A STUDY AND CRITIQUE OF THE MEAN POSITION CONCEPT

IN RELATIVISTIC WAVE MECHANICS

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THESIS

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CHAPTER I

INTRODUCTION

There is in the literature a large number of papers dealing with the question of a one-particle interpretation of the solutions of the equations of relativistic quantum theory. Generally, these go under the title of relativistic wave equations, as opposed to relativistic field equations. That is to say, the phrase "wave" implies attempts at a one-particle interpretation of the state functions. What is meant by this is that one would like to be able to talk about the probability (or density) of finding a particle at a certain point. As is well known, this is given non-relativistically by $|\psi(\vec{x},t)|^2$. The interpretation of $|\psi(\vec{x},t)|^2$ as the probability density of finding a particle at the point $\vec{x}$ at the time t comes from the general quantum theory assumption that

$$\left| \int u_\alpha^*(\vec{x})\, \psi(\vec{x},t)\, d\vec{x} \right|^2$$

is the probability of measuring some observable $\hat{A}$ of a complete compatible set of observables and finding the eigenvalue α when the system is in the state represented by ψ, with $\hat{A}\, u_\alpha = \alpha\, u_\alpha$. What is meant by the prior statement is that in the coordinate representation the eigenfunction of the point position operator is $\delta(\vec{X}-\vec{x}')$ for the eigenvalue $\vec{X}$. If $\hat{A}$ represents the point position operator, then the probability density is

$$\left|\int \delta(\vec{x}-\vec{x}')\, \psi(\vec{x}',t)\, d\vec{x}\right|^2 = \left|\psi(\vec{x},t)\right|^2.$$

The probability interpretation of $\psi(\vec{x},t)$ together with the assumption that its time variation is given by Schroedinger's equation is supported by the fact that

$$\int \left|\psi(\vec{x},t)\right|^2 d\vec{x}$$

is constant in time if $\psi(\vec{x},t)$ satifies Schroedinger's equation. In the non-relativistic case this is equivalent to the existence of a continuity equation. That is, letting $|\psi(\vec{x},t)|^2 = \rho$,

$$\frac{\partial}{\partial t}\int \rho \, d\vec{x} = \int \frac{\partial \rho}{\partial t}\, d\vec{x} = -\int div\, \vec{j}\, d\vec{x}$$

$$= -\int \vec{j}\cdot d\vec{\sigma} = 0$$

if it is assumed that $\psi(\vec{x},t) \rightarrow 0$ at the boundaries of the volume of integration. The other way of showing the constancy of $\int |\psi(\vec{x},t)|^2\, d\vec{x}$ is

$$\frac{\partial}{\partial t}\int \psi^* \psi \, d\vec{x} = \int \left(\frac{\partial \psi^*}{\partial t}\psi + \psi^* \frac{\partial \psi}{\partial t}\right) d\vec{x}$$

$$\frac{\partial}{\partial t}\int \psi^* \psi \, d\vec{x} = \frac{i}{\hbar}\left[\int (\hat{H}^+ \psi^*)\psi \, d\vec{x} - \int \psi^* (\hat{H}\psi) d\vec{x}\right]$$

$$= \frac{i}{\hbar}\int \left[(\hat{H}^+\psi^*)\psi - \psi^*(\hat{H}\psi)\right] d\vec{x}$$

since $\hat{H}$ is assumed Hermitian. These two ways are obviously equivalent when one uses Schroedinger's equation (and its complex conjugate) to derive $\frac{\partial \rho}{\partial t} + \mathrm{div}\, \vec{j}$.

The first relativistic wave equation was the Klein-Gordon (K. G.) equation, which, however, does not allow one to interpret $|\phi(\vec{x},t)|^2$, of its solutions, as a probability. It is possible to derive a continuity equation from the K. G. equation, but ρ is then not always positive or zero (4, p. 55), and this clearly precludes any attempt at letting ρ be a probability. Also, the manner in which $\rho = |\phi(\vec{x},t)|^2$ was obtained from $\left|\int \delta(\vec{x}-\vec{x}')\, \phi(\vec{x}',t)\, d\vec{x}'\right|^2$ appears, initially at least, to be unique. On the other hand, it is clear that one cannot possibly derive any useful information about $\frac{\partial}{\partial t}\int |\phi(\vec{x},t)|^2 d\vec{x}$ from the fact that $\phi(\vec{x},t)$ satisfies the K. G. equation. This is a consequence of the K. G. equation being second order in time, while the above expression involves only first time derivatives. Because of these problems the K. G. equation is generally considered a relativistic "field" equation and not a relativistic "wave" equation (2, p. 709).

There is an immediately apparent solution to these questions. It is to find a different equation that will give the time dependence of the state functions for the relativistic case, and from what has just been said it is clear that one wants an equation that is first order in the time derivative. Considering the manner in which Schroedinger originally derived the K. G. equation (3), one might choose to set

$$E = c\left[\vec{p}^{\,2} + c^2 m^2\right]^{1/2}$$

before making the replacements $E \to i\hbar \frac{\partial}{\partial t}$ and $\vec{p} \to \frac{\hbar}{i} \frac{\partial}{\partial \vec{x}}$. These substitutions give

$$i\hbar \frac{\partial \psi(\vec{x},t)}{\partial t} = c\left[-\hbar^2 \vec{\nabla}_x^{\,2} + m^2 c^2\right]^{1/2} \psi(\vec{x},t)$$

instead of the K. G. equation. However, several questions arise for this case also, the most obvious one being how to treat the square root of a differential operator, both in regard to its effect on $\psi(\vec{x},t)$ and its Hermitian or non-Hermitian nature. Another question is which approach should be used. Considering the necessary initial conditions in the two cases, it is clear that the first and second order equations are not equivalent. Yet, both cases come from the same classical expression for the energy. Some of these questions can and will be resolved later.

Dirac took the approach that the square-root operator could be linearized (1, p. 255) and obtained an equation that has satisfactory one-particle characteristics in many but not all aspects. In particular, the velocity operator has eigenvalues of $\pm\ c$ (the speed of light) and from one of the fundamental postulates of quantum theory these are the only measurable values of that observable; that is, a particle in a state represented by a solution of the Dirac equation can have velocity eigenvalues of only plus or minus the speed of light. This condition is clearly undesirable. The statements made here are the standard statements found in different textbooks but these statements are not entirely accurate. That is, the solutions of the Dirac equation do not explicitly enter the process of showing the eigenvalues of the velocity operator to be $\pm\ c$. The velocity operators are defined by $i\hbar\,\dot{\hat{x}}_i = [\hat{x}_i, \hat{H}]$, where $\hat{H}$ is the Dirac Hamiltonian; that is, only the Dirac Hamiltonian enters the proof and not the solutions of the Dirac equation. These considerations lead one to suggest that it is the operator representing the classical observable, in this case the velocity, that is the source of difficulties, and not the state functions. In fact, it is this idea that has motivated this study.

The basic concept to be used in studying the question of one-particle interpretations of relativistic wave equations is that of observables and operator representations that are different from the more usual classically motivated observables and representations. In particular, the concept of a mean-position observable will be used to determine to what extent the one-particle "problems" can be resolved.

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CHAPTER II

THE ORIGIN AND SIGNIFICANCE OF THE MEAN-POSITION OBSERVABLE

In this chapter some of the background and basic concepts of the mean-position observable will be developed. The literature is quite extensive on the mean-position concept. One of the earliest papers is by M. H. L. Pryce (5) on the definition of the center-of-mass in a relativistic system. In extending these classical relativistic ideas to quantum theory Pryce was led naturally to a mean-positiion observable. More recently, Newton and Wigner (3) used "natural"invariance requirements to obtain localized functions that were the eigenfunctions of the same operator that Pryce had found for the mean-position observable, thus furthering the significance of this concept. The idea in both cases is that as the name implies, the concept of the position of a relativistic particle is not as simple as in classical physics. It is obvious that a point particle in classical mechanics occupies a certain point in space at each instant of time. Even in non-relativistic quantum theory this may not be true; that is, since not all observables are simultaneously measurable, the position of a particle may not be definite under certain conditions.

