SPACES OF H-INTEGRABLE FUNCTIONS

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Chapter I

INTRODUCTION AND PRELIMINARY DISCUSSION

Introduction

In this thesis we consider integrals of a certain class of interval functions. Specifically we consider (Chapter II) a nondegenerate number interval [a,b], a real valued function m, defined and nondecreasing on [a,b], and the set H_m, of real valued functions f, defined on [a,b] such that

1) $f(a)=0$

2) for each subinterval [p,q] of [a,b], if $m(q) - m(p) = 0$, then $f(q) - f(p) = 0$

3) the set of all sums of the form $\sum_D \frac{(\Delta f)^2}{\Delta m}$ for subdivisions D of [a,b] is bounded above.

By means of a certain interval function integral, we define (Chapter II) an inner product $((\cdot,\cdot))_m$ for H_m. With respect to this inner product, we prove that H_m is a complete inner product space, in other words, a Hilbert space.

The remainder of the thesis is an examination of certain orthogonality and separability properties of H_m.

Preliminary Definitions and Theorems

Suppose that [a,b] is a number interval such that $a < b$.

<u>Definition 1</u>. - The statement "D is a subdivision of [a,b]" means

1) D is a finite set of number intervals [p,q] such that $a \leqq p < q \leqq b$

2) if I_1 and I_2 are distinct elements of D, then I_1 and I_2 have at most one point in common

3) if x is a number so that $a \leqq x \leqq b$, then x is in some element of D.

Definition 2. - The statement "D' is a refinement of a subdivision D of [a,b]" means that D' is a subdivision of [a,b], such that if x is an end point of some element of D, then x is an end point of some element of D'.

Suppose that [a,b] is a number interval and that H is a real valued function defined on $\{ I \mid I \text{ is a subinterval of } [a,b] \}$. We state the following theorem without proof.

Theorem 1. - If $a \leqq p < q \leqq b$, then there is no more than one number J, such that if c is a positive number, then there is a subdivision D of [p,q], such that if D' is a refinement of D, then $\left| J - \sum_{D'} H(I) \right| < c$.

If J is a number satisfying the conditions of Theorem 1 with respect to H and [p,q], then J will be called the integral of H on [p,q] and will be denoted by $\int_p^q H$. Throughout this thesis every integral considered will be the limit for refinements of subdivisions of the appropriate sums.

We also see that if $a \leqq r < w < s \leqq b$ and each of the integrals $\int_r^w H$ and $\int_w^s H$ exists (in the sense of Theorem 1), then $\int_r^s H$ exists and $\int_r^w H + \int_w^s H = \int_r^s H$.

At this point we adopt the convention that if each of x and y is a number, then $\frac{x}{y} = 0$ if $y = 0$ and $\frac{x}{y}$ has the usual meaning otherwise.

Theorem 2. - Suppose that [a,b] is a number interval and that each of f and g is a function, such that [a,b] is a subset of the common domain of f and g and such that g is non-decreasing on [a,b]. Suppose that if [p,q] is a subinterval of [a,b] and g(q) - g(p) = 0, then f(q) - f(p) = 0.

Then:

1) If E is a refinement of a subdivision D of a subinterval [p,q] of [a,b], then $\sum_D \frac{(\Delta f)^2}{\Delta g} \leqq \sum_E \frac{(\Delta f)^2}{\Delta g}$ (where $\sum_D \frac{(\Delta f)^2}{\Delta g}$ denotes the sum of $\frac{[f(s)-f(t)]^2}{g(s)-g(t)}$ over all elements [t,s] of D).

2) Suppose that [p,q] is a subinterval of [a,b]. The following three statements are equivalent:

a) There is a number M, such that if D is a subdivision of [p,q], then $\sum_D \frac{(\Delta f)^2}{\Delta g} \leqq M$.

b) There is a number J, such that if c is a positive number, then there is a subdivision D of [p,q], such that if E is a refinement of D, then $\left| J - \sum_E \frac{(\Delta f)^2}{\Delta g} \right| < c$. In this case by Theorem 1 there is only one such number J which in accordance with our convention we designate by $\int_p^q \frac{(df)^2}{dg}$.

c) There is a function h defined and nondecreasing on [p,q], such that if I is a subinterval of [p,q], then $(\Delta_I f)^2 \leqq (\Delta_I h)(\Delta_I g)$.

Proof - I. Suppose that [p,q] is a subinterval of [a,b] and that D is a subdivision of [p,q]. Suppose that E is a refinement of D and let $E_I = \{[s,t] \mid [s,t] \in E, [s,t] \subseteq I, I \in D\}$. Suppose that E_I has n elements. Let K = $= \{x \mid x \in [p,q], p \neq x \neq q, x$ is an end point of some element of $E_I\}$. K has n-1 elements. Let $k_1 = \min \{x \mid x \in K\}$.

There is a subdivision D_1 of $[p,q]$, such that $D_1 =$
$= \{[p,k_1], [k_1,q]\}$. Denote $[p,k_1]$ by I_1 and $[k_1 q]$
by I_1'.

a). Suppose that $\Delta_{I_1} g \, \Delta_{I_1'} g \neq 0$. Either
$\Delta_{I_1} g \, \Delta_{I_1'} f = \Delta_{I_1'} g \, \Delta_{I_1} f$ or $\Delta_{I_1} g \, \Delta_{I_1'} f \neq$
$\Delta_{I_1'} g \, \Delta_{I_1} f$. In either case

$$(\Delta_{I_1} g \, \Delta_{I_1'} f - \Delta_{I_1'} g \, \Delta_{I_1} f)^2 \geqq 0$$

$$(\Delta_{I_1} g \, \Delta_{I_1'} f)^2 - 2(\Delta_{I_1} g \, \Delta_{I_1'} f)(\Delta_{I_1'} g \, \Delta_{I_1} f) + \\ + (\Delta_{I_1'} g \, \Delta_{I_1} f)^2 \geqq 0$$

$$2(\Delta_{I_1} g \, \Delta_{I_1'} f)(\Delta_{I_1'} g \, \Delta_{I_1} f) \leqq (\Delta_{I_1} g \, \Delta_{I_1'} f)^2 + \\ + (\Delta_{I_1'} g \, \Delta_{I_1} f)^2$$

$$\Delta_{I_1} g \, \Delta_{I_1'} g \, (\Delta_{I_1} f)^2 + 2 \, (\Delta_{I_1} g \, \Delta_{I_1'} f)(\Delta_{I_1'} g \, \Delta_{I_1} f) + \\ + \Delta_{I_1'} g \, \Delta_{I_1} g \, (\Delta_{I_1'} f)^2 \leqq \Delta_{I_1} g \, \Delta_{I_1'} g \, (\Delta_{I_1} f)^2 + \\ + (\Delta_{I_1} g \, \Delta_{I_1'} f)^2 + (\Delta_{I_1'} g \, \Delta_{I_1} f)^2 + \\ + \Delta_{I_1'} g \, \Delta_{I_1'} g \, (\Delta_{I_1'} f)^2$$

$$(\Delta_{I_1} f + \Delta_{I_1'} f)^2 \, (\Delta_{I_1} g \, \Delta_{I_1'} g) \leqq \\ (\Delta_{I_1} g + \Delta_{I_1'} g) \, [\Delta_{I_1'} g(\Delta_{I_1} f)^2 + \Delta_{I_1} g(\Delta_{I_1'} f)^2]$$

$$\frac{(\Delta_{I_1} f + \Delta_{I_1'} f)^2}{\Delta_{I_1} g + \Delta_{I_1'} g} \leqq \frac{\Delta_{I_1'} g \, (\Delta_{I_1} f)^2 + \Delta_{I_1} g \, (\Delta_{I_1'} f)^2}{\Delta_{I_1} g \, \Delta_{I_1'} g}$$

$$\frac{(\Delta_{I_1} f + \Delta_{I_1'} f)^2}{\Delta_{I_1} g + \Delta_{I_1'} g} \leqq \frac{(\Delta_{I_1} f)^2}{\Delta_{I_1} g} + \frac{(\Delta_{I_1'} f)^2}{\Delta_{I_1'} g}$$

$$\frac{(\Delta_I f)^2}{\Delta_I g} \leqq \frac{(\Delta_{I_1} f)^2}{\Delta_{I_1} g} + \frac{(\Delta_{I_1'} f)^2}{\Delta_{I_1'} g}$$

b) Suppose that $\Delta_{I_1} g \; \Delta_{I_1'} g = 0$. One of the following is true:

i) $\Delta_{I_1} g = 0$, $\Delta_{I_1'} g \neq 0$,

ii) $\Delta_{I_1} g = 0$, $\Delta_{I_1'} g = 0$,

iii) $\Delta_{I_1} g \neq 0$, $\Delta_{I_1'} g = 0$. Due to the nature of Δf when $\Delta g = 0$, we have:

i) $$\frac{(\Delta_I f)^2}{\Delta_I g} = \frac{(\Delta_{I_1} f + \Delta_{I_1'} f)^2}{\Delta_{I_1} g + \Delta_{I_1'} g}$$

$$= \frac{(0 + \Delta_{I_1'} f)^2}{0 + \Delta_{I_1'} g}$$

$$\frac{(\Delta_I f)^2}{\Delta_I g} = \frac{(\Delta_{I_1'} f)^2}{\Delta_{I_1'} g}$$

ii) $$\frac{(\Delta_I f)^2}{\Delta_I g} = \frac{(0 + 0)^2}{0 + 0}$$

$$\frac{(\Delta_I f)^2}{\Delta_I g} = 0 \text{ by convention}$$

iii) $$\frac{(\Delta_I f)^2}{\Delta_I g} = \frac{(\Delta_{I_1} f + 0)^2}{\Delta_{I_1} g + 0}$$

$$\frac{(\Delta_I f)^2}{\Delta_I g} = \frac{(\Delta_I f)^2}{\Delta_I g} .$$

Now let $k_2 = \min \{ K - \{k_1\} \}$. There is a subdivision D_2 of $[k_1, q]$, such that $D_2 = \{ [k_1, k_2], [k_2, q] \}$. Denote $[k_1, k_2]$ by I_2 and $[k_2, q]$ by I_2'. Repeating a) and b) above for I_2 and I_2', we see that

$$\frac{(\Delta_{I_1'} f)^2}{\Delta_{I_1'} g} = \frac{(\Delta_{[k_1,q]} f)^2}{\Delta_{[k_1,q]} g} \leqq \frac{(\Delta_{I_2} f)^2}{\Delta_{I_2} g} + \frac{(\Delta_{I_2'} f)^2}{\Delta_{I_2'} g} .$$

Thus by induction we see that for $1 \leqq j \leqq n-1$ if $k_j = \min \{ K - \{k_1, \ldots, k_{j-1}\} \}$ and D_j is a subdivision of $[k_{j-1}, q]$ such that $D_j = \{ [k_{j-1}, k_j], [k_j, q] \}$, then $\dfrac{(\Delta_{I_{j-1}'} f)^2}{\Delta_{I_{j-1}'} g} \leqq \dfrac{(\Delta_{I_j} f)^2}{\Delta_{I_j} g} + \dfrac{(\Delta_{I_j'} f)^2}{\Delta_{I_j'} g}$. In addition

$$\frac{(\Delta_I f)^2}{\Delta_I g} \leqq \frac{(\Delta_{I_1} f)^2}{\Delta_{I_1} g} + \frac{(\Delta_{I_1'} f)^2}{\Delta_{I_1'} g} \leqq \cdots \leqq$$

$$\leqq \left[\sum_{j-1}^{n-1} \frac{(\Delta_{I_j} f)^2}{\Delta_{I_j} g} \right] + \frac{(\Delta_{I_{n-1}'} f)^2}{\Delta_{I_{n-1}'} g}$$

Therefore,

$\dfrac{(\Delta_I f)^2}{\Delta_I g} \leqq \sum_E \dfrac{(\Delta f)^2}{\Delta g}$. Now summing over all I in D we obtain $\sum_D \dfrac{(\Delta f)^2}{\Delta g} \leqq \sum_E \dfrac{(\Delta f)^2}{\Delta g}$.

II. Suppose that a) is true. Let $H = \left\{ z \mid z = \sum_D \frac{(\Delta f)^2}{\Delta g}\right.$ for some subdivision D of $[p,q] \Big\}$. H is bounded above by M. Thus there is a number J such that J is the least upper bound of H. Let c be a positive number. There is a subdivision D of [p,q], such that $\left| J - \sum_D \frac{(\Delta f)^2}{\Delta g} \right| < c.$

Let E be a refinement of D. By I $\sum_D \frac{(\Delta f)^2}{\Delta g} \leqq$

$\leqq \sum_E \frac{(\Delta f)^2}{\Delta g}$ and $\sum_E \frac{(\Delta f)^2}{\Delta g} \leqq J.$

Thus $\left| J - \sum_E \frac{(\Delta f)^2}{\Delta g} \right| < c.$

Suppose that b) is true. Let D be a subdivision of [p,q] and suppose that c is a positive number. There is a subdivision A of [p,q], such that if A' is a refinement of A, then

$\left| J - \sum_{A'} \frac{(\Delta f)^2}{\Delta g} \right| < c.$ Let B be the greatest common refinement of A and D. Then

$$\left| J - \sum_B \frac{(\Delta f)^2}{\Delta g} \right| < c$$

$$\left| \sum_B \frac{(\Delta f)^2}{\Delta g} - J \right| < c$$

$$\left| \sum_B \frac{(\Delta f)^2}{\Delta g} \right| - |J| < c$$

$$\left| \sum_B \frac{(\Delta f)^2}{\Delta g} \right| < |J| + c.$$

Since $\frac{(\Delta_I f)^2}{\Delta_I g} \geqq 0$ for all subintervals I of [p,q], it follows that

$\sum_{B} \frac{(\Delta f)^2}{\Delta g} \geqq 0$. Thus $\sum_{B} \frac{(\Delta_I f)^2}{\Delta_I g} < |J| + c$. Since B is a refinement of D, we see by I that $\sum_{D} \frac{(\Delta f)^2}{\Delta g} \leqq$

$\leqq \sum_{B} \frac{(\Delta f)^2}{\Delta g}$; therefore $\sum_{D} \frac{(\Delta f)^2}{\Delta g} < |J| + c$.

Let $|J| + c = M$.

