QUANTIZED HYDRODYNAMICS

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Landau's theory of quantized hydrodynamics is derived using the current algebra approach to nonrelativistic quantum mechanics. Upon reviewing Landau's theory, his result for the velocity-velocity commutator is shown to have the wrong sign, and his result for the equation of motion of the velocity operator is shown to be incorrect. The quantum mechanical Hamiltonian is written in terms of density and velocity. An explicit expression is obtained for the term in Landau's Hamiltonian which corresponds to the internal energy of the boson fluid. In addition to the terms in Landau's Hamiltonian, the quantum mechanical Hamiltonian contains a quantum pressure term. The equation of motion obtained for the velocity operator thus contains a quantum pressure term. This term has exactly the same form as that obtained using the Gross-Pitaevskii equation. The Fock space realization of the current algebra is used to show the nonexistence of the phase operator and the inverse density operator. The coherent state representation is used to show that the velocity operator is the gradient of a potential operator. The derivation of quantum hydrodynamics from the Gross-Pitaevskii equation is shown to give equations of the same form as the current algebra derivation, with the equations being written in terms of functions rather than operators.

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QUANTIZED HYDRODYNAMICS

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TABLE OF CONTENTS

TABLE OF CONTENTS--Continued

CHAPTER I

INTRODUCTION

In 1941 Landau developed a theory of quantized hydrodynamics.[1] This was included in the same paper with his famous successful phenomenological explanation of liquid helium, which predicted the existence of second sound in helium II. The two sections of his paper were apparently unrelated.

More recent attempts have been made to obtain a quantized theory of hydrodynamics similar to Landau's by approaching the problem through the many-particle Schroedinger equation. This renewed interest in quantum hydrodynamics has been stirred by the success of current algebra in describing strongly interacting particles, or hadrons.[2] In attempting to formulate theories of quantum mechanics in terms of densities and currents as the dynamical variables, authors[2] have chosen nonrelativistic quantum mechanics as a testing ground for their theories. If a theory is to be applied to high energy physics, it should first be shown to describe correctly nonrelativistic quantum mechanics, where the dynamical equation is the Schroedinger equation. This interest in quantum mechanics using currents and densities as variables naturally leads to a renewed interest in quantum hydrodynamics.

The quantized hydrodynamical theories are written in operator form. To obtain a complete description of quantum hydrodynamics, however, the theories must be cast in terms of expectation values. No one has yet done this with theories derived from the many-particle Schroedinger equation using the current algebra approach, because eigenstates of the Hamiltonian written in terms of the density and current or velocity are unknown. Hydrodynamical theories written in terms of operators are called quantized hydrodynamics in this paper, while theories written in terms of expectation values are called quantum hydrodynamics.

All of the theories of quantized hydrodynamics reviewed in this paper contain mathematical defects. The quantization procedure used by Landau[1] to obtain a quantum mechanical Hamiltonian from the expression for the energy of a classical fluid is dubious, since he did not use canonically conjugate coordinates and momenta. The derivations of quantized hydrodynamics from the many-particle Schroedinger equation rely upon the use of nonexistent inverse field and density operators.[3,4] Some of the theories are derived using manipulations of operators that are not even formally correct, such as replacing operators with functions to facilitate rearranging the order of the operators. No one yet has successfully derived Landau's quantized hydrodynamics from the Schroedinger equation.

The object of this paper is to derive Landau's theory of quantized hydrodynamics from the many-particle Schroedinger equation. Landau's results are obtained, together with an additional term in the Hamiltonian. This term leads to a quantum correction in the equation of motion for the velocity operator, which is a quantum pressure term in operator form. One term in Landau's equation of motion for the velocity operator[1] is shown to be incorrect. Although the derivation used contains the inverse density operator, the manipulations are formally correct.

The most satisfactory derivation of quantum hydrodynamics from the many-particle Schroedinger equation to date seems to be the derivation from the Gross-Pitaevskii equation.[5,6] This derivation gives hydrodynamical equations which are written in terms of functions. The Gross-Pitaevskii equation approach is a rigorous one.[7] Although the final result is an approximation, the neglected terms are known, and could in theory be calculated.

Landau's theory of quantized hydrodynamics[1] is reviewed in Section II, and a correction is made to his equation of motion for the velocity operator. The current algebra formulation of nonrelativistic quantum mechanics[2,8,9,10] is reviewed in Section III and shown to be of doubtful validity. Landau's quantized hydrodynamics is derived from the current algebra approach to the many-particle Schroedinger equation in Section IV. A theory of quantized hydrodynamics in terms of density and phase operators[3] is also reviewed in Section IV. In Section V the Fock space form of the hydrodynamic operators is shown, and the inverse field and inverse density operators are shown not to exist in Fock space. In Section VI an explicit form of the velocity operator as the gradient of a phase operator is derived in the coherent state representation.[11] Section VII is a review of the Gross-Pitaevskii equation approach to quantum hydrodynamics.[5,6,7] Section VIII presents the conclusions. The systems discussed in this paper are boson systems.

CHAPTER II

LANDAU'S QUANTIZED HYDRODYNAMICS

In this section a review of Landau's quantized hydrodynamics[1] is presented for completeness and for comparison with more recent work. Landau's quantization procedure is explained first, and then his operators are defined. The commutation relations for the operators are given, and these relations are used to develop the Heisenberg equations of motion for the operators.

Landau's quantization procedure consists essentially of substituting operators for the density and velocity functions in the classical expression for the energy of a unit volume of a liquid.[1] This substitution results in Landau's Hamiltonian density. The Hamiltonian operator is then the volume integral of the Hamiltonian density. This quantization procedure is somewhat dubious, for the reasons which are explained next.

The usual quantization procedure for quantizing a classical equation is to replace the canonical coordinates and momenta in the classical equation with the corresponding quantum operators which satisfy the canonical commutation relations. If the classical density is taken to be the generalized coordinate, then the velocity is not its conjugate momentum. The density and velocity operators are thus not canonical, so we cannot use the usual procedure of field quantization.

Landau defined the density operator $\hat{\rho}(\bar{x})$ for a system of N identical particles as[1]

$$\hat{\rho}(\bar{x}) = m \sum_{i=1}^{N} \delta(\bar{x}-\bar{r}_i) , \tag{2.1}$$

where m is the mass of a particle and $\bar{r}_i$ is the position coordinate of the ith particle. The current density operator $\hat{\bar{J}}(\bar{x})$ is defined as

$$\hat{\bar{J}}(\bar{x}) = \frac{1}{2} \sum_{j=1}^{N} \left[\bar{p}_j \, \delta(\bar{x}-\bar{r}_j) + \delta(\bar{x}-\bar{r}_j) \, \bar{p}_j \right] , \tag{2.2}$$

where $\bar{p}_j$ is the usual momentum operator,

$$\bar{p}_j = \frac{\hbar}{i} \bar{\nabla}_j .$$

In this paper the terms current and density will always refer to mass current and mass density, respectively, rather than to particle current and particle density. These definitions give operators which have the same form as the density and current operators of second quantization in the N-particle subspace of Fock space, which will be considered later.

The velocity operator $\hat{\bar{V}}(\bar{x})$ which Landau used in his Hamiltonian is defined as[1]

$$\hat{\bar{V}}(\bar{x}) = \frac{1}{2} \left[\hat{\rho}^{-1}(\bar{x}) \hat{\bar{J}}(\bar{x}) + \hat{\bar{J}}(\bar{x}) \hat{\rho}^{-1}(\bar{x}) \right] , \tag{2.3a}$$

which is manifestly Hermitian. This definition seems reasonable until one tries to write $\hat{\rho}^{-1}(\bar{x})$, which for a one-particle system would look like

$$\hat{\rho}^{-1}(\bar{x}) = \left[\delta(\bar{x}-\bar{r})\right]^{-1} ,$$

so that one could write

$$\delta(\bar{x}-\bar{r}_1)\left[\delta(\bar{x}-\bar{r}_1)\right]^{-1} = 1 .$$

Such a function would have to be infinity everywhere except at the point $\bar{x}=\bar{r}_1$, where it must be zero. Since such a function apparently does not exist, Landau's procedures which utilize $\hat{\bar{V}}$ or $\hat{\rho}^{-1}$ must be taken as merely formal procedures which are not rigorously established. It is nevertheless possible that meaningful physical results could ultimately be obtained by these formal procedures. Landau also wrote the current operator in terms of the density and velocity operators as

$$\hat{\bar{J}}(\bar{x}) = \frac{1}{2}\left[\hat{\rho}(\bar{x})\hat{\bar{V}}(\bar{x}) + \hat{\bar{V}}(\bar{x})\rho(\bar{x})\right] . \tag{2.3b}$$

This equation may be considered as an implicit definition of $\hat{\bar{V}}(\bar{x})$, and is consistent with Eq. (2.3a) and the commutation relations between the operators, as is shown in Appendix A. Once the density-density and current-density commutators are known, $\hat{\bar{J}}(\bar{x})$ may be replaced by Eq. (2.3b) in the current-density commutator to obtain the velocity-density commutator.

Then the current-current commutator may be found, and this gives an equation which may be solved for the velocity-velocity commutator upon replacing $\hat{\bar{\bar{J}}}(\bar{x})$ by Eq. (2.3b). This procedure avoids the use of the inverse density operator. The density-density commutator is obviously zero. The current-density and current-current commutators are calculated in Appendices B and C, respectively, and are found to be the same as those obtained when the density and current operators are written in terms of the field annihilation and creation operators of second quantization, as is discussed in Sections III and IV. Since the current operator has the same definition in terms of the density and velocity operators in Landau's theory and in second quantization, the equivalence of the velocity-density and velocity-velocity commutators in Landau's theory and second quantization follows immediately. The velocity-density and velocity-velocity commutators are given in Section IV. The values of the commutators are given below:

$$\left[\hat{\rho}(\bar{x}), \hat{\rho}(\bar{y})\right] = 0 , \tag{2.4a}$$

$$\left[\hat{\bar{\bar{J}}}(\bar{x}), \hat{\rho}(\bar{y})\right] = \frac{\hbar}{i} \hat{\rho}(\bar{x}) \bar{\nabla}_x \delta(\bar{x}-\bar{y}) , \tag{2.4b}$$

$$\left[\hat{\bar{V}}(\bar{x}), \hat{\rho}(\bar{y})\right] = \frac{\hbar}{i} \bar{\nabla}_x \delta(\bar{x}-\bar{y}) , \tag{2.4c}$$

$$\left[\hat{V}_k(\bar{x}), \hat{V}_\ell(\bar{y})\right] = -\frac{\hbar}{i} \delta(\bar{x}-\bar{y}) \left[\hat{\rho}(\bar{x})\right]^{-1} \left(\frac{\partial \hat{V}_\ell(\bar{x})}{\partial x_k} - \frac{\partial \bar{V}_k(\bar{x})}{\partial x_\ell} \right) , \tag{2.4d}$$

and

$$\left[\hat{J}_K(\bar{x}),\ \hat{J}_\ell(\bar{y})\right] = -i\hbar \frac{\partial}{\partial x_\ell}\left[\delta(\bar{x}-\bar{y})\, J_K(\bar{x})\right] + i\hbar \frac{\partial}{\partial y_K}\left[\delta(\bar{x}-\bar{y}) J_\ell(\bar{y})\right]. \tag{2.4e}$$

Equation (2.4d) differs in sign from Landau's result.[1]

Landau obtained the Hamiltonian density for his theory by substituting operators for functions in the classical expression for the energy of a unit volume of a liquid,

$$\mathcal{H} = \frac{1}{2}\rho(\bar{r})v^2(\bar{r}) + \rho(\bar{r})\mathcal{E}(\rho), \tag{2.5}$$

where $\rho(\bar{r})$ is the density of the liquid, $\bar{v}(\bar{r})$ is the velocity of the unit volume of the liquid, and $\mathcal{E}(\rho)$ is the internal energy of a unit mass of the liquid.[1] Upon substituting the operators $\hat{\rho}$, $\hat{\bar{V}}$, and $\mathcal{E}(\hat{\rho})$ for the classical function, Landau obtained the Hamiltonian density operator

$$\mathcal{H} = \frac{1}{2}\hat{\bar{V}}\cdot\hat{\rho}\,\hat{\bar{V}} + \hat{\rho}\mathcal{E}(\hat{\rho}), \tag{2.6}$$

which is written in a symmetric form. The Hamiltonian operator was then obtained by integration over the volume of the liquid.

$$\hat{H} = \int\!\!\int\left\{\frac{\hat{\bar{V}}\hat{\rho}\hat{\bar{V}}}{2} + \hat{\rho}\mathcal{E}(\hat{\rho})\right\} d^3r\,. \tag{2.7}$$

Landau then used the commutation relations in Eq. (2.4) and the Hamiltonian of Eq. (2.7) to derive the Heisenberg equations of motion for the density and velocity operators.[1] The equation of motion for the density operator is

$$\frac{\partial \hat{\rho}(\bar{x})}{\partial t} = -\bar{\nabla}\cdot\hat{\bar{J}}(\bar{x}) = -\bar{\nabla}\cdot\frac{1}{2}\left[\hat{\rho}(\bar{x})\hat{\bar{V}}(\bar{x}) + \hat{\bar{V}}(\bar{x})\hat{\rho}(\bar{x})\right], \tag{2.8}$$

which is the operator form of the equation of continuity and is derived in Appendix D. The equation of motion which Landau obtained for the velocity operator is

$$\frac{\partial \hat{V}_\ell(\bar{x})}{\partial t} + \frac{1}{2}\left[\hat{V}_k(\bar{x})\frac{\partial \hat{V}_\ell}{\partial x_k} + \frac{\partial \hat{V}_\ell(\bar{x})}{\partial x_k}\hat{V}_k(\bar{x})\right] = -\left[\hat{\rho}(\bar{x})\right]^{-1}\frac{\partial}{\partial x_\ell}\frac{d\mathcal{E}(\hat{\rho})}{d\hat{\rho}}, \tag{2.9}$$

which is Euler's equation in operator form.

Equation (2.9) is shown to be incorrect in Appendix E, where the following equation of motion for the velocity operator is derived:[12]

$$\frac{\partial \hat{V}_\ell(\bar{x})}{\partial t} + \frac{1}{2}\left[\left(\hat{\bar{V}}(\bar{x})\cdot\bar{\nabla}\right)\hat{V}_\ell(\bar{x}) + \left(\bar{\nabla}\hat{V}_\ell(\bar{x})\right)\cdot\hat{\bar{V}}(\bar{x})\right] = -\frac{\partial}{\partial x_\ell}\frac{d}{d\hat{\rho}}\left[\hat{\rho}\mathcal{E}(\hat{\rho})\right]. \tag{2.10}$$

A dimensional analysis of the term on the right side of Eq. (219) shows that it has dimensions of length^7 (mass^2 x time^2) which are not the required dimensions of acceleration. The dimensions in the term on the right of Eq. (2.10) are indeed those of acceleration as required.

Two derivations of Eq. (2.10) are given in Appendix E. In the first derivation, $\mathcal{E}(\hat{\rho})$ is treated as a function of $\hat{\rho}$ and expanded as a power series in $\hat{\rho}$ to facilitate calculating its commutator with $\hat{V}_\ell(\bar{x})$. In the second derivation, the term in the Hamiltonian involving $\mathcal{E}(\hat{\rho})$ is treated as a functional of $\hat{\rho}(\bar{x})$, $E[\hat{\rho}]$, to show explicitly the similarity to the results that are obtained in Section IV. The functional $E[\hat{\rho}]$ is defined as

$$E[\hat{\rho}] = \int \hat{\rho}(\bar{y})\,\mathcal{E}\big(\hat{\rho}(\bar{y})\big)\,d^3y\,. \tag{2.11}$$

The functional treatment is the more general of the two, and is shown to give the results of the first treatment as a special case. The more general result is

$$\frac{\partial \hat{V}_\ell(\bar{x})}{\partial t} + \frac{1}{2}\left\{ \hat{\bar{V}}(\bar{x})\cdot\bar{\nabla}\hat{V}_\ell(\bar{x}) + \left(\bar{\nabla}\hat{V}_\ell(\bar{x})\right)\cdot\hat{\bar{V}}(\bar{x})\right\} = -\frac{\partial}{\partial x_\ell}\,\frac{\delta E[\hat{\rho}]}{\delta\hat{\rho}(\bar{x})}\,. \tag{2.12}$$

Since neither ordinary differentiation nor functional differentiation with respect to an operator is well defined, the derivatives with respect to the density operator in Eqs. (2.10) and (2.11) must be interpreted as formal manipulations in which the operators are replaced by functions, the differentiation is done, and the operators are reinserted into the resulting expression.