In fact, unless the state has been "prepared" by measuring either the position observable or other compatible observable, only the probability of measuring and finding the particle at a certain point is known. The mean-position concept requires a modification of even this probability statement. That is, if the point-position observable is not a measurable quantity for a relativistic particle but if the mean position is measurable, then a measurement of the mean position that finds the eigenvalue $\vec{X}$ "fixes" the system in the eigenstate of the mean-position operator with the eigenvalue $\vec{X}$. The essence of the present discussion is the answer to the question: what does a sufficiently quickly repeated measurement of the mean position find and is the particle at the point $\vec{X}$ immediately after the first measurement? The answers, as will become clear, depend on the meaning of the mean position as an observable. A measurement of the mean position that finds $\vec{X}$ does not tell one that the particle, considered as a point, is at the point $\vec{X}$, but only tells one that the particle is definitely in a region of the order of magnitude of the particle's Compton wavelength about $\vec{X}$. The sufficiently quickly repeated measurement of the mean position will, in fact, not necessarily find $\vec{X}$ again but as will be shown in chapter III, will find some other value. The point-position concept is considered a nonmeasurable quantity in relativistic quantum theory and is, consequently, not an observable. It is this viewpoint that will be followed in the present treatment and its consequences for relativistic wave mechanics is the object. One might summarize by saying that not only is measurement of the particle position no longer a classical concept (that is, it is not simultaneously measurable with momentum), but now the very observable itself must be modified from a point concept to what one might call a non-local observable. This is not to say that the mean position is not a point function of space, but rather to say that what is measured at a point in space is not the particle being at that point. It is as if a second uncertainty has been introduced, not by some other measurement, but as an intrinsic property of nature. An interesting fact is that Pauli (4) shows by a semi-classical analysis of the measurement process of the position of a particle tnat one should expect to be able to define a point probability density for the case where $v \ll c$,that is, for the non-relativistic case. There is no proof given that one could not do so for the relativistic case, but the arguments do not indicate that one should exist.

All this leads one to suggest that $\hat{\vec{x}}$ is not an observable; that is, representing $\hat{\vec{x}}$ by $\vec{x}'$ and its immediate consequence that $\hat{\vec{x}}|\vec{x}'\rangle = \vec{x}'|\vec{x}'\rangle$ and $|\vec{x}'\rangle \rightarrow \delta(\vec{x}-\vec{x}')$ is not meaningful in the relativistic region. This then allows one to explain why $|\psi(\vec{x},t)|^2$ should not be the probability of finding the particle at $\vec{x}$, even if the system is in the state represented by ψ . That is, even if

$$\left|\int \delta(\vec{x}-\vec{x}')\,\psi(\vec{x}',t)\,d\vec{x}'\right|^2 = |\psi(\vec{x},t)|^2$$

is obviously true, it has no physical significance if $\delta(\vec{x}-\vec{x}')$ is not the eigenfunction of any measurable quantity. Thus, one of the most perplexing problems of interpretation is removed. It should be noted that Pauli (4) appears to refrain from making

$$\int u_\alpha^* \,\psi\, d\tau$$

the probability amplitude, a fundamental postulate of quantum theory, thus keeping the problem from ever arising. This, however, is not the standard approach.

In the approach taken here the probability amplitude for finding the mean position of the particle to be $\vec{X}$ (in the coordinate or $\vec{x}$ -representation) is

$$\int u_{\vec{X}}^*(\vec{x}')\,\psi(\vec{x}',t)\,d\vec{x}'$$

where $u_{\vec{X}}(\vec{x})$ is the $\vec{x}$ -representation of the mean-position eigenfunction corresponding to the eigenvalue $\vec{X}$. In what is to follow it will be shown that

$$u_{\vec{X}}(\vec{x}) = \frac{1}{(2\pi)^3}\int \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{X})}}{(\vec{k}^2+m^2)^{1/4}}\,d\vec{k}$$

and not the Dirac delta function represents the eigenfunction of the position observable. This expression for $\mathcal{U}_{\hat{x}}(\vec{x})$ is not a simple function, but as will be shown in the next chapter, $\mathcal{U}_{\hat{x}}(\vec{x})$ can be written in a tractable form in a certain limit. It is worth noting that now a clear-cut distinction should be made between representations and probability amplitudes; that is, usually the two are identical because any observable in its own representation ($\hat{\vec{x}}$, in the $\vec{x}$ -representation; $\hat{\vec{p}}$, in the $\vec{p}$ -representation, etc.) has for its eigenfunction the Dirac delta function ($\delta(\vec{x}-\vec{x}')$, $\delta(\vec{p}-\vec{p}')$, etc.). This means that

$$\int \mathcal{U}_\alpha^*(\tau)\,\psi(\tau)\,d\tau$$

goes to (in the α-representation)

$$\int \mathcal{U}_\alpha^*(\alpha')\,\psi(\alpha')\,d\alpha' = \int \delta(\alpha-\alpha')\,\psi(\alpha')\,d\alpha' = \psi(\alpha)$$

which since α is arbitrary, is just the α'-representation of ψ . For this reason the Dirac notation $\langle\alpha|\psi\rangle = \psi(\alpha)$ tends to be somewhat misleading; that is, generally $\langle\alpha|\psi\rangle$ is considered the inner product of $|\alpha\rangle$ with $|\psi\rangle$ which by definition is the probability amplitude of finding α when the system is in the state ψ . If one defines this inner product by the symbol $(|\alpha\rangle, |\psi\rangle)$ and means by $\langle\alpha|\psi\rangle$ the α-representation of ψ , then one must be careful not to equate $(|\alpha\rangle, |\psi\rangle)$ with $\langle\alpha|\psi\rangle$. The general question of when can an operator be represented simply by multiplication by that variable appears to be unresolved and is possibly related to the objections raised by von Neumann (2). A clarification of whether an operator in its own representation is represented by multiplication by the eigenvalue would be most desirable. Part of the problem can be seen by considering the mean-position eigenvalue problem

$$\hat{\vec{X}}\,|\vec{X}'\rangle = \vec{X}'|\vec{X}'\rangle$$

where the capitals have been used to denote mean quantities. If one were to take the "mean position" representation of this eigenvalue problem, then by the standard approach $\hat{\vec{X}}$ would be replaced by $\vec{X}'$ and $|\vec{X}'\rangle$ by $\delta(\vec{X}-\vec{X}')$, so it is clear that

$$u_{\vec{X}}(\vec{X}') = \delta(\vec{X}-\vec{X}') .$$

At first sight this seems quite all right; that is, the Dirac delta eigenfunction would appear to be in keeping with the statement that a sufficiently quickly repeated measurement of $\hat{\vec{X}}$ should find $\vec{X}'$ again. That is, the probability of finding any other value than $\vec{X}'$ should be zero and

$$\int_{\text{all space}} |u_{\vec{X}}(\vec{X}')|^2 d\vec{X}'$$

should be one, which it clearly is. The problem is how do the points of space represented by $\vec{X}$ differ from those represented by $\vec{X}$? There is a distinction between $\hat{\vec{X}}$ and $\hat{\vec{x}}$; that is, they represent different observables or one is measurable and, hence, observable while the other is not observable. Perhaps the difference in the physical quantities is sufficient, but one might want some distinction other than just using small and capital letters for the points in space.

