V \cong \mathbb{C}^4$ *by the regular representation, and let* g *be a generator of* G. *Then*

$$\mathrm{HH}^2_0(S(V)\#G) \cong \mathrm{HH}^2_0(1) \oplus \mathrm{HH}^2_0(g^2),$$

where $\mathrm{HH}^2_0(1) \cong \mathrm{HH}^2_0(g^2) \cong \mathbb{C}\alpha_g$ *with* $\alpha_g = v_1^* \wedge v_g^* + v_g^* \wedge v_{g^2}^* + v_{g^2}^* \wedge v_{g^3}^* + v_{g^3}^* \wedge v_1^*$. *The parameter space of graded Hecke algebras for* $S(V)\#G$ *is 2-dimensional.*

Proof. Each element of G is in its own conjugacy class, so there are three nonidentity components to compute. Since g and g^{-1} are order four, their eigenvalues in the regular representation are $1, i, -1, -i$. Hence $\mathrm{codim}(g) = \mathrm{codim}(g^{-1}) = 3$, and $\mathrm{HH}^2_0(g) = \mathrm{HH}^2_0(g^{-1}) = 0$

by Observation 6.0.4. However, the eigenvalues of g^2 are $1, 1, -1, -1$, so $\operatorname{codim}(g) = 2$ and

$$\mathrm{HH}_0^2(g^2) \cong \left(\bigwedge^2((V^{g^2})^\perp)^*\right)^{Z(g^2)}.$$

Note that

$$V^{g^2} = \mathrm{Span}_{\mathbb{C}}\{v_1 + v_{g^2}, v_g + v_{g^3}\} \qquad \text{and} \qquad (V^{g^2})^\perp = \mathrm{Span}_{\mathbb{C}}\{v_1 - v_{g^2}, v_g - v_{g^3}\}.$$

The two-form $\alpha_g = v_1^* \wedge v_g^* + v_g^* \wedge v_{g^2}^* + v_{g^2}^* \wedge v_{g^3}^* + v_{g^3}^* \wedge v_1^*$ is a basis of $\bigwedge^2((V^{g^2})^\perp)^*$ and is invariant under $Z(g) = G$. Thus, $\mathrm{HH}_0^2(g^2) \cong \mathbb{C}\alpha_g$. By Lemma 6.1.2, α_g also spans $\mathrm{HH}_0^2(1)$. □

6.3. From Cohomology to Graded Hecke Algebras

In this section, we use Theorem 11.4 and Proposition 11.5 of Shepler and Witherspoon [25] to convert each element of $\mathrm{HH}_0^2(S(V)\#G)$ into a skew-symmetric bilinear form

$$\kappa : V \times V \to \mathbb{C}G$$

that defines a graded Hecke algebra

$$\mathcal{H}_\kappa = T(V)\#G/I_\kappa,$$

where I_κ is the two-sided ideal $I_\kappa = \langle v \otimes w - w \otimes v - \kappa(v, w) : v, w \in V\rangle$. The work of Shepler and Witherspoon shows that this conversion yields *all* graded Hecke algebras that are deformations of the skew group algebra $S(V)\#G$.

Before proceeding, we establish some notational conventions. First, note that for a given map $\kappa : V \times V \to \mathbb{C}G$, there exist unique maps $\kappa_g : V \times V \to \mathbb{C}$ such that κ decomposes as the sum

$$\kappa = \sum_{g\in G} \kappa_g g.$$

Secondly, recall that if W is any finite dimensional vector space, then the space $\bigwedge^2 W^*$ identifies with the space of skew-symmetric bilinear forms from $W \times W$ into $\mathbb{C}$. After choosing a basis of W, say $w_1, \ldots, w_n$, a bilinear form $\alpha : W \times W \to \mathbb{C}$ can be described by a matrix $[\alpha]$ with (i, j)-entry equal to $\alpha(w_i, w_j)$.

6.3.1. Lemma. *Let G be a finite group. Fix an ordering of the basis $\{v_g : g \in G\}$ of $V \cong \mathbb{C}^{|G|}$, and let $[g]_R$ denote the matrix of the group element g acting on V by the right regular representation: $\vec{g}(v_x) = v_{xg^{-1}}$. Then the matrix of the two-form $\alpha_g = \sum_{x \in G} v_x^* \wedge v_{xg}^*$ in $\bigwedge^2 V^*$ is*

$$[\alpha_g] = [g]_R - [g]_R^\intercal.$$

Proof. If $g^2 = 1$, then $[\alpha_g] = 0$ by part (i) of Lemma 6.1.2. On the other hand, $[g]_R$ is a permutation matrix and $g = g^{-1}$, so $[g]_R - [g]_R^\intercal = [g]_R - [g^{-1}]_R = 0$, as required.