It is at this point that Landau's theory of quantized hydrodynamics stops.[1] It is thus an incomplete theory, since a complete theory of observables should have its dynamical equations in expectation value form. As far as we know, the only useful application of Landau's quantized hydrodynamics was obtained by Pitaevskii,[13] who rederived Feynman's theory for the energy spectrum of superfluid helium from Landau's quantized hydrodynamics.

Landau's original 1941 paper goes on to discuss other aspects of Helium II, such as its energy spectrum, heat capacity, and heat conductivity.[1] It was in this paper and another[14] in 1946 that Landau developed his famous excitation spectrum for Helium II which includes the "roton dip." It was also in this paper that Landau made the famous prediction of the existence of second sound in Helium II.

CHAPTER III

CURRENT ALGEBRA APPROACH TO NONRELATIVISTIC QUANTUM MECHANICS

In this section the current algebra approach to nonrelativistic quantum mechanics is reviewed. First the original motivation for the theory is given. Then the current and density operators are defined and their commutation relations in second quantization given. The Hamiltonian operator is then defined, and a functional representation of nonrelativistic quantum mechanics is developed.

Current algebras have been used to describe strongly interacting particles, or hadrons.[2] The formulation of a complete dynamical theory of hadrons in terms of currents is more physical than is using the underlying fields, since the currents are more closely related to experimental quantities. A nonrelativistic quantum system of many particles was first investigated as a proving ground for the current algebra, because of its simple nature and because its dynamical behavior is well known.

The current algebra approach to nonrelativistic quantum mechanics is related to Landau's quantized hydrodynamics by its use of density and current operators and its formulation of the Hamiltonian in terms of density and current operators.

It was not originally developed as a hydrodynamical theory, however. Thus one is lead to the possibility of deriving Landau's theory from the current algebra approach. The theory is written in the formalism of second quantization, which is natural for systems of many particles in which the number of particles can vary.

Current Algebra

The mass density operator for a system of several interacting particles is written as[2]

$$\hat{\rho}(\bar{x}) = m\hat{\psi}^{+}(\bar{x})\hat{\psi}(\bar{x}), \tag{3.1}$$

where $\hat{\psi}^{+}(\bar{x})$ and $\hat{\psi}(\bar{x})$ are the field creation and field annihilation operators, respectively, of second quantization for bosons satisfying the commutation relations

$$\left[\hat{\psi}^{+}(\bar{x}), \hat{\psi}^{+}(\bar{y})\right] = \left[\hat{\psi}(\bar{x}), \hat{\psi}(\bar{y})\right] = 0, \tag{3.2}$$

$$\left[\hat{\psi}(\bar{x}),\ \hat{\psi}^{+}(\bar{y})\right] = \delta(\bar{x}-\bar{y}). \tag{3.3}$$

The mass current density operator is defined as

$$\hat{\bar{J}}(\bar{x}) = \frac{\hbar}{2i}\left[\hat{\psi}^{+}(\bar{x})\left(\bar{\nabla}\hat{\psi}(\bar{x})\right) - \left(\bar{\nabla}\hat{\psi}^{+}(\bar{x})\right)\hat{\psi}(\bar{x})\right]. \tag{3.4}$$

The commutation relations for these operators are the same as those for the corresponding operators in Landau's theory.[2] The density-density commutator is

$$\left[\hat{\rho}(\bar{x}), \hat{\rho}(\bar{y})\right] = 0, \tag{3.5}$$

which is derived in Appendix F. The density-current commutator is

$$\left[\hat{\rho}(\bar{x}), \hat{J}_\ell(\bar{y})\right] = -i\hbar \frac{\partial}{\partial x_\ell}\left[\hat{\rho}(\bar{x})\,\delta(\bar{x}-\bar{y})\right], \tag{3.6}$$

which is derived in Appendix G. The current-current commutator is

$$\left[\hat{J}_K(\bar{x}), \hat{J}_\ell(\bar{y})\right] = -i\hbar\frac{\partial}{\partial x_\ell}\left[\delta(\bar{x}-\bar{y})\hat{J}_k(\bar{x})\right] + i\hbar\frac{\partial}{\partial y_K}\left[\delta(\bar{x}-\bar{y})\hat{J}_\ell(\bar{y})\right], \tag{3.7}$$

which is derived in Appendix H. In deriving Eq. (3.6) and Eq. (317), use is made of the identity[15]

$$F(\bar{x})\left[\bar{\nabla}_y\,\delta(\bar{x}-\bar{y})\right] = F(\bar{y})\left[\bar{\nabla}_y\,\delta(\bar{x}-\bar{y})\right] + \delta(\bar{x}-\bar{y})\left[\bar{\nabla}_y F(\bar{y})\right], \tag{3.8}$$

which is obtained by differentiating the equation

$$F(\bar{x})\,\delta(\bar{x}-\bar{y}) = F(\bar{y})\,\delta(\bar{x}-\bar{y}) \;. \tag{3.9}$$

The identity

$$\left[\bar{\nabla}_x F(\bar{x})\right]\cdot\left[\bar{\nabla}_y\,\delta(\bar{x}-\bar{y})\right] + F(\bar{x})\left[\bar{\nabla}_x\cdot\bar{\nabla}_y\,\delta(\bar{x}-\bar{y})\right]$$

$$= F(\bar{y}) \; \bar{\nabla}_x \cdot \bar{\nabla}_y \, \delta(\bar{x}-\bar{y}) \; + \; \bar{\nabla}_x \, \delta(\bar{x}-\bar{y}) \; \cdot \; \bar{\nabla}_y \, F(\bar{y}) \tag{3.10}$$

is also used, which is obtained by differentiating Eq. (3.8). Equations (3.8) and (3.10) are quite useful in dealing with current and density commutators in general.

The Hamiltonian in Terms of Current
and Density Operators

The Hamiltonian may be written in terms of the field annihilation and creation operators as[16]

$$\hat{H} = \hat{T} + \hat{U} + \hat{V} \; , \tag{3.11}$$

where the kinetic energy is

$$\hat{T} \equiv -\frac{\hbar^2}{2m} \int \hat{\psi}^+(\bar{x}) \, \nabla^2 \, \hat{\psi}(\bar{x}) \, d^3x \, , \tag{3.12a}$$

the external potential is

$$\hat{U} \equiv \int \hat{\psi}^+(\bar{x}) \, U(\bar{x}) \, \hat{\psi}(\bar{x}) \; , \tag{3.12b}$$

and the two-body potential is

$$\hat{V} \equiv \frac{1}{2} \int \hat{\psi}^+(\bar{x}) \hat{\psi}^+(\bar{y}) \, V(|\bar{x}-\bar{y}|) \, \hat{\psi}(\bar{y}) \hat{\psi}(\bar{x}) \, d^3x \, d^3y \; . \tag{3.12c}$$

$V(|x-y|)$ is the two-body potential between a pair of particles, and $U(\bar{x})$ is the external potential at the point $\bar{x}$.

The form of the Hamiltonian written in terms of current and density operators of Eq. (3.1) and Eq. (3.4) is derived from the Hamiltonian of Eq. (3.11) in Appendix I and is shown below:[2]

$$\hat{H} = \hat{T}' + \hat{U}' + \hat{V}' , \tag{3.13}$$

where the kinetic energy is

$$\hat{T}' \equiv \frac{\hbar^2}{8m} \int \left[\frac{1}{m}\left(\overline{\nabla}\hat{\rho}(\overline{x})\right) - \frac{2i}{\hbar}\hat{\overline{J}}(\overline{x}) \right] \cdot \hat{m}\hat{\rho}^{-1}(\overline{x}) \left[\left(\frac{1}{m}\overline{\nabla}\hat{\rho}(\overline{x})\right) + \frac{2i}{\hbar} J(\overline{x}) \right] d^3x , \tag{3.14a}$$

the external potential is

$$\hat{U}' \equiv \frac{1}{m} \int \hat{\rho}(\overline{x}) U(|x|)\, d^3x , \tag{3.14b}$$

and the two-body potential is

$$\hat{V}' \equiv \frac{1}{2m^2} \int V(|x-y|)\, \hat{\rho}(\overline{x})\hat{\rho}(\overline{y})\, d^3x\, d^3y + \frac{1}{2m} V(0) \int \hat{\rho}(\overline{y})\, d^3y . \tag{3.14c}$$

The second term in the operator $\hat{V}$ is a (possibly infinite) constant which may be subtracted from the Hamiltonian without changing the form of any subsequent equations in the theory.

The presence of the inverse density operator in Eq. (3.14a) renders the Hamiltonian invalid, since the inverse density operator does not exist in Fock space, as will be shown in Section V. Further calculations involving the Hamiltonian must therefore be regarded as merely formal.

Functional Representation of the Current Algebra

The basic formulation of the theory is completed with the introduction of a functional representation of the commutator algebra given by Eqs. (3.5), (3.6), and (3.7).[8] It is assumed (erroneously) that eigenvectors of the density operator exist and form a complete set in terms of which any state in Hilbert space may be expanded. Eigenvectors of the density operator are labeled by their eigenvalues:

$$\hat{\rho}(\bar{x})\left|\hat{\rho}\right\rangle = \rho(\bar{x})\left|\rho\right\rangle . \tag{3.15}$$

The set of components of an arbitrary vector $|\Psi\rangle$ along the basis formed by the eigenvectors of $\hat{\rho}(\bar{x})$ is then a wave functional.

$$\Psi(\rho) = \langle \rho | \Psi \rangle . \tag{3.16}$$

It is shown in Appendix J that the commutation relations of Eqs. (3.6) and (3.7) are satisfied if the actions of $\hat{J}_k(\bar{x})$ and $\hat{\rho}(\bar{x})$ on the wave functional in the density eigenvector basis have the following realizations:

$$\hat{\rho}(\bar{x}) \longrightarrow \rho(\bar{x}), \tag{3.17a}$$

and

$$\hat{J}_K(\bar{x}) \longrightarrow -i\hbar\rho(\bar{x})\frac{\partial}{\partial x_K}\,\frac{\delta}{\delta\rho(\bar{x})}\,, \tag{3.17b}$$

where $\frac{\delta}{\delta\rho(\bar{x})}$ denotes the functional derivative with respect to the eigenfunction of $\hat{\rho}(\bar{x})$.

The total momentum operator is given by[2]

$$P = \int \hat{\bar{J}}(\bar{x})\, d^3x. \tag{3.18}$$

The energy spectrum of the system is determined by the Schroedinger equation in the functional representation,

$$\hat{H}\Psi(\rho) = E\Psi(\rho)\,, \tag{3.19}$$

where the current and density operators in the Hamiltonian operate according to Eq. (3.17). This form of Schroedinger's equation is obtained by noting that the Hamiltonian whose operators have the realizations given by Eqs. (3.17a) and (3.17b) has the same effect when operating on the wave functional $\Psi(\rho)$ as when the Hamiltonian operates in the usual sense upon a wave vector $|\Psi\rangle$, and the inner product of the result is taken with an eigenvector of the density operator,

$$\hat{H}\Psi(\rho) = \langle\rho|\hat{H}|\Psi\rangle = E\Psi(\rho)\,.$$

The scalar product in the functional representation is given by

$$\langle \Psi | \Phi \rangle = \int \Psi^*(\rho)\, \Phi(\rho)\, D(\rho) \,, \tag{3.20}$$

where $\int D(\rho)$ signifies a functional integral over all functions $\rho(\bar{x})$ such that $\rho(\bar{x}) \geqslant 0$ and

$$\int \frac{1}{m} \rho(\bar{x})\, d^3x = N \,,$$

where N is the number of particles in the system.

Thus a complete functional representation of nonrelativistic quantum mechanics has been developed. The next step is to obtain solutions to Eq. (3.19) for various systems. It should be noted, however, that the assumed eigenstates of $\hat{\rho}(\bar{x})$ in Eq. (3.15) do not have the desired properties of a basic set. The properties of the eigenstates of $\hat{\rho}(\bar{x})$ are demonstrated in Section V.

The "eigenvalues" of $\hat{\rho}(\bar{x})$ are shown by Gross[10] to be

$$\rho(\bar{x}) = m \sum_{i=1}^{N} \delta(\bar{x} - \bar{r}_i) \tag{3.21}$$

for a system of N particles. As is pointed out in Section V, however, this is just the form of the operator in the N-particle subspace of Fock space. Furthermore, the "eigenvalues" of $\hat{\rho}(\bar{x})$ given in Eq. (3.21) consist of only one eigenvalue. Since Pardee, et al.[9] claim that the functional integral in Eq. (3.19) is over the eigenvalues of $\hat{\rho}(\bar{x})$, it is not a valid functional integral. The scalar product in the functional representation is therefore undefined. The functional representation is thus of doubtful value.

CHAPTER IV

DERIVATION OF QUANTIZED HYDRODYNAMICS FROM THE MANY-PARTICLE SCHROEDINGER EQUATION

Landau's theory of quantized hydrodynamics[1] is derived from the many-particle Schroedinger equation in this section. In practice, the derivation of the theory from the many-particle Schroedinger equation means that second quantization is used. The density and current operators are first defined in terms of the field operators, and their commutation relations are derived. The velocity operator is then defined in terms of the density and current operators, and its commutation relations are derived. These commutation relations between the density, current, and velocity operators are the same as those obtained in Section II. An explicit form of the velocity operator in terms of the field operators which is used in one theory[3] is given and shown to be incorrect.

The Hamiltonian written in terms of the density and current operators is cast in a form which is equivalent to Landau's Hamiltonian except for a term which may be identified as a quantum pressure term. This Hamiltonian contains the inverse density operator and was derived using the inverse field operators. These operators are shown in Section V not to exist, so the derivation of Landau's theory must be considered as merely a formal one. The Heisenberg equations of motion for the density and velocity operators derived from the many-particle Schroedinger equation are therefore the same as the equations of motion obtained by Landau, except for the quantum pressure term in the velocity equation of motion.