This question of representations can be further clarified by the following observations. If one demands that $\hat{\vec{x}}$ and $\hat{\vec{p}}$ satisfy the usual commutation relations

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$$

then one obvious representation is $\hat{\vec{x}} \rightarrow \vec{x}$ and $\hat{\vec{p}} \rightarrow -i\hbar\vec{\nabla}_{\vec{x}}$, and this leads inevitably to a delta function for the eigenfunction of $\hat{\vec{x}}$. Furthermore, von Neumann (1, p. 570) has shown that if there is a finite number of degrees of freedom in the system (not a field), then any other representation of the commutation relations is unitarily equivalent and, hence, can give no different physical results for quantum mechanics. Yet, the $\vec{k}$ -representation of $\hat{\vec{x}}$ is $i\vec{\nabla}_{\vec{k}}$ or $i\left(\vec{\nabla}_{\vec{k}} - \frac{1}{2}\frac{\vec{k}}{(\vec{k}^2+m^2)}\right)$ depending on whether one is talking about the point-position observable or the mean-position observable. Clearly, both of the operators satisfy the above commutation relations (with $\hat{\vec{p}} \rightarrow \hat{\vec{k}}$), hence, by von Neumann's theorem these operators must be either (1) unitarily equivalent, or (2) there must be an infinite number of degrees of freedom in the system. The first alternative seems unlikely in that the physical results that come from using the mean-position representation appear to be quite different from those for the point-position representation. The second would then appear as the correct one, except that the idea arises that perhaps one is working with a new observable, $\hat{\vec{\mathrm{X}}}$, not just a new representation, which satisfies the same commutation relations as $\hat{\vec{X}}$. This latter concept is the logical choice as long as one wants to stay within a "relativistic wave mechanics".

One further alternative is that the above commutation relations are not the correct ones for the mean-position observable. In fact, a new set of commutation relations will be derived in the next chapter using the analogy between infinitesimal canonical transformations in classical mechanics and infinitesimal unitary transformations in quantum mechanics. The new commutation relations have strong consequences since all attempts at formulating new position observables and their representations use as a guide the requirement that they satisfy the standard commutation relations.

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CHAPTER III

INTERPRETATIONS AND VARIATIONS OF THE KLEIN-GORDON EQUATION

As has been pointed out previously, the standard Klein-Gordon equation is unsatisfactory as a relativistic one-particle equation for several reasons. There have, however, been several variations of the standard approach that have met with some success. In the treatment summarized by Feshbach and Villars (3, p. 26) the Klein-Gordon equation is replaced by two differential equations, and the wave function and its first time derivative are considered as the components of a two-component wave function. The two equations are thus combined into one that is very analogous to the Dirac equation. The two degrees of freedom are interpreted as being the positive and negative charge of the associated particle. The density, ρ , and continuity equation are then related to the charge of the particle and not its position. Further, the two component wave functions require an extension of the inner product that leads naturally to a new position operator. The new position operator is a consequence of the fact that the usual one, $\hat{X} \rightarrow i \vec{\nabla}_k$, is not Hermitian within the new inner product. It is also possible to take the approach of Schweber (5, p. 56) in which one agrees to work only with positive energy states, in an attempt to have a reasonable ρ, and one is led in an undefined way to introduce a new inner product so that, ". . . a linear vector space can be made into a Hilbert space . . ." The manner in which the use of only positive energy states makes ρ satisfactory is not declared and is not easily deducible from what is stated in Schweber's book. The equation of continuity that is derivable from the Klein-Gordon equation gives

$$\rho = \frac{i}{2m}\left[\varphi^* \frac{\partial\varphi}{\partial t} - \frac{\partial\varphi^*}{\partial t}\varphi\right]$$

and if one arbitrarily sets $E\varphi = i\frac{\partial\varphi}{\partial t}$ and $E\varphi^* = -i\frac{\partial\varphi^*}{\partial t}$, then

$$\rho = \frac{E}{m}\varphi^*\varphi$$

which is greater than or equal to zero if $E > 0$. Relativistically, one might take $E = (\vec{k}^2 + m^2)^{1/2}$, and so find that the choice of the positive sign makes $E > 0$. This approach appears good at first sight, but the following after-thoughts arise. Since the Klein-Gordon equation is of second order in time, φ and $\frac{\partial\varphi}{\partial t}$ are arbitrary and, hence, ρ may be either positive, zero, or negative. The above demonstration of when $\rho \geq 0$ depends on the replacement of $i\frac{\partial\varphi}{\partial t}$ by $E\varphi$, and clearly the two differential equations for the time dependence of the state functions are basically different. Hence, using both of them in the same argument would seem, at best, rather dubious. No such objections can be made to the approach of Feshbach and Villars, but they make no claims about the position of the relativistic spin-zero particle and its relation to ρ. Consequently, in the following it is shown how one may define and interpret the position concept in the relativistic realm, not for the Klein-Gordon equation, but for

$$i \frac{\partial |\psi\rangle}{\partial t} = (\hat{\vec{k}}^2 + m^2)^{1/2} |\psi\rangle \qquad \text{(III-1)}$$

where either the positive sign or the negative sign will be used.

If one takes the $\vec{k}$ -representation of the above wave equation, it becomes

$$i \frac{\partial |\psi\rangle}{\partial t} = (\vec{k}^2 + m^2)^{1/2} |\psi\rangle \qquad \text{(III-2)}$$

with an arbitrary choice of the plus sign. Now

$$\int \psi^*(\vec{k},t)\, \psi(\vec{k},t)\, d\vec{k}$$

is constant in time since

$$\begin{aligned}\frac{\partial}{\partial t}\int \psi^*(\vec{k},t)\, \psi(\vec{k},t)\, d\vec{k} &= \int\left(\frac{\partial \psi^*(\vec{k},t)}{\partial t}\psi(\vec{k},t) + \psi^*(\vec{k},t)\frac{\partial \psi(\vec{k},t)}{\partial t}\right) d\vec{k} \\ &= \int\Big\{ i\left[(\vec{k}^2+m^2)^{1/2}\psi^*(\vec{k},t)\right]\psi(\vec{k},t) \\ &\qquad - i\,\psi^*(\vec{k},t)\left[(\vec{k}^2+m^2)^{1/2}\psi(\vec{k},t)\right]\Big\}\, d\vec{k} \\ &= 0.\end{aligned}$$

Further, in the $\vec{x}$ -representation (III-1) becomes

$$i\,\frac{\partial\psi(\vec{x},t)}{\partial t} = \left(-\vec{\nabla}_x^{\,2} + m^2\right)^{1/2}\psi(\vec{x},t)$$

and one needs to know what effect the square-root operator has on $\psi(\vec{x},t)$. This can be found from

$$(\hat{k} + m^2)^{1/2}\,|\psi\rangle = (\hat{k}^2 + m^2)^{1/2}\,|\psi\rangle$$

$$\langle\vec{x}|(\hat{k}^2 + m^2)^{1/2}|\psi\rangle = \int \langle\vec{x}|(\hat{k}^2+m^2)^{1/2}|\vec{k}\rangle\langle\vec{k}|\psi\rangle\, d\vec{k}$$

$$(-\vec{\nabla}_x^{\,2} + m^2)^{1/2}\,\psi(\vec{x},t) = \int (\vec{k}^2+m^2)^{1/2}\langle\vec{x}|\vec{k}\rangle\,\psi(\vec{k},t)\, d\vec{k}$$

where $\langle\vec{x}|\vec{k}\rangle = (2\pi)^{-3/2}\, e^{i\vec{k}\cdot\vec{x}}$, which is derivable from the fact that $\vec{k}$ generates an infinitesimal change δx . Thus

$$(-\vec{\nabla}_x^{\,2} + m^2)^{1/2}\,\psi(\vec{x},t) = (2\pi)^{-3/2}\int (\vec{k}^2+m^2)^{1/2} e^{i\vec{k}\cdot\vec{x}}\,\psi(\vec{k},t)\, d\vec{k}$$