If $g^2 \neq 1$, then

$$\alpha_g(v_x, v_y) = \begin{cases} 1 & \text{if } y = xg \\ -1 & \text{if } x = yg \\ 0 & \text{otherwise.} \end{cases}$$

The (v_x, v_y)-entry of $[g]_R$ is 1 if $v_{yg^{-1}} = v_x$ and 0 otherwise. Transposing, the (v_x, v_y)-entry of $[g]_R^\intercal$ is 1 if $v_{xg^{-1}} = v_y$ and 0 otherwise. It follows that the (v_x, v_y)-entry of $[g]_R - [g]_R^\intercal$ is precisely $\alpha_g(v_x, v_y)$. □

6.3.2. Example. Let the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ act on $V \cong \mathbb{C}^8$ by the left regular representation, and let $v_1, v_{-1}, v_i, v_{-i}, v_j, v_{-j}, v_k, v_{-k}$ be an ordered basis of V. In Proposition 6.2.1, we showed

$$\mathrm{HH}_0^2(S(V)\#Q_8) \cong \mathrm{HH}_0^2(1) \cong \mathrm{Span}_{\mathbb{C}}\{\alpha_i, \alpha_j, \alpha_k\},$$

so, by Proposition 11.5 of [25], the maps $\kappa : V \times V \to \mathbb{C}Q_8$ that define a graded Hecke algebra $\mathcal{H}_\kappa$ are precisely the maps

$$\kappa = a\alpha_i + b\alpha_j + c\alpha_k,$$

where a, b, c are any complex scalars. The matrix of κ is

$$[\kappa] = [\kappa_1] = \begin{pmatrix} 0 & 0 & a & -a & b & -b & c & -c \\ 0 & 0 & -a & a & -b & b & -c & c \\ -a & a & 0 & 0 & -c & c & b & -b \\ a & -a & 0 & 0 & c & -c & -b & b \\ -b & b & c & -c & 0 & 0 & -a & a \\ b & -b & -c & c & 0 & 0 & a & -a \\ -c & c & -b & b & a & -a & 0 & 0 \\ c & -c & b & -b & -a & a & 0 & 0 \end{pmatrix}.$$

The parameter space of graded Hecke algebras is 3-dimensional.

6.3.3. Example. Let the cyclic group $G = \{1, g, g^2\}$ act on $V \cong \mathbb{C}^3$ by the left regular representation, and let v_1, v_g, v_{g^2} be an ordered basis of $V \cong \mathbb{C}^3$. In Proposition 6.2.3, we showed that

$$\mathrm{HH}_0^2(1) \cong \mathrm{HH}_0^2(g) \cong \mathrm{HH}_0^2(g^2) \cong \mathbb{C}\alpha_g,$$

so, by Proposition 11.5 of [25], the maps $\kappa : V \times V \to \mathbb{C}G$ that define a graded Hecke algebra $\mathcal{H}_\kappa$ are precisely the maps

$$\kappa = a\alpha_g + b\alpha_g g + c\alpha_g g^2,$$

where a, b, c are any complex scalars. The matrix of α_g is

$$[\alpha_g] = [g]_R - [g]_R^\intercal = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix},$$

so each element $a + bg + cg^2$ in the group algebra determines a graded Hecke algebra with commutator relations

$$[v_1, v_g] = [v_g, v_{g^2}] = [v_{g^2}, v_1] = a + bg + cg^2.$$

The parameter space of graded Hecke algebras is 3-dimensional.

6.3.4. Example. Let the cyclic group $G = \{1, g, g^2, g^3\}$ act on $V \cong \mathbb{C}^4$ by the left regular representation, and let $v_1, v_g, v_{g^2}, v_{g^3}$ be an ordered basis of V. In Proposition 6.2.4, we showed

$$\mathrm{HH}^2_0(1) \cong \mathrm{HH}^2_0(g^2) \cong \mathbb{C}\alpha_g \text{ and } \mathrm{HH}^2_0(g) \cong \mathrm{HH}^2_0(g^3) = 0,$$

so, by Proposition 11.5 of [25], the maps $\kappa : V \times V \to \mathbb{C}G$ that define a graded Hecke algebra $\mathcal{H}_\kappa$ are precisely the maps

$$\kappa = a\alpha_g + b\alpha_g g^2,$$

where a and b are any complex scalars. The matrix of α_g is

$$[\alpha_g] = [g]_R - [g]_R^\intercal = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix},$$

so each element $a + bg^2$ in the group algebra determines a graded Hecke algebra with commutator relations

$$[v_1, v_g] = [v_g, v_{g^2}] = [v_{g^2}, v_{g^3}] = [v_{g^3}, v_1] = a + bg^2$$

and

$$[v_1, v_{g^2}] = [v_g, v_{g^3}] = 0.$$

The parameter space of graded Hecke algebras is 2-dimensional.