Suppose that c) is true. Let D be a subdivision of [p,q]. For each I in D, $(\Delta_I f)^2 \leqq \Delta_I h \ \Delta_I g$. Thus

$\frac{(\Delta_I f)^2}{\Delta_I g} \leqq \Delta_I h$. Summing over all I in D, we have

$$\sum_{D} \frac{(\Delta f)^2}{\Delta g} \leqq \sum_{D} \Delta h$$

$$\sum_{D} \frac{(\Delta f)^2}{\Delta g} \leqq h(q) - h(p).$$

Denote h(q) - h(p) by M. Then a) is true.

Suppose that a) is true. Since a) is equivalent to b), $\int_s^t \frac{(df)^2}{dg}$ exists for every subinterval [s,t] of [a,b] and, therefore, also for every subinterval of [p,q]. If x is in [p,q], let h be the function defined by

$$h(x) = \begin{cases} 0, \text{ if } x = p \\ \int_p^x \frac{(df)^2}{dg}, \text{ if } p < x \leqq q. \end{cases}$$

Suppose that each of x and y is in [p,q], such that x < y.

$$h(y) - h(x) = \int_p^y \frac{(df)^2}{dg} - \int_p^x \frac{(df)^2}{dg}$$

$h(y) - h(x) = \int_x^y \frac{(df)^2}{dg} \geqq 0$, for if c is a positive number, then there are subdivisions A and B of [p,y] and [p,x] respectively, such that if A' and B' are refinements of A and B respectively, then

$$\left| J_1 - \sum_{A'} \frac{(\Delta f)^2}{\Delta g} \right| < c/2 \text{ and } \left| \sum_{B'} \frac{(\Delta f)^2}{\Delta g} - J_2 \right| < c/2,$$

where $J_1 = \int_p^y \frac{(df)^2}{dg}$ and $J_2 = \int_p^x \frac{(df)^2}{dg}$.

There is a refinement F of A such that x is an end point of some element of F. Let

$A_x = \{ I \mid I \in F, I \subseteq [p,x] \}$. Let B* be a common refinement of B and A_x, and let $D = B^* \cup [F - A_x]$. Suppose that D' is a refinement of D. If $D'_x = \{ I \mid I \in D', I \subseteq [p,x] \}$ and $D'_y = \{ I \mid I \in D', I \subseteq [x,y] \}$, then

$$\left| J_1 - \sum_{D'_x} \frac{(\Delta f)^2}{\Delta g} - \sum_{D'_y} \frac{(\Delta f)^2}{\Delta g} \right| < \frac{c}{2}, \text{ and}$$

$$\left| \sum_{D'_x} \frac{(\Delta f)^2}{\Delta g} - J_2 \right| < \frac{c}{2}.$$

Thus

$$\left| (J_1 - J_2) - \sum_{D'_y} \frac{(\Delta f)^2}{\Delta g} \right| < c, \text{ and}$$

$$\int_x^y \frac{(df)^2}{dg} = \int_p^y \frac{(df)^2}{dg} - \int_p^x \frac{(df)^2}{dg}.$$

Let $J_1 - J_2 = J$. Since each of the sums approximating J is a sum of nonnegative terms, $J \geqq 0$. $\{[x,y]\}$ is a subdivision of $[x,y]$, so that by a)

$$\frac{(f(y) - f(x))^2}{g(y) - g(x)} \leqq J = h(y) - h(x). \quad \text{Thus}$$

$$(f(y) - f(x))^2 \leqq (h(y) - h(x)) \; (g(y) - g(x)).$$

The following corollary is a consequence of Theorem 2.

Corollary 1. - If $\int_p^q \frac{(df)^2}{dg}$ exists, then f is of bounded variation on $[p,q]$.

Proof. - Suppose that D is a subdivision of $[p,q]$. Then by Theorem 2, there is a function h defined and nondecreasing on $[p,q]$, such that if I is in D, then $(\Delta_I f)^2 \leqq \Delta_I h \, \Delta_I g$. Since each of h and g is nondecreasing on $[p,q]$, $\Delta_I h \geqq 0$ and $\Delta_I g \geqq 0$ for each I in D. Therefore, $\Delta_I h \, \Delta_I g \geqq 0$. In addition, $(\Delta_I f)^2 \geqq 0$ for each I in D. Thus

$$|\Delta_I f| = \sqrt{(\Delta_I f)^2} \leqq \sqrt{\Delta_I h \, \Delta_I g} = \sqrt{\Delta_I h} \, \sqrt{\Delta_I g}.$$

Summing over all I in D, we have

$$\sum_D |\Delta f| \leqq \sum_D \sqrt{\Delta h} \, \sqrt{\Delta g} \text{ and then}$$

$$\left(\sum_D |\Delta f| \right)^2 \leqq \left(\sum_D \sqrt{\Delta h} \, \sqrt{\Delta g} \right)^2. \quad \text{By the}$$

Schwarz inequality

$$\left(\sum_D \sqrt{\Delta h} \, \sqrt{\Delta g} \right)^2 \leqq \sum_D \left(\sqrt{\Delta h} \right)^2 \sum_D \left(\sqrt{\Delta g} \right)^2,$$

or

$$\left(\sum_D |\Delta f| \right)^2 \leqq \sum_D \Delta h \sum_D \Delta g. \quad \text{Now}$$

$\sum_D \Delta h = h(q) - h(p)$ and $\sum_D \Delta g = g(q) - g(p)$. Let $h(q) - h(p) = J_1$ and $g(q) - g(p) = J_2$. Then

$(\sum_D |\Delta f|)^2 \leqq J_1 J_2$ and since $0 \leqq (\sum_D |\Delta f|)^2$,

$$\sum_D |\Delta f| \leqq \sqrt{J_1 J_2}.$$

Theorem 3. Suppose that $[a,b]$ is a number interval and that each of m, f and g is a function, such that $[a,b]$ is a subset of the common domain of m, f and g, with m non-decreasing on $[a,b]$, such that if $a \leqq p < q \leqq b$ and $m(q) - m(p) = 0$, then $f(q) - f(p) = 0$ and $g(q) - g(p) = 0$. If $a \leqq s < t \leqq b$ and each of $\int_s^t \frac{(df)^2}{dm}$ and $\int_s^t \frac{(dg)^2}{dm}$ exist, then $\int_s^t \frac{[d(f+g)]^2}{dm}$ exists.

Proof.- There are numbers J_1 and J_2, such that if D is a subdivision of $[s,t]$, then $\sum_D \frac{(\Delta f)^2}{\Delta m} \leqq J_1$ and $\sum_D \frac{(\Delta g)^2}{\Delta m} \leqq J_2$. For each I in D, $\Delta m_I \geqq 0$. Thus there is a number $\sqrt{\Delta m_I} \geqq 0$, such that $(\sqrt{\Delta m_I})^2 = \Delta m_I$. Then $\left[\sum_D \left(\frac{\Delta f}{\sqrt{\Delta m}}\right)^2\right] \leqq J_1$ and $\left[\sum_D \left(\frac{\Delta g}{\sqrt{\Delta m}}\right)^2\right] \leqq J_2$. Thus

$$\left[\sum_D \left(\frac{\Delta f}{\sqrt{\Delta m}}\right)^2\right] \left[\sum_D \left(\frac{\Delta g}{\sqrt{\Delta m}}\right)^2\right] \leqq J_1 \left[\sum_D \left(\frac{\Delta g}{\sqrt{\Delta m}}\right)^2\right],$$

$J_1 \left[\sum_D \left(\frac{\Delta g}{\sqrt{\Delta m}}\right)^2\right] \leqq J_1 J_2$ and $\left[\sum_D \left(\frac{\Delta f}{\sqrt{\Delta m}}\right)^2\right]\left[\sum_D \left(\frac{\Delta g}{\sqrt{\Delta m}}\right)^2\right] \leqq$

$\leqq J_1 J_2$. By the Schwarz inequality, $\left[\sum_D \left(\frac{\Delta f}{\sqrt{\Delta m}}\right) \left(\frac{\Delta g}{\sqrt{\Delta m}}\right)\right]^2 \leqq$

$\leqq \left[\sum_D \left(\frac{\Delta f}{\sqrt{\Delta m}}\right)^2\right] \left[\sum_D \left(\frac{\Delta g}{\sqrt{\Delta m}}\right)^2\right]$. Therefore $\left[\sum_D \frac{\Delta f \Delta g}{\Delta m}\right]^2 \leqq$

$\leqq J_1 J_2$. Since each side of the preceeding inequality is nonnegative,

$$\left|\sum_D \frac{\Delta f \, \Delta g}{\Delta m}\right| \leqq \sqrt{J_1 J_2}. \text{ Let } J_3 = \sqrt{J_1 J_2}.$$

Let A be a subdivision of [s,t]. Consider the sum

$$\sum_A \frac{[\Delta(f+g)]^2}{\Delta m}.$$

$$\sum_A \frac{[\Delta(f+g)]^2}{\Delta m} = \sum_A \frac{(\Delta f + \Delta g)^2}{\Delta m}$$

$$\sum_A \frac{[\Delta(f+g)]^2}{\Delta m} = \sum_A \frac{(\Delta f)^2 + 2\,\Delta f\,\Delta g + (\Delta g)^2}{\Delta m}$$

$$\sum_A \frac{[\Delta(f+g)]^2}{\Delta m} = \sum_A \frac{(\Delta f)^2}{\Delta m} + 2 \sum_A \frac{\Delta f\,\Delta g}{\Delta m} + \sum_A \frac{(\Delta g)^2}{\Delta m}$$

$$\sum_A \frac{[\Delta(f+g)]^2}{\Delta m} \leqq \sum_A \frac{(\Delta f)^2}{\Delta m} + 2 \left|\sum_A \frac{\Delta f\,\Delta g}{\Delta m}\right| + \sum_A \frac{(\Delta g)^2}{\Delta m}$$

$$\sum_{A} \frac{[\Delta(f+g)]^2}{\Delta m} \leqq J_1 + 2\, J_3 + J_2.$$

Thus by theorem 2, $\int_s^t \frac{[d(f+g)]^2}{dm}$ exists.

Corollary 2. - Under the hypothesis of Theorem 2, there is a number J, such that if $0 < c$, then there is a subdivision D of [s,t], such that if D' is a refinement of D, then

$$\left| J - \sum_{D'} \frac{\Delta f \, \Delta g}{\Delta m} \right| < c.$$ In this case J is unique.

Proof. - Let c be a positive number. Suppose that $\int_s^t \frac{(df)^2}{dm} = J_2$ and $\int_s^t \frac{(dg)^2}{dm} = J_3$. By Theorem 3,

$\int_s^t \frac{[d(f+g)]^2}{dm}$ exists and has the value J_1.

There are subdivisions A, B, and C of [s,t], such that if A', B', and C' are refinements of A, B, and C respectively, then

$$\left| J_1 - \sum_{A'} \frac{(\Delta f + \Delta g)^2}{\Delta m} \right| < \frac{2c}{3}$$

$$\left| \sum_{B'} \frac{(\Delta f)^2}{\Delta m} - J_2 \right| < \frac{2c}{3}$$

$$\left| \sum_{C'} \frac{(\Delta g)^2}{\Delta m} - J_3 \right| < \frac{2c}{3}$$

. Let D be the greatest common refinement of A, B, and C. Then if D' is a refinement of D,

$$\left| J_1 - \sum_{D'} \frac{(\Delta f + \Delta g)^2}{\Delta m} \right| + \left| \sum_{D'} \frac{(\Delta f)^2}{\Delta m} - J_2 \right| +$$

$$+ \left| \sum_{D'} \frac{(\Delta g)^2}{\Delta m} - J_3 \right| < \frac{2c}{3} + \frac{2c}{3} + \frac{2c}{3} = 2c.$$

$$\left| J_1 - J_2 - J_3 - \sum_{D'} \frac{(\Delta f + \Delta g)^2}{\Delta m} + \sum_{D'} \frac{(\Delta f)^2}{\Delta m} + \right.$$

$$\left. + \sum_{D'} \frac{(\Delta g)^2}{\Delta m} \right| < 2c.$$

$$\left| J_1 - J_2 - J_3 - \right.$$

$$\left. - \sum_{D'} \frac{(\Delta f)^2 + 2 \Delta g \Delta f + (\Delta g)^2 - (\Delta f)^2 - (\Delta g)^2}{\Delta m} \right| < 2c$$

$$\left| J_1 - J_2 - J_3 - 2 \sum_{D'} \frac{\Delta f \Delta g}{\Delta m} \right| < 2c$$

$$\frac{1}{2} \left| J_1 - J_2 - J_3 - 2 \sum_{D'} \frac{\Delta f \Delta g}{\Delta m} \right| < c$$

$$\left| \frac{1}{2} (J_1 - J_2 - J_3) - \sum_{D'} \frac{\Delta f \Delta g}{\Delta m} \right| < c.$$

Therefore let $J = \frac{1}{2} (J_1 - J_2 - J_3)$ and $\int_s^t \frac{df dg}{dm} =$

$$= \frac{1}{2} \left[\int_s^t \frac{[d(f+g)]^2}{dm} - \int_s^t \frac{(df)^2}{dm} - \int_s^t \frac{(dg)^2}{dm} \right]$$

exists. Uniqueness follows from Theorem 1.

Chapter II

CHARACTERIZATION OF THE CLASS H_m OF INTERVAL FUNCTIONS

Suppose that [a,b] is a number interval and that m is a real valued function defined and nondecreasing on [a,b] such that $m(a) \neq m(b)$.

Definition 3 - If m is the function with the properties specified above, then H_m denotes the set of all real valued functions f defined on [a,b] such that

1) $f(a)=0$

2) if [p,q] is a subinterval of [a,b] and $m(q)-m(p)=0$, then $f(q)-f(p)=0$

3) the set of all sums of the form $\sum_D \frac{(\Delta f)^2}{\Delta m}$ for subdivisions D of [a,b] is bounded above.

We note that by Theorem 2, if f is in H_m, then $\int_p^q \frac{(df)^2}{dm}$ exists for each subinterval [p,q] of [a,b].

Definition 4 - If each of f and g is in H_m, then f+g is the function whose domain contains [a,b] such that for each x in [a,b], $(f+g)(x)=f(x)+g(x)$.