and now if one forms

$$\frac{\partial}{\partial t}\int \psi^*(\vec{x})\psi(\vec{x})\, d\vec{x} = \int\left(\frac{\partial\psi^*(\vec{x})}{\partial t}\,\psi(\vec{x}) + \psi^*(\vec{x})\,\frac{\partial\psi(\vec{x})}{\partial t}\right) d\vec{x}$$

$$= \int i\left[(-\vec{\nabla}_x^{\,2}+m^2)^{1/2}\psi^*(\vec{x})\right]\psi(\vec{x})\; d\vec{x}$$

$$- \int i\,\psi^*(\vec{x})\left[(-\vec{\nabla}_x^{\,2}+m^2)^{1/2}\psi(\vec{x})\right]\, d\vec{x}$$

$$= \frac{i}{(2\pi)^3}\int d\vec{x}\,d\vec{k}\,d\vec{k}'\, e^{-i\vec{k}\cdot\vec{x}}(\vec{k}^2+m^2)^{1/2} e^{i\vec{k}'\cdot\vec{x}}\,\psi^*(\vec{k})\psi(\vec{k}')$$

$$- \frac{i}{(2\pi)^3}\int d\vec{x}\,d\vec{k}\,d\vec{k}'\, e^{-i\vec{k}\cdot\vec{x}}(\vec{k}'^2+m^2)^{1/2} e^{i\vec{k}'\cdot\vec{x}}\,\psi^*(\vec{k})\,\psi(\vec{k}'),$$

and using the standard integral representation of the delta function

$$\frac{1}{(2\pi)^3}\int e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}}\,d\vec{x} = \delta(\vec{k}-\vec{k}')$$

the above becomes

$$i\int d\vec{k}\int d\vec{k}'\,\delta(\vec{k}-\vec{k}')(\vec{k}^2+m^2)^{\frac{1}{2}}\,\psi^*(\vec{k})\,\psi(\vec{k}')$$

$$-i\int d\vec{k}\int d\vec{k}'\,\delta(\vec{k}-\vec{k}')(\vec{k}'^2+m^2)^{\frac{1}{2}}\,\psi^*(\vec{k})\,\psi(\vec{k}')$$

which is equal to zero. Therefore,

$$\int \psi^*(\vec{x},t)\,\psi(\vec{x},t)\,d\vec{x}$$

is constant in time. At first sight this result appears very desirable; that is, it removes the major objection to interpreting $|\psi(\vec{x},t)|^2$ as a probability density. Therefore, one might feel that if one only works with equation (III-1), then one has a sensible relativistic wave equation. Another major objection, the zitterbewegung, which is the high frequency oscillatory motion, is also removed. Since it occurs as a consequence of the interference between the positive and negative energy states (4, p. 90), choosing only the positive or only the negative will surely remove this paradoxical result. These successes are only partial, however, because of the fact that using only positive energy states one can not form a delta function in coordinate space, that is,

$$\psi(\vec{x},t) = \frac{1}{(2\pi)^{3/2}} \int e^{i\vec{k}\cdot\vec{x}}\ \psi(\vec{k},t)\, d\vec{k}$$

and since $\psi(\vec{x},t)$ satisfies (III-2),

$$\psi(\vec{k},t) = \varphi(\vec{k})\, e^{-i(\vec{k}^2+m^2)^{\frac{1}{2}}t}$$

or

$$\psi(\vec{x},t) = \frac{1}{(2\pi)^{3/2}} \int \varphi(\vec{k})\, e^{i[\vec{k}\cdot\vec{x} - (\vec{k}^2+m^2)^{\frac{1}{2}}t]}\, d\vec{k}.$$

This should in general contain

$$\varphi_+(\vec{k})\, e^{i[\vec{k}\cdot\vec{x} - (\vec{k}^2+m^2)^{\frac{1}{2}}t]} + \varphi_-(\vec{k})\, e^{i[\vec{k}\cdot\vec{x} + (\vec{k}^2+m^2)^{\frac{1}{2}}t]}$$

and unless these negative energy states are included the value of $\psi(\vec{x},t)$ can not be made as small as desired (1, p. 38). An interesting consequence of this can be seen as follows. Consider

$$|\psi\rangle = \int |\vec{k}\rangle \langle \vec{k}|\psi\rangle\, d\vec{k}$$

and

$$\langle \vec{x}'|\psi\rangle = \int \langle \vec{x}'|\vec{k}\rangle \langle \vec{k}|\psi\rangle\, d\vec{k}$$

or for $|\psi\rangle = |\vec{x}\rangle$,

$$\langle \vec{x}'|\vec{x}\rangle = \int \langle \vec{x}'|\vec{k}\rangle \langle \vec{k}|\vec{x}\rangle\, d\vec{k}$$

and since $\langle \vec{x}'|\vec{k}\rangle = (2\pi)^{-3/2} e^{i\vec{k}\cdot\vec{x}'}$ one might write $\langle \vec{k}|\vec{x}\rangle = (2\pi)^{-\frac{3}{2}} e^{-i\vec{k}\cdot\vec{x}}$ and obtain

$$\langle \vec{x}' | \vec{x} \rangle = \frac{1}{(2\pi)^3} \int e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')} d\vec{k}$$

which is the previously used expression for the delta function. However, this has been done using $\langle \vec{x} | \vec{k} \rangle^* = \langle \vec{k} | \vec{x} \rangle$, which is true only if the representations and inner products are identical, that is, one of the basic properties of an inner product is that $(|\alpha\rangle, |\beta\rangle)^* = (|\beta\rangle, |\alpha\rangle)$. If the inner product of two elements is not equal to the complex conjugate of the inner product of the two elements in reverse order, then $\langle \vec{k} | \vec{x} \rangle$ can not be found from $\langle \vec{x} | \vec{k} \rangle$ and, in fact, is not necessarily equal to $(2\pi)^{-3/2} e^{-i\vec{k}\cdot\vec{x}}$. This principle is in keeping with the previous statement that delta functions can not be built up out of only positive energy states. The point of interest here is the manner in which this fact is shown. Consider the derivation of

$$\langle \vec{x} | \vec{k} \rangle = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}},$$

which follows from

$$\hat{\vec{k}} | \vec{k}' \rangle = \vec{k}' | \vec{k}' \rangle$$

and in the $\vec{x}$ -representation this equation becomes

$$-i \vec{\nabla}_x \langle \vec{x} | \vec{k}' \rangle = \vec{k}' \langle \vec{x} | \vec{k}' \rangle$$

hence,

$$\langle \vec{x} | \vec{k}' \rangle = C e^{i\vec{k}'\cdot\vec{x}},$$

and C is $(2\pi)^{-3/2}$ for the proper normalization. This derivation depends on $\hat{\vec{k}}$ being $-i\vec{\nabla}_{\vec{x}}$ in the $\vec{x}$ -representation which follows from $\hat{\vec{k}}$ being the generator of a unitary δx displacement. Now if one attempts a similar derivation for $\langle \vec{k} | \vec{x} \rangle$, one starts with

$$\hat{\vec{x}} \, | \vec{x}' \rangle = \vec{x}' \, | \vec{x}' \rangle$$

and a question then arises as to the $\vec{k}$ -representation of $\hat{\vec{x}}$. Normally, it is taken to be $i\vec{\nabla}_{\vec{k}}$, but if the mean-position concept is used instead of the point-position, then $\hat{\vec{x}}$ goes to $i\left(\vec{\nabla}_{\vec{k}} - \frac{1}{2}\frac{\vec{k}}{(\vec{k}^2+m^2)}\right)$. Therefore, the position-eigenfunction problem becomes

$$i\left(\vec{\nabla}_{\vec{k}} - \frac{1}{2}\frac{\vec{k}}{(\vec{k}^2+m^2)}\right)\langle \vec{k} | \vec{x}' \rangle = \vec{x}' \langle \vec{k} | \vec{x}' \rangle$$