APPENDIX A

REFLECTION LENGTH AND CODIMENSION CODE

To compute reflection length and atoms in the codimension poset (see Chapter 2), we wrote the following code that runs using the software GAP [19]. The code requires the CHEVIE package [9] to load complex reflection groups and their character tables. Note, however, that our functions `LengthPartition` and `LengthClassFunction` do not require the group to be a reflection group and can be used to compute length with respect to any set of generators closed under conjugation.

```
###########################################################################
# lengthcodim.gap

###########################################################################
#                    WANT TO COMPUTE REFLECTION LENGTH?
# 1. Start a GAP3 session and load chevie
#    gap> LoadPackage("chevie");
# 2. Read in this file lengthcodim.gap from wherever you saved it, e.g.
#    gap> Read("mygapfolder/lengthcodim.gap");
# 3. Define your complex reflection group G4-G37, e.g.
#    gap> G:=ComplexReflectionGroup(4);
# 4. Get the character table for your group
#    gap> t:=CharTable(G);
# 5. If your group is a Coxeter group (23,28,30,35,36,37), then you
#    need to attach the group to the character table manually:
#    gap> t.group:=G;
# 6. Determine which character corresponds to the reflection rep
#    gap> refrep:=GetRefRep(t);
# 7. Determine which conjugacy classes contain reflections
#    gap> refclasses:=RefClasses(t,refrep);
# 8. Find absolute reflection length!
#    gap> LengthClassFunction(t,refclasses);
#
# NOTE: Steps 6-8 may be done all at once:
#    gap> L:=LengthClassFunction(t,RefClasses(t,GetRefRep(t)));
```

```
##########################################################################
#                         FUNCTIONS IN THIS FILE
# This file contains functions for comparing reflection length and
# codimension posets for complex reflection groups.
#-------------------------------------------------------------------------
# ClassNmbrToName:=function(charTable,classNmbr)
# ClassToNmbr:=function(charTable,class)
# ClassToName:=function(charTable,class)
# ConjClassMultTable:=function(charTable,classList1,classList2)
# IrrCodim:=function(charTable,charNmbr,classNmbr)
# CodimClassFunction:=function(charTable,charNmbr)
# CodimPartition:=function(charTable,charNmbr)
# IsMinimalInP:=function(charTable,charNmbr,classk,P)
# MinimalClassFunction:=function(charTable,charNmbr)
# LengthPartition:=function(charTable, genClassNums)
# LengthClassFunction:=function(charTable,genClassNums)
# RefClasses:=function(charTable,charNmbr)
# RefClassFunction:=function(charTable,charNmbr)
# GetRefRep:=function(charTable)
#-------------------------------------------------------------------------

RequirePackage("chevie");

##########################################################################
#                           CLASS NMBR TO NAME
# This function converts the CLASSNMBR of a conjugacy class (position in
# list of conjugacy classes attached to CHARTABLE into its name used in
# the display of CHARTABLE
#-------------------------------------------------------------------------
ClassNmbrToName:=function(charTable,classNmbr)

   ClassNamesCharTable(charTable);

   return charTable.classnames[classNmbr];

end;

##########################################################################
#                             CLASS TO NMBR
# This function returns the position of a conjugacy CLASS in the list of
# conjugacy classes attached to CHARTABLE
#-------------------------------------------------------------------------
ClassToNmbr:=function(charTable,class)

   return Position(ConjugacyClasses(charTable.group),class);
```

```
end;

##########################################################################
#                              CLASS TO NAME
# This function returns the class name for a conjugacy CLASS in the list
# of conjugacy classes attached to CHARTABLE
#-------------------------------------------------------------------------
ClassToName:=function(charTable,class)

   return ClassNmbrToName(charTable,ClassToNmbr(charTable,class));

end;

##########################################################################
#                  CONJUGACY CLASS MULTIPLICATION TABLE
# Input:  CHARTABLE - the character table of a group G
#         CLASSLIST1 - a list of positions of conjugacy classes
#                      in charTable.classes
#         CLASSLIST2 - a second list of positions of conjugacy classes in
#                      charTable.classes
#
# Output: this function prints out the multiplication table for
#         multiplying conjugacy classes in the first list with the
#         conjugacy classes in the second list
#-------------------------------------------------------------------------
ConjClassMultTable:=function(charTable,classList1,classList2)

   local c1, c2, result, k;

   for c1 in classList1 do

      for c2 in classList2 do

         # pick out the class numbers k having nonzero
         # ClassAlgConst(c1,c2,k)
         result:=Filtered([1..Length(charTable.irreducibles)],
                       k->ClassMultCoeffCharTable(charTable,c1,c2,k)>0);

         Print(ClassNmbrToName(charTable,c1),"*",
               ClassNmbrToName(charTable,c2),":",
               List(result,k->ClassNmbrToName(charTable,k)),"\n");

      od;

      Print("\n");
```

```
   od;

end;

###########################################################################
#                             IRR  CODIM
# Input:  CHARTABLE - the character table of a group G
#         CHARNMBR - position of an irreducible character in
#                    charTable.irreducibles
#         CLASSNMBR - position of a conjugacy class in charTable.classes
#
# Output:  codimension of subspace fixed by an elt in the conjugacy class
#          given by CLASSNMBR in representation with specified character
#
# WARNING:  THIS FUNCTION ASSUMES THE IDENTITY IS IN THE FIRST COLUMN
#           OF THE CHARTABLE
#--------------------------------------------------------------------------
IrrCodim:=function(charTable,charNmbr,classNmbr)

   local evals;

   evals:=Eigenvalues(charTable,