Definition 5 - If f is in H_m and k is a real number, then kf is the function whose domain contains [a,b] such that for each x in [a,b], $(kf)(x)=k(f(x))$.

We now show that H_m is a linear space with operations of addition and scalar multiplication as defined in definitions 4 and 5 and with the set of real numbers as its scalar field.

Theorem 4 - If each of f, g, and h is in H_m and each of k, k_1, and k_2 is a real number, then the following statements are true:

1) $(f+g) \epsilon H_m$

2) $f+g=g+f$

3) $f+(g+h)=(f+g)+h$

4) there is an element θ in H_m such that if f is in H_m, then $f+\theta=f$.

5) $kf \ \epsilon \ H_m$

6) $k(f+g)=kf+kg$

7) $k_1(k_2f)=k_1k_2f$

8) $(k_1+k_2)f=k_1f+k_2f$

9) the following two statements are equivalent:

i) $kf=\theta$

ii) $k=0$ or $f=\theta$, where 0 has the usual meaning.

Proof -

1) $f(a)=0$ and $g(a)=0$ so that $(f+g)(a)=0$. Suppose that $[p,q]$ is a subinterval of $[a,b]$ such that $m(q)-m(p)=0$. Then $f(q)-f(p)=0$ and $g(q)-g(p)=0$ so that $(f(q)-f(p))-(g(q)-g(p))=0$ and $(f(q)+g(q))-(f(p)+g(p))=(f+g)(q) - (f+g)(p)=0$. By Theorem 3, $\int_a^b \frac{[d(f+g)]^2}{dm}$ exists and by Theorem 2 the set of all sums of the form $\sum_D \frac{[\Delta(f+g)]^2}{\Delta m}$ for subdivisions D of $[a,b]$ is bounded above. Thus f+g is in H_m. Uniqueness follows from the fact that each of f, g, and f+g is real

valued. Statements 2) and 3) follow directly from the commutative and associative properties respectively of the real numbers.

4) Let $\Theta(x)=0$ for every x in [a,b]. 1) and 2) of Definition 3 are obviously satisfied. Let D be a subdivision of [a,b]. $\sum_D \frac{[\Delta\Theta]^2}{\Delta m} = \sum_D 0=0$. Thus Θ is in H_m.

$$(f+g)(x)=f(x)+\Theta(X)$$
$$=f(x)+0$$
$$(f+g)(x)=f(x).$$

5) Suppose that k is a real number and that f is in H_m. Consider the function kf. $(kf)(a)=k(f(a))$
$=k(0)$
$(kf)(a)=0$. Suppose that [p,q] is a subinterval of [a,b] such that $m(q)-m(p)=0$. Then

$$kf(q)-kf(p)=k(f(q)-f(p))$$
$$=k(0)$$

$kf(q)-kf(p)=0$. Suppose that D is a subdivision of [a,b]. Consider $\sum_D \frac{[\Delta(kf)]^2}{\Delta m}$. There is a number M such that if A is a subdivision of [a,b], $\sum_A \frac{[\Delta f]^2}{\Delta m} \leqq M$. Then

$$\sum_A \frac{[\Delta(kf)]^2}{\Delta m} = \sum_{[s,t]\epsilon A} \frac{[kf(t)-kf(s)]^2}{m(t)-m(s)}$$
$$= \sum_{[s,t]\epsilon A} \frac{k^2[f(t)-f(s)]^2}{m(t)-m(s)}$$
$$= k^2\sum_A \frac{[\Delta f]^2}{\Delta m} \leqq k^2M.$$ Thus kf is in H_m

Properties 6), 7), and 8) follow from the parallel properties of the real numbers.

9) Suppose that ii) is true. If $k = 0$, then for any x in [a,b] $kf(x)=0(f(x))=0$. If $f=\Theta$, then $kf(x)=k\Theta(x)=k(0)=0$. In either case $kf(x)=\Theta(x)$ for every x in [a,b]. Suppose that i) is true. If $k=0$, then ii) is true. Suppose that $k\neq 0$. Then since $kf(x)=0$ for each x in [a,b], $f(x)=0$ for each x in [a,b] which implies that $f=\Theta$.

Definition 6 - If each of f and g is in H_m, we define $\int_a^b \frac{dfdg}{dm}$ to be the inner product of f and g with respect to m and denote the integral by $((f,g))_m$.

The following theorem justifies the preceeding definition and establishes the fact that H_m is an inner product space.

Theorem 5 - If each of f and g is in H_m and k is a number, then the following statements are true:

1) $((f,g))_m$ is a real number
2) $((f,f))_m \geqq 0$ and $((f,f))_m=0$ if and only if $f=\Theta$
3) $((f,g))_m=((g,f))_m$
4) $((f+g,h))_m=((f,h))_m+((g,h))_m$
5) $((f,kg))_m=k((f,g))_m$.

Proof - 1) is true since $\int_a^b \frac{dfdg}{dm}$ is a real number.

2) $((f,f))_m=\int_a^b \frac{dfdf}{dm} = \int_a^b \frac{(df)^2}{dm}$. Since for any subdivision D of [a,b], $\sum_D \frac{(\Delta f)^2}{\Delta m}$ is nonnegative, we see by the proof of Theorem 2 that $0 \leqq \int_a^b \frac{(df)^2}{dm}$. Suppose that $f=\Theta$. Then if D is a subdivision of [a,b], $\sum_D \frac{(\Delta f)^2}{\Delta m} = \sum_D \frac{0}{\Delta m} = 0$ from which we deduce that $\int_a^b \frac{(df)^2}{dm}=0$. Suppose that

$((f,f))_m=0$, that $a<x\leqq b$, and let D be a subdivision of [a,b] such that $[a,x]\epsilon D$. Since $0 \leqq \sum_D \frac{(\Delta f)^2}{\Delta m} \leqq \int_a^b \frac{(df)^2}{dm} = 0$ we see that each term is identically zero so that

$\frac{(f(x)-f(a))^2}{m(x)-m(a)} = 0$. If $m(x)-m(a)=0$, then $f(x)-f(a)=0$ and $f(x)=f(a)=0$. If $m(x)-m(a)\neq 0$, then $(f(x)-f(a))^2=0$ and $f(x)-f(a)=0$ which means that $f(x)=0$. Thus $f(x)$ is identically zero for all x in [a,b].

3) Statement 3) follows directly from the commutative property of the real numbers.

4) By Theorem 2 each of $\int_a^b \frac{(df)^2}{dm}$, $\int_a^b \frac{(dg)^2}{dm}$, and $\int_a^b \frac{(dh)^2}{dm}$ exists and by Theorem 3 each of $\int_a^b \frac{[d(f+g)]^2}{dm}$, $\int_a^b \frac{dfdh}{dm}$, $\int_a^b \frac{dgdh}{dm}$ and $\int_a^b \frac{d(f+g)dh}{dm}$ exists. Suppose that c is a positive number. There are subdivisions A, B, and C of [a,b] such that if A', B', and C' are refinements of A, B, and C respectively, then $\left|\int_a^b \frac{dfdh}{dm} - \sum_{A'} \frac{\Delta f\Delta h}{\Delta m}\right| < c/3$, $\left|\int_a^b \frac{dgdh}{dm} - \sum_{B'} \frac{\Delta g\Delta h}{\Delta m}\right| < c/3$, and $\left|\int_a^b \frac{d(f+g)dh}{dm} - \sum_{C'} \frac{\Delta(f+g)\Delta h}{\Delta m}\right| < c/3$. Let D be a common refinement of A, B, and C and suppose that D' is a refinement of D. Then

$$\left|\int_a^b \frac{dfdh}{dm} - \sum_{D'} \frac{\Delta f\Delta h}{\Delta m}\right| + \left|\int_a^b \frac{dgdh}{dm} - \sum_{D'} \frac{\Delta g\Delta h}{\Delta m}\right| + \left|\sum_{D'} \frac{\Delta(f+g)\Delta h}{\Delta m} - \int_a^b \frac{d(f+g)dh}{dm}\right| < c$$

and

$$\left| \left(\int_a^b \frac{df\,dh}{dm} + \int_a^b \frac{dg\,dh}{dm} - \int_a^b \frac{d(f+g)\,dh}{dm} \right) - \left(\sum_{D'} \frac{\Delta f \Delta h}{\Delta m} + \sum_{D'} \frac{\Delta g \Delta h}{\Delta m} - \sum_{D'} \frac{\Delta (f+g) \Delta h}{\Delta m} \right) \right| < c$$

Since $\sum_{D'} \frac{\Delta f \Delta h}{\Delta m} + \sum_{D'} \frac{\Delta g \Delta h}{\Delta m} = \sum_{D'} \frac{\Delta f \Delta h + \Delta g \Delta h}{\Delta m} = \sum_{D'} \frac{\Delta (f+g) \Delta h}{\Delta m}$,

$$\left| \int_a^b \frac{df\,dh}{dm} + \int_a^b \frac{dg\,dh}{dm} - \int_a^b \frac{d(f+g)\,dh}{dm} \right| < c, \text{ therefore}$$

$$\int_a^b \frac{df\,dh}{dm} + \int_a^b \frac{dg\,dh}{dm} = \int_a^b \frac{d(f+g)\,dh}{dm} .$$

5) Consider $\int_a^b \frac{df\,d(kg)}{dm}$. There are subdivisions A and B of [a,b] such that if A' and B' are refinements of A and B respectively, then $\left| \int_a^b \frac{df\,d(kg)}{dm} - \sum_{A'} \frac{\Delta f \Delta (kg)}{\Delta m} \right| < c/2$ and $\left| \sum_{B'} \frac{\Delta f \Delta g}{\Delta m} - \int_a^b \frac{df\,dg}{dm} \right| < c/[2(|k|+1)]$. Let D be a common refinement of A and B and suppose that D' is a refinement of D. $|k| \left| \sum_{D'} \frac{\Delta f \Delta g}{\Delta m} - \int_a^b \frac{df\,dg}{dm} \right| < \frac{c\,|k|}{2(|k|+1)} \leq c/2$, so that $\left| \sum_{D'} \frac{\Delta f \Delta (kg)}{\Delta m} - k \int_a^b \frac{df\,dg}{dm} \right| < c/2$. Then

$$\left| \int_a^b \frac{df\,d(kg)}{dm} - \sum_{D'} \frac{\Delta f \Delta (kg)}{\Delta m} \right| + \left| \sum_{D'} \frac{\Delta f \Delta (kg)}{\Delta m} - k \int_a^b \frac{df\,dg}{dm} \right| < c \text{ and } \left| \int_a^b \frac{df\,d(kg)}{dm} - k \int_a^b \frac{df\,dg}{dm} \right| < c.$$

Therefore $\int_a^b \frac{df\,d(kg)}{dm} = k \int_a^b \frac{df\,dg}{dm}$.

<u>Definition 7</u> - If f is in H_m, we define the norm of f with respect to m, denoted by $\|f\|_m$, by $\|f\|_m = \sqrt{((f,f))_m}$.

It is a well known consequence of the properties of a linear space in which an inner product and a norm have been defined that the following inequalities are true for elements f, g, and h of the space:

1) Schwarz inequality: $|((f,g))_m| \leqq \|f\|_m \|g\|_m$
2) Minkowski inequality: $\|f+g\|_m \leqq \|f\|_m + \|g\|_m$
3) Triangle inequality: $\|f-g\|_m \leqq \|f-h\|_m + \|h-g\|_m$
4) $\left| \|f\|_m - \|g\|_m \right| \leqq \|f-g\|_m$.

<u>Lemma 1</u> - Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions in H_m such that if D is a subdivision of [a,b], then

$$\sum_D |\Delta(f_p-f_q)| \rightarrow 0 \text{ as } \min\{p,q\} \rightarrow \infty. \text{ Then } \{f_n\}_{n=1}^{\infty}$$

converges pointwise for each x in [a,b].

<u>Proof</u> - Let x be an element of [a,b]. If x=a, then for all positive integers n, $f_n(x)=f_n(a)=0$ which gives us convergence trivally for x=a. Suppose that $a < x \leqq b$ and let c be a positive number. There is a subdivision D of [a,b] such that [a,x] ϵ D. There is a positive number N such that if each of p and q is a positive integer, and $N < \min\{p,q\}$, then $\left|\sum_D |\Delta(f_p-f_q)| - 0\right| < c$ or since the sum is nonnegative, $\sum_D |\Delta(f_p-f_q)| < c$. Since $|(f_p(x)-f_q(x))-(f_p(a)-f_q(a))|$ is a term of the previous sum, $|(f_p(x)-f_q(x))-(f_p(a)-f_q(a))| < c$. Now $f_p(a)-f_q(a)=$ 0-0=0 so that $|f_p(x)-f_q(x)| < c$. Thus for each x we conclude that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence and has a limit. Therefore there is a function g whose domain contains [a,b] such that $f_n(x) \rightarrow g(x)$ as $n \rightarrow \infty$ for each x in [a,b].

Lemma 2 - Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of elements of H_m such that $\|f_p-f_q\|_m \to 0$ as $\min\{p,q\}\to\infty$. Then the set $R=\{z \mid z=\|f_n\|_m$, n a positive integer, $f_n \in H_m\}$ is bounded.

Proof - Since for each positive integer n, $\|f_n\|_m \geqq 0$, R is bounded below by 0. There is a positive number N such that if each of p and q is a positive integer and $N < \min\{p,q\}$, then $\left|\|f_p-f_q\|_m - 0\right| = \|f_p-f_q\|_m < 1$. Let p* be the least positive integer greater than N and q be any positive integer greater than N. Then $\left|\|f_q\|_m - \|f_{p*}\|_m\right| \leqq \|f_q-f_{p*}\|_m < 1$ and therefore $\|f_q\|_m < \|f_{p*}\|_m + 1$.
Let $M=\max\{\|f_1\|_m, \|f_2\|_m, \ldots, \|f_{p*-1}\|_m, \|f_{p*}\|_m+1\}$. R is bounded above by M.

Theorem 6 - Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of elements of H_m such that $\|f_p-f_q\|_m \to 0$ as $\min\{p,q\}\to\infty$. Then there is a function g in H_m such that $\|f_p-g\|_m \to 0$ as $p\to\infty$.