which has as a solution

$$\langle \vec{k} | \vec{x}' \rangle = \frac{1}{(2\pi)^{3/2}} (\vec{k}^2+m^2)^{\frac{1}{4}} e^{-i\vec{k}\cdot\vec{x}'}$$

with the proper normalization. Therefore,

$$\langle \vec{x} | \vec{x}' \rangle = \frac{1}{(2\pi)^3} \int e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} (\vec{k}^2+m^2)^{\frac{1}{4}} \, d\vec{k}$$

which is not a delta function and clearly does not represent a particle localized at the point $\vec{x}'$; in fact, $\langle \vec{x} | \vec{x}' \rangle$ is not, by the present interpretation, the probability amplitude for finding the particle at the point $\vec{x}'$ but is the $\vec{x}$ -representation of $|\vec{x}'\rangle$. The above equation for $\langle \vec{x} | \vec{x}' \rangle$ is not readily evaluated and, hence, is of no physical interest. Corinaldesi and Strocchi (2, p. 62) use a slightly different $\vec{k}$ -representation for the mean-position operator

$$\hat{\vec{x}} \to i\left(\vec{\nabla}_{k} + \frac{1}{2}\,\frac{\vec{k}}{(\vec{k}^2+m^2)}\right),$$

but it has far reaching consequences in the present treatment. It is worth noting that the representation with the minus sign (Schweber's) appears to agree with that of Pryce's and Newton and Wigner's. The $\langle \vec{k} | \vec{x}' \rangle$ found by using the above representation of $\hat{\vec{x}}$ is

$$\langle \vec{k} | \vec{x}' \rangle = B\, e^{-i\vec{k}\cdot\vec{x}'}(\vec{k}^2+m^2)^{-1/4}$$

and B is chosen, for a particular normalization, to be $(2\pi)^{-3/2}$. Once again this isolated function has no direct physical significance. It is, however, the state function in the $\vec{k}$ -representation for a particle prepared in the state in which the mean position of the particle is $\vec{x}'$. Using this latter form of $\langle \vec{k} | \vec{x}' \rangle$

$$\langle \vec{x} | \vec{x}' \rangle = \frac{1}{(2\pi)^3}\int e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}(\vec{k}^2+m^2)^{-1/4}\,d\vec{k}$$

and this will be the expression used in this treatment. Expressions that do have physical significance can be formed from these representations by taking the inner products of state functions with eigenfunctions. In other words, instead of being concerned about whether $\int |\psi(\vec{x},t)|^2\, d\vec{x}$ is constant in time, one should be interested in expressions like

$$\int \langle \vec{x} | \vec{x}' \rangle^{*} \langle \vec{x} | \psi \rangle \, d\vec{x}$$

or

$$\int \langle \vec{k} | \vec{x}' \rangle^{*} \langle \vec{k} | \psi \rangle \, d\vec{k}$$

and in integrals over all space of the absolute-value-squared of these probability amplitudes (inner products), in the $\vec{x}$ - and $\vec{k}$ -representations respectively. In particular, consider

$$\int \langle \vec{k} | \vec{x}' \rangle^{*} \langle \vec{k} | \vec{x} \rangle \, d\vec{k}$$

which is the probability amplitude for finding the mean position to be $\vec{x}'$ if the mean position of the particle has been found to be $\vec{x}$ from an instantly previous measurement. The probability amplitude can be evaluated by the following analysis.

Carrying out the integration in spherical coordinates with $\vec{k} \cdot (\vec{x} - \vec{x}') = - k \, | \vec{x} - \vec{x}' | \cos\theta$, one has

$$\int \langle \vec{k} | \vec{x}' \rangle^{*} \langle \vec{k} | \vec{x} \rangle \, d\vec{k} = \frac{1}{(2\pi)^3} \int e^{i \vec{k} \cdot (\vec{x}' - \vec{x})} (\vec{k}^2 + m^2)^{-\frac{1}{2}} \, d\vec{k}$$

$$= \frac{1}{(2\pi)^3} \int_0^{\infty} \int_0^{\pi} \int_0^{2\pi} \frac{e^{i k R \cos\theta}}{(k^2 + m^2)^{\frac{1}{2}}} \, k^2 \sin\theta \, dk \, d\theta \, d\varphi$$

$$= - \frac{(2\pi)^{-2}}{i R} \int_0^{\infty} \int_0^{\pi} \frac{k \, dk}{(k^2 + m^2)^{\frac{1}{2}}} \frac{\partial}{\partial\theta} e^{i k R \cos\theta} \, d\theta$$

$$\int \langle \vec{k}|\vec{x}'\rangle^* \langle \vec{k}|\vec{x}\rangle \, d\vec{k} = \frac{2(2\pi)^{-2}}{R} \int_0^\infty \frac{k \sin kR}{(k^2+m^2)^{\frac{1}{2}}} \, dk$$

$$= -\frac{2(2\pi)^{-2}}{R} \frac{\partial}{\partial R} \int_0^\infty \frac{\cos kR}{(k^2+m^2)^{\frac{1}{2}}} \, dk$$

where $R = |\vec{x} - \vec{x}'|$. The Basset function of order n has the following integral representation (2, p. 63)

$$K_n(\rho) = \frac{2^n \Gamma(n+1)}{\rho^n \Gamma(\frac{1}{2})} \int_0^\infty (z^2+1)^{-n-\frac{1}{2}} \cos \rho z \, dz .$$

Furthermore, the Basset function obeys the following recursion relation (2, p. 63)

$$\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}\right)^m \left(\rho^n K_n(\rho)\right) = (-1)^m \rho^{n-m} K_{n-m}(\rho),$$

where

$$K_n(\rho) = \frac{i\pi}{2} e^{i\frac{n\pi}{2}} H_n^1(i\rho) = \frac{i\pi}{2} e^{-i\frac{n\pi}{2}} H_{-n}^1(i\rho)$$

and $H_n^1(i\rho)$ is the Hankel function of the first kind of order n with argument $i\rho$. Letting $z = \frac{k}{m}$ and $\rho = mR$, then

$$\int \langle \vec{k}|\vec{x}'\rangle^* \langle \vec{k}|\vec{x}\rangle \, d\vec{k} = -2(2\pi)^{-2} m^2 \left(\frac{1}{\rho}\frac{\partial}{\partial \rho}\right) \int_0^\infty \frac{\cos \rho z}{(z^2+1)^{\frac{1}{2}}} \, dz$$

$$= -\frac{2(m)}{(2\pi)^2} \left(\frac{1}{\rho}\frac{\partial}{\partial \rho}\right) K_0(\rho)$$

$$= -\frac{2m^2}{(2\pi)^2} \left[(-1)\rho^{-1} K_{-1}(\rho)\right]$$

$$= \frac{m^2}{4\pi} \frac{H_1^1(i\rho)}{\rho} .$$

The Basset function has the following limits as the argument ρ approaches large ($\rho \to \infty$) values and small ($\rho \to 0$) values respectively, for order $n \geq 0$,

$$K_n(\rho) \sim \left(\frac{\pi}{2\rho}\right)^{\frac{1}{2}} e^{-\rho} \qquad (\rho \to \infty)$$

and

$$K_n(\rho) \sim \frac{\Gamma(n)}{2} \left(\frac{\rho}{2}\right)^{-n} \qquad (\rho \to 0).$$

Therefore, as ρ approaches large values

$$\int \langle \vec{k}|\vec{x}'\rangle^* \langle \vec{k}|\vec{x}\rangle \, d\vec{k} \sim \frac{m^2}{4\pi\rho}\left[-\frac{i2}{\pi} e^{-i\frac{\pi}{2}} \left(\frac{\pi}{2\rho}\right)^{\frac{1}{2}} e^{-\rho}\right]$$

$$= -\frac{m^2}{(2\pi)^{3/2}} \quad \frac{e^{-\rho}}{\rho^{3/2}}$$

for $\rho = mR \to \infty$, and for ρ approaching small values

$$\int \langle \vec{k}|\vec{x}'\rangle^* \langle \vec{k}|\vec{x}\rangle \, d\vec{k} \sim \frac{m^2}{4\pi\rho}\left[-\frac{i2}{\pi} e^{-i\frac{\pi}{2}} \frac{\Gamma(1)}{2} \left(\frac{\rho}{2}\right)^{-1}\right]$$

$$= -\frac{m^2}{2\pi^2} \; \rho^{-1}.$$

Using the limited form for large ρ one observes that the integral

$$\int \langle \vec{k}|\vec{x}'\rangle^* \langle \vec{k}|\vec{x}\rangle \, d\vec{k}$$

is reduced by e^{-1} of its original value for $R = \frac{1}{m} \left(= \frac{\hbar}{mc}\right)$; that is, the value of the integral is appreciable only for values of the order of the Compton wavelength of the particle. The physical significance and its unusual consequences will be examined shortly.