A formulation of the theory in terms of density and phase operators is also given,[3] although it is shown in Section V that the phase operator does not exist in this formulation. Finally, mention is made, at the end of this section, of a theory in which the boson fluid is assumed to have a rotational and an irrotational part.[17]

Hydrodynamic Operators in Second Quantization

The first step in the derivation of Landau's theory of quantized hydrodynamics from the many-particle Schroedinger equation is to show that the density, current, and velocity operators of second quantization have the same commutation relations as do those of Landau's theory. The density and current operators were defined in terms of the field operators in Eqs. (3.1) and (3.4) in Section III, and their commutation relations were also given there in Eqs. (3.5), (3.6), and (3.7). These are repeated for convenience below:

$$\left[\hat{\rho}(\bar{x}),\, \hat{J}_{\ell}(\bar{y})\right] = -i\hbar \frac{\partial}{\partial x_{\ell}} \left[\rho(\bar{x})\, \delta(\bar{x}-\bar{y})\right] , \tag{4.1a}$$

$$\left[\hat{\rho}(\bar{x}),\, \hat{\rho}(\bar{y})\right] = 0, \tag{4.1b}$$

and

$$\left[\hat{J}_k(\bar{x}), \hat{J}_\ell(\bar{x})\right] = -i\hbar \frac{\partial}{\partial x_\ell}\left[\delta(\bar{x}-\bar{y})\hat{J}_k(\bar{x})\right] + i\hbar \frac{\partial}{\partial y_k}\left[\delta(\bar{x}-\bar{y}) J_\ell(\bar{y})\right] , \tag{4.1c}$$

where the density is

$$\hat{\rho}(\bar{x}) = m\hat{\psi}^+(\bar{x})\,\hat{\psi}(\bar{x}) , \tag{4.1d}$$

and the current is

$$\hat{\bar{J}}(\bar{x}) = \frac{\hbar}{i}\left[\hat{\psi}^+(\bar{x})\left(\bar{\nabla}\hat{\psi}(\bar{x})\right) - \left(\bar{\nabla}\hat{\psi}^+(\bar{x})\right)\hat{\psi}(\bar{x})\right] . \tag{4.1e}$$

The velocity operator may be defined implicitly in terms of density and current operators as

$$\hat{\bar{J}}(\bar{x}) = \frac{1}{2}\left[\hat{\rho}(\bar{x})\hat{\bar{V}}(\bar{x}) + \hat{\bar{V}}(\bar{x})\hat{\rho}(\bar{x})\right] . \tag{4.2}$$

This definition avoids the use of the inverse density operator, and is used by several authors.[11,17]

The commutation relations between $\hat{\rho}(\bar{x})$ and $\hat{\bar{V}}(\bar{x})$ can be found using the above definition of $\hat{\bar{V}}(\bar{x})$ without recourse to the use of any nonexistent inverse operator. The velocity-density commutator is

$$\left[\hat{\rho}(\bar{x}), \hat{V}_\ell(\bar{y})\right] = i\hbar\left[\frac{\partial}{\partial y_\ell}\delta(\bar{x}-\bar{y})\right] , \tag{4.3}$$

which is found using the known value of the density-current commutator in Appendix K.

The velocity-velocity commutator may then be calculated using Eq. (4.3) and the known value of the current-current commutators. The result is

$$\hat{\rho}(\bar{x})\left[\hat{V}_k(\bar{x}), \hat{V}_\ell(\bar{y})\right] = i\hbar\delta(\bar{x}-\bar{y})\left[\frac{\partial \hat{V}_\ell(\bar{x})}{\partial x_k} - \frac{\partial \hat{V}_k(\bar{x})}{\partial x_\ell}\right] , \qquad (4.4)$$

which is calculated in Appendix L. All the commutators given so far in this section, with the exception of Eq. (4.4), agree with the commutators obtained by Landau for the density, current, and velocity operators in his theory.[1] Equation (4.4) differs in sign from Landau's result, but agrees with other authors.[11,17]

Derivation of Landau's Hamiltonian

The second step in the derivation of Landau's theory from second quantization is to derive Landau's Hamiltonian of Eq. (2.7) from the Hamiltonian of second quantization. The first step in the derivation of the Hamiltonian is taken in Section III, where the Hamiltonian written in terms of the density and current operators is derived from the Hamiltonian of second quantization and is shown in Eqs. (3.13) and (3.14). The terms $\hat{U}'$ and $\hat{V}'$ in Eq. (3.13) are already in the form of the first two terms in a series expansion for the term $E\left[\hat{\rho}\right]$ in Eq. (2.11).

If $E[\hat{\rho}]$ is expanded as

$$E[\hat{\rho}] = \sum_{n=1}^{\infty} \int f(\bar{x}_1 \dots \bar{x}_n)\, \hat{\rho}(\bar{x}_1) \dots \hat{\rho}(\bar{x}_n)\, d^3x_1 \dots d^3x_n ,$$

then the first two terms are given by Eq. (3.14) as

$$E[\hat{\rho}] = \int \left[\frac{1}{m} U(\bar{x}) + \frac{1}{2m} V(0) \right] \hat{\rho}(\bar{x})\, d^3x_1 + \int \frac{1}{2m^2} V(\bar{x},\bar{y})\, \hat{\rho}(\bar{x})\, \hat{\rho}(\bar{y})\, d^3x\, d^3y, \qquad (4.5a)$$

and the second quantization approach has given an explicit form for $E[\hat{\rho}]$. The kinetic energy term in the second quantized Hamiltonian may be manipulated to obtain the term in Landau's Hamiltonian involving the velocities plus a term which is a function of the density operator and may be considered a quantum correction term since it contains a factor of $\hbar^2$. The result is

$$\hat{T} = \int \left(\frac{1}{2m} \hat{\bar{V}}(\bar{x}) \cdot \hat{\rho}(\bar{x})\, \hat{\bar{V}}(\bar{x}) + \frac{\hbar^2}{8m^2} \hat{\rho}^{-1}(\bar{x}) \cdot \bar{\nabla} \hat{\rho}(\bar{x}) \cdot \bar{\nabla} \hat{\rho}(\bar{x}) \right) d^3x , \qquad (4.5b)$$

which is derived in Appendix M. The full Hamiltonian derived from the second quantized Hamiltonian is thus

$$\hat{H} = \int \left(\frac{1}{2m} \hat{\bar{V}}(\bar{x}) \cdot \hat{\rho}(\bar{x})\, \hat{\bar{V}}(\bar{x}) + \frac{\hbar^2}{8m^2} \hat{\rho}^{-1}(\bar{x}) \left(\bar{\nabla} \hat{\rho}(\bar{x}) \right) \cdot \bar{\nabla} \hat{\rho}(\bar{x}) \right) d^3x + E[\hat{\rho}], \qquad (4.6)$$

where $E[\hat{\rho}]$ is given in Eq. (4.5a). This is the same as Landau's Hamiltonian except for the quantum correction term $\hat{H}_Q$.

$$\hat{H}_Q \equiv \int \frac{\hbar^2}{8m^2} \hat{\rho}^{-1}(\bar{x}) \left(\bar{\nabla}\hat{\rho}(\bar{x})\right) \cdot \left(\bar{\nabla}\hat{\rho}(\bar{x})\right) d^3x \,, \qquad (4.7)$$

which explicitly involves $\hbar$. $\hat{H}_Q$ also contains a divergent term which may be neglected, since it is a constant and does not enter into the equations of motion.

This form of the Hamiltonian has been derived by some authors[3] by using an incorrect formulation of the second quantization Hamiltonian in terms of density and phase operator and then replacing operators by complex functions. Other authors[4] have replaced the operators in the Hamiltonian of Eq. (3.12) with complex functions and reordered the functions to obtain a Hamiltonian of the form of the one in Eq. (4.6). Although the inverse density operator in Eq. (4.6) renders that form of the Hamiltonian invalid, the derivation in this paper is at least formally correct, and is done without recourse to replacing operators with functions.

Equations of Motion in the Many-Particle Theory

The commutation relations between the density and velocity operators are the same in Landau's theory and in the second quantization formulation. The Hamiltonians are also the same, except for the term $\hat{H}_Q$ in the Hamiltonian derived from the second quantization Hamiltonian. The equations of motion for the density and velocity operators should therefore be the same in both cases, except for terms due to $\hat{H}_Q$ in the theory derived from the many-particle Schroedinger equation. Since the density operator should commute with functions of the density operator, the operator form of the equation of continuity obtained by Landau follows immediately when the Hamiltonian of Eq. (4.6) is substituted into the Heisenberg equation of motion for the density operator,

$$\frac{\partial \hat{\rho}(\bar{x})}{\partial t} = \frac{i}{\hbar}\left[\hat{H}\hat{\rho}(\bar{x}) - \hat{\rho}(\bar{x})\hat{H}\right] = -\bar{\nabla}\cdot\hat{\bar{J}}(\bar{x}) . \tag{4.8}$$

The equation of motion for a component of the velocity operator in second quantization is the same as that of Eq. (2.12) with the addition of a term given by

$$\frac{1}{\hbar}\left[\hat{H}_Q, \hat{V}_\ell(\bar{x})\right] = \frac{\hbar^2}{4m^2}\frac{\partial}{\partial x_\ell}\bar{\nabla}\cdot\left(\hat{\rho}^{-1}(\bar{x})\bar{\nabla}\hat{\rho}(\bar{x})\right) + \frac{\hbar^2}{8m^2}\frac{\partial}{\partial x_\ell}\left[\bar{\nabla}\hat{\rho}(\bar{x})\cdot\hat{\rho}^{-1}(\bar{x})\hat{\rho}^{-1}(\bar{x})\bar{\nabla}\hat{\rho}(\bar{x})\right] , \tag{4.9}$$

which is derived in Appendix N.

This term may be manipulated to obtain a term of exactly the same form as that of the quantum pressure term in the form of Euler's equation which is derived from the Gross-Pitaevskii equation in Section VII. The entire equation of motion for the velocity operator is

$$\frac{\partial \hat{V}_\ell(\bar{x})}{\partial t} = -\frac{1}{2}\hat{\bar{V}}(\bar{x})\cdot\bar{\nabla}\hat{V}_\ell(\bar{x}) - \frac{1}{2}\left(\nabla\hat{V}_\ell(\bar{x})\right)\cdot\hat{\bar{V}}(\bar{x}) + \frac{\hbar^2}{4m^2}\frac{\partial}{\partial x_\ell}\bar{\nabla}\cdot\left(\hat{\rho}^{-1}(\bar{x})\bar{\nabla}\hat{\rho}(\bar{x})\right)$$

$$+\frac{\hbar^2}{8m^2}\frac{\partial}{\partial x_\ell}\left[\left(\bar{\nabla}\hat{\rho}(\bar{x})\right)\cdot\hat{\rho}^{-1}(\bar{x})\hat{\rho}^{-1}(\bar{x})\,\bar{\nabla}\hat{\rho}(\bar{x})\right]-\frac{\partial}{\partial x_\ell}\;\frac{\delta}{\delta\hat{\rho}(\bar{x})}\;\mathcal{E}\left[\hat{\rho}\right], \qquad (4.10)$$

where $\mathcal{E}\left[\hat{\rho}\right]$ is defined in Eq. (4.5a) in terms of the microscopic theory. Equation (4.10) is a fundamental result of this paper.

An Incorrect Form of the Velocity Operator

It would be desirable, if only for completeness, to write the velocity operator explicitly in terms of the field operators of second quantization. Fanelli and Struzynski write the velocity operator in terms of the field annihilation and creation operators as[3]

$$\hat{\bar{V}}(\bar{x}) = \frac{\hbar}{2mi}\left[\hat{\Psi}^{-1}(\bar{x})\left(\bar{\nabla}\hat{\Psi}(\bar{x})\right)-\left(\bar{\nabla}\hat{\Psi}^{+}(\bar{x})\right)\left(\hat{\Psi}^{+}(x)\right)^{-1}\right] . \qquad (4.11)$$

This must be considered as merely a formal equation, since the inverse field operators do not exist, as is shown in Section V. It is shown in Appendix O, however, that this expression for $\hat{\bar{V}}$ is inconsistent with the definition of the velocity operator which is written in terms of the density and current operators in Eq. (2.3), when the density and current operators are written in terms of the field operators of Eqs. (3.1) and (3.4). The succeeding derivations of Fanelli and Struzynski which depend on the form of the velocity operator given in Eq. (4.11) are thus not even formally correct. They do obtain correct results for the density-velocity commutator, however.

They show that the velocity-velocity commutator must be zero, and that the curl of the velocity operator is also zero. This makes their zero result for the commutator consistent with the results of Landau and others.[4] This would also make Yee's assumption[17] of the existence of a rotational component of the fluid unnecessary. However, they make use of inverse density and field operators to obtain their zero results. It has not been possible in this paper to obtain a zero result for the velocity-velocity commutator rigorously. It is suspected that although the curl of the velocity can be taken to be zero as shown by Yee[17] and Turski,[11] this does not necessarily follow from the commutation relations.

The Phase Operator

An interesting aspect of the theory of quantized hydrodynamics as derived from the many-particle Schroedinger equation is the result that the velocity operator is the gradient of a velocity potential operator. Fanelli and Struzynski[3] arrive at this result by defining what they call Hermitian operators $\hat{n}(\bar{x})$ and $\hat{\phi}(\bar{x})$ by the following equations:[3]

$$\hat{\psi}(\bar{x}) = \exp\left[i\hat{\phi}(\bar{x})\right]\sqrt{\hat{n}(\bar{x})} \quad , \tag{4.12a}$$

and

$$\hat{\psi}^{+}(\bar{x}) = \sqrt{\hat{n}(\bar{x})}\ \exp\left[-i\hat{\phi}(\bar{x})\right], \tag{4.12b}$$

where

$$\hat{n}(\bar{x}) = \frac{1}{m}\ \hat{\rho}(\bar{x}) = \hat{\psi}(\bar{x})\,\psi(\bar{x}) \tag{4.13}$$

is the particle density operator.

It must be noted, however, that the phase operator $\hat{\phi}(\bar{x})$ is not correctly defined by Eqs. (4.2a) and (4.2b). This is shown in Section V. The result that the velocity operator is the gradient of a velocity potential operator[11] is obtained more rigorously in Section VI, however. The coherent state representation is used to obtain that result.

The following commutation rules are postulated for $\hat{n}(\bar{x})$ and $\hat{\phi}(\bar{x})$:[3]

$$\left[\hat{n}(\bar{x}), \hat{\phi}(\bar{y})\right] = i\delta(\bar{x}-\bar{y}), \tag{4.14a}$$

$$\left[\hat{n}(\bar{x}), \hat{n}(\bar{y})\right] = \left[\hat{\phi}(\bar{x}), \hat{\phi}(\bar{y})\right] = 0. \tag{4.14b}$$

That is, $\hat{n}$ and $\hat{\phi}$ are assumed to be canonical variables. From Eqs. (4.14a) and (4.14b) additional commutation relations may be obtained:

$$\left[\sqrt{\hat{n}(\bar{x})}, \hat{\phi}(\bar{y})\right] = \frac{1}{2}\left(\hat{n}(\bar{x})\right)^{-1/2}\ \delta(\bar{x}-y), \tag{4.15a}$$

and

$$\left[\sqrt{\hat{n}(\bar{x})}\,,\ \bar{\nabla}\hat{\phi}(\bar{y})\right] = \frac{1}{2}\left(\hat{n}(\bar{y})\right)^{-1/2}\left[\bar{\nabla}_y\,\delta(\bar{x}-\bar{y})\right]. \tag{4.15b}$$

When these forms of the field operators are substituted into Eq. (3.4) for the current density operator, the result is

$$\hat{\bar{J}}(\bar{x}) = \frac{\hbar}{2i}\Big\{\sqrt{\hat{n}(\bar{x})}\,\exp\left[-i\hat{\phi}(\bar{x})\right]\bar{\nabla}\left(\exp\left[i\phi(\bar{x})\right]\ \hat{n}(\bar{x})\right)$$

$$- \bar{\nabla}\left(\sqrt{\hat{n}(\bar{x})}\ \exp\left[-i\hat{\phi}(\bar{x})\right]\right)\exp\left[i\phi(\bar{x})\right]\sqrt{\hat{n}(\bar{x})}\Big\}\,, \tag{4.16}$$

$$= \hbar\sqrt{\hat{n}(\bar{x})}\ \left(\bar{\nabla}\hat{\phi}(\bar{x})\right)\sqrt{\hat{n}(\bar{x})}\ . \tag{4.17}$$

Applying the commutation relation, Eq. (4.15), we obtain

$$\hat{\bar{J}}(\bar{x}) = \frac{\hbar}{2}\left[\hat{n}(\bar{x})\left(\bar{\nabla}\hat{\phi}(\bar{x})\right) + \left(\bar{\nabla}\hat{\phi}(\bar{x})\right)\hat{n}(\bar{x})\right]\ . \tag{4.18}$$

Comparison with Eq. (4.2) leads one to assume that the velocity operator may be written as

$$\hat{\bar{V}}(\bar{x}) = \frac{\hbar}{m}\,\bar{\nabla}\hat{\phi}(\bar{x})\ . \tag{4.19}$$

Thus the phase operator $\hat{\phi}(\bar{x})$ serves as a velocity potential operator.