                      charTable.irreducibles[charNmbr],classNmbr);

   return Eigenvalues(charTable,charTable.irreducibles[charNmbr],1)[1]
            -evals[Length(evals)];

end;

###########################################################################
#                          CODIM CLASS FUNCTION
# Input:  CHARTABLE - character table for a group
#         CHARNMBR - position of an irreducible character in
#                    charTable.irreducibles
#
# Output:  a class function with g -> codim(V^g),
#          where V^g=subspace fixed by g
#--------------------------------------------------------------------------
CodimClassFunction:=function(charTable,charNmbr)

   local c;

   return ClassFunction(charTable,List([1..Length(charTable.irreducibles)],
            c->IrrCodim(charTable,charNmbr,c)));

end;
```

```
##########################################################################
#                              CODIM PARTITION
# Input: CHARTABLE - character table for a group
#        CHARNMBR - position of an irreducible character in
#                   charTable.irreducibles
#
# Output:  a list P with each entry a list of conjugacy class numbers
#          P[1] holds the class numbers of elts g with codim(V^g)=0
#          P[2] holds the class numbers of elts g with codim(V^g)=1
#           :
#          P[dimV+1] holds the class numbers of elts with codim(V^g)=dimV
#
# WARNING: THIS FUNCTION ASSUMES THE IDENTITY IS IN THE FIRST COLUMN
#          OF THE CHARTABLE
#
#          THE FUNCTIONS IsMinimalInP AND MinimalClassFunction DEPEND
#          ON THE OUTPUT OF THIS FUNCTION BEING A LIST AS DESCRIBED ABOVE
#-------------------------------------------------------------------------
CodimPartition:=function(charTable,charNmbr)

   local P, P_i, dimChar, classNmbr;

   # get the dimension of the representation
   dimChar:=Eigenvalues(charTable,charTable.irreducibles[charNmbr],1)[1];

   # make a list with dimV+1 slots (one for each possible codimension),
   # each slot holding an empty list []
   P:=List([1..(dimChar+1)],P_i->[]);

   # for each conjugacy class number, get codim(g) for elts g in
   # that class, and store the class number in the corresponding slot
   # (note: the numbering is offset by 1 since GAP labels slots in a list
   #        starting with 1 instead of 0)
   for classNmbr in [1..Length(charTable.irreducibles)] do

      Add( P[IrrCodim(charTable,charNmbr,classNmbr)+1], classNmbr);

   od;

   return P;

end;
```

```
##########################################################################
#                           IS MINIMAL IN P
# Input:  CHARTABLE - character table of a reflection group
#         CHARNMBR - position of a reflection representation in
#                    CHARTABLE.irreducibles
#         CLASSK - position of a conjugacy class in CHARTABLE.classes
#         P - partition of conjugacy classes according to codimension
#
# NOTE:  This function assumes that P is a list of length dimV+1 with
#        P[0+1]=list of positions of classes with codimension 0
#        P[1+1]=list of positions of classes with codimension 1
#          :
#        P[n+1]=list of positions of classes with codimension n=dimV
#-------------------------------------------------------------------------
IsMinimalInP:=function(charTable,charNmbr,classk,P)

   local d, smallercount, flag, i, classi, classj;

   d:=IrrCodim(charTable,charNmbr,classk);

   # if class consists of reflections, it is minimal
   if d=1 then

      return true;

   fi;

   # if class consists of elts acting as identity, it is not minimal
   if d=0 then

      return false;

   fi;

   # set up a counter to see if there are any classes below classk
   # in the poset
   smallercount:=0;

   # compute ClassAlgConst(classi,classj,classk) for nontrivial
   # classes classi and classj having codimensions adding to codim(classk)
   for i in [1..(d-1)] do

      for classi in P[i+1] do

         for classj in P[(d-i)+1] do
```

```
          smallercount:=smallercount
              +ClassMultCoeffCharTable(charTable,classi,classj,classk);

        od;

     od;

   od;

   if smallercount > 0 then

      return false;

   fi;

   return true;

end;

##########################################################################
#                          MINIMAL CLASS FUNCTION
# Input:  CHARTABLE - character table of a reflection group
#         CHARNMBR - position of a reflection representation in
#                    CHARTABLE.irreducibles
#
# Output: a class function with g->1 if g is minimal in the codim poset
#                                g->0 otherwise
#-------------------------------------------------------------------------
MinimalClassFunction:=function(charTable,charNmbr)

   local P, minClassList, i;

   P:=CodimPartition(charTable,charNmbr);
   minClassList:=List([1..Length(charTable.irreducibles)],i->0);

   for i in [1..Length(charTable.irreducibles)] do

      if IsMinimalInP(charTable,charNmbr,i,P) then

         minClassList[i]:=1;

      fi;

   od;
```

```
   return ClassFunction(charTable,minClassList);

end;

##########################################################################
#                             LENGTH PARTITION
# Input:  CHARTABLE - character table of a group G
#         GENCLASSNUMS - conjugacy class numbers for a set of generators
#                        of the group G
#                        (ASSUMING 1 is not in GENCLASSNUMS)
#
# Output: a partition of the conjugacy classes of group G according to
#         length with respect to the conjugacy classes given in
#         GENCLASSNUMS
#
#         Definition of length(g):
#         length(g)=min number of elts needed to express g as a product
#                   of elts from the conjugacy classes in GENCLASSNUMS
#         This length function is constant on conjugacy classes.
#
#         The partition is returned as a list.  Each element of the list
#         is itself a list:  the first entry is a length (possibly
#         "infinity"); the second entry is a list of class numbers of
#         elts with that length.
#-------------------------------------------------------------------------