To show the constancy in time of

$$G = \int \left| \int \langle \vec{x} | \vec{x}' \rangle^* \langle \vec{x} | \psi \rangle \, d\vec{x} \right|^2 d\vec{x}',$$

consider

$$G = \int d\vec{x}\, d\vec{x}'\, d\vec{x}'' \langle \vec{x} | \vec{x}' \rangle \langle \vec{x} | \psi \rangle^* \langle \vec{x}'' | \vec{x}' \rangle^* \langle \vec{x}'' | \psi \rangle .$$

Then

$$G = \frac{1}{(2\pi)^3} \int d\vec{x}\, d\vec{x}'\, d\vec{x}''\, d\vec{k}\, d\vec{k}' \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\, e^{-i\vec{k}'\cdot(\vec{x}''-\vec{x}')}}{(\vec{k}^2+m^2)^{\frac{1}{4}} (\vec{k}'^2+m^2)^{\frac{1}{4}}} \psi^*(\vec{x},t)\, \psi(\vec{x}'',t)$$

and the exponentials can be regrouped to

$$e^{i\vec{x}'\cdot(\vec{k}'-\vec{k})}\; e^{i\vec{k}\cdot\vec{x}}\; e^{-i\vec{k}'\cdot\vec{x}''}$$

so that

$$\frac{1}{(2\pi)^3} \int e^{-i\vec{x}'\cdot(\vec{k}-\vec{k}')}\, d\vec{x}'$$

gives $\delta(\vec{k}-\vec{k}')$ and

$$G = \int d\vec{x}\, d\vec{x}''\, d\vec{k}\, \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}'')}}{(\vec{k}^2+m^2)^{\frac{1}{2}}} \psi^*(\vec{x},t)\, \psi(\vec{x}'',t)$$

after an integration over $\vec{k}'$. The integration over $\vec{k}$ can not be performed to give a $\delta(\vec{x}-\vec{x}'')$ because of the denominator, so one expands the ψ's to obtain

$$G = (2\pi)^{-3}\int d\vec{x}\, d\vec{x}''\, d\vec{k}\, d\vec{k}'\, d\vec{k}''\, \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}'')}}{(\vec{k}^2+m^2)^{\frac{1}{2}}}\, e^{-i\vec{k}'\cdot\vec{x}}\psi^*(\vec{k}')\, e^{i\vec{k}''\cdot\vec{x}''}\psi(\vec{k}'')$$

$$= (2\pi)^{-3}\int d\vec{x}\, d\vec{x}''\, d\vec{k}\, d\vec{k}'\, d\vec{k}''\, \frac{e^{i\vec{x}\cdot(\vec{k}-\vec{k}')}\, e^{i\vec{x}''\cdot(\vec{k}''-\vec{k})}}{(\vec{k}^2+m^2)^{\frac{1}{2}}}\, \psi^*(\vec{k}')\psi(\vec{k}'')$$

and integrating over $\vec{x}$ and $\vec{x}''$ gives $(2\pi)^3\delta(\vec{k}-\vec{k}')\,\delta(\vec{k}-\vec{k}'')$, or

$$G = (2\pi)^3 \int \frac{\psi^*(\vec{k})\,\psi(\vec{k})}{(\vec{k}^2+m^2)^{\frac{1}{2}}}\, d\vec{k}$$

after the integrations over $\vec{k}'$ and $\vec{k}''$. The constancy in time follows from

$$\frac{\partial G}{\partial t} = (2\pi)^3 \int \frac{d\vec{k}}{(\vec{k}^2+m^2)^{\frac{1}{2}}}\left[\frac{\partial\psi^*(\vec{k},t)}{\partial t}\psi(\vec{k},t) + \psi^*(\vec{k},t)\frac{\partial\psi(\vec{k},t)}{\partial t}\right]$$

$$= (2\pi)^3 \int \frac{d\vec{k}}{(\vec{k}^2+m^2)^{\frac{1}{2}}}\left[\left\{i(\vec{k}^2+m^2)^{\frac{1}{2}}\psi^*(\vec{k},t)\right\}\psi(\vec{k},t) + \psi^*(\vec{k},t)\left\{-i(\vec{k}^2+m^2)^{\frac{1}{2}}\psi(\vec{k},t)\right\}\right]$$

$$= i(2\pi)^3 \int d\vec{k}\left[\psi^*(\vec{k},t)\psi(\vec{k},t) - \psi^*(\vec{k},t)\psi(\vec{k},t)\right]$$

$$= 0.$$

In a similar, but simpler, manner the constancy in time of

$$\int \left| \int \langle \vec{k}|\vec{x}'\rangle^* \langle \vec{k}|\psi\rangle\, d\vec{k} \right|^2 d\vec{x}'$$

can be shown. An interesting consequence of the relations shown above is that

$$\langle \vec{x}' | \vec{x}'' \rangle ,$$

$$\int \langle \vec{k} | \vec{x}' \rangle^* \langle \vec{k} | \vec{x}'' \rangle \, d\vec{k} ,$$

and

$$\int \langle \vec{x} | \vec{x}' \rangle^* \langle \vec{x} | \vec{x}'' \rangle \, d\vec{x}$$

are all equal; that is, they are all given by

$$\frac{1}{(2\pi)^3} \int \frac{e^{i\vec{k}\cdot(\vec{x}' - \vec{x}'')}}{(\vec{k}^2 + m^2)^{\frac{1}{2}}} \, d\vec{k} .$$

This fact would seem to suggest that (1) $\langle \vec{x}' | \vec{x}'' \rangle$ is, in fact, a probability amplitude, and (2) one might write $\langle \vec{k} | \vec{x}' \rangle^* = \langle \vec{x}' | \vec{k} \rangle$ ($\langle \vec{x} | \vec{x}' \rangle^* = \langle \vec{x}' | \vec{x} \rangle$) and use the completeness of $\int |\vec{k}\rangle d\vec{k} \langle \vec{k}|$ ($\int |\vec{x}\rangle d\vec{x} \langle \vec{x}|$) to show their equivalence. Since these interpretations and manipulations can cause difficulties in other cases, it seens better not to accept these as legitimate operations. The analogy to ordinary quantum mechanics is, however, so striking that something of significance might be involved.