As the gradient of a potential, the velocity operator would have a zero curl. The zero result of Fanelli and Struzynski[3] for the velocity-velocity commutator is thus compatible in this formulation with the commutator of Landau[1] and Bierter and Morrison,[4] which containsthe curl of the velocity operator as a factor.

Rotational and Irrotational Current

Yee develops a theory of quantized hydrodynamics in which the current is assumed to have a rotational and an irrotational part.[17] He then assumes that Eq. (4.4) for the velocity-velocity commutator may be satisfied by two separate components of the velocity operator. One component has the form

$$\hat{\bar{V}}_1(\bar{x}) = -\bar{\nabla}\hat{\phi}(\bar{x}), \tag{4.20}$$

so that

$$\bar{\nabla}_x \hat{\bar{V}}_1(\bar{x}) = 0. \tag{4.21}$$

This accounts for the irrotational component of the current. Yee then assumes as second component $\hat{\bar{V}}_2(\bar{x})$ such that

$$\bar{\nabla}\times\hat{\bar{V}}_2(\bar{x}) \neq 0. \tag{4.22}$$

This accounts for a rotational component of the current.

A Treatment of the Inverse Density Operator

Yee[17] also treats the inverse density operator in an interesting manner. He writes the density operator as an average density, which is a function, plus an operator which represents the deviation of the density from the average,

$$\hat{\rho}(\bar{x}) = \rho_o(\bar{x}) + \hat{\varepsilon}(\bar{x}) \,, \tag{4.23}$$

where $\hat{\varepsilon}(\bar{x})$ is considered small. He then expands the inverse density operator as

$$\left(\hat{\rho}(\bar{x})\right)^{-1} = \frac{1}{\rho_o(\bar{x}) + \hat{\varepsilon}(\bar{x})} \,, \tag{4.24}$$

$$= \frac{\hat{I}}{\rho_o(\bar{x})} - \frac{\hat{\varepsilon}(\bar{x})}{\rho_o^2(\bar{x})} + \frac{\hat{\varepsilon}^2(\bar{x})}{\rho_o^3(\bar{x})} + \cdots \,. \tag{4.25}$$

He is then able to write equations without resorting to explicit use of the inverse density operator.

This procedure is simply complicating the confusion. A nonconvergent series is not very useful, and the nonexistence of the inverse density operator implies that the series, Eq. (4.25), is not convergent.

Although the formal derivation of Landau's quantized hydrodynamics from the many-particle Schroedinger equation accomplished in this section is satisfying, it should be re-emphasized that it is only a formal derivation. The Hamiltonian which was used was derived using inverse field operators, and contains the inverse density operator. In the next section these operators are shown not to exist.

Also, the equations of motion derived in this section are in operator form, and are not written in terms of functions. The equations of motion for the density and velocity are derived in terms of functions in Section VII.

CHAPTER V

FOCK SPACE FORMULATION OF THE CURRENT ALGEBRA

In this section the hydrodynamic operators of density, current, velocity and phase dealt with in the previous section are shown in their Fock space forms. Fock space is the natural space of the second quantization formulation of many-particle quantum mechanics. It is also the space in which the field annihilation and creation operators, $\hat{\psi}(\bar{x})$ and $\hat{\psi}^{+}(\bar{x})$, are defined. Since the hydrodynamic operators of the theories previously reviewed, except Landau's theory, are defined in terms of the field operators of second quantization, they operate in Fock space.[18]

It is shown that the density operator does not have an inverse in Fock space. The velocity-density and velocity-velocity commutators are derived in the formalism of second quantization without the use of inverse field operators or the inverse density operator. The Fock space appearance of the density-phase formulation of the hydrodynamic operators given in Section IV is used to show that the phase operator[3] cannot be defined.

Fock Space

A general Fock space vector may be written as[16]

$$\Psi\rangle \qquad \begin{bmatrix} \Psi^{(0)} \\ \Psi^{(1)}\ (\bar{r}_1) \\ \Psi^{(2)}\ (\bar{r}_1, \bar{r}_2) \\ \vdots \\ \Psi^{(N)}\ (\bar{r}_1, \ldots, \bar{r}_N) \\ \vdots \end{bmatrix} , \tag{5.1}$$

where the superscripts denote the number of particles in the subspace, the symbol $\longleftrightarrow$ indicates that the column vector is a particular realization of the general Fock space vector, and $\Psi^{(N)}$ is the N-particle wave function. The Fock space is the direct sum of the Hilbert spacesfor zero, one, two, up to an infinite number of particles.

The inner product of two Fock space vectors is defined as[16]

$$\langle \Psi | \Phi \rangle \equiv \sum_{n=0}^{\infty} \left(\Psi^{(n)}, \Phi^{(n)} \right) , \tag{5.2}$$

where $\left(\Psi^{(n)}, \Phi^{(n)}\right)$ is the inner product in the n-particle subspace. The vectors are normalized,

$$\langle \Psi | \Psi \rangle = \sum_{n=0}^{\infty} \left(\Psi^{(n)}, \Psi^{(n)} \right) = \sum_{n=0}^{\infty} \left\| \Psi^{(n)} \right\|^2 = 1, \tag{5.3}$$

so that $\left\| \Psi^{(n)} \right\|^2$ is the probability of finding the system with N particles.

The annihilation operator annihilates a particle at a point,[16] and when operating on a Fock space vector has the form

$$\hat{\Psi}(\bar{x})\,|\,\Psi\rangle \longleftrightarrow$$

$$\begin{bmatrix} 0 & \left(\sqrt{1}\int d^3r_1\,\delta(\bar{x}-\bar{r}_1)\right) & 0 & 0 & 0\ldots & \Psi^{(0)} \\ 0 & 0 & \left(\sqrt{2}\int d^3r_2\,\delta(\bar{x}-\bar{r}_2)\right) & 0 & 0\ldots & \Psi^{(1)}(\bar{r}_1) \\ 0 & 0 & 0 & \left(\sqrt{3}\int d^3r_3\,\delta(\bar{x}-\bar{r}_3)\right) & 0\ldots & \Psi^{(2)}(\bar{r}_1,\bar{r}_2) \\ 0 & 0 & 0 & 0 & \ddots & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \end{bmatrix} \tag{5.4}$$

$$= \begin{bmatrix} \sqrt{1} & \Psi^{(1)}(\bar{x}) \\ \sqrt{2} & \Psi^{(2)}(\bar{r}_1,\bar{x}) \\ \sqrt{3} & \Psi^{(3)}(\bar{r}_1,\bar{r}_2,\bar{x}) \\ & \vdots \\ \sqrt{N} & \Psi^{(N)}(\bar{r}_1,\ldots,\bar{r}_{(N-1)}\bar{x}) \\ & \vdots \end{bmatrix} , \tag{5.5}$$

where $\bar{x}$ is considered a fixed parameter rather than a variable in the argument of the wave functions. The annihilation operator thus reduces by one the number of particles in each subspace, and shifts each element in the Fock space state vector up one place. In particular, there exists a vacuum state such that

$$\hat{\psi}(\bar{x})\,|\,\mathrm{vac}\rangle = 0, \tag{5.6}$$

and $|\mathrm{vac}\rangle$ is realized in Fock space by

$$|\mathrm{vac}\rangle \quad \begin{bmatrix} \Psi^{(0)} \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \tag{5.7}$$

where $\Psi^{(0)}$ is a complex number of unit modulus. Noting that $\Psi(\bar{x})$ has zeros all along the diagonal in its matrix form, it is apparent that $\hat{\psi}^{-1}(\bar{x})$ does not exist.

The creation operator creates a particle at a point.[16] Operating on a Fock space vector it has the form

$$\hat{\psi}^{+}(\bar{x})\,|\Psi\rangle \longleftrightarrow$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & \Psi^{(0)} \\ \sqrt{1}\,\hat{S}_1\delta(\bar{x}-\bar{r}_1) & 0 & 0 & 0 & \cdots & \Psi^{(1)}_{(\bar{r}_1)} \\ 0 & \sqrt{2}\,\hat{S}_2\delta(\bar{x}-\bar{r}_2) & 0 & 0 & \cdots & \Psi^{(2)}_{(\bar{r}_1,\bar{r}_2)} \\ 0 & 0 & \sqrt{3}\,\hat{S}_3\delta(\bar{x}-\bar{r}_3 & 0 & \cdots & \vdots \\ 0 & 0 & 0 & \sqrt{N}\,\hat{S}_N\delta(\bar{x}-\bar{r}_N) \quad 0 & \cdots & \Psi^{(N)}_{(\bar{r}_1,\ldots,\bar{r}_N)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix} \tag{5.8}$$

$$
= \begin{bmatrix} 0 \\ \sqrt{1} \quad \hat{S}_1 \quad \delta(\bar{x}-\bar{r}_1)\, \Psi^{(0)} \\ \sqrt{2} \quad \hat{S}_2 \quad \delta(\bar{x}-\bar{r}_2)\, \Psi^{(1)}(\bar{r}_1) \\ \vdots \\ \sqrt{N} \quad \hat{S}_N \quad \delta(\bar{x}-\bar{r}_N)\, \Psi^{(N-1)}(\bar{r}_1 \ldots \bar{r}_{N-1}) \\ \vdots \end{bmatrix} , \tag{5.9}
$$

where $\hat{S}_N$ is the symmetrizer for bosons or antisymmetrizer for fermions. $\hat{S}_N$ is defined as

$$
\hat{S}_N = \frac{1}{N!} \sum_{\substack{P \\ (1 \ldots N)}} \sigma^{|P|} \, P \quad , \tag{5.10}
$$

where the sum is over all the permutations of the N particles, $\hat{P}$ is the operator which carries out the permutations, σ is one for bosons and minus one for fermions, and $|P|$ is the order of the permutation. The creation operator thus adds a particle with a delta function wave function to the system, and symmetrizes or antisymmetrizes the resulting wave function. Since the determinant of the matrix form of $\hat{\psi}^+(\bar{x})$ is zero, $\hat{\psi}^+(\bar{x})$ also has no inverse in Fock space.

Density and Current Operators in Fock Space

The density operator in Fock space has the form[16]

$$\hat{\rho}(\bar{x}) = m\,\hat{\Psi}^{+}(\bar{x})\,\hat{\Psi}(\bar{x}) \longleftrightarrow$$

$$\longleftrightarrow m \begin{bmatrix} 0 & & & & \\ & \delta(\bar{x}-\bar{r}_1) & & 0 & \\ & & \delta(\bar{x}-\bar{r}_1)+\delta(\bar{x}-\bar{r}_2) & & \\ & & & \ddots & \\ & 0 & & \sum_{i=1}^{N}\delta(\bar{x}-\bar{r}_i) & \\ & & & & \ddots \end{bmatrix} . \qquad (5.11)$$

Although the density operator is diagonal, its determinant is zero, and it thus has no inverse in Fock space. The element of the density operator which operates in the N-particle subspace is the same as the density operator given by Landau for an N-particle system in Eq. (2.1).

If the form of the density operator for a one-particle system is investigated, it is easy to see that eigenfunctions of the density operator do not have the necessary properties to serve as a basis set. Since $\hat{\rho}(\bar{x})$ merely multiplies the wave function by a delta function in the one-particle subspace, the operator is apparently the same as its "eigenvalue,"

$$\left[\hat{\rho}(\bar{x})\right]_{11} = m\,\delta(\bar{x}-\bar{r}_1) \quad . \qquad (5.12)$$

This form of the "eigenvalue" of $\hat{\rho}(\bar{x})$, which is assumed by Pardee, <u>et</u> <u>al</u>.,[9] is responsible for some of the difficulties with nonrelativistic quantum mechanics formulated in terms of currents and densities, which are discussed in Section III. When one attempts to calculate the expectation value of the density, the result for a one-particle system is

$$\left\langle \Psi^{(1)} \right| \hat{\rho}(\bar{x}) \left| \Psi^{(1)} \right\rangle = m \int \delta(\bar{x}-\bar{r}_1) \left| \Psi(\bar{r}_1) \right|^2 d^3 r_1 = m \left| \Psi^{(1)}(\bar{x}) \right|^2 . \tag{5.13}$$

Since the expectation value must be equal to an eigenvalue if taken for an eigenstate, the above results apparently require

$$\left| \Psi^{(1)}(\bar{x}) \right|^2 = \delta(\bar{x}-\bar{r}_1) \ . \tag{5.14}$$

Thus although any wave function seems to satisfy the eigenvalue equation for $\hat{\rho}(\bar{x})$,

$$\delta(\bar{x}-\bar{r}_1) \Psi^{(1)}(\bar{r}_1) = \delta(\bar{x}-\bar{r}_1) \Psi^{(1)}(\bar{r}_1) \ , \tag{5.15}$$

to give the correct expectation values, the function must apparently have the form

$$\Psi^{(1)}(\bar{r}_1) = \sqrt{\delta(0)} \ , \tag{5.16}$$

from Eq. (5.14), which is hardly suitable to serve as a basis set.

The form of the current density operator in Fock space is given below:

$$\hat{\bar{J}}(\bar{x}) = \frac{\hbar}{2i}\left[\hat{\psi}^{+}(\bar{x})\left(\bar{\nabla}\hat{\psi}(\bar{x})\right) - \left(\bar{\nabla}\hat{\psi}^{+}(\bar{x})\right)\hat{\psi}(\bar{x})\right] \leftrightarrow \tag{5.17}$$

$$\leftrightarrow \frac{\hbar}{2i}\begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & \left(\bar{\nabla}_1\delta(\bar{x}-\bar{r}_1)+\delta(\bar{x}_1-\bar{r}_1)\bar{\nabla}_1\right) & 0 & \cdots \\ & & \ddots & \\ 0 & 0 & \sum_{i=1}^{N}\bar{\nabla}_i\delta(\bar{x}-\bar{r}_i)+\delta(\bar{x}-\bar{r}_i)\bar{\nabla}_i & 0\cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{5.18}$$

An element of this diagonal matrix operates on the N-particle subspace of Fock space. An element has the same form as does the current operator of Eq. (2.2) for a system of N-particles in Landau's theory. This form of the current operator can be derived by substituting the forms of the field operators given in Eqs. (5.4) and (5.8) into Eq. (5.17) and letting them operate on a general Fock space vector.

The Density-Phase Formulation in Fock Space

The Fock space form of the creation and annihilation operators may be used to show that these operators may not be written in the density-phase formulation of Section IV.

In that formulation, the creation and annihilation operators were written as[3]

$$\hat{\psi}(\bar{x}) = \exp\left[i\hat{\phi}(\bar{x})\right]\sqrt{\hat{n}(\bar{x})} \quad , \tag{5.19a}$$

$$\hat{\psi}^{+}(\bar{x}) = \sqrt{\hat{n}(\bar{x})}\, \exp\left[-i\,\hat{\phi}(\bar{x})\right] \quad . \tag{5.19b}$$

Equations (5.19a) and (5.19b) are supposed to define Hermitian operators $\hat{\phi}(x)$ and $\hat{n}(\bar{x})$, where

$$\hat{n}(\bar{x}) = \frac{1}{m}\,\hat{\rho}(\bar{x}) . \tag{5.20}$$

In the one-particle subspace of Fock space, $\sqrt{\hat{n}(\bar{x})}$ would have the form

$$\left[\sqrt{\hat{n}(\bar{x})}\right]_{11} = \sqrt{\delta(\bar{x}-\bar{r}_1)} \quad . \tag{5.21}$$

Even if this strange function and its inverse exist, the phase operator $\hat{\phi}(\bar{x})$ is not properly defined by Eqs. (5.19a) and (5.19b).

The phase operator must satisfy both Eqs. (5.19a) and (5.19b), and the unitarity condition

$$e^{i\hat{\phi}(\bar{x})}\, e^{-i\hat{\phi}(\bar{x})} = e^{-i\hat{\phi}(\bar{x})}\, e^{i\hat{\phi}(\bar{x})} = \hat{I} \quad . \tag{5.22}$$

It is shown that the phase operator cannot satisfy both of these conditions.