Proof - Let c be a positive number. There is a positive number N such that if each of p and q is a positive integer such that $N < \min\{p,q\}$, then

$$\|f_p-f_q\|_m < \frac{c}{\sqrt{m(b)-m(a)}}.$$

By theorem 2 there is a function h such that

$$h(x) = \begin{cases} 0, & \text{if } x=a \\ \int_a^x \frac{[d(f_p-f_q)]^2}{dm}, & \text{if } a < x \leqq b. \end{cases}$$

By the corollary of Theorem 2, for any subdivision D of [a,b],

$$\sum_D \left|\Delta(f_p-f_q)\right| \leqq \sqrt{[m(b)-m(a)] \int_a^b \frac{[d(f_p-f_q)]^2}{dm}} =$$

$$= \sqrt{m(b)-m(a)}\ \|f_p-f_q\|_m \quad \text{where} \int_a^b \frac{[d(f_p-f_q)]^2}{dm} = h(b)-h(a).$$

Thus

$$\frac{1}{\sqrt{m(b)-m(a)}} \sum_D \left|\Delta(f_p-f_q)\right| \leqq \|f_p-f_q\|_m < \frac{c}{\sqrt{m(b)-m(a)}}$$

so that $\sum_D \left|\Delta(f_p-f_q)\right| < c$, which implies by Lemma 1 that if $a \leqq x \leqq b$, then $\left|f_p(x)-f_q(x)\right| \to \theta(x)$ as $\min\{p,q\} \to \infty$. Thus there is a function g such that if x is in [a,b], then $f_n(x) \to g(x)$ as $n \to \infty$. Since $f_n(a)=0$ for all positive integers n, it follows that $g(a)=0$. Suppose that [s,t] is a subinterval of [a,b] such that $m(t)-m(s)=0$. For each positive integer n, $f_n(t)-f_n(s)=0$. There are positive numbers N_s and N_t such that $\left|f_j(s)-g(s)\right| < c/2$ and $\left|g(t)-f_k(t)\right| < c/2$ if $N_t < k$ and $N_s < j$. Let $N=\max\{N_t,N_s\}$. If r is a positive integer and $N < r$, then $\left|f_r(s)-g(s)\right| < c/2$ and $\left|g(t)-f_r(t)\right| < c/2$ so that

$$\left|g(t)-f_r(t)\right| + \left|f_r(s)-g(s)\right| < c \text{ and}$$

$$\left|g(t)-g(s)\right| = \left|g(t)-f_r(t)+f_r(s)-g(s)\right| < c. \text{ Thus}$$

$$g(t)-g(s)=0.$$

Let D be a subdivision of [a,b] and d the number of elements of D. Suppose that I=[s,t] is an element of D and $0 < \Delta_I m$.

There are positive numbers N_1 and N_2 such that if each of p and q is a positive integer and $N_1 < p$ and $N_2 < q$, then $\left| g(s) - f_p(s) \right| < \frac{W^{1/2}}{2}$ and $\left| f_q(t) - g(t) \right| < \frac{W^{1/2}}{2}$, where $W = \frac{c\,\Delta_I m}{d}$. Let $N_I = \max\{N_1, N_2\}$. If n_I is a positive integer and $N_I < n_I$, then

$$\left|\Delta_I g - \Delta_I f_{n_I}\right| = \left|(g(s)-g(t))-(f_{n_I}(s)-f_{n_I}(t))\right|$$

$$\leqq \left|g(s)-f_{n_I}(s)\right| + \left|f_{n_I}(t)-g(t)\right|$$

$\left|\Delta_I g - \Delta_I f_{n_I}\right| < W^{1/2}$, from which we obtain

$\left|\Delta_I g\right| - \left|\Delta_I f_{n_I}\right| < W^{1/2}$ or $\left|\Delta_I g\right| < \left|\Delta_I f_{n_I}\right| + W^{1/2}$ and

$$\frac{(\Delta_I g - \Delta_I f_{n_I})^2}{\Delta_I m} < \frac{W}{\Delta_I m} = \frac{c}{d}.$$ Now

$$\frac{(\Delta_I g - \Delta_I f_{n_I})^2}{\Delta_I m} = \frac{(\Delta_I g)^2}{\Delta_I m} - \frac{2(\Delta_I g)(\Delta_I f_{n_I})}{\Delta_I m} + \frac{(\Delta_I f_{n_I})^2}{\Delta_I m} < \frac{c}{d}$$

$$\frac{(\Delta_I g)^2}{\Delta_I m} < \frac{2(\Delta_I g)(\Delta_I f_{n_I})}{\Delta_I m} - \frac{(\Delta_I f_{n_I})^2}{\Delta_I m} + \frac{c}{d}$$

$$< \frac{2\left|\Delta_I g\right| \left|\Delta_I f_{n_I}\right|}{\Delta_I m} - \frac{(\Delta_I f_{n_I})^2}{\Delta_I m} + \frac{c}{d}$$

$$< \frac{2\left|\Delta_I f_{n_I}\right| \left(\left|\Delta_I f_{n_I}\right| + W^{1/2}\right)}{\Delta_I m} - \frac{(\Delta_I f_{n_I})^2}{\Delta_I m} + \frac{c}{d}$$

$$\frac{(\Delta_I g)^2}{\Delta_I m} < 2\frac{(\Delta_I f_{n_I})^2}{\Delta_I m} + 2\left(\frac{c}{d}\right)^{1/2}\frac{|\Delta_I f_{n_I}|}{(\Delta_I m)^{1/2}} - \frac{(\Delta_I f_{n_I})^2}{\Delta_I m} + \frac{c}{d}$$

Let $N_D = \max\{N_I \mid I \in D\}$. Then if n is a positive integer and $N_D < n$,

$$\sum_D \frac{(\Delta g)^2}{\Delta m} < \sum_D \frac{(\Delta f_n)^2}{\Delta m} + [2(c)^{1/2}] \sum_D \frac{|\Delta f_n|}{(\Delta m)^{1/2}(d)^{1/2}} + \sum_D \frac{c}{d}$$

which by the Schwarz inequality does not exceed

$$\sum_D \frac{(\Delta f_n)^2}{\Delta m} + [2(c)^{1/2}]\sqrt{\sum_D \frac{(\Delta f_n)^2}{\Delta m}}\sqrt{\sum_D \frac{1}{d}} + c.$$ Then

$$\sum_D \frac{(\Delta g)^2}{\Delta m} < \sum_D \frac{(\Delta f_n)^2}{\Delta m} + [2(c)^{1/2}]\sqrt{\sum_D \frac{(\Delta f_n)^2}{\Delta m}} + c$$ so that

$$\sum_D \frac{(\Delta g)^2}{\Delta m} < (\|f_n\|_m)^2 + [2(c)^{1/2}]\,\|f_n\|_m + c.$$

By Lemma 2 there is a number M such that $\|f_n\|_m \leqq M$ for every n. Thus

$$\sum_D \frac{(\Delta g)^2}{\Delta m} < M^2 + 2M(c)^{1/2} + c.$$ Therefore g is in H_m.

Suppose that c is a positive number. There is a positive number N' such that if each of p and q is a positive integer and $N' < \min\{p,q\}$, then

$$\sqrt{\int_a^b \frac{[d(f_p-f_q)]^2}{dm}} < c/2 \text{ so that } \int_a^b \frac{[d(f_p-f_q)]^2}{dm} < c^2/4.$$

Thus for any subdivision B of [a,b].

$$\sum_B \frac{[\Delta(f_p-f_q)]^2}{\Delta m} < c^2/4.$$ Let D be a subdivision of [a,b]

and consider $\sum_D \frac{[\Delta(g-f_n)]^2}{\Delta m}$ for $n > N'$. For each I in D let

N_I be a positive number such that if n_I is a positive integer and $n_I > N_I$ then $\frac{[\Delta(g-f_{n_I})]^2}{\Delta_I{}^m} < \frac{c^2}{4d}$ where d is the number of elements in D. If $N_D = \max\left\{N_I \mid I \epsilon D\right\}$, then for $n' > N_D$

$$\sum_D \frac{[\Delta(g-f_{n'})]^2}{\Delta m} < \frac{c^2}{4}.$$

Let n* be a positive integer such that $n^* > \max\{N', N_D\}$ Then

$$\sum_D \frac{[\Delta(g-f_n)]^2}{\Delta m} = \sum_{[s,t]\epsilon D} \frac{[(g(t)-f_n(t))-(g(s)-f_n(s))]^2}{\Delta m}$$

$$= \sum [(g(t)-f_n(t))+(f_{n*}(t)-f_{n*}(t)) -$$

$$-(g(s)-f_n(s))+(f_{n*}(s)-f_{n*}(s))]^2/\Delta m$$

$$= \sum [(g(t)-f_{n*}(t))-(g(s)-f_{n*}(s)) +$$

$$+(f_{n*}(t)-f_n(t))-(f_{n*}(s)-f_n(s))]^2/\Delta m$$

$$= \sum_D \frac{[\Delta(g-f_{n*}) + \Delta(f_{n*}-f_n)]^2}{\Delta m}$$

$$= \sum_D \frac{[\Delta(g-f_{n*})]^2}{\Delta m} + 2 \sum_D \frac{[\Delta(g-f_{n*})][\Delta(f_{n*}-f_n)]}{\Delta m} +$$

$$+ \sum_D \frac{[\Delta(f_{n*}-f_n)]^2}{\Delta m} \leqq$$

$$\leqq \sum_D \frac{[\Delta(g-f_{n*})]^2}{\Delta m} + 2 \left| \sum_D \frac{[\Delta(g-f_{n*})][\Delta(f_{n*}-f_n)]}{\Delta m} \right| +$$

$$+ \sum_D \frac{[\Delta(f_{n*}f_n)]^2}{\Delta m},$$ which

by the Schwarz inequality, does not exceed

$$\sum_D \frac{[\Delta(g-f_{n*})]^2}{\Delta m} + 2\sqrt{\sum_D \frac{[\Delta(g-f_{n*})]^2}{\Delta m}}\sqrt{\sum_D \frac{[\Delta(f_{n*}-f_n)]^2}{\Delta m}} +$$

$$+ \sum_D \frac{[\Delta(f_{n*}-f_n)]^2}{\Delta m} <$$

$$< \frac{c^2}{4} + (2)\frac{c^2}{4} + \frac{c^2}{4} = c^2$$

Therefore, $\int_a^b \frac{[d(g-f_n)]^2}{dm} \leqq c^2$ and therefore $\|g-f_n\|_m \leqq c$ for $n > N$. Thus $\|g-f_n\|_m \to 0$ as $n \to \infty$.

From Theorems 4, 5, and 6 we see that H_m is a Hilbert space.

Chapter III

DISCUSSION PRELIMINARY TO THE PROOF OF SEPARABILITY

The statement "f is H-integrable on [a,b]" means that $\int_b^a \frac{(df)^2}{dm}$ exists in the sense of Theorem 2.

Theorem 7 - Suppose that each of g^* and m is a function defined on [a,b], where m is defined as before and g^* is continuous. If h is the function defined by

$$h(x)=\begin{cases} 0, \text{ if } a=x \\ \int_a^x g^*(t)dm(t), \text{ if } a < x \leq b, \end{cases}$$

then h is H-integrable.

Proof - m nondecreasing on [a,b], implies that m is of bounded variation on [a,b]. Thus, since g^* is continuous on [a,b], $\int_p^q g^*(t)dm(t)$ exists for every subinterval [p,q] of [a,b].

1) Suppose that [p,q] is a subinterval of [a,b], such that $m(q)-m(p)=0$. Let $\int_p^q g^*(t)dm(t)=J_{[p,q]}$ and suppose that c is a positive number. There is a subdivision D of [p,q], such that if D' is a refinement of D and r is a function whose domain is D', such that r(I) is in I for each I in D', then $\left| J_{[p,q]} - \sum_{D'} g^*(r(I))\Delta m \right| < c$. Now $m(v)-m(u)=0$ for each

subinterval [u,v] of [p,q], so that $\sum_{D'} g^*(r(I))\,\Delta m = 0$, which implies that $J_{[p,q]} = 0$.

2) By the proof of part one of Theorem 2, if D is a subdivision of [a,b] and E is a refinement of D, then

$$\sum_{D} \frac{(\Delta h)^2}{\Delta m} \leqq \sum_{E} \frac{(\Delta h)^2}{\Delta m} .$$

3) Suppose that D is a subdivision of [a,b] and has d elements. Since g* is continuous and m is of bounded variation on [a,b], there are numbers G and M, such that $|g^*(x)| \leqq G$ for every x in [a,b], and if E is a subdivision of [a,b], $\sum_{E} |\Delta m| \leqq M$.

Consider the sum

$$\sum_{D} \frac{(\Delta h)^2}{\Delta m} = \sum_{D} \frac{\left[\int_p^q g^*(t)dm(t)\right]^2}{m(q)-m(p)} .$$

For notation, let [p,q]=I. For each I in D, there is a subdivision E_I of I, such that if E_I' is a refinement of E_I, and $r_{E_I'}$ is a function whose domain is E_I' such that $r_{E_I'}(U)$ is in U for every U in E_I', then

$$\int_p^q g^*(t)dm(t) \leqq \left| \sum_{E_I'} g^*(r_{E_I'}(U))(\Delta_U m) \right| + k$$

$$\leqq \sum_{E_I'} \left| g^*(r_{E_I'}(U)) \right| \Delta_U m + k, \text{ where } k^2 = \frac{M}{d^2} .$$

Then

$$\sum_D \frac{(\Delta h)^2}{\Delta m} \leqq \sum_D \left(\frac{\left[\sum_{E_I'} \left| g^*(r_{E_I'}(U)) \right| \Delta_U m + k \right]^2}{\Delta_I m} \right)$$

$$\leqq \sum_D \left(\frac{\left[\sum_{E_I'} \left| g^*(r_{E_I'}(U)) \right| \Delta_U m \right]^2}{\Delta_I m} \right) + 2k \sum_D \left[\frac{\sum_{E_I'} \left| g^*(r_{E_I'}(U)) \right| \Delta_U m}{\Delta_I m} \right] +$$

$$+ \sum_D \frac{k^2}{\Delta_I m}$$

$$\leqq \sum_D \left(\frac{G^2 \left[\sum_{E_I'} \Delta_U m \right]^2}{\Delta_I m} \right) + 2kG \sum_D \left[\frac{\sum_{E_I'} \Delta_U m}{\Delta_I m} \right] + \sum_D \frac{k^2}{\Delta_I m}$$

$$\leqq G^2 \sum_D \frac{(\Delta_I m)^2}{\Delta_I m} + 2kG \sum_D 1 + \sum_D \frac{1}{d^2}$$

$$\leqq G^2 \sum_D \Delta m + \sum_D 2kG + \sum_D \frac{1}{d}$$

$$\leqq G^2 M + \sum_D \frac{2kdG + 1}{d} = G^2 M + 2kdG + 1$$

$\sum_D \frac{(\Delta h)^2}{\Delta m} \leqq G^2 M + 2G\sqrt{M} + 1.$ Thus h is H-integrable.