The above expression for the probability amplitude for repeating the mean-position measurement has an interesting physical interpretation. It is that if one measures the mean position and finds $\vec{x}'$, then a sufficiently quickly repeated measurement of the mean position will not find $\vec{x}'$ with certainty, but the probability is only significant (greater than $\sim e^{-1}$) for distances less than the Compton wavelength of the particle. The usual statements (5, p. 61) concerning the physical interpretation of the $\vec{X}$ -representation of the mean-position eigenfunction would, at first thought, appear to be a contradiction, but because of the above equivalences this might indeed be a correct result for the wrong reason. The idea of not being able to repeat a measurement sufficiently quickly and get the same value needs clarification since it is generally considered to be basic to the concept of measurement. If one imagines a moving marble on a pool table and an array of joined identical boxes that just covers the table, then the mean-position measurement might be considered to be the process of placing the array of boxes on the table and noting what box the marble is in. Clearly, if one repeats this sufficiently quickly (before the marble can move) one would expect to find it in the same box, thus apparently contradicting the above result for

$$\int \langle \vec{x} | \vec{X}' \rangle^{+} \langle \vec{x} | \vec{X}'' \rangle \, d\vec{x} .$$

The reason that this pool table result is incorrect is that the joined array of boxes is fixed with respect to space, that is, the surface of the table. For two real measurements the boxes may overlap, that is, even though the particle does not move before the second measurement, it may be found in another box. If this were not true, the boxes would give a type of background for an absolute reference system.

One other unusual consequence of the mean-position concept will now be derived. The commutation relations $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$ can be obtained, as is well-known, by subjecting the operators $\hat{x}_i$ or $\hat{p}_j$ to similarity transformations with the unitary operators $\hat{U}_{\delta x}$ and $\hat{U}_{\delta p}$, respectively. A similar process will now be carried out using the mean-position observable.

If one takes the $\vec{k}$ -representation of $\hat{x}_i$, the mean-position operator, to be

$$i\left(\frac{\partial}{\partial k_i} + \frac{1}{2}\frac{k_i}{(\vec{k}^2+m^2)}\right)$$

then

$$\langle k_i' | \hat{x}_i | k_i \rangle = i\left(\frac{\partial}{\partial k_i} + \frac{1}{2}\frac{k_i}{(\vec{k}^2+m^2)}\right)\delta(k_i - k_i')$$

or

$$\hat{x}_i | k_i \rangle = i\left(\frac{|k_i + \delta k_i\rangle - |k_i\rangle}{\delta k_i} + \frac{1}{2}\frac{k_i}{(\vec{k}^2+m^2)}|k_i\rangle\right)$$

$$|k_i + \delta k_i\rangle = \left(\hat{I} - i\,\hat{x}_i\,\delta k_i - \frac{1}{2}\frac{k_i}{(\vec{k}^2+m^2)}\delta k_i\right)|k_i\rangle$$

but $|k_i + \delta k_i\rangle = \hat{U}_{\delta k_i}|k_i\rangle$, so

$$\hat{U}_{\delta k_i} = \hat{I} - i\,\hat{x}_i\,\delta k_i - \frac{1}{2}\frac{k_i}{(\vec{k}^2+m^2)}\delta k_i$$

and

$$\hat{U}^{+}_{\delta k_i} = \hat{I} + i\,\hat{x}_i\,\delta k_i - \frac{1}{2}\frac{k_i}{(\vec{k}^2+m^2)}\,\delta k_i .$$

Therefore,

$$\hat{k}_j \rightarrow \hat{k}_j + \hat{I}\,\delta_{ij}\,\delta\hat{k}_j = \hat{U}_{\delta k_i}\,\hat{k}_j\,\hat{U}^{+}_{\delta k_j}$$

and so,

$$\hat{k}_j + \hat{I}\,\delta_{ij}\,\delta\hat{k}_j = \left(\hat{I} - i\hat{x}_i\,\delta\hat{k}_i - \frac{1}{2}\frac{\hat{k}_i}{(\vec{k}^2+m^2)}\delta\hat{k}_i\right)\hat{k}_j\left(\hat{I} + i\,\hat{x}_i\,\delta\hat{k}_i - \frac{1}{2}\frac{\hat{k}_i}{(\vec{k}^2+m^2)}\delta\hat{k}_i\right)$$

$$= \hat{k}_j - i\,[\hat{x}_i,\hat{k}_j]\,\delta\hat{k}_i - \frac{\hat{k}_i\hat{k}_j}{(\vec{k}^2+m^2)}\,\delta\hat{k}_i + O(\delta\hat{k}_i)$$

which to first order terms in $\delta\hat{k}_i$ gives,

$$[\,\hat{x}_i,\hat{k}_j] = i\left(\delta_{ij} + \frac{\hat{k}_i\hat{k}_j}{(\vec{k}^2+m^2)}\right)$$

rather than the usual commutation relations. The absence of the additional term in non-relativistic quantum theory might be accounted for by noting that the expectation value of this term, in $\vec{k}$ -eigenstates, is

$$\frac{k_i\,k_j}{(\vec{k}^2+m^2)}$$

which for the non-relativistic region, $|\vec{k}|^2 << m^2$, gives a term of the order of zero, or at least very much less than one. The interesting feature is that even for i=j, $\hat{x}_i$ and $\hat{p}_j$ do not commute in the relativistic region. Normally, one says that $\hat{X}_i$ and $\hat{p}_j$ should commute because a measurement of the X coordinate does not affect the

y momentum, but this is now no longer correct. In fact, the non-commutation might be considered to be in harmony with the concept of the mean-position measurement, for a measurement of the mean position that finds X' implies only that the particle is in a region of the order of the Compton wavelength around X'; that is, it is fixed to some extent in the y and z directions as well. This statement actually goes somewhat beyond what the equations show. It is, however, in accord with the concept of the mean-position observable.

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CHAPTER IV

CONCLUSION

In the previous chapters an attempt has been made to make a consistent one-particle interpretation of relativistic quantum theory using the mean-position observable exclusively. This work was done only for a very special type of wave equation, that is,

$$i\hbar \frac{\partial \psi}{\partial t} = (\hat{p}^2 + m^2)^{\frac{1}{2}} \psi$$

and one would like to pursue this approach for other types of equations, for instance the Dirac equation. In addition some very special assumptions were made concerning the form of the inner product and the expansion postulate. Several other possibilities exist and will now be enumerated and their consequences described. Instead of choosing the $\mathbf{k}$ -representation of the inner product as

$$(\psi, \varphi) = \int \psi^*(\mathbf{k},t)\, \varphi(\mathbf{k},t)\, d\mathbf{k}$$

one might choose the Lorentz invariant form

$$(\psi, \varphi) = \int \frac{\psi^*(\mathbf{k},t)\, \varphi(\mathbf{k},t)}{(\mathbf{k}^2 + m^2)^{1/2}}\, d\mathbf{k}$$

and this Lorentz invariant inner product would clearly change the results of this investigation significantly. In particular, if one uses

$$\langle \vec{k} | \vec{x} \rangle = \frac{1}{(2\pi)^{3/2}} e^{-i\vec{k}\cdot\vec{x}} (\vec{k}^2+m^2)^{-\frac{1}{4}}$$

for the $\vec{k}$ -representation of the mean-position eigenfunction, the probability of finding the mean position of the particle at the point $\vec{x}'$ if the mean position was measured and found to be $\vec{x}$ previously is

$$\frac{m\pi^2}{2(2\pi)^4} \left(\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})}\right)^2 \frac{|H'_{1/2}(i\rho)|^2}{\rho},$$

which appears quite different from the expression for this quantity derived in the previous chapter, but this probability has, in fact, the same qualitative behavior for large values of $\rho = m|\vec{x} - \vec{x}'|$ as the probability found in Chapter III. That is to say, it decreases rapidly for distances beyond the Compton wavelength of the particle. Consequently, the above change in the inner product would not change markedly the results described in the previous chapter.