LengthPartition:=function(charTable, genClassNums)

   local P, G, classes, remainingClasses, i, classi, classj, classk,
         previouslength, currentlength;

   G:=charTable.group;
   classes:=ConjugacyClasses(G);

   # P will be the partition:
   #     [0,[1]] says the identity has length zero
   #     [1, genClassNums] says the generating classes have length 1
   # IT IS ASSUMED THAT GENCLASSNUMS DOES NOT CONTAIN THE IDENTITY
   P:=[ [0,[1]], [1,genClassNums] ];

   # list all the class numbers [1,2,3,...]
   remainingClasses:=Set(List([1..Length(classes)],i->i));

   # remove 1 and all the genClassNums from remainingClasses
   #   (since their length is already known)
   # note: SubtractSet is destructive
   SubtractSet(remainingClasses,P[1][2]);
```

```
SubtractSet(remainingClasses,P[2][2]);

previouslength:=1;
currentlength:=2;

while Length(P[previouslength+1][2])>0 and
              Size(remainingClasses)>0         do

   # There were some classes with previouslength, so now we need
   # to look for classes of currentlength, which will be stored in
   # slot currentlength+1 of P

   Add(P,[currentlength,[]]);

   for classi in genClassNums do

      for classj in P[previouslength+1][2] do

         for classk in remainingClasses do

            if (not classk in P[currentlength+1][2]) and
               ClassMultCoeffCharTable(charTable,
                                       classi,classj,classk)>0 then

               # can multiply an element from generating class with
               # an element with previouslength and get an element
               # in classk, so classk has currentlength
               AddSet(P[currentlength+1][2],classk);

            fi;

         od;

      od;

   od;

   SubtractSet(remainingClasses,P[currentlength+1][2]);

   previouslength:=currentlength;
   currentlength:=currentlength+1;

od;

if Length(P[previouslength+1][2])=0 and Size(remainingClasses)>0 then
```

```
      P[previouslength+1][1]:="infinity";
      AddSet(P[previouslength+1][2],remainingClasses);

   fi;

   return P;

end;

###########################################################################
#                          LENGTH CLASS FUNCTION
# Input:  CHARTABLE - character table of a group
#         GENCLASSNUMS - positions of conjugacy classes whose elements
#                        generate the group (the identity should NOT be
#                        in this list)
#
# Output: a class function with g->length(g)
#         [length(g)=min number of elts needed to express g as a product
#                    of elts from the conjugacy classes in GENCLASSNUMS]
#         if the classes specified in GENCLASSNUMS do not generate the
#         group, an error message is printed and nothing is returned
#--------------------------------------------------------------------------
LengthClassFunction:=function(charTable,genClassNums)

   local G, classes, lengthlist, i, P, Pd, c;

   G:=charTable.group;
   classes:=ConjugacyClasses(G);

   # store lengths of elts in each conj class
   lengthlist:=List([1..Length(classes)],i->-1);

   P:=LengthPartition(charTable,genClassNums);

   # each Pd in the length partition P is a list [d,[c_1,...,c_r]]
   #   -the first slot gives a length d
   #   -the second slot is the list of positions of the conjugacy classes
   #    having length d
   # if a class is not generated, it will be in a slot ["infinity",[...]]
   for Pd in P do

      for c in Pd[2] do

         lengthlist[c]:=Pd[1];

      od;
```

```
   od;

   # < evaluates to true when objects of different types are compared
   if Maximum(lengthlist) < "infinity" then

      return ClassFunction(charTable,lengthlist);

   fi;

   Print("Some classes not generated.  Could not make class function.\n");

end;

###########################################################################
#                          REFLECTION CLASSES
# Input:   CHARTABLE - character table for a reflection group
#          CHARNMBR - position of a reflection representation in the
#                     the list CHARTABLE.irreducibles
#
# Output:  list of positions in CHARTABLE.classes corresponding to
#          conjugacy classes of reflections
#--------------------------------------------------------------------------
RefClasses:=function(charTable,charNmbr)

   return CodimPartition(charTable,charNmbr)[2];

end;

###########################################################################
#                          REFLECTION CLASS FUNCTION
# Input: CHARTABLE - character table of a reflection group
#        CHARNMBR - position of a reflection representation in
#                   CHARTABLE.irreducibles
#
# Output: a class function with g->1 if g is a reflection
#                               g->0 if g is not a reflection
#--------------------------------------------------------------------------
RefClassFunction:=function(charTable,charNmbr)

   local charfunctionrefs, refs, r, i;

   charfunctionrefs:=List([1..Length(charTable.irreducibles)],i->0);

   refs:=RefClasses(charTable,charNmbr);
```

```
   for r in refs do

      charfunctionrefs[r]:=1;

   od;

   return ClassFunction(charTable,charfunctionrefs);

end;

###########################################################################
#                              GET REF REP
# Input:  CHARTABLE - character table of a reflection group
#
# Output: position of a reflection representation in the list
#         CHARTABLE.irreducibles of irreducible characters of the
#         reflection group
#--------------------------------------------------------------------------
GetRefRep:=function(charTable)

   local classes, i, refrepCharValueList;

   classes:=ConjugacyClasses(charTable.group);

   refrepCharValueList:=List([1..Length(classes)],
       i->ReflectionCharValue(charTable.group,classes[i].representative));

   for i in [1..Length(charTable.irreducibles)] do

      if ScalarProduct(ClassFunction(charTable,charTable.irreducibles[i]),
                       ClassFunction(charTable,refrepCharValueList))=1 then

        return i;

      fi;

   od;

end;

###########################################################################
#EOF
```