Lemma 3 - If each of f and g is a function defined on [a,b], such that f is continuous and g is H-integrable, let h be the function defined by

$$h(x) = \begin{cases} 0, \text{ if } x=a \\ \int_a^x f(t)dm(t), \text{ if } a < x \leq b. \end{cases}$$

$\int_a^b \frac{dhdg}{dm}$ exists.

Proof - By Theorem 7, h is H-integrable. Thus by the corollary of Theorem 2, $\int_a^b \frac{dhdg}{dm}$ exists.

Lemma 4 - If each of f and m is a function defined on [a,b], such that f is continuous and m in nondecreasing with $m(a) \neq m(b)$, then for each positive number c there is a positive number d, such that if D is a subdivision of [a,b], such that $|f(x)-f(y)| < d$ for x and y in an element of D, then for each I in D, such that $\Delta_I m \neq 0$

$$\frac{\left| \int_I f(t)dm(t) - f(r)\Delta_I m \right|}{\Delta_I m} < c, \text{ where } r \text{ is in } I.$$

Proof - Suppose that c is a positive number. There is a subdivision E of [a,b], such that if I is in E and each of x and y is in I, then $|f(x)-f(y)| < \frac{c}{2}$. For each I in E for which $\Delta_I m \neq 0$, there is a subdivision F_I of I, such that if F_I' is a refinement of F_I and r' is a function whose domain is F_I', such that r'(U) is in U for each U in F_I', then

$$\left|\int_I fdm - \sum_{F_I'} f(r'(U))\Delta_U m\right| < \frac{c\Delta_I m}{2}$$, where $\int_I fdm$ denotes $\int_p^q f(t)dm(t)$ if $I=[p,q]$. Thus

$$\frac{\left|\int_I fdm - \sum_{F_I'} f(r'(U))\Delta_I m\right|}{\Delta_I m} < \frac{c}{2}.$$ Now

$$\frac{\left|\sum_{F_I'} f(r'(U))\Delta_U m - f(r)\Delta_I m\right|}{\Delta_I m}$$ does not exceed $\frac{c}{2}$, for if

r is in I, then

$$\frac{\left|\sum_{F_I'} f(r'(U))\Delta_U m - f(r)\Delta_I m\right|}{\Delta_I m} = \frac{\left|\sum_{F_I'} f(r'(U))\Delta_U m - \sum_{F_I'} f(r)\Delta_U m\right|}{\Delta_I m}$$

$$= \frac{\left|\sum_{F_I'} (f(r'(U))-f(r))\Delta_U m\right|}{\Delta_I m}$$

$$\leq \frac{\sum_{F_I'} \left|f(r'(U))-f(r)\right| \Delta_U m}{\Delta_I m}$$

$$\leq \frac{c}{2}\left[\frac{\sum_{F_I'} \Delta_U m}{\Delta_I m}\right] = \frac{c}{2}.$$

Thus

$$\frac{\left|\int_I fdm - f(r)\Delta_I m\right|}{\Delta_I m} \leq \frac{\left|\int_I fdm - \sum_{F_I'} f(r'(U))\Delta_U m\right|}{\Delta_I m} +$$

$$+ \frac{\left| \sum_{F'_I} f(r'(U))\Delta_U m - f(r)\Delta_I m \right|}{\Delta_I m}$$

$< \frac{c}{2} + \frac{c}{2} = c$. Thus we obtain the desired result if we take $d=\frac{c}{2}$.

<u>Theorem 8</u> - Suppose that each of f and g is a function defined on [a,b], such that f is continuous and g is H-integrable. If h is the function defined by

$$h(x) = \begin{cases} 0, \text{ if } x=a \\ \int_a^x f(t)dm(t), \text{ if } a < x \leq b, \end{cases}$$

then $\int_a^b \frac{dhdg}{dm} = \int_a^b f(t)dg(t)$.

<u>Proof</u> - g is of bounded variation on [a,b], and f is continuous on [a,b], so that $\int_a^b f(t)dg(t)$ exists. By Lemma 3, $\int_a^b \frac{dhdg}{dm}$ exists. Let $\int_a^b \frac{dhdg}{dm} = J_1$ and $\int_a^b f(t)dg(t) = J_2$. Suppose that c is a positive number. There is a subdivision E of [a,b], such that if E' is a refinement of E, and r' is a function, such that E' is the domain of r', and r'(I) is in I for each I in E', then

$\left| J_2 - \sum_{E'} f(r'(I))\Delta_I g \right| < \frac{c}{3}$. There is a subdivision F of [a,b], such that if F' is a refinement of F, then

$\left| J_1 - \sum_{F'} \frac{\Delta h \Delta g}{\Delta m} \right| < \frac{c}{3}$. There is a subdivision G of [a,b], such that if I is in G, and each of x and y is in I, then $\left|f(x)-f(y)\right| < \frac{c}{6(L+1)}$, where $L = \int_a^b dg$. Let D be a common refinement of E, F, and G. If D' is a refinement of D, let $D^* = \left\{ I \mid I \in D', \Delta_I m \neq 0 \right\}$. Then if r is a function whose domain is D', such that r(I) is in I for each I in D',

$$\left| J_1 - \sum_{D'} \frac{\left[\int_I f\,dm\right] \Delta_I g}{\Delta_I m} \right| < \frac{c}{3}.$$

Since $\int_I f\,dm = 0$ and $\Delta_I g = 0$ if $\Delta_I m = 0$, $\sum_{D'} \frac{\left[\int_I f\,dm\right] \Delta_I g}{\Delta_I m} = \sum_{D^*} \frac{\left[\int_I f\,dm\right] \Delta_I g}{I^m}$.

Thus

$$\left| J_1 - \sum_{D^*} \frac{\left[\int_I f\,dm\right] \Delta_I g}{\Delta_I m} \right| + \left| J_2 - \sum_{D^*} f(r(I))\Delta_I g \right| < \frac{2c}{3}$$

$$\left| J_1 - J_2 + \sum_{D^*} f(r(I))\Delta_I g - \sum_{D^*} \frac{\left[\int_I f\,dm\right] \Delta_I g}{\Delta_I m} \right| < \frac{2c}{3}$$

$$\left| J_1 - J_2 \right| - \left| \sum_{D^*} \frac{\left[\int_I f\,dm\right] \Delta_I g - f(r(I))\Delta_I m \Delta_I g}{\Delta_I m} \right| < \frac{2c}{3}$$

$$\left| J_1 - J_2 \right| < \frac{2c}{3} + \left| \sum_{D^*} \Delta_I g \left[\frac{\int_I f\,dm - f(r(I))\Delta_I m}{\Delta_I m} \right] \right|$$

$$\left| J_1 - J_2 \right| < \frac{2c}{3} + \sum_{D^*} \left| \Delta_I g \right| \frac{\left| \int_I f\,dm - f(r(I))\Delta_I m \right|}{\Delta_I m},$$

which by Lemma 4 does not exceed $\frac{2c}{3} + \frac{c}{3(L+1)} \sum_{D^*} \left| \Delta_I g \right| < c$.

Thus $J_1 = J_2$.

Theorem 9 - Suppose that each of f, g, and m is a function defined on [a,b], such that f and g are each continuous, and m is nondecreasing with $m(b) \neq m(a)$. Let h_1 and h_2 be the functions defined by

$$h_1(x) = \begin{cases} 0, \text{ if } x=a \\ \int_a^x f(t)dm(t), \text{ if } a < x \leq b, \end{cases}$$

and

$$h_2(x) = \begin{cases} 0, \text{ if } x=a \\ \int_a^x g(t)dm(t), \text{ if } a < x \leq b. \end{cases}$$

$$\int_a^b \frac{dh_1 dh_2}{dm} = \int_a^b f(t)g(t)dm(t).$$

Proof - By Theorem 7, each of h_1 and h_2 is H-integrable, so that by the corollary to Theorem 2, $\int_a^b \frac{dh_1 dh_2}{dm}$ exists. By Theorem 8, $\int_a^b \frac{dh_1 dh_2}{dm} = \int_a^b f(t)dh_2(t)$. Since each of f and g is continuous, fg is continuous, so that $\int_a^b f(t)g(t)dm(t)$ exists. Let $\int_a^b f(t)dh_2(t) = J_1$ and $\int_a^b f(t)g(t)dm(t) = J_2$. Suppose that c is a positive number. There is a subdivision D of [a,b], such that if D' is a refinement of D, and r is a function whose domain is D', such that r(I) is in I for each

I in D', then $\left| J_1 - \sum_{D'} f(r(I))\Delta_I h_2 \right| < \frac{c}{3}$. There is a subdivision E of [a,b], such that if E' is a refinement of E, and r' is a function whose domain is E', such that r'(I) is in I for each I in E', then

$$\left| J_2 - \sum_{E'} f(r'(I))g(r'(I))\Delta_I m \right| < \frac{c}{3}.$$

There is a subdivision F of [a,b], such that if I is in F, and each of x and y is in I, then $\left| g(x)-g(y) \right| < \frac{c}{6(LM+1)}$, where $L= \text{lub}\left\{ z \mid z= \left| f(x) \right| ,\ x\epsilon[a,b] \right\}$ and $M= m(b)-m(a)$. Let G be a common refinement of D, E, and F. If G' is a refinement of G, and s is a function whose domain is G', such that s(I) is in I for each I in G', then

$$\left| J_1 - \sum_{G'} f(s(I))\Delta_I h_2 \right| + \left| J_2 - \sum_{G'} f(s(I))g(s(I))\Delta_I m \right| < \frac{2c}{3}.$$

$$\left| J_1 - J_2 + \sum_{G'} f(s(I))g(s(I))\Delta_I m - \sum_{G'} f(s(I))\Delta_I h_2 \right| < \frac{2c}{3}$$

$$\left| J_1 - J_2 \right| - \left| \sum_{G'} f(s(I))(\Delta_I h_2 - g(s(I))\Delta_I m) \right| < \frac{2c}{3}$$

$$\left| J_1 - J_2 \right| < \frac{2c}{3} + \left| \sum_{G'} f(s(I)) \left(\int_I g dm - g(s(I))\Delta_I m \right) \right|$$

$$< \frac{2c}{3} + \sum_{G'} \left| f(s(I)) \right| \left| \int_I g dm - g(s(I))\Delta_I m \right|.$$

$\int_I g dm = 0$, if I is in G', such that $\Delta_I m = 0$. Thus if $G^* = \left\{ I \mid I\epsilon G', \Delta_I m \neq 0 \right\}$,

$$\left| J_1 - J_2 \right| < \frac{2c}{3} + \sum_{G^*} \left(\left| f(s(I)) \right| \Delta_I m \right) \frac{\left| \int_I g dm - g(s(I))\Delta_I m \right|}{\Delta_I m},$$

so that by Lemma 4,

$$|J_1-J_2| < \frac{2c}{3} + \sum_{G^*} |f(s(I))| \Delta_I m \left(\frac{c}{3(LM+1)}\right)$$

$$< \frac{2c}{3} + \frac{c}{3M} \sum_{G^*} \frac{|f(s(I))|}{L} \Delta_I m \leqq \frac{2c}{3} + \frac{c}{3M} \sum_{G^*} \Delta_I m$$

$|J_1-J_2| < \frac{2c}{3} + \frac{c}{3} = c$. Thus $J_1=J_2$.

The following theorem is stated without proof.

<u>Theorem 10</u> - If f is a nondecreasing function defined on [a,b], then f is quasi-continuous on [a,b]. That is, if x is in [a,b], then the limit from the right, $f(x^+)$, exists for $a \leqq x < b$, and the limit from the left, $f(x^-)$, exists for $a < x \leqq b$.

<u>Theorem 11</u> - Suppose that g is a function defined on [a,b], such that g is of bounded variation on [a,b], g(a) = 0, and if f is a continuous function defined on [a,b], then $\int_a^b f(t)dg(t) = 0$. If $a < x < b$, then $g(x^-)=g(x^+)= 0$.

<u>Proof</u> - Under the above conditions g(b) = 0, for if f(x) = 1 for every x in [a,b], then there is a subdivision D of [a,b], such that if D' is a refinement of D, and r is a function whose domain is D', such that r(I) is in I for each I in D, then $|g(b)-g(a)| = \left|\sum_{D'} \Delta g\right| = \left|\sum_{D'} f(r)\Delta g\right| < c$.

Thus g(b)=g(b)=0. Since g is of bounded variation on [a,b], g may be expressed as the difference of two nondecreasing functions. Each of these functions is quasi-continuous, so that g is also quasi-continuous.