A second alternative to what was done in the previous chapter, which would produce profound changes, is the replacement of

$$\langle \vec{k} | \vec{x} \rangle = \frac{1}{(2\pi)^{3/2}} e^{-i\vec{k}\cdot\vec{x}} (\vec{k}^2+m^2)^{-1/4}$$

by

$$\langle \vec{k} | \vec{x} \rangle = \frac{1}{(2\pi)^{3/2}} e^{-i\vec{k}\cdot\vec{x}} (\vec{k}^2+m^2)^{\frac{1}{4}},$$

which is the form used by Schweber (3, p. 62). In the version of the inner product used in Chapter III, this probability using Schweber's eigenfunction of the mean-position operator gives

$$\int \langle \vec{k} | \vec{x}' \rangle^* \langle \vec{k} | \vec{x} \rangle \, d\vec{k} = \frac{1}{(2\pi)^3} \int (\vec{k}^2 + m^2)^{\frac{1}{2}} e^{i \vec{k} \cdot (\vec{x}' - \vec{x})} \, d\vec{k}$$

and leads to divergencies, which is clearly undesirable. However, if one combines Schweber's mean-position eigenfunction with the Lorentz invariant inner product, the result is

$$\int \frac{\langle \vec{k} | \vec{x}' \rangle^* \langle \vec{k} | \vec{x} \rangle}{(\vec{k}^2 + m^2)^{\frac{1}{2}}} \, d\vec{k} = \frac{1}{(2\pi)^3} \int \frac{e^{i \vec{k} \cdot (\vec{x}' - \vec{x})}}{(\vec{k}^2 + m^2)^{\frac{1}{2}}} (\vec{k}^2 + m^2)^{\frac{1}{2}} \, d\vec{k} = \delta(\vec{x} - \vec{x}')$$

which implies that the result of repeating the measurement sufficiently quickly gives the same value. This is in agreement with standard quantum theory, but in contradistinction to the results presented in the previous chapter. If one wished to keep this repeatability of a measurement, the above argument is a possible approach. The previous chapter, however, has given a physical argument for not necessarily being able to repeat a measurement with unlimited accuracy.

There was still another assumption made in this work that could be modified with significant consequences. This is the expansion postulate

$$|\psi\rangle = \int |\vec{k}\rangle \langle \vec{k} | \psi \rangle \, d\vec{k}.$$

In this work there has been a restriction to the positive sheet of the hyperboloid $k_0^2 - \vec{k}^2 = m^2$, where $k_0 = E$, by taking the positive sign for $(\vec{k}^2 + m^2)^{\frac{1}{2}}$ in the wave equation. Hence, one might put this into the expansion postulate by writing

$$|\psi\rangle = \int |\vec{k}\rangle \; \delta(\vec{k}^2 + m^2)\, \eta(k_0) \langle \vec{k}|\psi\rangle \, d\vec{k}\, dk_0$$

with $\eta(k_0) = 1$ for $k_0 > 0$ and $\eta(k_0) = 0$ for $k_0 < 0$. This leads in a natural manner to

$$|\psi\rangle = \int \frac{|\vec{k}\rangle \langle \vec{k}|\psi\rangle}{2(\vec{k}^2 + m^2)^{\frac{1}{2}}}\, d\vec{k}$$

and in the $\vec{x}$ -representation gives

$$\langle \vec{x}|\psi\rangle = \frac{1}{2(2\pi)^{\frac{3}{2}}} \int \frac{e^{i\vec{k}\cdot\vec{x}} \langle \vec{k}|\psi\rangle}{(\vec{k}^2 + m^2)^{\frac{1}{2}}}\, d\vec{k}$$

which is the form used by several authors for the relativistic Fourier transform. This seems to be an inconsistent approach from the present standpoint for the following reasons. If in

$$\langle \vec{x}|\psi\rangle = \frac{1}{2} \int \frac{\langle \vec{x}|\vec{k}\rangle \langle \vec{k}|\psi\rangle}{(\vec{k}^2 + m^2)^{\frac{1}{2}}}\, d\vec{k} = \frac{1}{2(2\pi)^{3/2}} \int \frac{e^{i\vec{k}\cdot\vec{x}} \langle \vec{k}|\psi\rangle}{(\vec{k}^2 + m^2)}\, d\vec{k}$$

one replaces $|\psi\rangle$ by $|\vec{k}'\rangle$, then

$$\langle \vec{x}|\vec{k}'\rangle = \frac{1}{2} \int \frac{\langle \vec{x}|\vec{k}\rangle \langle \vec{k}|\vec{k}'\rangle}{(\vec{k}^2 + m^2)^{\frac{1}{2}}}\, d\vec{k} = \frac{1}{2(2\pi)^{3/2}} \frac{e^{i\vec{k}'\cdot\vec{x}}}{(\vec{k}'^2 + m^2)^{\frac{1}{2}}}$$

which is a contradiction with what was used for $\langle \vec{x}|\vec{k}\rangle$ above. That is, $\langle \vec{x}|\vec{k}'\rangle$ should be equal to $(2\pi)^{-\frac{3}{2}} e^{i\vec{k}'\cdot\vec{x}}$ from $\hat{\vec{k}}$ being the generator of an infinitesimal $\delta\vec{x}$

displacement. Further, if one takes $|\psi\rangle = |\vec{x}'\rangle$, then in the $\vec{k}$ -representation

$$\langle\vec{k}'|\vec{x}'\rangle = \int \frac{\langle\vec{k}'|\vec{k}\rangle\langle\vec{k}|\vec{x}'\rangle}{2(\vec{k}^2+m^2)^{\frac{1}{2}}}\, d\vec{k} = \frac{\langle\vec{k}'|\vec{x}'\rangle}{2(\vec{k}^2+m^2)^{\frac{1}{2}}} ,$$

which is also a contradiction. Therefore, it seems preferable not to use the above form of the expansion postulate but to use $|\psi\rangle = \int |\vec{k}\rangle\langle\vec{k}|\psi\rangle\, d\vec{k}$.

In addition to the above alternatives it should be pointed out that one could also investigate the use of other representations of the mean-position observable. Numerous other possibilities exist in the works of Pryce (2) and Mathews (1).

The result of Chapter III concerning the new commutation relations has some interesting implications. It is well known that the mean-position observable satisfies the standard commutation relations and, yet, it has been shown in Chapter III that the mean-position observable generates new commutation relations if one considers the mean-position observable as the generator of $\delta\vec{X}$. These two facts are contradictory. The reason for this is that the point-position operator and the mean-position operator are related by

$$\hat{X}_i = \hat{x}_i + \frac{i}{2}\frac{\hat{k}_i}{(\hat{\vec{k}}^2+m^2)}$$

and if $\hat{X}_i$ is Hermitian, that is, $\hat{X}_i = \hat{X}_i^+$, it is clear that $\hat{x}_i \neq \hat{x}_i^+$. Hence, in the derivation of

$$[\hat{x}_i, k_j] = i\ \delta_{ij},$$

one uses the Hermitian nature of $\hat{x}_i$ and if $\hat{x}_i$ is no longer Hermitian, then one does not have the usual commutation relations for $\hat{x}_i$ with k_j . In particular, if $\hat{x}_i$ is not assumed Hermitian, then the resulting commutation relations are

$$[\hat{x}_i, \hat{k}_j] = i\ \delta_{ij} \quad ,$$

which is equivalent to the commutation relations for $\hat{X}_i$ with $\hat{k}_j$. There is still the rather surprising fact that the generator, $\hat{\mathcal{U}}_{\delta k}$, of a useful representation of the mean-position observable produces a condition on the mean-position observable that the representation does not satisfy. The origin of this, as already stated, is in $\hat{x}_i$ not being Hermitian and there there does not seem to be any possible alternative to this result.

In conclusion it may be said that it is possible to have a sensible, self-consistent, one-particle relativistic wave equation for

$$i\hbar\frac{\partial\psi}{\partial t} = (\hat{p}^2 + m^2)^{\frac{1}{2}}\psi$$

if one takes as fundamental the mean-position observable. Whether this can also be carried out for other relativistic wave equations is still undecided and, in fact, some of the implications from this case require further clarification, in particular the new commutation relations.

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