APPENDIX B

POINCARÉ POLYNOMIALS FOR ABSOLUTE REFLECTION LENGTH AND CODIMENSION

Tables B.1 and B.2 display, for each exceptional complex reflection group $G_4 - G_{37}$, a two-variable polynomial whose $x^l y^d$ term indicates the number of elements in the group that have absolute reflection length l and codimension d. The polynomials for the Coxeter groups G_{23}, G_{28}, G_{30}, and $G_{35} - G_{37}$ are included for completeness, though their elegant factorization has long been known (see Shephard-Todd [21]). We computed the polynomials for the non-Coxeter groups using the code in Appendix A.

G_4	$1 + 8xy + (14x^2 + x^3)y^2$
G_5	$1 + 16xy + (52x^2 + 3x^3)y^2$
G_6	$1 + 14xy + (31x^2 + 2x^3)y^2$
G_7	$1 + 22xy + (85x^2 + 36x^3)y^2$
G_8	$1 + 18xy + (69x^2 + 8x^3)y^2$
G_9	$1 + 30xy + (145x^2 + 16x^3)y^2$
G_{10}	$1 + 34xy + (211x^2 + 42x^3)y^2$
G_{11}	$1 + 46xy + (327x^2 + 202x^3)y^2$
G_{12}	$1 + 12xy + (23x^2 + 12x^3)y^2$
G_{13}	$1 + 18xy + (47x^2 + 30x^3)y^2$
G_{14}	$1 + 28xy + (101x^2 + 14x^3)y^2$
G_{15}	$1 + 34xy + (157x^2 + 96x^3)y^2$
G_{16}	$1 + 48xy + (448x^2 + 102x^3 + x^4)y^2$
G_{17}	$1 + 78xy + (875x^2 + 246x^3)y^2$
G_{18}	$1 + 88xy + (1240x^2 + 471x^3)y^2$
G_{19}	$1 + 118xy + (1853x^2 + 1628x^3)y^2$
G_{20}	$1 + 40xy + (244x^2 + 75x^3)y^2$
G_{21}	$1 + 70xy + (505x^2 + 144x^3)y^2$
G_{22}	$1 + 30xy + (119x^2 + 90x^3)y^2$