1) Suppose that $a \leqq x < b$, and that c is a positive number. There is a positive number d^*, such that if $a \leqq x < y \leqq b$ and $|x-y| < d^*$, then $|g(x^+)-g(y)| < \frac{c}{2}$. Let $d=\min\{d^*, b-x\}$. Let f be the function defined by

$$f(t)= \begin{cases} 0, \text{ if } a \leqq t < x \\ \frac{t-x}{d}, \text{ if } x \leqq t < x+d \\ 1, \text{ if } x+d \leqq t \leqq b. \end{cases}$$

Obviously f is continuous on [a,b]. Since $g(a)=g(b)=0$, and $\int_a^b f(t)dg(t)=f(b)g(b)-f(a)g(a)-\int_a^b g(t)df(t)$, then

$$\int_a^b f(t)dg(t)=\int_a^b g(t)df(t)=\int_a^x g(t)df(t) + \int_x^{x+d} g(t)df(t) + \int_{x+d}^b g(t)df(t) = 0.$$

Each of $\int_a^x g(t)df(t)$ and $\int_{x+d}^b g(t)df(t)$ is zero, since f is constant on each of the intervals [a,x] and [x+d,b]. Thus $\int_x^{x+d} g(t)df(t) = 0$. There is a subdivision D of [x,x+d], such that if D' is a refinement of D, and r^* is a function whose domain is D', such that $r^*(I)$ is in I for each I in D, then $\left|\sum_{D'} g(r^*)\Delta f\right| < \frac{c}{2}$. For each I in D',

$$\text{let } r(I)= \begin{cases} r^*(I), \text{ if } x \text{ is not in } I \\ z,\ z\in I,\ z\neq x, \text{ if } x\in I. \end{cases}$$

Thus for each I in D', $g(x^+) - \frac{c}{2} < g(r(I)) < g(x^+) + \frac{c}{2}$, so that $g(r(I))=g(x^+) + k(r(I))$, where $|k(r(I))| < \frac{c}{2}$. Then

$$\frac{c}{2} > \left|\sum_{D'} g(r(I))\Delta_I f\right| = \left|\sum_{D'} [g(x^+) + k(r(I))]\Delta_I f\right|$$

$$> \left|\sum_{D'} g(x^+)\Delta f\right| - \left|\sum_{D'} k(r(I))\Delta_I f\right| \text{, so that}$$

$$\left|\sum_{D'} g(x^+)\Delta f\right| < \frac{c}{2} + \left|\sum_{D'} k(r(I))\Delta_I f\right|.$$ Now, since for

each I in D', $\Delta_I f \geqq 0$, and $f(x+d)-f(x)=1$, it follows that

$$g(x^+) = \left|\sum_{D'} g(x^+)\Delta f\right| < \frac{c}{2} + \sum_{D'}|k(r(I))|\,\Delta_I f < \frac{c}{2} + $$

$$+ \sum_{D'} \frac{c}{2}\Delta f$$

$$< \frac{c}{2} + \frac{c}{2} = c.$$ Therefore $g(x^+) = 0$.

2) Suppose that $a < x \leqq b$, and that c is a positive number. There is a positive number d^*, such that if $a \leqq y < x \leqq b$ and $|y-x| < d^*$, then $|g(x^-)-g(y)| < \frac{c}{2}$. Let $d=\min\{x-a, d^*\}$.

Let f be the function defined by

$$f(t)= \begin{cases} 1, \text{ if } a \leqq t \leqq x-d \\ 1- \frac{t-(x-d)}{d}, \text{ if } x-d < t \leqq x \\ 0, \text{ if } x < t \leqq b. \end{cases}$$

As in part 1)

$$\int_a^b f(t)dg(t) = \int_a^b g(t)df(t) = \int_a^{x-d} g(t)df(t) + \int_{x-d}^x g(t)df(t) +$$

$$+ \int_x^b g(t)df(t) = 0.$$ Each of

$\int_a^{x-d} g(t)df(t)$ and $\int_x^b g(t)df(t)$ is zero, since f is constant on each of the intervals [a,x-d] and [x,b]. Thus $\int_{x-d}^{x} g(t)df(t) = 0$. There is a subdivision D of [x-d,x], such that if D' is a refinement of D, and r* is a function whose domain is D', such that r*(I) is in I for each I in D', then $\left| \sum_{D'} g(r^*)\Delta f \right| < \frac{c}{2}$.

For each I in D', let $r(I) = \begin{cases} r^*(I), \text{ if } x \text{ is not in } I \\ z,\ z\ I,\ z \neq x, \text{ if } x\ I. \end{cases}$

Thus for each I in D', $g(x^-) - \frac{c}{2} < g(r(I)) < g(x^-) + \frac{c}{2}$,

so that $g(r(I)) = g(x^-) + k(r(I))$, where $\left| k(r(I)) \right| < \frac{c}{2}$.

Then

$$\frac{c}{2} > \left| \sum_{D'} g(r(I))\Delta_I f \right| = \left| \sum_{D'} [g(x^-) + k(r(I))]\Delta_I f \right|$$

$$> \left| \sum_{D'} g(x^-)\Delta f \right| - \left| \sum_{D'} k(r(I))\Delta_I f \right|, \text{ so that}$$

$$\left| \sum_{D'} g(x^-)\Delta f \right| < \frac{c}{2} + \left| \sum_{D'} k(r(I))\Delta_I f \right|. \text{ Now, since}$$

$$\left| \sum_{D'} \Delta f \right| = \left| f(x)-f(x-d) \right| = \left| -1 \right|, \text{ it follows that}$$

$$\left|g(x^-)\right| = \left|g(x^-)\right| \left| \sum_{D'} \Delta f \right| < \frac{c}{2} + \left| k(r(I)) \right| \left| \sum_{D'} \Delta f \right|$$

$$< \frac{c}{2} + \frac{c}{2} = c. \text{ Therefore } g(x^-) = 0.$$

We see that if the condition that either g is left continuous at each x, such that $a < x \leqq b$ or g is right continuous at each x, such that $a \leq x < b$ is added to the hypothesis of Theorem 11, then $g(x) = 0$ for every x in $[a,b]$.

Suppose that V is an inner product space with inner product $((\cdot,\cdot))$ and zero element θ.

Lemma 5 - If $\{\phi_1,\phi_2,\ldots,\phi_k\}$ is an orthonormal set of elements of V, then $((u-\sum_{i=1}^{k}((u,\phi_i))\phi_i,\phi_j))=0$ for $j=1, 2,\ldots,k$ and any u in V.

Proof -

$$((u-\sum_{j-1}^{k}((u,\phi_i))\phi_i,\phi_j))=((u,\phi_j))-((\sum_{i=1}^{k}((u,\phi_i))\phi_i,\phi_j))$$

$$=((u,\phi_j))-\sum_{i=1}^{k}((u,\phi_i))((\phi_i,\phi_j))$$

$$=((u,\phi_j))-((u,\phi_j))((\phi_j,\phi_j))$$

$$=((u,\phi_j))-((u,\phi_j))$$

$$((u-\sum_{i=1}^{k}((u,\phi_i))\phi_i,\phi_j))= 0 .$$

Theorem 12 - If $A = \{u_i\}_{i=1}^{\infty}$ is a linearly independent sequence of elements of V, then there is an orthonormal sequence $B= \{\phi_i\}_{i=1}^{\infty}$ of elements V, such that if y is a linear combination of the first n elements of A, then y is a linear combination of the first n elements of B, and if x is a linear combination of the first n elements of B, then x is a linear combination of the first n elements of A.

Proof - $u_1 \neq \Theta$, for otherwise A is linearly dependent. Thus $\|u_1\| \neq 0$. Define $\phi_1 = \frac{u_1}{\|u_1\|}$.

$$((\phi_1,\phi_1)) = \left(\left(\frac{u_1}{\|u_1\|}, \frac{u_1}{\|u_1\|}\right)\right) = \left(\frac{1}{\|u_1\|}\right)^2((u_1,u_1)) = \left(\frac{\|u_1\|}{\|u_1\|}\right)^2 = 1.$$

Thus ϕ_1 is orthonormal. Let $v_2 = u_2 - ((u_2,\phi_1))\phi_1$. By Lemma 5, v_2 is orthogonal to ϕ_1. Thus since ϕ_1 is a linear combination of u_1, v_2 is a linear combination of $\{u_1,u_2\}$ and cannot be Θ. Define $\phi_2 = \frac{u_2-((u_1,\phi_1))\phi_1}{\|u_2-((u_1,\phi_1))\phi_1\|}$.

$\{\phi_1,\phi_2\}$ is orthonormal, since ϕ_2 is a scalar multiple of v_2, which is orthogonal to ϕ_1 and $((\phi_2,\phi_2)) =$

$$= \left(\frac{1}{\|u_2-((u_2,\phi_1))\phi_1\|}\right)^2 \left(\|u_2-((u_2,\phi_1))\phi_1\|\right)^2 = 1.$$

We note u_1 and u_2 are linear combinations of ϕ_1 and $\{\phi_1,\phi_2\}$ respectively. In general, if k is a positive integer, let $v_k = u_k - \sum_{i=1}^{k-1}((u_k,\phi_i))\phi_i$. By Lemma 5, v_k is orthogonal to each of $\phi_1,\ldots,\phi_{k-1}$. Since each ϕ_i is a linear combination of $\{u_1,\ldots,u_i\}$, v_k is a linear combination of $\{u_1,\ldots,u_k\}$ and cannot be Θ. Define

(1) $$\phi_k = \frac{u_k - \sum_{i=1}^{k-1}((u_k,\phi_i))\phi_i}{\|u_k - \sum_{i=1}^{k-1}((u_k,\phi_i))\phi_i\|}.$$ Suppose that each

of i and j is a positive integer less than k.
Since $\{\phi_1,\ldots,\phi_{k-1}\}$ is orthonormal,

$$((\phi_i,\phi_j))=\begin{cases}0, \text{ if } i\neq j\\ 1, \text{ if } i=j.\end{cases}$$

$$((\phi_j,\phi_k))=((\phi_j, \frac{v_k}{\|v_k\|}))=\frac{1}{\|v_k\|}((\phi_j,v_k))=\frac{0}{\|v_k\|}=0$$

$$((\phi_k,\phi_k))=\left(\left(\frac{v_k}{\|v_k\|},\frac{v_k}{\|v_k\|}\right)\right)=\left(\frac{1}{\|v_k\|}\right)^2((v_k,v_k))=\left(\frac{\|v_k\|}{\|v_k\|}\right)^2=1.$$

Thus $\{\phi_1,\ldots,\phi_k\}$ is orthonormal. From (1), we see that u_k is a linear combination of $\{\phi_1,\ldots,\phi_k\}$.

The sequence $\{\phi_i\}_{i=1}^{\infty}$ formed in this manner is orthonormal. Since each ϕ_i is a linear combination of $\{u_1,\ldots,u_i\}$, and each u_i is a linear combination of $\{\phi_1,\ldots,\phi_i\}$, any linear combination of $\{\phi_1,\ldots,\phi_n\}$ is a linear combination of $\{u_1,\ldots,u_n\}$ and conversely.

Suppose that H is a Hilbert space with inner product $((\cdot,\cdot))$. The following theorem is stated without proof.

Theorem 13 - The union of a countable collection of countable sets is countable.

Theorem 14 - Suppose that $\{\phi_i\}_{i=1}^{\infty}$ is an orthonormal sequence of elements of H. The following four statements are equivalent:

1) The set of all finite linear combinations of the ϕ_i's is dense in H.

2) If z is in H and $((z,\phi_n))=0$ for every n, then $z=\Theta$.

3) If x is in H, then $\|x-\sum_{i=1}^{n}((x,\phi_i))\phi_i\|\rightarrow 0$ as $n\rightarrow\infty$.

4) There is a countable set T of elements of H, such that H is separable with respect to T.

<u>Proof</u> - I. Suppose that 1) is true and that z is in H, such that $((z,\phi_i))=0$ for every i. Let c be a positive number. There is a positive integer n and a sequence of scalars $\{a_i\}_{i=1}^{n}$, such that $c>\|z-\sum_{i=1}^{n}a_i\phi_i\|$.

$$
\begin{aligned}
c^2>\|z-\textstyle\sum_{i=1}^{n}a_i\phi_i\|^2 &=((z-\textstyle\sum_{i=1}^{n}a_i\phi_i,z-\textstyle\sum_{i=1}^{n}a_i\phi_i))\\
&=((z,z))-2((z,\textstyle\sum_{i=1}^{n}a_i\phi_i)) +\\
&\quad+\textstyle\sum_{i=1}^{n}a_i^{\,2}((\phi_i,\phi_i))\\
&=((z,z))-2\textstyle\sum_{i=1}^{n}a_i((z,\phi_i))+\textstyle\sum_{i=1}^{n}a_i^2\\
&=((z,z))+\textstyle\sum_{i=1}^{n}a_i^2 \quad \text{, so that}
\end{aligned}
$$

$c^2>((z,z))=\|z\|^2$. Thus $c>\|z\|$ and $\|z\|=0$. Thus $z=\Theta$.

II. Suppose that 2) is true and that x is in H. Suppose that p is a positive integer, and that c is a positive number.

$$0 \leqq \|x-\sum_{i=1}^{p} ((x,\phi_i))\phi_i\|^2$$

$$\leqq ((x-\sum_{i=1}^{p}((x,\phi_i))\phi_i, x-\sum_{i=1}^{p}((x,\phi_i))\phi_i))$$

$$\leqq ((x,x))-2\sum_{i=1}^{p}((x,\phi_i))^2 +\sum_{i=1}^{p}((x,\phi_i))^2$$

$0 \leqq ((x,x))-\sum_{i=1}^{p}((x,\phi_i))^2$. Thus for each positive integer p, $((x,x)) \geqq \sum_{i=1}^{p}((x,\phi_i))^2$, which implies that there is a number J, such that $\sum_{i=1}^{p}((x,\phi_i))^2 \rightarrow J$ as $p \rightarrow \infty$. There is a positive number N, such that if each of m and n is a positive integer, such that $N < \min\{m,n\}$, then

$$\left| \sum_{i=1}^{m}((x,\phi_i))^2-\sum_{i=1}^{n} ((x,\phi_i))^2 \right| < c.$$

For each positive integer p, let $y_p=\sum_{i=1}^{p}((x,\phi_i))\phi_i$. Consider $\|y_m-y_n\|^2$ where $N < \min\{m,n\}$ and assume for convenience that $m \geqq n$.