The Fock space form of $\sqrt{\hat{n}(\bar{x})}$ must be

$$\sqrt{\hat{n}(\bar{x})} \longleftrightarrow \begin{bmatrix} 0 & & & & & \\ & 0 & & & & 0 \\ & & \sqrt{\delta(\bar{x}-\bar{r}_1)} & & & \\ & & & \sqrt{\delta(\bar{x}-\bar{r}_1)+\delta(\bar{x}-\bar{r}_2)} & & \\ & 0 & & & \ddots & \\ & & & & & \sqrt{\sum_{i=1}^{N} \delta(\bar{x}-\bar{r}_i)} \\ & & & & & & \ddots \end{bmatrix} . \qquad (5.23)$$

Therefore to satisfy Eq. (5.19a), exp $\left[i\hat{\phi}(\bar{x})\right]$ must have the Fock space form

$$\exp\left[i\hat{\phi}(\bar{x})\right] \longleftrightarrow$$

$$\longleftrightarrow \begin{bmatrix} \phi_{00} & \left(\hat{\psi}(\bar{x})\right)_{01} \left(\hat{n}(\bar{x})\right)_{11}^{-1/2} & \phi_{02} & \phi_{03} & \cdots \\ \phi_{10} & \phi_{11} & \left(\hat{\psi}(\bar{x})\right)_{12} \left(\hat{n}(\bar{x})\right)_{22}^{-1/2} & \phi_{13} & \cdots \\ \phi_{20} & \phi_{21} & \phi_{22} & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \end{bmatrix} , \qquad (5.24)$$

where $\left(\hat{\psi}(\bar{x})\right)_{01}$ for example, is the element of the Fock space matrix for $\hat{\psi}(\bar{x})$ in the zeroth row and the first column shown in Eq. (5.4), and the ϕ_{ij}'s are as yet undetermined elements of exp $\left[i\hat{\phi}(\bar{x})\right]$. Similarly, the adjoint operator,

$$\exp\left[-i\hat{\phi}(\bar{x})\right] = \left(\exp\left[i\hat{\phi}(\bar{x})\right]\right)^{+} ,$$

must have the Fock space form

$$\exp\left[-i\hat{\phi}(\bar{x})\right] \longleftrightarrow$$

$$\longleftrightarrow \begin{bmatrix} \phi^*_{00} & \phi^*_{10} & \phi^*_{20} & \cdots \\ \left(\hat{n}(\bar{x})\right)_{11}^{-1/2}\left(\hat{\psi}^+(\bar{x})\right)_{10} & \phi^*_{11} & \phi^*_{21} & \cdots \\ \phi^*_{02} & \left(\hat{n}(\bar{x})\right)_{22}^{-1/2}\left(\hat{\psi}^+(\bar{x})\right)_{21} & \phi^*_{22} & \cdots \\ & & & \ddots \\ \phi^*_{03} & \phi^*_{13} & \left(\hat{n}(\bar{x})\right)_{33}^{-1/2}\left(\hat{\psi}^+(\bar{x})\right)_{32} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} . \qquad (5.25)$$

If $\left(\hat{n}(\bar{x})\right)^{-1}$ is written formally as

$$\left[\hat{n}(\bar{x})\right]^{-1} = \hat{\psi}^{-1}(\bar{x})\left[\hat{\psi}^+(\bar{x})\right]^{-1} , \qquad (5.26)$$

then the formal result of the multiplication $\exp\left[i\hat{\phi}(\bar{x})\right] \times \exp\left[-i\hat{\phi}(\bar{x})\right]$ is the unit operator only if all the ϕ_{ij}'s are zero. When the order of multiplication is reversed, the result is

$$\exp\left[-i\hat{\phi}(\bar{x})\right]\exp\left[i\hat{\phi}(\bar{x})\right] \leftrightarrow$$

$$\leftrightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & & \\ 0 & 0 & \ddots & \ddots & \\ \vdots & \vdots & & & \end{bmatrix} . \tag{5.27}$$

Therefore $\exp\left[i\hat{\phi}(\bar{x})\right]$ is not unitary, and a phase operator $\hat{\phi}(\bar{x})$ has not been properly defined.

The preceeding proof of the nonunitarity of $\exp\left[i\hat{\phi}(\bar{x})\right]$ closely parallels the treatment of the phase operator for a quantum mechanical single harmonic oscillator given by Susskind and Glogower.[19] They show that the phase operator for a quantum mechanical single harmonic oscillator is not correctly defined by the exponential form. They then define Hermitian operators

$$\hat{\cos}\,\phi(\bar{x}) \equiv \frac{1}{2}\left[\hat{e}^{\,i\phi(\bar{x})} + \hat{e}^{\,-i\phi(\bar{x})}\right],$$

$$\hat{\sin}\,\phi(\bar{x}) \equiv \frac{1}{2i}\left[\hat{e}^{\,i\phi(\bar{x})} - \hat{e}^{\,-i\phi(\bar{x})}\right],$$

where the exp $i\phi(\bar{x})$ operators are no longer considered to define a phase operator. They then go on to show that $\hat{\cos}\,\phi(\bar{x})$ and $\hat{\sin}\,\phi(\bar{x})$ are observable dynamical variables. In doing so, they make use of the existence of the inverse of the number operator for a single harmonic oscillator.

The number operator for a single harmonic oscillator in the treatment of Susskind and Glogower enters their calculations in the same way as does the operator $\hat{n}(\bar{x})$ in the treatment in this paper.[19] The operator $\hat{n}(\bar{x})$ is the number density operator for Fock space. In the Fock space treatment, operators corresponding to the $\hat{\cos}\,\phi$ and $\hat{\sin}\,\phi$ have not been defined. This is due to the presence of terms such as

$$\left[\delta\,(\bar{x}-\bar{r}_1)\right]^{-1/2},$$

which would appear in the definition.

Since the phase operator is not properly defined, the derivation by Fanelli and Struzynski[3] showing that the velocity operator is the gradient of a phase operator is invalid.

Similarly, the curl of the velocity operator, which is the value of the velocity-velocity commutator, is not necessarily zero.

In Section VI a form of the velocity operator which overcomes some of these difficulties is developed, in which the velocity operator is the gradient of a velocity potential operator.[11] This velocity potential operator is not necessarily a phase operator.

CHAPTER VI

THE VELOCITY POTENTIAL OPERATOR IN THE COHERENT STATES REPRESENTATION

In this section a form of the velocity operator as the gradient of a potential operator is rigorously derived. These operators are written in terms of the coherent states. A brief summary of the properties of the coherent states is first given.

Coherent States

The coherent states used in this derivation are the eigenstates of the field annihilation operator.[20] They are the tensor products of the coherent states developed by Glauber as eigenstates of the particle annihilation operator. The particle annihilation operator eigenstates satisfy the equation

$$\hat{a}_{\bar{K}} \left| \{\alpha_{\bar{K}}\} \right\rangle = \alpha_{\bar{K}} \left| \{\alpha_{\bar{K}}\} \right\rangle , \tag{6.1}$$

where $\hat{a}_{\bar{K}}$ is the particle annihilation operator, $\alpha_{\bar{K}}$ is a complex eigenvalue, and $\{\alpha_{\bar{K}}\}$ is the set of all $\alpha_{\bar{K}}$. The particle annihilation operators may be written in terms of the field annihilation operators as

$$\hat{a}_{K} = \int d^3x \, \phi^*_{\bar{K}}(\bar{x}) \, \hat{\Psi}(\bar{x}) , \tag{6.2}$$

where $\phi_{\bar{k}}(\bar{x})$ is the wave function of the annihilated particle. The states $\big|[\alpha]\big\rangle \equiv \big|\{\alpha_{\bar{k}}\}\big\rangle$ satisfy the equation,

$$\hat{\psi}(\bar{x})\big|[\alpha]\big\rangle = \sum_{\bar{k}} |\alpha_{\bar{k}}|\, e^{i\bar{k}\cdot\bar{x}} \big|\{\alpha_{\bar{k}}\}\big\rangle \equiv \alpha(\bar{x})\big|[\alpha]\big\rangle \;. \tag{6.3}$$

Their form in occupation number space is

$$\big|\{\alpha_{\bar{k}}\}\big\rangle = \exp\Big(-\frac{1}{2}\sum_{\bar{k}} |\alpha_{\bar{k}}|^2\Big) \sum_{\{n_{\bar{k}}\}=0}^{\infty} \prod_{\bar{k}} \frac{\alpha_{\bar{k}}^{\,n_k}}{\sqrt{n_{\bar{k}}!}} \big|\{n_{\bar{k}}\}\big\rangle , \tag{6.4}$$

where the symbol $\{n_{\bar{k}}\}$ denotes the set of occupation number states of the system, and the sum is over all the occupation numbers from zero to infinity.

The coherent states are over complete and nonorthogonal, although they are normalized.[20] The scalar product of two coherent states never vanishes and is given by

$$\big\langle [\alpha] \,\big|\, [\beta] \big\rangle = \exp\Big[(\alpha|\beta) - \frac{1}{2}\|\alpha\|^2 - \frac{1}{2}\|\beta\|^2 \;, \tag{6.5}$$

where the inner product is

$$(\alpha|\beta) = \sum_{\bar{k}} \alpha^*_{\bar{k}}\, \beta_{\bar{k}} = \int dx\, \alpha^*(\bar{x})\, \beta(\bar{x}) \tag{6.6}$$

and

$$\|\alpha\|^2 = (\alpha|\alpha) \,.$$

The unit operator may be written in terms of the coherent states as

$$\hat{I} = \int \left| \{\alpha_{\bar{k}}\} \right\rangle \left\langle \{\alpha_{\bar{k}}\} \right| \prod_{\bar{k}} \frac{d^2\alpha_{\bar{k}}}{\pi} \equiv \int \big| |\alpha| \big\rangle \big\langle |\alpha| \big| D(\alpha) , \qquad (6.7)$$

where

$$d^2\alpha_{\bar{k}} \equiv d\,Re(\alpha_{\bar{k}})\, d\, I_m(\alpha_{\bar{k}}) ,$$

and $D(\alpha)$ indicates that the integral is a functional integral over all complex functions $\alpha(\bar{x})$. The states $|\alpha\rangle$ shall be written $|\alpha\rangle$ hereafter to simplify the notation.

The Velocity Operator in the Coherent State Representation

The matrix elements of the density and current operators are given in the coherent states as[11]

$$\langle \alpha | \hat{\rho}(\bar{x}) | \beta \rangle = \langle \alpha | m \hat{\Psi}^{+}(\bar{x}) \hat{\Psi}(\bar{x}) | \beta \rangle = m \alpha^{*}(\bar{x}) \beta(\bar{x}) , \qquad (6.8)$$

and

$$\langle \alpha | \hat{\bar{J}}(\bar{x}) | \beta \rangle = \frac{1}{2} \frac{\hbar}{i} \langle \alpha | \left[\hat{\Psi}^{+}(\bar{x}) \bar{\nabla} \hat{\Psi}^{+}(\bar{x}) - \left(\bar{\nabla} \hat{\Psi}(\bar{x}) \right) \hat{\Psi}(\bar{x}) \right] | \beta \rangle$$

$$= \frac{1}{2} \frac{\hbar}{i} \left[\alpha^{*}(\bar{x}) \bar{\nabla} \beta(\bar{x}) - \left(\bar{\nabla} \alpha^{*}(\bar{x}) \right) \beta(\bar{x}) \right] . \qquad (6.9)$$

As the first step in finding the coherent state representation of the velocity operator, it is shown in Appendix P that an arbitrary operator $\hat{\Theta}$ may be written using the resolution of the unit operator given by Eq. (6.7), as[20]

$$\hat{\Theta} = \int \exp\left(-\frac{1}{2}\|\alpha\|^2 - \frac{1}{2}\|\beta\|^2\right) \Theta(\alpha^*, \beta) |\alpha\rangle\langle\beta| \; D(\alpha) D(\beta), \tag{6.10}$$

where $\Theta(\alpha^*, \beta)$ is defined in terms of the matrix element of $\hat{\Theta}$ as

$$\Theta(\alpha^*, \beta) = \langle\alpha|\hat{\Theta}|\beta\rangle \exp\left(\frac{1}{2}\|\alpha\|^2 + \frac{1}{2}\|\beta\|^2\right). \tag{6.11}$$

The functions Θ corresponding to the operators $\hat{\rho}(\bar{x})$ and $\hat{\bar{J}}(\bar{x})$ have the following forms:

$$\hat{\rho}(\bar{x}) \longrightarrow \mathcal{R}(\alpha^*, \beta; \bar{x}) = m\alpha^*(\bar{x})\beta(\bar{x}) \exp\left|(\alpha|\beta)\right| , \tag{6.12a}$$

$$\hat{\bar{J}}(\bar{x}) \longrightarrow \mathcal{J}(\alpha^*, \beta; \bar{x}) = \frac{1}{2} \frac{\hbar}{i} \Big|\alpha^*(\bar{x})\left(\bar{\nabla}\beta(\bar{x})\right)$$

$$- \left(\bar{\nabla}\alpha^*(\bar{x})\right)\beta(\bar{x})\Big| \; \exp\left|(\alpha|\beta)\right| . \tag{6.12b}$$

The product of two operators $\hat{\Theta}_1$ and $\hat{\Theta}_2$ may be written as[20]

$$\hat{\Theta} = \hat{\Theta}_1 \hat{\Theta}_2 . \tag{6.13}$$

The functions Θ, Θ_1, and Θ_2, defined as in Eq. (6.11), obey the equation

$$\Theta(\alpha^*, \beta) = \int \Theta_1(\alpha^*, \gamma)\, \Theta_2(\gamma^*, \beta) \exp\left(-\|\gamma\|^2\right) D(\gamma) , \tag{6.14}$$

which is derived in Appendix Q. Turski states that Eq. (6.14) determines the function Θ_2 if the functions Θ_1 and Θ are known. If $\hat{\Theta}_2$ is an unknown operator, then the function Θ_2 may be inserted into Eq. (6.12) to give the form of the unknown operator Θ_2.

This is the way in which the form of the velocity operator will be found. The velocity operator is defined implicitly by

$$\hat{\bar{J}}(\bar{x}) = \frac{1}{2}\left|\hat{\rho}(\bar{x})\,\hat{\bar{V}}(\bar{x}) + \hat{\bar{V}}(\bar{x})\,\hat{\rho}(\bar{x})\right| . \tag{6.15}$$

Since the form of the functions $\bar{\mathcal{J}}(\alpha^*; \beta; \bar{x})$ and $\mathcal{R}(\alpha^*; \beta; \bar{x})$ are known, the function $\bar{\mathcal{V}}(\alpha^*; \beta; \bar{x})$ which satisfies the equation of the form of Eq. (6.14),

$$\bar{\mathcal{J}}(\alpha^*; \beta; \bar{x}) = \frac{1}{2}\int \mathcal{R}(\alpha^*; \gamma; (\bar{x}))\bar{\mathcal{V}}(\gamma^*; \beta; \bar{x}) \exp(-\|\gamma\|^2)\, D(\gamma)$$
$$+ \frac{1}{2}\int \bar{\mathcal{V}}(\alpha^*; \gamma; \bar{x})\,\mathcal{R}(\gamma^*; \beta; \bar{x}) \exp(-\|\gamma\|^2)\, D(\gamma) , \tag{6.16}$$

may be found and may be substituted into Eq. (6.11) to give the operator $\hat{\bar{V}}(\bar{x})$.

It is worthwhile to note that if Eq. (6.14) determine the form of Θ_2 uniquely, then the form obtained for $\hat{\bar{V}}(\bar{x})$ would rule out the existence of a rotational component for the boson system. Yee's assumption[17] of such a rotational component would then be unnecessary. It is not apparent, though, that the form obtained for $\hat{\bar{V}}(\bar{x})$ in this section is necessarily unique.