TABLE B.1. Reflection Length and Codimension Polynomials for Groups $G_4 - G_{22}$

G_{23}	$(1+xy)(1+5xy)(1+9xy)$
G_{24}	$1+21xy+119x^2y^2+(147x^3+48x^4)y^3$
G_{25}	$1+24xy+(174x^2+9x^3)y^2+(368x^3+72x^4)y^3$
G_{26}	$1+33xy+(318x^2+9x^3)y^2+(812x^3+123x^4)y^3$
G_{27}	$1+45xy+519x^2y^2+(1033x^3+560x^4+2x^5)y^3$
G_{28}	$(1+xy)(1+5xy)(1+7xy)(1+11xy)$
G_{29}	$1+40xy+530x^2y^2+(2600x^3+120x^4)y^3+(3187x^4+1200x^5+2x^6)y^4$
G_{30}	$(1+xy)(1+11xy)(1+19xy)(1+29xy)$
G_{31}	$1+60xy+(1210x^2+60x^3)y^2+(9300x^3+1800x^4)y^3+(18747x^4+13620x^5+1282x^6)y^4$
G_{32}	$1+80xy+(2220x^2+90x^3)y^2+(24800x^3+3600x^4)y^3+(89082x^4+35646x^5+x^6)y^4$
G_{33}	$1+45xy+750x^2y^2+(5670x^3+80x^4)y^3+(18609x^4+1440x^5)y^4+(18685x^5+6480x^6+80x^7)y^5$
G_{34}	$1+126xy+6195x^2y^2+(149940x^3+1120x^4)y^3+(1834119x^4+70560x^5)y^4$
	$+(10121454x^5+1428000x^6+10080x^7)y^5+(15821171x^6+9243108x^7+504912x^8+252x^9+2x^{10})y^6$
G_{35}	$(1+xy)(1+4xy)(1+5xy)(1+7xy)(1+8xy)(1+11xy)$
G_{36}	$(1+xy)(1+5xy)(1+7xy)(1+9xy)(1+11xy)(1+13xy)(1+17xy)$
G_{37}	$(1+xy)(1+7xy)(1+11xy)(1+13xy)(1+17xy)(1+19xy)(1+23xy)(1+29xy)$

TABLE B.2. Reflection Length and Codimension Polynomials for Groups $G_{23}-G_{37}$