$$\|y_m-y_n\|^2=((y_m-y_n,y_m-y_n))$$

$$=((y_m,y_m))-2((y_m,y_n)) + ((y_n,y_n))$$

$$=((\sum_{i=1}^{m}((x,\phi_i))\phi_i,\sum_{i=1}^{m}((x,\phi_i))\phi_i)) -$$

$$-2((\sum_{i=1}^{m}((x,\phi_i))\phi_i,\sum_{j=1}^{n}((x,\phi_j))\phi_j)) +$$

$$+((\sum_{j=1}^{n}((x,\phi_j))\phi_j,\sum_{j=1}^{n}((x,\phi_j))\phi_j))$$

$$=\sum_{i=1}^{m}((x,\phi_i))^2 -$$

$$-2\sum_{i=1}^{m}((x,\phi_i))((\phi_i,\sum_{j=1}^{n}((x,\phi_j))\phi_j)) +$$

$$+\sum_{j=1}^{m}((x,\phi_j))^2$$

$$=\sum_{i=1}^{m}((x,\phi_i))^2 -$$

$$-2\sum_{i=1}^{m}((x,\phi_i))[\sum_{j=1}^{n}((x,\phi_j))((\phi_i,\phi_j))] +$$

$$+\sum_{j=1}^{n}((x,\phi_j))^2$$

$$=\sum_{i=1}^{m}((x,\phi_i))^2 -$$

$$-2\sum_{i=1}^{n}((x,\phi_i))[\sum_{j=1}^{n}((x,\phi_j))((\phi_i,\phi_j))] -$$

$$-2\sum_{i=n+1}^{m}((x,\phi_i))[\sum_{j=1}^{n}((x,\phi_j))((\phi_i,\phi_j))] +$$

$$+\sum_{i=1}^{n}((x,\phi_j))^2$$

$$=\sum_{i=1}^{m}((x,\phi_i))^2-2\sum_{i=1}^{n}((x,\phi_i))^2 +\sum_{j=1}^{n}((x,\phi_j))^2$$

$\|y_m-y_n\|^2 =\sum_{i=1}^{m}((x,\phi_i))^2 -\sum_{i=1}^{n}((x,\phi_i))^2 < c$. Thus since H is complete, the sequence $\{y_i\}_{i=1}^{\infty}$ converges to some element y of H. Consider $((x-y,\phi_k))$ for some positive integer k. Since $\{\sum_{i=1}^{n}((x,\phi_i))^2\}_{n=1}^{\infty}$ converges, $((x,\phi_i)) \to 0$ as $i \to \infty$. Thus if c is a positive number, there is a positive number N', such that if q is a positive integer and $q > N'$, then $|((x,\phi_q))| < c$. Consider the sequence $\{((x-\sum_{i=1}^{n}((x,\phi_i))\phi_i,\phi_k))\}_{n=1}^{\infty}$.

$$\left|((x-\sum_{i=1}^{q}((x,\phi_i))\phi_i,\phi_k))\right| = \left|((x,\phi_k))-((\sum_{i=1}^{q}((x,\phi_i))\phi_i,\phi_k))\right|$$

$$= \left|((x,\phi_k))-\sum_{i=1}^{q}((x,\phi_i))((\phi_i,\phi_k))\right|$$

$$= \begin{cases} \left|((x,\phi_k))-((x,\phi_k))\right|, & \text{if } k \leqq q \\ \left|((x,\phi_k))\right|, & \text{if } k > q \end{cases}$$

$\left|((x-\sum_{i=1}^{q}((x,\phi_i))\phi_i,\phi_k))\right| < c$. Thus $((x-\sum_{i=1}^{n}((x,\phi_i))\phi_i,\phi_k))$ $\to 0$ as $n \to \infty$. If $\{f_n\}_{n=1}^{\infty}$ is a sequence of elements of H, such that $f_n \to f$ as $n \to \infty$, then $((f_n,\phi_k)) \to ((f,\phi_k))$ as $n \to \infty$, for if c is a positive number, there is a positive number N'', such that if s is a positive integer, such that $s > N''$, then $\| f_s - f \| < c$. Since f is in H, $((f,\phi_k))$ exists. Now $\left|((f_n,\phi_k))-((f,\phi_k))\right| = \left|((f_n-f,\phi_k))\right|$, which by the Schwarz inequality does not exceed $\| f_n-f \| \; \| \phi_k \| = \| f_n - f \| < c$. Thus if

$f_n = x-\sum_{i=1}^{n}((x,\phi_i))\phi_i$ for each n, $((x-\sum_{i=1}^{n}((x,\phi_i))\phi_i,\phi_k))$

$\to ((x-y,\phi_k))$ as $n \to \infty$. Since previously we saw that

$((x-\sum_{i=1}^{n}((x,\phi_i))\phi_i,\phi_k)) \to 0$ as $n \to \infty$, we conclude that

$((x-y,\phi_k)) = 0$ for each k. By 2), $x-y = \theta$, so that $x=y$.

Then since $\sum_{i=1}^{n}((x,\phi_i))\phi_i \to x$ as $n \to \infty$, $x-\sum_{i=1}^{n}((x,\phi_i))\phi_i$ $\to 0$ as $n \to \infty$.

III. Suppose that 3) is true and that c is a positive number. Let x be an element of H. There is a positive number

N such that if n is a positive integer such that $N < n$, then $\| x-\sum_{i=1}^{n}((x,\phi_i))\phi_i \| < c$. Since for each positive integer i less than or equal to n, $((x,\phi_i))$ is a real number, $\sum_{i=1}^{n}((x,\phi_i))\phi_i$ is a finite linear combination of the ϕ_i 's. Thus the set of all finite linear combinations of the ϕ_i 's is dense in H.

IV. Suppose that 1) is true and that c is a positive number. If x is an element of H, there is a positive integer n and a sequence of scalars $\{a_i\}_{i=1}^{n}$, such that $\| x-\sum_{i=1}^{n}a_i\phi_i \| < \frac{c}{2}$. Now if b_i is a rational number, then

$$\| a_i\phi_i-b_i\phi_i\|^2=((a_i\phi_i-b_i\phi_i,a_i\phi_i-b_i\phi_i))$$
$$=((a_i\phi_i,a_i\phi_i))-2((a_i\phi_i,b_i\phi_i)) + ((b_i\phi_i,b_i\phi_i))$$
$$= a_i^2 - 2a_ib_i + b_i^2 = (a_i-b_i)^2.$$

Thus for each a_i, let b_i be a rational number, such that $|a_i-b_i| < \frac{c}{2n}$.

Then $\| a_i\phi_i-b_i\phi_i\|^2 < \left(\frac{c}{2n}\right)^2$ and $\|a_i\phi_i-b_i\phi_i\| < \frac{c}{2n}$.

$$\frac{c}{2} = \sum_{i=1}^{n} \frac{c}{2n} > \sum_{i=1}^{n}\| a_i\phi_i-b_i\phi_i\| \geq \|\sum_{i=1}^{n}a_i\phi_i-\sum_{i=1}^{n}b_i\phi_i\|,$$

so that

$$c = \frac{c}{2} + \frac{c}{2} > \| x - \sum_{i=1}^{n} a_i \phi_i \| + \| \sum_{i=1}^{n} a_i \phi_i - \sum_{i=1}^{n} b_i \phi_i \|$$

$$> \| x - \sum_{i=1}^{n} a_i \phi_i + \sum_{i=1}^{n} a_i \phi_i - \sum_{i=1}^{n} b_i \phi_i \|$$

$$> \| x - \sum_{i=1}^{n} b_i \phi_i \| .$$

Thus the set of all linear combinations of the ϕ_i 's with rational coefficients is dense in H. The set of all rational linear combinations of ϕ_1 is countable. The set of all rational linear combinations of ϕ_2 is countable. Thus the set of all rational linear combinations of $\{\phi_1, \phi_2\}$ is countable. In general the set of all rational linear combinations of $\{\phi_1, \dots, \phi_n\}$ is countable for each n. Let $T_n = \{z \mid z \text{ is a rational linear combination of } \{\phi_1, \dots, \phi_n\}\}$. $T' = \{T_1, \dots, T_n, \dots\}$ is countable, so that $T = \bigcup_{T_i \in T'} T_i$ is countable. Thus H is separable with respect to T.

V. Suppose that 4) is true. Let $\{t_1, \dots, t_n, \dots\}$ (1) be an ordering of T. Let $T^* = \{t_1^*, \dots, t_n^*, \dots\}$ be a linearly independent set selected from T by eleminating those elements in the ordering (1) that are linear combinations of their predecessors. We see that any finite subset of T* is linearly independent. By Theorem 12, there is an orthonormal sequence $\{\phi_i\}_{i=1}^{\infty}$ of elements of H, such that if f is a linear combination of the first n elements of

T^*, then f is a linear combination of the first n elements of $\{\phi_i\}_{i=1}^{\infty}$. Suppose that x is an element of H. Let $\sum_{i=1}^{n} b_i t_i^*$ be a linear combination of the first n elements of T^*, such that $\| x - \sum_{i=1}^{n} b_i t_i^* \| < c$. Let $\{a_i\}_{i=1}^{n}$ be a sequence of scalars, such that $\sum_{i=1}^{n} a_i \phi_i = \sum_{i=1}^{n} b_i t_i^*$. Then $\| x - \sum_{i=1}^{n} a_i \phi_i \| < c$. Obviously $\sum_{i=1}^{n} a_i \phi_i$ is a finite linear combination of the ϕ_i 's. Thus the set of all finite linear combinations of the ϕ_i 's is dense in H.

Chapter IV

SEPARABILITY OF H_m

Throughout this chapter, we assume that m is a function defined on [a,b], such that m is strictly increasing, and either m is left continuous at each x, such that $a < x \leqq b$, or m is right continuous at each x, such that $a \leqq x < b$.

Theorem 15 - If f is in H_m, then either f is left continuous at each x, such that $a < x \leqq b$, or f is right continuous at each x, such that $a \leqq x < b$.

Proof - In the proof of Theorem 2, we saw that if f is in H_m, then for each subinterval [p,q] of [a,b],

$$(f(q)-f(p))^2 \leqq \int_p^q \frac{(df)^2}{dm} (m(q)-m(p)) .$$

Let $J=\int_a^b \frac{(df)^2}{dm}$.

I. Suppose that m is left continuous at each x, such that $a < x \leqq b$ and that $a < y \leqq b$. m is left continuous at y. There is a subinterval [z,y] of [a,b], such that if x is in [z,y], then $m(y)-m(x) < \frac{c^2}{J+1}$. For each x in [z,y],

$$\begin{aligned}(f(y)-f(x))^2 &\leqq \int_x^y \frac{(df)^2}{dm} \ (m(y)-m(x)) \\ &\leqq J(m(y)-m(x)) \\ &< J \ \frac{c^2}{J+1} \\ &< c^2 .\end{aligned}$$

Thus $|f(y)-f(x)| < c$ for

each x in [z,y], which implies that f is left continuous at y.

II. Suppose that m is right continuous at each x, such that $a \leq x < b$ and that $a \leq y < b$. m is right continuous at y. There is a subinterval [y,z] of [a,b], such that if x is in [y,z], then $m(x)-m(y) < \frac{c^2}{J+1}$. For each x in [y,z],

$$(f(x)-f(y))^2 \leq \int_y^x \frac{(df)^2}{dm} (m(x)-m(y))$$

$$\leq J(m(x)-m(y))$$

$$< J \frac{c^2}{J+1}$$

$$< c^2 .$$ Thus $|f(x)-f(y)| < c$

for each x in [y,z], which implies that f is right continuous at y.

If [p,q] is an interval, then the length of [p,q] is the number q-p.

<u>Definition 8</u> - For each positive integer n, let D_n be a subdivision of [a,b] containing exactly n+1 elements each of which has length $\frac{b-a}{n+1}$. Let $K_n = \{x_0, x_1, \ldots, x_{n+1}\}$ denote the set of all endpoints of the elements of D_n, where

$$a = x_0 < x_1 < \ldots < x_n < x_{n+1} = b.$$

Let F_n denote the set of all functions h defined on [a,b], such that

$$h(x) = \begin{cases} \text{a rational number, if } x \epsilon K_n \\ h(x_{i-1}) + \frac{x-x_{i-1}}{x_i-x_{i-1}}(h(x_i)-h(x_{i-1})), \text{if } x \epsilon [x_{i-1}x_i], \text{for} \\ \qquad i=1,\ldots,n+1;\ x \not\epsilon K_n. \end{cases}$$

For each h in F_n, the (n+2)-tuple $(h(x_0),h(x_1),\ldots,h(x_{n+1}))$ is called the n<u>th</u> order coordinate sequence of h.

There is exactly one n<u>th</u> order coordinate sequence corresponding to each h in F_n. If A is an (n+2)-tuple of rational numbers, then A completely determined some function in F_n.

<u>Theorem 16</u> - $F=\bigcup_{i=1}^{\infty}F_i$ for F_i defined in Definition 8 is countable.

<u>Proof</u> - Suppose that n is a positive integer and consider F_n. For each function h in F_n, there is exactly one n<u>th</u> order coordinate sequence $(a_0,a_1,\ldots,a_{n+1})$. For each n<u>th</u> order coordinate sequence of rational numbers $(b_0,b_1,\ldots,b_{n+1})$, there is exactly one function h in F_n, such that $h(x_i)=b_i$, for each i such that $i=0,1,\ldots,n+1$. Thus F_n contains as many unique functions as there are unique (n+2)-tuples of rational numbers. If $(c_0,c_1,\ldots,c_{n+1})$ is an n<u>th</u> order coordinate sequence of rational numbers,

then for each c_i there is only a countable number of values that c_i may have. Thus since there is only a finite number of c_i's to be determined in each coordinate sequence, there is a countable number of nth order coordinate sequences of rational numbers. Therefore F_n is countable. The set of all sets F_n is countable, so that the union $F=\bigcup_{i=1}^{\infty} F_i$ is countable by Theorem 13.

Theorem 17 - Let S denote the set of all functions defined and continuous on [a,b]. If c is a positive number and f is an element of S, then there is a sequence $\{h_i\}_{i=1}^{\infty}$ of elements of F, such that there is a positive number N, such that if n is a positive integer and $n > N$, then $|f(x)-h_n(x)| < c$, for every x in [a,b].