In order to eliminate divergent terms of the form of $\sum_{\bar{K}} -i\bar{K}$ which would appear in the solution of Eq. (6.17), the velocity operator is redefined as

$$\hat{\bar{J}}(\bar{x}) = \lim_{\bar{\epsilon}\to 0} \frac{1}{2}\left[\hat{\rho}(\bar{x})\,\hat{\bar{V}}(\bar{x}+\bar{\epsilon}) + \hat{\bar{V}}(\bar{x}+\bar{\epsilon})\,\hat{\rho}(\bar{x})\right]. \tag{6.17}$$

The operators defined as in Eq. (6.17) obey the same commutation relations as when they are defined by Eq. (6.15). It is shown in Appendix R that when the form of $\hat{\bar{J}}(\bar{x})$ given in Eq. (6.17) is used, Eq. (6.16) may be written as

$$\begin{aligned}\bar{\mathcal{J}}(\alpha^*;\beta;\bar{x}) = {} & \lim_{\bar{\epsilon}\to 0}\int \mathcal{R}(\alpha^*;\gamma;\bar{x})\,\bar{\Upsilon}(\gamma^*;\beta;\bar{x}+\bar{\epsilon})\exp\left(-\|\gamma\|^2\right)D(\gamma)\\ & + \frac{1}{2}\frac{\hbar}{i}\exp\left[(\alpha|\beta)\right]\lim_{\bar{\epsilon}\to 0}\bar{\nabla}_{\bar{\epsilon}}\,\delta(\bar{\epsilon}).\end{aligned} \tag{6.18}$$

In solving Eq. (6.18) for $\bar{\Upsilon}(\alpha^*;\beta;\bar{x})$, the following identity and its complex conjugate are used:[20]

$$\int \exp\left(\alpha^*_{\bar{K}}\gamma_{\bar{K}} - |\gamma_{\bar{K}}|^2\right)\left(\gamma_{\bar{K}}\right)^n \frac{d^2\gamma_K}{\pi} = \left(\frac{\partial}{\partial\alpha^*_{\bar{K}}}\right)^n F\left(\alpha^*_{\bar{K}}\right). \tag{6.19}$$

This identity is proven in Appendix S.

It is shown in Appendix T that the function $\bar{\Upsilon}(\alpha^*;\beta;\bar{x})$ given below in Eq. (6.20) is a solution[11] to Eq. (6.18):

$$\bar{\gamma}(\alpha^*;\beta;\bar{x}) = \frac{1}{2}\frac{\hbar}{im}\exp\left[(\alpha|\beta)\right]\left[\frac{1}{\beta(\bar{x})}\ \bar{\nabla}\beta(\bar{x}) - \frac{1}{\alpha^*(\bar{x})}\ \bar{\nabla}\alpha^*(\bar{x})\right] . \tag{6.20}$$

It is shown in Appendix U that if $\alpha(\bar{x})$ is written in polar form as

$$\alpha(\bar{x}) = f_\alpha(\bar{x}) \exp\left[i\phi_\alpha(\bar{x})\right] , \tag{6.21}$$

then $\bar{\gamma}(\alpha^*;\beta;\bar{x})$ may be written as

$$\bar{\gamma}(\alpha^*;\beta;\bar{x}) = \frac{1}{2}\frac{\hbar}{m}\exp\left[(\alpha|\beta)\right]\left[\bar{\nabla}\left(\phi_\alpha(\bar{x})+\phi_\beta(\bar{x})\right) + i\bar{\nabla}\ln\left(f_\alpha(\bar{x})/f_\beta(\bar{x})\right)\right] . \tag{6.22}$$

When the form of $\bar{\gamma}(\alpha^*;\beta;\bar{x})$ given by Eq. (6.22) is inserted into Eq. (6.20), the result is

$$\bar{V}_\alpha(\bar{x}) \equiv \langle\alpha|\hat{\bar{V}}(\bar{x})|\alpha\rangle = \frac{\hbar}{m}\ \bar{\nabla}\phi_\alpha(\bar{x}) . \tag{6.23}$$

The operator $\hat{\bar{V}}(\bar{x})$ may now be found by substituting $\bar{\gamma}(\alpha^*;\beta;\bar{x})$ into Eq. (6.10). The result, which is calculated in Appendix V, is[11]

$$\hat{\bar{V}}(\bar{x}) = \frac{\hbar}{m}\int \bar{\nabla}\phi_\alpha(\bar{x})\,|\alpha\rangle\langle\alpha|\,D(\alpha) . \tag{6.24}$$

The velocity operator $\hat{\bar{V}}(\bar{x})$ has thus been shown to be the gradient of a velocity potential operator $\hat{\phi}(\bar{x})$,

$$\hat{\bar{V}}(\bar{x}) = \frac{\hbar}{m}\ \bar{\nabla}\hat{\phi}(\bar{x}) , \tag{6.25}$$

where

$$\hat{\Phi}(\bar{x}) = \int \phi_\alpha(\bar{x}) \,|\alpha\rangle\langle\alpha|\, D(\alpha) \ . \tag{6.26}$$

Equations of Motion

Turski writes the expectation value of the Hamiltonian in the coherent states as[11]

$$\langle\alpha|\hat{H}|\alpha\rangle = \frac{1}{2}\frac{\hbar^2}{m}\int \rho_\alpha(\bar{x})\,\bar{\nabla}\phi_\alpha(\bar{x})\cdot\bar{\nabla}\phi_\alpha(\bar{x})\,d^3x$$

$$+\frac{1}{2}\frac{\hbar^2}{m}\int \bar{\nabla}\left(\rho_\alpha(\bar{x})\right)^{1/2}\cdot\bar{\nabla}\left(\rho_\alpha(\bar{x})\right)^{1/2}\,d^3x \ +$$

$$+\frac{1}{m^2 2}\iint V(\bar{x},\bar{y})\,\rho_\alpha(\bar{x})\rho_\alpha(\bar{y})\,d^3x\,d^3y \equiv H(\rho_\alpha,\phi_\alpha)\,, \tag{6.27}$$

where $\hat{H}$ is the Hamiltonian given by Eq. (3.13), $V(x,\bar{y})$ is the two-body potential, and $\rho_\alpha(\bar{x})$ is defined by

$$\alpha(\bar{x}) = \left(\rho_\alpha(\bar{x})\right)^{1/2}\frac{1}{\sqrt{m}}\,\exp\left[i\phi_\alpha(\bar{x})\right] \ . \tag{6.28}$$

He then says that by functionally differentiating $H(\rho_\alpha,\phi_\alpha)$ with respect to ϕ_α and ρ_α the continuity equation and the Bernoulli equation for the many-boson system in the state $|\alpha\rangle$ may be obtained. These are the equations of motion for $\rho_\alpha(\bar{x})$ and $\phi_\alpha(\bar{x})$ and are shown below:

$$\frac{\partial}{\partial t}\rho_\alpha(\bar{x}) + \frac{\hbar}{m}\bar{\nabla}\cdot\left(\rho(\bar{x})\bar{\nabla}\phi_\alpha(\bar{x})\right) = 0,$$

$$\frac{\hbar}{m}\frac{\partial\phi_\alpha(\bar{x})}{\partial t} + \frac{1}{2}\frac{\hbar^2}{m^2}\left(\bar{\nabla}\phi_\alpha(\bar{x})\right)^2 = -\frac{1}{m}\int V(\bar{x},\bar{y})\,\rho_\alpha(\bar{y})\,d^3y$$

$$-\frac{1}{2}h^2\left(\rho_\alpha(x)\right)^{-1/2}\nabla^2\left(\rho_\alpha(\bar{x})\right)^{1/2}. \tag{6.29}$$

Equations (6.28) and (6.29) are the same results which are given by the Gross-Pitaevskii equation approach to quantum hydrodynamics. However, the means Turski used to obtain these equations is incorrect, because the coherent states are time independent. Therefore, the expectation values of the operators in the coherent states are time independent. The functional differentiation will not give the time dependent equations, but will give only their time independent form.

The coherent states approach has rigorously shown the velocity operator to be the gradient of a velocity potential operator. The equations of quantum hydrodynamics, i.e., the time dependent continuity equation and the Bernoulli equation in expectation value form, have still not been rigorously derived. It is shown in Section VII how they may be rigorously derived from the Gross-Pitaevskii equation.

CHAPTER VII

THE GROSS-PITAEVSKII EQUATION AND QUANTUM HYDRODYNAMICS

The Gross-Pitaevskii equation is derived in this section. It is obtained by making a simple-shift canonical transformation on the field operators. Equating the coefficients of the unit operator in the Heisenberg equation of motion for the transformed field operator gives the time-dependent Gross-Pitaevskii equation.[21] Since it is not mathematically rigorous to equate the coefficients of the unit operator, mention is made of a more rigorous derivation.

Euler's equation and the Bernoulli equation for functions are derived from the Gross-Pitaevskii equation. The condensate wave function is written in hydrodynamical form by writing it in terms of a modulus which is the square root of the density and a phase which turns out to be proportional to the velocity potential. Equating the real and imaginary parts of the resulting equation gives the equation of continuity and the Bernoulli equation.

The Gross-Pitaevskii equation approach to quantum hydrodynamics has several advantages over the current algebra approach. The main advantage is that the hydrodynamical equations which are obtained are for functions rather than for operators. The quantization of circulation condition also follows more readily from the Gross-Pitaevskii approach. Although the Gross-Pitaevskii equation is an approximation, the terms neglected in making the approximation are known.[7] These terms could be calculated in theory.

Gross-Pitaevskii Equation

The first step in obtaining the Gross-Pitaevskii equation is to make a canonical transformation on the field operators.[6] The transformation has the form

$$\hat{\Psi}(\bar{x},t) = \phi(\bar{x},t) + \hat{\chi}(\bar{x},t), \tag{7.1}$$

where $\phi(\bar{x},t)$ is a function which measures the "average value," in some sense, of the field operator, and $\hat{\chi}(\bar{x},t)$ is an operator which measures the deviation of the field operator from the average. The time dependence of the operators depends on their being in the Heisenberg picture;

$$\hat{\Psi}(\bar{x},t) = e^{i\hat{H}t/\hbar}\,\hat{\Psi}(\bar{x})\,e^{-i\hat{H}t/\hbar} \;.$$

The function $\phi(\bar{x},t)$ is defined as

$$\phi(\bar{x},t) = \langle \hat{\Psi}(\bar{x},t) \rangle \;. \tag{7.2}$$

Since $\hat{\chi}(\bar{x},t)$ measures the deviation of $\hat{\Psi}(\bar{x},t)$ from the average, it satisfies

$$\langle \hat{\chi}(\bar{x},t) \rangle = 0. \tag{7.3}$$

The field creation operator is obtained by taking the Hermitian conjugate of Eq. (7.1):

$$\hat{\psi}^{+}(\bar{x},t) = \phi^{*}(\bar{x},t) + \hat{\chi}^{+}(\bar{x},t), \tag{7.4}$$

where

$$\phi^{*}(\bar{x},t) = \langle \hat{\psi}^{+}(\bar{x},t) \rangle , \tag{7.5}$$

and

$$\langle \hat{\chi}^{+}(\bar{x},t) \rangle = 0 . \tag{7.6}$$

If the transformation is to be canonical, the operators $\hat{\chi}$ and $\hat{\chi}^{+}$ must satisfy the same canonical commutation relations in Eqs. (3.2) and (3.3) as do the field operators. If the form of the field operators given by Eqs. (7.1) and (7.4) is inserted into Eqs. (3.2) and (3.3), the result is

$$\left[\hat{\chi}(\bar{x},t), \hat{\chi}(\bar{y},t)\right] = 0 , \tag{7.7}$$

and

$$\left[\hat{\chi}(\bar{x},t), \hat{\chi}^{+}(y,t)\right] = \delta(\bar{x}-\bar{y}) . \tag{7.8}$$

Thus the operators $\hat{\chi}$ and $\hat{\chi}^{+}$, which are called devion operators, are canonical. It should be noted that the commutation relations Eqs. (7.7) and (7.8) are "equal time" commutation relations.

The "average value" of the field operator can be calculated between the (n-1) and n particle subspaces of Fock space[7]:

$$\phi(\bar{x},t) = \langle \Psi^{(n-1)} | \hat{\psi}(\bar{x},t) | \Psi^{(n)} \rangle , \qquad (7.9)$$

where $\Psi^{(n)}$ is the n-particle wave function. The order parameter can also be calculated for a general Fock space state:

$$\phi(\bar{x},t) = \langle \Psi | \hat{\psi}(\bar{x},t) | \Psi \rangle .$$

$\phi(\bar{x},t)$ is known as the order parameter of the system, or the "condensate wave function." If the expectation value of the density is calculated using the transformed field operators, the result is

$$\langle \Psi | \hat{\rho}(\bar{x},t) | \Psi \rangle = \rho(\bar{x},t) = m \langle \Psi | \hat{\psi}^{+}(\bar{x},t)\, \hat{\psi}(\bar{x},t) | \Psi \rangle$$

$$= m \langle \Psi | \left(\phi^{*}(\bar{x},t) + \hat{\chi}^{+}(\bar{x},t)\right) \left(\phi(\bar{x},t) + \hat{\chi}(\bar{x},t)\right) | \Psi \rangle$$

$$= m\, \phi^{*}(\bar{x},t)\, \phi(\bar{x},t) + m \langle \hat{\chi}(\bar{x},t) \rangle\, \phi^{*}(\bar{x},t) + m \langle \hat{\chi}^{+}(\bar{x},t) \rangle\, \phi(\bar{x},t) + m \langle \hat{\chi}^{+}(\bar{x},t)\, \hat{\chi}(\bar{x},t) \rangle . \qquad (7.10)$$

Applying Eqs. (7.3) and (7.6) to Eq. (7.10), we obtain the final result

$$\rho(\bar{x},t) = m\, |\phi(\bar{x},t)|^{2} + m \langle \hat{\chi}^{+}(\bar{x},t)\, \hat{\chi}(\bar{x},t) \rangle . \qquad (7.11)$$

These results are not used in the derivation given in this section, but are used in the mathematically rigorous derivation which will be mentioned later.

The Hamiltonian for a system of many identical bosons is

$$\hat{H} = \int \hat{\psi}^{+}(\bar{x},t)T(\bar{x})\hat{\psi}(\bar{x},t)\, d^3x + \frac{1}{2}\iint \hat{\psi}^{+}(\bar{x},t)\hat{\psi}^{+}(\bar{y},t)V(\bar{x},\bar{y})\hat{\psi}(\bar{y},t)\hat{\psi}(\bar{x},t)\, d^3x\, d^3y, \tag{7.12}$$

where

$$T(\bar{x}) = -\frac{\hbar}{2m}\bar{\nabla}_x^2 + U(\bar{x}), \tag{7.13}$$

and $U(\bar{x})$ is the external potential. When this Hamiltonian is inserted into the Heisenberg equation of motion for the field annihilation operator, the result is

$$i\hbar\frac{\partial\hat{\psi}(\bar{x},t)}{\partial t} = \hat{\psi}(\bar{x},t),\hat{H} = \hat{T}(\bar{x})\hat{\psi}(\bar{x},t) + \int \hat{\psi}^{+}(\bar{y},t)V(\bar{x},\bar{y})\hat{\psi}(\bar{y},t)\hat{\psi}(\bar{x},t)\, d^3y, \tag{7.14}$$

which is shown in Appendix W. When the transformed field operators of Eqs. (7.1) and (7.4) are substituted into Eq. (7.14), the result is

$$i\hbar\frac{\partial\phi(\bar{x},t)}{\partial t} = T(\bar{x})\phi(\bar{x},t) + \int \phi^{*}(\bar{y},t)V(x,\bar{y})\phi(\bar{y},t)\phi(\bar{x},t)\, d^3y$$

$+$ (terms involving the devion operators). (7.15)

Equating the coefficients of the unit operator in Eq. (7.15) gives

$$i\hbar \frac{\partial \phi(\bar{x},t)}{\partial t} = \frac{-\hbar^2}{2m} \nabla^2 \phi(\bar{x},t) + U(\bar{x}) \phi(\bar{x},t)$$

$$+ \int \phi^*(\bar{y},t) V(\bar{x},\bar{y}) \phi(\bar{y},t) \phi(\bar{x},t) d^3y \, . \tag{7.16}$$

Equation (7.16) is the Gross-Pitaevskii equation.