APPENDIX C

HOCHSCHILD COHOMOLOGY TABLES FOR GROUP G6

In this appendix, we summarize the Hochschild cohomology of the skew group algebra $S(V)\#G$, where V is any irreducible representation of the reflection group G_6 (as numbered in the Shephard-Todd classification). The character table as displayed by the software GAP [19] is included here in order to give context for the numbering of the characters in the cohomology tables. The cohomology tables are explained in Example 4.2.1.

```
##########################################################
# CHARACTER TABLE OF REFLECTION GROUP G6

gap> DisplayCharTable(t6);
G6
      2  4  3  4   2   2   2  2   2   2   2  2  3  4  4
      3  1  .  1   1   1   1  1   1   1   1  1  .  1  1

        1a 2a 2b  3a 12a 12b 6a  3b 12c 12d 6b 4a 4b 4c
     2P 1a 1a 1a  3b  6b  6b 3b  3a  6a  6a 3a 2b 2b 2b
     3P 1a 2a 2b  1a  4b  4c 2b  1a  4c  4b 2b 4a 4c 4b
     5P 1a 2a 2b  3b 12d 12c 6b  3a 12b 12a 6a 4a 4b 4c
     7P 1a 2a 2b  3a 12b 12a 6a  3b 12d 12c 6b 4a 4c 4b
    11P 1a 2a 2b  3b 12c 12d 6b  3a 12a 12b 6a 4a 4c 4b

X.1      1  1  1   1   1   1  1   1   1   1  1  1  1  1
X.2      1 -1  1   1  -1  -1  1   1  -1  -1  1  1 -1 -1
X.3      1 -1  1   A  -A  -A  A  /A -/A -/A /A  1 -1 -1
X.4      1 -1  1  /A -/A -/A /A   A  -A  -A  A  1 -1 -1
X.5      1  1  1   A   A   A  A  /A  /A  /A /A  1  1  1
X.6      1  1  1  /A  /A  /A /A   A   A   A  A  1  1  1
X.7      2  . -2  -A   B  -B  A -/A  /B -/B /A  .  D -D
X.8      2  . -2  -A  -B   B  A -/A -/B  /B /A  . -D  D
X.9      2  . -2  -1   C  -C  1  -1  -C   C  1  .  D -D
X.10     2  . -2  -1  -C   C  1  -1   C  -C  1  . -D  D
X.11     2  . -2 -/A -/B  /B /A  -A  -B   B  A  .  D -D
X.12     2  . -2 -/A  /B -/B /A  -A   B  -B  A  . -D  D
X.13     3 -1  3   .   .   .  .   .   .   .  . -1  3  3
X.14     3  1  3   .   .   .  .   .   .   .  . -1 -3 -3

A = E(3)^2 = (-1-ER(-3))/2 = -1-b3
B = E(12)^11
C = E(4) = ER(-1) = i
D = 2*E(4) = 2ER(-1) = 2i
```

```
###########################################################
# HOCHSCHILD COHOMOLOGY TABLES FOR GROUP G6
#
# NOTE: det(g) line displays 1 if det(g)=1
#                              . otherwise
-----------------------------------------------------------
G6 with 2-dimensional rep X.7,8,11,12

      2  4  3  4   2   2   2  2   2   2   2  2  3  4  4
      3  1  .  1   1   1   1  1   1   1   1  1  .  1  1

        1a 2a 2b  3a 12a 12b 6a  3b 12c 12d 6b 4a 4b 4c

HH^0(g)  *  0  0   0   0   0  0   0   0   0  0  0  0  0
HH^1(g)  *  0  0   0   0   0  0   0   0   0  0  0  0  0
HH^2(g)  *  0  0   0   0   0  0   0   0   0  0  0  0  0

* = (S(V) tensor \bigwedge^k(V^*))^G

det(g)   1  .  1   .   .   .  .   .   .   .  .  1  .  .
codim(g) 0  1  2   1   2   2  2   1   2   2  2  2  2  2


-----------------------------------------------------------
G6 with 2-dimensional rep X.9,10

      2  4  3  4   2   2   2  2   2   2   2  2  3  4  4
      3  1  .  1   1   1   1  1   1   1   1  1  .  1  1

        1a 2a 2b  3a 12a 12b 6a  3b 12c 12d 6b 4a 4b 4c

HH^0(g)  *  0  0   0   0   0  0   0   0   0  0  0  0  0
HH^1(g)  *  0  0   0   0   0  0   0   0   0  0  0  0  0
HH^2(g)  *  0  0   0   0   0  0   0   0   0  0  0  0  0

* = (S(V) tensor \bigwedge^k(V^*))^G

det(g)   1  .  1   1   .   .  1   1   .   .  1  1  .  .
codim(g) 0  1  2   2   2   2  2   2   2   2  2  2  2  2
```

```
-----------------------------------------------------------
G6 with 3-dimensional rep X.13

      2  4   3  4   2   2   2   2   2   2   2   2   3  4  4
      3  1   .  1   1   1   1   1   1   1   1   1   .  1  1

        1a  2a 2b  3a 12a 12b  6a  3b 12c 12d  6b  4a 4b 4c

HH^0(g)  *   0  *   0   0   0   0   0   0   0   0   0  *  *
HH^1(g)  *   0  *   0   0   0   0   0   0   0   0   0  *  *
HH^2(g)  * 2m1  * 2m1 2m1 2m1 2m1 2m1 2m1 2m1 2m1 2m1  *  *
HH^3(g)  * 3m1  * 3m1 3m1 3m1 3m1 3m1 3m1 3m1 3m1 3m1  *  *

* = (S(V) tensor \bigwedge^k(V^*))^G
2m1 = S(V^g) tensor C
      [require poly part to have degree 2 mod 1]
3m1 = S(V^g) tensor (V^g)^*
      [require poly part to have degree 3 mod 1]

det(g)   1   1  1   1   1   1   1   1   1   1   1   1  1  1
codim(g) 0   2  0   2   2   2   2   2   2   2   2   2  0  0



-----------------------------------------------------------
G6 with 3-dimensional rep X.14

      2  4   3  4   2   2   2   2   2   2   2   2   3  4  4
      3  1   .  1   1   1   1   1   1   1   1   1   .  1  1

        1a  2a 2b  3a 12a 12b  6a  3b 12c 12d  6b  4a 4b 4c

HH^0(g)  *   0  *   0   0   0   0   0   0   0   0   0  0  0
HH^1(g)  *   0  *   0   0   0   0   0   0   0   0   0  0  0
HH^2(g)  *   0  * 2m2   0   0 2m2 2m2   0   0 2m2 2m2  0  0
HH^3(g)  *   0  * 3m2   0   0 3m2 3m2   0   0 3m2 3m2  0  0

* = (S(V) tensor \bigwedge^k(V^*))^G
2m2 \subset S(V^g) tensor C
            [require poly part to have degree 2 mod 2]
3m2 \subset S(V^g) tensor (V^g)^*
            [require poly part to have degree 3 mod 2]

det(g)   1   .  1   1   .   .   1   1   .   .   1   1  .  .
codim(g) 0   1  0   2   3   3   2   2   3   3   2   2  3  3

############################################################
```

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