Proof - Suppose that c is a positive number and that f is an element of S. Let $D_1 = \{[a,x_1],[x_1,b]\}$ be a subdivision of [a,b], such that $x_1=a+\frac{b-a}{2}$. Let $K_1 = \{x_0,x_1,x_2\}$ denote the set of all endpoints of the elements of D_1, where

$$a=x_0 < x_1 < x_2 =b.$$

Let h_1 be the function defined by

$$h_1(x)=\begin{cases} \text{a rational number } p \text{ such that } |f(x)-p|< \frac{c}{6}, & \text{if } x\epsilon K_1 \\ h_1(x_{i-1}) + \frac{x-x_{i-1}}{x_i-x_{i-1}}(h_1(x_i)-h_1(x_{i-1})), & \text{if } x\epsilon[x_{i-1},x_i], \\ & \text{for } i=1,2;\ x \not\epsilon K_1. \end{cases}$$

h_1 is continuous and therefore is in S. In general, if n is a positive integer, let $D_n = \{[a,x_1],[x_1,x_2],\ldots,[x_n,b]\}$ be a subdivision of [a,b], such that $x_i = a + i\,\frac{b-a}{n+1}$ for $i=1,2,\ldots,n$. Let $K_n = \{x_0, x_1, \ldots, x_{n+1}\}$ denote the set of all endpoints of the elements of D_n, where

$$a = x_0 < x_1 < \ldots < x_n < x_{n+1} = b.$$

Let h_n be the function defined by

$$h_n(x) = \begin{cases} \text{a rational number } p, \text{ such that } f(x)-p < \frac{c}{6}, & \text{if } x \in k_n \\ h_n(x_{i-1}) + \frac{x-x_{i-1}}{x_i - x_{i-1}}\,(h_n(x_i) - h_n(x_{i-1})), & \text{if } x \in [x_{i-1}, x_i], \\ & \text{for } i=1,\ldots,n+1\ ; \\ & x \notin K_n. \end{cases}$$

h_n is continuous and therefore is in S.

There is a positive number d, such that if each of x and y is in [a,b] and $|x-y| < d$, then $|f(x)-f(y)| < \frac{c}{6}$. Let N be the least positive integer, such that $\frac{b-a}{d} \leqq N$. Consider h_n for $n > N$. $D_n = \{[a,x_1],[x_1,x_2],\ldots,[x_n,b]\}$ is a subdivision of [a,b], such that $x_i = a + i\,\frac{b-a}{n+1}$ for $i=1,2,\ldots,n$. The set of all endpoints of the elements of D_n is denoted by

$$K_n = \{x_0, x_1, \ldots, x_n, x_{n+1}\}, \text{ where}$$
$$a = x_0 < x_1 < \ldots < x_n < x_{n+1} = b.$$

Each element of D_n is of length $\frac{b-a}{n+1} < \frac{b-a}{N} \leqq d$. Thus if each of z_1 and z_2 is in $[x_{i-1}, x_i]$, then $|f(z_1)-f(z_2)| < \frac{c}{6}$.

Suppose that $a \leqq x \leqq b$. If x is in K_n, then $|h_n(x)-f(x)| < \frac{c}{6}$. If x is not in K_n, let $[x_{i-1}, x_i]$ be that element of D_n that contains x. Each of the following three statements is true:

1) $|h_n(x_i)-f(x_i)| < \frac{c}{6}$.
2) $|h_n(x_{i-1})-f(x_{i-1})| < \frac{c}{6}$.
3) $|f(x_i)-f(x_{i-1})| < \frac{c}{6}$.

Thus

$$\begin{aligned}\frac{c}{3} &> |h_n(x_i)-f(x_i)| + |f(x_{i-1})-h_n(x_{i-1})| \\ &> |h_n(x_i)-f(x_i) + f(x_{i-1})-h_n(x_{i-1})| \\ &> |h_n(x_i)-h_n(x_{i-1})| - |f(x_i)-f(x_{i-1})|\end{aligned}$$

$|h_n(x_i)-h_n(x_{i-1})| < \frac{c}{3} + |f(x_i)-f(x_{i-1})| < \frac{c}{3} + \frac{c}{6} = \frac{c}{2}$. Thus since $|h_n(x)-h_n(x_i)| \leqq |h_n(x_{i-1})-h_n(x_i)|$,

$$|h_n(x)-h_n(x_i)| + |f(x)-f(x_i)| < \frac{c}{2} + \frac{c}{6}$$

$$|h_n(x)-h_n(x_i) + f(x_i)-f(x)| < \frac{2c}{3}$$

$$|h_n(x)-f(x)| - |f(x_i)-h_n(x_i)| < \frac{2c}{3}$$

$$\left| h_n(x)-f(x) \right| < \frac{2c}{3} + \left| f(x_I) - h_n(x_I) \right| < \frac{2c}{3} + \frac{c}{6} = \frac{5c}{6} < c .$$

Let θ denote the function defined on [a,b], such that $\theta(x) = 0$ for every x in [a,b].

Theorem 18 - There is a linearly independent subset F* of F, such that the set of all finite linear combinations of the elements of F* is dense in S.

Proof - By Theorem 16, F is countable. Let

$$(1) \qquad \{h_1, \ldots, h_n, \ldots\}$$

be an ordering of F. Let $F^* = \{h_1^*, \ldots, h_n^*, \ldots\}$ be a linearly independent set selected from F by eliminating those elements in the ordering (1) that are linear combinations of their predecessors. We see that any finite subset of F* is linearly independent. For each h in F, h is in F* or h is a linear combination of elements in F*.

Suppose that f is an element of S. If c is a positive number, there is an element h_n of F, such that $\left| f(x)-h_n(x) \right| < c$ for every x in [a,b]. If h_n is in F*, then $h_n = h_m^*$ for some $m \leqq n$. If h_n is not in F*, there is a linear combination A of elements of F*, such that $h_n = A$. In either case, there is some linear combination B of elements of F*, such that $B = h_n$, so that $\left| f(x)-B(x) \right| < c$. Thus F* is dense in S.

Definition 9 - Let S be the set of all continuous functions defined on [a,b]. If each of f and g is in S, define

$_m((f,g)) = \int_a^b f(t)g(t)dm(t)$.

Theorem 19 - If each of f, g, and h is in S and k is a number, then the following statements are true:

1) $_m((f,g))$ is a real number.

2) $_m((f,f)) \geqq 0$ and $_m((f,f))=0$ if and only if $f=\theta$.

3) $_m((f,g)) = {}_m((g,f))$.

4) $_m((f+g,h)) = {}_m((f,h)) + {}_m((g,h))$.

5) $_m((f,kg)) = k({}_m((f,g)))$.

Proof - Suppose that each of f, g, and h is an element of S and that k is a number.

I. Since each of f and g is a continuous function, the product fg is also continuous. Thus the integral $\int_a^b f(t)g(t)dm(t)$, which is a real number, exists.

II. Suppose that f is a continuous function. Then $\int_a^b (f(t))^2 dm(t) = {}_m((f,f))$ exists. Let D be a subdivision of [a,b]. Let r be a function whose domain is D, such that r(I) is in I for every I in D. Consider $\sum_D (f(r))^2 \Delta m$. For each I in D, $\Delta_I m > 0$. In addition, $(f(r))^2 \geqq 0$. Thus $\sum_D (f(r))^2 \Delta m$ is nonnegative. Therefore since every approximating sum of $\int_a^b (f(t))^2 dm(t)$

is nonnegative, ${}_m((f,f)) \geqq 0$.

Suppose that $f(x) = 0$ for every x in $[a,b]$. Then for every subdivision D of $[a,b]$,

$$\sum_D (f(r))^2\Delta m = \sum_D (0)\Delta m = 0$$

regardless of the function r. Thus ${}_m((f,f))=0$ if $f = \theta$.

Suppose that f is a continuous function, such that ${}_m((f,f)) = 0$. Suppose that for some q in $[a,b]$, $f(q)\neq 0$. Then $(f(q))^2 > 0$. There is a subdivision D of $[a,b]$, such that if I is in D, and each of x and y is in I, then $\left|(f(x))^2-(f(y))^2\right| < \frac{(f(q))^2}{2}$. Suppose that $[s,t]$ is that element of D that contains q. Let E be a subdivision of $[s,t]$ and r a function whose domain is E, such that r(I) is in I for every I in E. Consider the sum

$$\sum_E (f(r))^2\Delta m.$$

$$\sum_E (f(r))^2\Delta m \geqq \sum_E \frac{(f(q))^2}{2} \Delta m = \frac{(f(q))^2}{2} \sum_E \Delta m$$

$$\geqq \frac{(f(q))^2}{2} (m(t)-m(s))$$. Since m is strictly increasing, $m(t)-m(s) > 0$. Since, for every subdivision E of $[s,t]$, $\sum_E (f(r))^2\Delta m \geqq \frac{(f(q))^2}{2} (m(t)-m(s))$, then

$$\int_s^t (f(t))^2 dm(t) > 0.$$

Now $\int_a^b (f(t))^2 dm(t) \geqq \int_s^t (f(t))^2 dm(t)$, so that

$\int_a^b (f(t))^2 dm(t) > 0$, which is a contradiction of the assumption that ${}_m((f,f)) = 0$. Thus $f(x) = 0$ for every x in $[a,b]$.

III. $${}_m((f,g)) = \int_a^b f(t)g(t)dm(t)$$
$$= \int_a^b g(t)f(t)dm(t)$$
$$= {}_m((g,f)) .$$

IV. $${}_m((f+g,h)) = \int_a^b (f(t) + g(t))h(t)dm(t)$$
$$= \int_a^b (f(t)h(t) + g(t)h(t))dm(t)$$
$$= \int_a^b f(t)h(t)dm(t) + \int_a^b g(t)h(t)dm(t)$$
$$= {}_m((f,h)) + {}_m((g,h)).$$

V. $${}_m((f,kg)) = \int_a^b f(t)(kg(t))dm(t)$$
$$= k \int_a^b f(t)g(t)dm(t)$$
$$= k({}_m((f,g))) .$$

<u>Theorem 20</u> - There is a sequence $\{\phi_i\}_{i=1}^{\infty}$ of elements of S, such that

1) g is a linear combination of the first n elements of F* if and only if g is a linear combination of the first n elements of $\{\phi_i\}_{i=1}^{\infty}$, and

2) $_m((\emptyset_i,\emptyset_j))=\begin{cases}0, \text{ if } i\neq j\\ 1, \text{ if } i=j\end{cases}$.

Proof - If we replace the general inner product $((.,.))$ in Theorem 12 with the inner product $_m((.,.))$, we obtain the required sequence $\{\emptyset_i\}_{i=1}^{\infty}$ from the linearly independent set F* where each $\emptyset_k$ is given by

$$\emptyset_k = \frac{h_k^* - \sum_{i=1}^{k-1} {}_m((h_k^*,\emptyset_i))\emptyset_i}{\left\| h_k^* - \sum_{i=1}^{k-1} {}_m((h_k^*,\emptyset_i))\emptyset_i \right\|} .$$

Definition 10 - Since each $\emptyset_i$ obtained in Theorem 20 is a linear combination of continuous functions, $\emptyset_i$ is continuous on [a,b]. For each $\emptyset_i$ define

$u_i(x)=\int_a^x \emptyset_i(t)dm(t)$.

Theorem 21 - The sequence $\{u_i\}_{i=1}^{\infty}$ is an orthonormal sequence with respect to the inner product $((.,.))_m$.

Proof - Suppose that each of u_s and u_t is an element of u_i $_{i=1}$. By Theorem 7, each u_i is H-integrable. Thus every u_i is in H_m.

I. Suppose that $s = t$. Then $((u_s,u_t))_m = ((u_s,u_s))_m$.

$((u_s,u_s))_m=\int_a^b \frac{du_s du_s}{dm}$. By Theorem 9,

$\int_a^b \frac{du_s du_s}{dm} = \int_a^b (\emptyset_s(x))^2 dm(x)$.

Since $\{\phi_i\}_{i=1}^{\infty}$ is orthonormal,

$$\int_a^b (\phi_s(x))^2 dm(x) = {}_m((\phi_s,\phi_s)) = 1, \text{ so that } ((u_s,u_s))_m = 1.$$

II. Suppose that $s \neq t$. Then $((u_s,u_t))_m = \int_a^b \frac{du_s du_t}{dm}$. By Theorem 9 $\int_a^b \frac{du_s du_t}{dm} = \int_a^b \phi_s(x)\phi_t(x)dm(x)$. Since $\{\phi_i\}_{i=1}^{\infty}$ is orthonormal, $\int_a^b \phi_s(x)\phi_t(x)dm(x) =$

$= {}_m((\phi_s,\phi_t)) = 0$, so that $((u_s,u_t))_m = 0$.

Theorem 22 - If g is in H_m, such that $((g,u_i))_m = 0$ for all i, then $g = \Theta$.

Proof - Suppose that g is in H_m, such that $((g,u_i))_m = 0$ for all i. Suppose that c is a positive number and that f is a continuous function defined on [a,b]. By Theorem 8, $\int_a^b \frac{du_i dg}{dm} = \int_a^b \phi_i(t)dg(t)$, so that $\int_a^b \phi_i(t)dg(t) = 0$ for every positive integer i. g is of bounded variation, so that there is a number M, such that if D is a subdivision of [a,b], then $M > \sum_D |\Delta g|$. There is a positive integer n and a sequence of scalars $\{a_i\}_{i=1}^{n}$, such that

$\left| f(x) - \sum_{i=1}^{n} a_i\phi_i(x) \right| < \frac{c}{M+1}$ for every x in [a,b].

Thus $f(x) = k(x) + \sum_{i=1}^{n} a_i\phi_i(x)$ where $|k(x)| < \frac{c}{M+1}$ for every x in [a,b]. Consider $\int_a^b f(t)dg(t)$.

$$\left|\int_a^b f(t)dg(t)\right| = \left|\int_a^b (k(t) + \sum_{i=1}^n a_i \emptyset_i(t))dg(t)\right|$$
$$= \left|\int_a^b k(t)dg(t) + \sum_{i=1}^n \int_a^b \emptyset_i(t)dg(t)\right|$$
$$= \left|\int_a^b k(t)dg(t)\right| < \frac{c}{M+1} M < c.$$

Therefore $\int_a^b f(t)dg(t) = 0$. By the proof of Theorem 11, each of $g(a)$, $g(b)$, $g(x^+)$, and $g(x^-)$ is zero. By Theorem 15, either g is left continuous at each x, such that $a < x \leq b$ or g is right continuous at each x, such that $a \leq x < b$, so that $g(x) = 0$ for every x in [a,b].

<u>Theorem 23</u> - H_m is separable.

<u>Proof</u> - By Theorem 22, if g is an element of H_m, such that $((g,u_i))_m=0$ for all i, then $g = \Theta$. By Theorem 14, this is equivalent to the statement that H_m is separable. By the proof of Theorem 14, we see that H_m is separable with respect to the set of all finite rational linear combinations of the sequence $\{u_i\}_{i=1}^{\infty}$.