Although the derivation of the Gross-Pitaevskii equation given above is straightforward and reasonable, it is not mathematically rigorous.[7] The terms in Eq. (7.15) involving the devion operators modify the average value ϕ of the field operator. The Gross-Pitaevskii equation also has the disadvantage that it can not be applied to a system where the two-body potential $V(\bar{x},\bar{y})$ has a strongly repulsive core, as is the case with liquid helium. This is because the last term, which is the average field due to all the other particles, becomes divergent. Equation (7.16) is a Hartree-like equation.

A modified Gross-Pitaevskii equation may be derived rigorously, however, which is applicable to systems with two-body potentials with strongly repulsive cores.[7] This is a correction to the Gross-Pitaevskii equation given by Eq. (7.16), which is then the first order approximation to the exact equation. The modified Gross-Pitaevskii equation is obtained by first finding the equation of motion of the order parameter. The term in that equation which describes the average potential due to the other particles is then expanded in a perturbation series in which the coherent state representation is used. The perturbation expansion is then partially resummed to give an average field in which the potential has been replaced by the T matrix. This gives a finite average effective field.[7]

Quantum Hydrodynamics

The equation of continuity and the Bernoulli equation are derived from the Gross-Pitaevskii equation given by Eq. (7.16) by first assuming that the second term on the right in Eq. (7.11) may be neglected,[6]

$$m\left\langle \hat{\chi}^{+}(\bar{x},t)\,\hat{\chi}(\bar{x},t)\right\rangle \simeq 0\,. \tag{7.17}$$

This is equivalent to assuming that the two-body interaction is small. In this case, the density expectation value is

$$\rho(\bar{x},t) \simeq m\ \phi(\bar{x},t)^{2}\,. \tag{7.18}$$

The order parameter may then be written as

$$\phi(\bar{x},t) = \frac{1}{\sqrt{m}}\sqrt{\rho(\bar{x},t)}\;e^{iR(\bar{x},t)}\,. \tag{7.19}$$

where $R(\bar{x},t)$ is a real function.

The expectation value of the current is approximately

$$\bar{j}(\bar{x},t) = \frac{1}{2}\left[\phi^{*}(\bar{x},t)\frac{\hbar}{i}\bar{\nabla}\phi(\bar{x},t) - \phi(\bar{x},t)\frac{\hbar}{i}\bar{\nabla}\phi^{*}(\bar{x},t)\right], \tag{7.20}$$

so that

$$\bar{j}(\bar{x},t) = \rho(\bar{x},t)\frac{\hbar}{m}\bar{\nabla}R(\bar{x},t)\,, \tag{7.21}$$

when use is made of Eq. (7.19).

The current is related to the velocity $\bar{V}(\bar{x},t)$ by

$$\bar{j}(\bar{x},t) = \rho(\bar{x},t)v(\bar{x},t)\,. \tag{7.22}$$

Equations (7.21) and (7.22) imply that the phase $R(\bar{x},t)$ is the velocity potential,

$$v(\bar{x},t) = \frac{\hbar}{m}\bar{\nabla}R(\bar{x},t)\,. \tag{7.23}$$

Identifying the velocity as the gradient of a phase implies that its curl must be zero. Stokes' theorem then states that the line integral of the velocity around a closed path is zero,

$$\int \bar{\nabla}_x \bar{\nabla}R\cdot\hat{n}\,da = \oint \bar{\nabla}R\cdot d\bar{l} = 0.$$

However, when the integral is around a singular point, the result is

$$\oint \bar{V}_\alpha\cdot d\bar{l} = \frac{\hbar}{m}\oint\bar{\nabla}R\cdot d\bar{l} = \frac{\hbar}{m}2\pi n, \qquad n = 0,1,2\ldots\ .$$

which follows from the single-valuedness of the order parameter. This is the quantization of circulation condition for a boson system.

It is shown in Appendix X that substituting $\phi(\bar{x},t)$ given by Eq. (7.19) into Eq. (7.16) gives the result

$$i\hbar \frac{\partial \sqrt{\frac{\rho(\bar{x},t)}{m}}}{\partial t} - \sqrt{\frac{\rho(\bar{x},t)}{m}}\ \frac{\partial R(\bar{x},t)}{\partial t} = -\hbar^2\left[\nabla^2 \frac{\rho(\bar{x},t)}{m} + i\left(\bar{\nabla} R(\bar{x},t)\right)\cdot\left(\bar{\nabla}\left(\frac{\rho(\bar{x},t)}{m}\right)^{1/2}\right)\right.$$

$$\left. + i\left(\bar{\nabla} R(\bar{x},t)\right)\cdot\left(\bar{\nabla}\left(\frac{\rho(\bar{x},t)}{m}\right)^{1/2}\right) - \sqrt{\frac{\rho(\bar{x},t)}{m}}\left(\bar{\nabla} R(\bar{x},t)\right)^{1/2} + i\sqrt{\frac{\rho(\bar{x},t)}{m}}\ \nabla^2 R(\bar{x},t)\right]$$

$$+ U(\bar{x})\sqrt{\frac{\rho(\bar{x},t)}{m}} + \sqrt{\frac{\rho(\bar{x},t)}{m}}\int V(\bar{x},t)\ \frac{\rho(\bar{y},t)}{m}\ d^3y\ . \tag{7.24}$$

Equating the imaginary parts of the above equation, we obtain the continuity equation

$$\frac{\partial \rho(\bar{x},t)}{\partial t} + \bar{\nabla}\cdot\bar{j}(\bar{x},t) = 0, \tag{7.25}$$

which is derived in Appendix Y. Equating the real parts of the above equation and taking the gradient of the resulting equation gives Euler's equation for a boson fluid,

$$\frac{\partial \bar{v}(\bar{x},t)}{\partial t} + \left(\bar{v}(\bar{x},t)\cdot\bar{\nabla}\right)\bar{v}(\bar{x},t) - \frac{1}{2}\frac{\hbar^2}{m^2}\bar{\nabla}\left(\frac{\nabla^2\left(\rho(\bar{x},t)\right)^{1/2}}{\sqrt{\rho(\bar{x},t)}}\right) = \frac{1}{m}\bar{F}_{ext.} + \frac{1}{m}\bar{F}_{int.}, \tag{7.26}$$

where the third term on the left side is the quantum correction term. The terms $\bar{F}_{ext.}$ and $\bar{F}_{int.}$ are the external and internal forces on the fluid. They are defined as

$$\bar{F}_{ext.} \equiv -\bar{\nabla} U(\bar{x}), \tag{7.27}$$

and

$$\bar{F}_{int.} \equiv -\bar{\nabla}\frac{1}{m}\int V(\bar{x},\bar{x}')\,\rho(\bar{x}')\,d^3x' \,. \tag{7.28}$$

Equation (7.26) is the force equation for the boson fluid. The term involving $\hbar$ is known as the quantum pressure term, since it may be written in terms of a pressure.

The essential equations of quantum hydrodynamics have thus been derived in expectation value form and the quantization of circulation condition has been derived. These equations have been shown to be approximations, but the terms neglected in the approximations are known. The mathematically dubious operations used in the operator derivations described in the previous sections have been avoided.

CHAPTER VIII

CONCLUSIONS

Although it has been satisfying to derive Landau's theory of quantized hydrodynamics from the many-particle Schroedinger equation in a manner that is at least formally correct, we seem to be no closer to a useful theory of quantized hydrodynamics. The use of the inverse density operator renders the derivation doubtful. Also, the equations obtained are in terms of operators, not functions. The Gross-Pitaevskii equation approach still seems the most fruitful, since it rigorously leads to the same form of hydrodynamical equations as does the current algebra approach. However, the results are equations which are written in terms of functions and can therefore be in principle solved. The similarity of the results given by the two approaches is nevertheless interesting. Perhaps a current algebra approach could be formulated which would not require the inverse density or inverse field operators, or a mathematical justification might be given for spaces other than Fock space.[18]

APPENDICES

APPENDIX A

DEMONSTRATION OF THE CONSISTENCY OF THE TWO DEFINITIONS OF THE VELOCITY OPERATOR

Equations (2.3a) and (2.3b) are shown to be consistent with each other and with the commutation relations between the operators as follows:

$$\hat{\bar{J}} = \frac{1}{2}\left[\hat{\rho}\hat{\bar{V}} + \hat{\bar{V}}\hat{\rho}\right]$$

$$\hat{\bar{V}} = \frac{1}{2}\left[\frac{1}{\hat{\rho}}\hat{\bar{J}} + \hat{\bar{J}}\frac{1}{\hat{\rho}}\right]$$

$$= \frac{1}{4}\left\{\frac{1}{\hat{\rho}}\hat{\rho}\hat{\bar{V}} + \frac{1}{\hat{\rho}(\bar{x})}\hat{\bar{V}}(\bar{x})\hat{\rho}(\bar{x}) + \hat{\rho}\hat{\bar{V}}\frac{1}{\hat{\rho}} + \hat{\bar{V}}\hat{\rho}\frac{1}{\hat{\rho}}\right\}$$

$$= \frac{1}{4}\left\{\hat{\bar{V}} + \hat{\bar{V}} + \frac{1}{\hat{\rho}(\bar{x})}\left[\hat{\rho}(\bar{x})\hat{\bar{V}}(\bar{x}) + \frac{\hbar}{i}\bar{\nabla}\delta(\bar{x}-\bar{y})\right]_{\bar{x}=\bar{y}}\right.$$

$$\left. + \left[\hat{\bar{V}}(\bar{x})\hat{\rho}(\bar{x}) - \frac{\hbar}{i}\bar{\nabla}_{\bar{x}}\delta(\bar{x}-\bar{y})\Big|_{\bar{x}=\bar{y}}\right]\frac{1}{\hat{\rho}(\bar{x})}\right\}$$

$$= \frac{1}{4}\left\{2\hat{\bar{V}} + \hat{\bar{V}} + \hat{\bar{V}} + \frac{1}{\hat{\rho}(\bar{x})}\left[\frac{\hbar}{i}\bar{\nabla}_{\bar{x}}\delta(\bar{x}-\bar{y})\Big|_{\bar{x}=\bar{y}} - \frac{\hbar}{i}\bar{\nabla}_{\bar{x}}\delta(\bar{x}-\bar{y})\Big|_{\bar{x}=\bar{y}}\right]\right\}$$

$$= \hat{\bar{V}} \quad , \qquad \text{(A1)}$$

and

$$\hat{\bar{\bar{J}}} = \frac{1}{2}\left[\hat{\rho}\hat{\bar{V}} + \hat{\bar{V}}\hat{\rho}\right] =$$

$$= \frac{1}{4}\left[\hat{\rho}\frac{1}{\hat{\rho}}\hat{\bar{\bar{J}}} + \hat{\rho}\hat{\bar{\bar{J}}}\frac{1}{\hat{\rho}} + \frac{1}{\hat{\rho}}\hat{\bar{\bar{J}}}\hat{\rho} + \hat{\bar{\bar{J}}}\frac{1}{\hat{\rho}}\hat{\rho}\right]$$

$$= \frac{1}{4}\left[2\hat{\bar{\bar{J}}} + \frac{1}{\hat{\rho}}\left[\hat{\rho}\hat{\bar{\bar{J}}} + \frac{\hbar}{i}\hat{\rho}\bar{\nabla}_{\bar{x}}\,\delta(\bar{y}-\bar{x})\Big|_{\bar{x}=\bar{y}}\right]\right.$$

$$\left. + \left[\hat{\bar{\bar{J}}}\hat{\rho} - \frac{\hbar}{i}\hat{\rho}\bar{\nabla}_{\bar{x}}\,\delta(\bar{y}-\bar{x})\Big|_{\bar{x}=\bar{y}}\right]\frac{1}{\hat{\rho}}\right]$$

$$= \frac{1}{4}\left[2\hat{\bar{\bar{J}}} + 2\hat{\bar{\bar{J}}}\right]$$

$$= \hat{\bar{\bar{J}}}\,.$$

The two definitions of the velocity operator are thus consistent. (A2)

APPENDIX B

CALCULATION OF LANDAU'S CURRENT DENSITY COMMUTATOR

Equation (2.9b) is calculated by substituting Eqs. (2.1) and (2.2) into the current-density commutator as follows:

$$\left[\hat{J}_K(\bar{x}), \rho(\bar{y})\right] = \frac{1}{2}\frac{\hbar}{i}\sum_{i=1}^{N}\left[\frac{\partial}{\partial r_{Ki}}\delta(\bar{x}-\bar{r}_i) + \delta(\bar{x}-\bar{r}_i)\frac{\partial}{\partial r_{Ki}}\right] \times$$

$$\sum_{j=1}^{N}\delta(\bar{y}-\bar{r}_j) - \sum_{j=1}^{N}\delta(\bar{y}-\bar{r}_j)J_K(\bar{x})$$

$$= \frac{1}{2}\frac{\hbar}{i}\sum_{i=1}^{N}\left\{\left[\frac{\partial}{\partial r_{Ki}}\delta(\bar{x}-\bar{r}_i)\right] + 2\delta(\bar{x}-\bar{r}_i)\frac{\partial}{\partial r_{Ki}}\right\}\sum_{j=1}^{N}\delta(\bar{y}-\bar{r}_j)$$

$$-\rho(\bar{y})J_K(\bar{x})$$

$$= \left\{\frac{1}{2}\frac{\hbar}{i}\sum_{i=1}^{N}\left\{\left[\frac{\partial}{\partial r_{Ki}}\delta(\bar{x}-\bar{r}_i)\right]\rho(\bar{y}) + \rho(\bar{y})2\delta(\bar{x}-\bar{r})\frac{\partial}{\partial r_{Ki}}\right.\right.$$

$$+ \ldots 2\delta(\bar{x}-\bar{r}_i)\left[\frac{\partial}{\partial r_{Ki}}\delta(\bar{y}-\bar{r}_i)\right]$$

$$- \frac{1}{2} \frac{\hbar}{i} \sum_{i=1}^{N} \rho(\bar{y}) \left[\frac{\partial}{\partial r_{Ki}} \delta(\bar{x}-\bar{r}_i) + 2\,\delta(\bar{x}-\bar{r}_i) \frac{\partial}{\partial r_{Ki}} \right] \Bigg\}$$

$$= \frac{1}{2} \frac{\hbar m}{i} \sum_{i=1}^{N} 2\,\delta(\bar{x}-\bar{r}_i) \left[\frac{\partial}{\partial r_{Ki}} \delta(\bar{y}-\bar{r}_i) \right]$$

$$= - \frac{\hbar m}{i} \sum_{i=1}^{N} \delta(\bar{x}-\bar{r}_i) \frac{\partial}{\partial y_K} \delta(\bar{y}-\bar{r}_i)$$

$$= - \frac{\hbar}{i} \sum_{i=1}^{N} \delta(\bar{x}-\bar{r}_i) \frac{\partial}{\partial y_K} \delta(\bar{y}-\bar{x})$$

$$= - \frac{\hbar}{i} \rho(\bar{x}) \frac{\partial}{\partial y_K} \delta(\bar{y}-\bar{x})$$

$$= \frac{\hbar}{i} \rho(\bar{x}) \frac{\partial}{\partial x_K} \delta(\bar{y}-\bar{x}) \quad , \tag{B1}$$

which is Eq. (2.4b).

APPENDIX C

THE CURRENT-CURRENT COMMUTATOR FOR LANDAU'S CURRENT OPERATOR

The current-current commutator of Eq. (2.9e) is calculated in this Appendix by substituting Landau's definition for the current operator, Eq. (2.2), into the commutator as follows:

$$\left[\hat{J}_k(\bar{x}), \hat{J}_\ell(\bar{y})\right] = \frac{1}{4}\left[\sum_{i=1}^{N}\left\{\frac{\hbar}{i}\frac{\partial}{\partial r_i^{k}}\delta(\bar{x}-\bar{r}_i) + \delta(\bar{x}-\bar{r}_i)\frac{\hbar}{i}\frac{\partial}{\partial r_i^{k}}\right\},\right.$$

$$\left.\sum_{j=1}^{N}\left\{\frac{\hbar}{i}\frac{\partial}{\partial r_j^{\ell}}\delta(\bar{y}-\bar{r}_j) + \frac{\hbar}{i}\delta(\bar{y}-\bar{r}_j)\frac{\partial}{\partial r_j^{\ell}}\right\}\right]$$

$$= \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\left[\left\{\frac{\hbar}{i}\frac{\partial}{\partial r_i^{k}}\delta(\bar{x}-\bar{r}_i) + \delta(\bar{x}-\bar{r}_i)\frac{\hbar}{i}\frac{\partial}{\partial r_i^{k}}\right\},\right.$$

$$\left.\left\{\frac{\hbar}{i}\frac{\partial}{\partial r_j^{\ell}}\delta(\bar{y}-\bar{r}_j) + \delta(\bar{y}-\bar{r}_j)\frac{\hbar}{i}\frac{\partial}{\partial r_j^{\ell}}\right\}\right]$$

$$= \frac{1}{4}\sum_{i=1}^{N}\left\{\left[\frac{\hbar}{i}\frac{\partial}{\partial r_i^{k}}\delta(\bar{x}-\bar{r}_i), \frac{\hbar}{i}\frac{\partial}{\partial r_i^{\ell}}\delta(\bar{y}-\bar{r}_i)\right]\right.$$

$$+\left[\frac{\hbar}{i}\frac{\partial}{\partial r_i^k}\,\delta(\bar{x}-\bar{r}_i),\ \frac{\hbar}{i}\,\delta(\bar{y}-\bar{r})\frac{\partial}{\partial r_i^\ell}\right]$$

$$+\left[\frac{\hbar}{i}\,\delta(\bar{x}-\bar{r}_i)\frac{\partial}{\partial r_i^k},\ \frac{\hbar}{i}\frac{\partial}{\partial r_i^\ell}\,\delta(\bar{y}-\bar{r})\right]$$

$$+\left[\frac{\hbar}{i}\,\delta(\bar{x}-\bar{r}_i)\frac{\partial}{\partial r_i^k},\ \frac{\hbar}{i}\,\delta(\bar{y}-\bar{r}_i)\frac{\partial}{\partial r_i^\ell}\right]\Bigg\}$$

$$=\frac{1}{4}\sum_{i=1}^{N}\Bigg\{\frac{\hbar}{i}\frac{\partial}{\partial r_i^k}\,\delta(\bar{x}-\bar{r}_i)\frac{\hbar}{i}\frac{\partial}{\partial r_i^\ell}\,\delta(\bar{x}-\bar{r}_i)$$

$$-\frac{\hbar}{i}\frac{\partial}{\partial r_i^\ell}\,\delta(\bar{y}-\bar{r}_i)\frac{\hbar}{i}\frac{\partial}{\partial r_i^k}\,\delta(\bar{x}-\bar{r}_i)+\frac{\hbar}{i}\frac{\partial}{\partial r_i^k}\,\delta(\bar{x}-\bar{r}_i)\frac{\hbar}{i}\,\delta(\bar{y}-\bar{r}_i)\frac{\partial}{\partial r_i^\ell}$$

$$-\frac{\hbar}{i}\,\delta(\bar{y}-\bar{r}_i)\frac{\partial}{\partial r_i^\ell}\frac{\hbar}{i}\frac{\partial}{\partial r_i^k}\,\delta(\bar{x}-\bar{r}_i)+\frac{\hbar}{i}\,\delta(\bar{x}-\bar{r}_i)\frac{\partial}{\partial r_i^k}\frac{\hbar}{i}\frac{\partial}{\partial r_i^\ell}\,\delta(\bar{y}-\bar{r}_i)$$

$$-\frac{\hbar}{i}\frac{\partial}{\partial r_i^\ell}\,\delta(\bar{y}-\bar{r}_i)\frac{\hbar}{i}\,\delta(\bar{x}-\bar{r}_i)\frac{\partial}{\partial r_i^k}+\frac{\hbar}{i}\,\delta(\bar{x}-\bar{r})\frac{\partial}{\partial r_i^k}\frac{\hbar}{i}\,\delta(\bar{y}-\bar{r}_i)\frac{\partial}{\partial r_i^\ell}$$

$$-\frac{\hbar}{i}\,\delta(\bar{y}-\bar{r}_i)\frac{\partial}{\partial r_i^\ell}\frac{\hbar}{i}\,\delta(\bar{x}-\bar{r}_i)\frac{\partial}{\partial r_i^k}\Bigg\}$$

$$= \frac{1}{4}\sum_{i=1}^{N}\Bigg\{ -\hbar^2\delta(\bar{x}-\bar{r}_i)\left[\frac{\partial^2\delta(\bar{y}-\bar{r}_i)}{\partial r_i^k\,\partial r_i^\ell}\right] + \hbar^2\delta(\bar{y}-\bar{r}_i)\left[\frac{\partial^2\delta(\bar{x}-\bar{r}_i)}{\partial r_i^\ell\,\partial r_i^k}\right]$$

$$-\hbar^2\delta(\bar{x}-\bar{r}_i)\left[\frac{\partial\delta(\bar{y}-\bar{r}_i)}{\partial r_i^k}\right]\frac{\partial}{\partial r_i^\ell} + \hbar^2\delta(\bar{y}-\bar{r}_i)\left[\frac{\partial\delta(\bar{x}-\bar{r}_i)}{\partial r_i^\ell}\right]\frac{\partial}{\partial r_i^k}$$

$$-\hbar^2\delta(\bar{x}-\bar{r}_i)\left[\frac{\partial\delta(\bar{y}-\bar{r}_i)}{\partial r_i^k}\right]\frac{\partial}{\partial r_i^\ell} + \hbar^2\delta(\bar{y}-\bar{r}_i)\left[\frac{\partial\delta(\bar{x}-\bar{r}_i)}{\partial r_i^\ell}\right]\frac{\partial}{\partial r_i^k}$$

$$+\hbar^2\delta(\bar{y}-\bar{r}_i)\left[\frac{\partial^2\delta(\bar{x}-\bar{r}_i)}{\partial r_i^\ell\,\partial r_i^k}\right] - \hbar^2\delta(\bar{x}-\bar{r}_i)\left[\frac{\partial^2\delta(\bar{y}-\bar{r}_i)}{\partial r_i^k\,\partial r_i^\ell}\right]$$

$$-\hbar^2\delta(\bar{x}-\bar{r}_i)\left[\frac{\partial\delta(\bar{y}-\bar{r}_i)}{\partial r_i^k}\right]\frac{\partial}{\partial r_i^\ell} \quad \hbar^2\delta(\bar{y}-\bar{r}_i)\left[\frac{\partial\delta(\bar{x}-\bar{r}_i)}{\partial r_i^\ell}\right]\frac{\partial}{\partial r_i^k}$$

$$-\hbar^2\delta(\bar{x}-\bar{r}_i)\left[\frac{\partial\delta(\bar{y}-\bar{r}_i)}{\partial r_i^k}\right]\frac{\partial}{\partial r_i^\ell} + \hbar^2\delta(\bar{y}-\bar{r}_i)\left[\frac{\partial\delta(\bar{x}-\bar{r}_i)}{\partial r_i^\ell}\right]\frac{\partial}{\partial r_i^k}\Bigg\}\,, \tag{C1}$$

where the brackets indicate that the derivatives are no longer operators, but operate only on the terms within the brackets. Applying Eqs. (3.8) and (3.10) to Eq. (C1) gives

$$= \frac{1}{4}\sum_{i=1}^{N}\Bigg\{ 2\hbar^2\left[\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y^k}\right]\left[\frac{\partial\delta(\bar{y}-\bar{r})}{\partial r_i^\ell}\right] + 2\hbar^2\delta(\bar{x}-\bar{y})\left[\frac{\partial^2\delta(\bar{y}-\bar{r}_i)}{\partial y^k\,\partial r_i^\ell}\right]$$

$$- 2\hbar^2\left[\frac{\partial\delta(\bar{y}-\bar{r}_i)}{\partial y^k}\right]\left[\frac{\partial\delta(\bar{x}-\bar{r}_i)}{\partial r_i^\ell}\right] - 4\hbar^2\delta(\bar{x}-\bar{y})\left[\frac{\partial\delta(\bar{y}-\bar{r}_i)}{\partial r_i^k}\right]\frac{\partial}{\partial r_i^\ell}$$

$$+ 4\hbar^2\left[\frac{\partial\delta(\bar{x}-\bar{r}_i)}{\partial r_i^k}\right]\delta(\bar{y}-\bar{r}_i)\frac{\partial}{\partial r_i'^\ell} + 4\hbar^2\delta(\bar{y}-\bar{x})\left[\frac{\partial\delta(\bar{x}-\bar{r}_i)}{\partial r_i^\ell}\right]\frac{\partial}{\partial r_i^k}$$

$$- 4\hbar^2\left[\frac{\partial\delta(\bar{y}-\bar{r}_i)}{\partial r_i^\ell}\right]\delta(\bar{x}-\bar{r}_i)\frac{\partial}{\partial r_i^k} - 2\hbar^2\left[\frac{\partial\delta(\bar{y}-\bar{x})}{\partial x^\ell}\right]\left[\frac{\partial\delta(\bar{x}-\bar{r}_i)}{\partial r_i^k}\right]$$

$$- 2\hbar^2\delta(\bar{y}-\bar{x})\left[\frac{\partial^2\delta(\bar{x}-\bar{r}_i)}{\partial x^\ell\,\partial r_i^k}\right] + 2\hbar^2\left[\frac{\partial\delta(\bar{y}-\bar{r}_i)}{\partial r_i^k}\right]\left[\frac{\partial}{\partial x^\ell}\delta(\bar{x}-\bar{r}_i)\right]\Bigg\}$$

$$= \sum_{i=1}^{N}\Bigg\{ i\hbar\left[\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y^k}\right]\frac{1}{2}\frac{\hbar}{i}\left[\frac{\partial\delta(\bar{y}-\bar{r}_i)}{\partial r_i^\ell}\right]$$

$$+ i\hbar\delta(\bar{x}-\bar{y})\frac{\hbar}{i}\left[\frac{\partial\delta(\bar{y}-\bar{r}_i)}{\partial y^k}\right]\frac{\partial}{\partial r_i^\ell} + i\hbar\delta(\bar{x}-\bar{y})\left[\frac{\partial}{\partial y^k}\frac{\hbar}{2i}\left[\frac{\partial\delta(\bar{y}-\bar{r}_i)}{\partial r_i^\ell}\right]\right]$$

$$- i\hbar\left[\frac{\partial\delta(\bar{x}-\bar{y})}{\partial x^\ell}\right]\frac{1}{2}\frac{\hbar}{i}\left[\frac{\partial\delta(\bar{x}-\bar{r}_i)}{\partial r_i^k}\right] - i\hbar\delta(\bar{x}-\bar{y})\frac{\hbar}{i}\left[\frac{\partial\delta(\bar{x}-\bar{r}_i)}{\partial x^\ell}\right]\frac{\partial}{\partial r_i^k}$$

$$- i\hbar\delta(\bar{x}-\bar{y})\left[\frac{\partial}{\partial x}\frac{\hbar}{2i}\left[\frac{\partial\delta(\bar{x}-\bar{r}_i)}{\partial r^k}\right]\right]$$

$$+ i\hbar \left[\frac{\partial \delta(\bar{x}-\bar{y})}{\partial y^{k}}\right] \frac{\hbar}{i}\, \delta(\bar{y}-\bar{r}_i) \frac{\partial}{\partial r_i^{\ell}} - i\hbar \left[\frac{\partial \delta(\bar{x}-\bar{y})}{\partial x^{\ell}}\right] \frac{\hbar}{i}\, \delta(\bar{x}-\bar{r}_i) \frac{\partial}{\partial r_i^{k}} \Bigg\}$$

$$= \sum_{i=1}^{N} \left\{ -i\hbar \frac{\partial}{\partial x^{\ell}} \left[\delta(\bar{x}-\bar{y})\, \hat{J}_{ki}(\bar{x})\right] + i\hbar \frac{\partial}{\partial y^{k}} \left[\delta(\bar{x}-\bar{y})\, \hat{J}_{\ell i}(\bar{y})\right] \right\}$$

$$= i\hbar \frac{\partial}{\partial x^{\ell}} \left[\delta(\bar{x}-\bar{y})\, \hat{J}_{k}(\bar{x})\right] + i\hbar \frac{\partial}{\partial y^{k}} \left[\delta(\bar{x}-\bar{y})\, \hat{J}_{\ell}(\bar{y})\right], \qquad \text{(C2)}$$

which is the same as Eq. (2.4e).

APPENDIX D

CALCULATION OF THE EQUATION OF MOTION FOR THE DENSITY OPERATOR

The Hamiltonian, (2.6) is inserted into the Heisenberg equation of motion,

$$\frac{\partial \hat{\rho}(\bar{x})}{\partial t} = \frac{i}{\hbar}\left[\hat{H}, \hat{\rho}(\bar{x})\right] , \tag{D1}$$

to obtain

$$\frac{\partial \hat{\rho}(\bar{x})}{\partial t} = \frac{1}{2}\frac{i}{\hbar}\int\Big\{\hat{\bar{V}}(\bar{y})\cdot\hat{\rho}(\bar{y})\hat{\bar{V}}(\bar{y})\hat{\rho}(\bar{x}) - \hat{\rho}(\bar{x})\hat{\bar{V}}(\bar{y})\cdot\hat{\rho}(\bar{y})\hat{\bar{V}}(\bar{y})$$

$$+ 2\hat{\rho}(\bar{y})\hat{\mathcal{E}}(\hat{\rho})\hat{\rho}(\bar{x}) - 2\hat{\rho}(\bar{x})\hat{\rho}(\bar{y})\hat{\mathcal{E}}(\hat{\rho})\Big\} d^3y \ . \tag{D2}$$

Since $\left[\hat{\rho}, \hat{\rho}\right] = 0$, the last two terms in the integral cancel, giving

$$\frac{\partial \hat{\rho}(\bar{x})}{\partial t} = \frac{i}{\hbar}\int\frac{1}{2}\Big\{\hat{\bar{V}}(\bar{y})\cdot\hat{\rho}(\bar{y})\left[\hat{\bar{V}}(\bar{y})\hat{\rho}(\bar{x}) - \hat{\rho}(\bar{x})\hat{\bar{V}}(\bar{y})\right]$$

$$+ \left[\hat{\bar{V}}(\bar{y})\hat{\rho}(\bar{x}) - \hat{\rho}(\bar{x})\hat{\bar{V}}(\bar{y})\right]\cdot\hat{\rho}(\bar{y})\hat{\bar{V}}(\bar{y})\Big\} d^3y$$

$$= \frac{i}{\hbar} \int \frac{\hbar}{2i} \left[\bar{\nabla}_y \, \delta(\bar{y}-\bar{x}) \right] \cdot \left[\hat{\bar{V}}(\bar{y}) \hat{\rho}(\bar{y}) + \hat{\rho}(\bar{y}) \, \hat{\bar{V}}(\bar{y}) \right] d^3y$$

$$= \int \left[\bar{\nabla}_y \, \delta(\bar{y}-\bar{x}) \right] \cdot \hat{\bar{J}}(\bar{y}) \, d^3y$$

$$= - \bar{\nabla} \circ \hat{\bar{J}}(\bar{x}) \,, \qquad \text{(D3)}$$

which then gives Eq. (2.8).

APPENDIX E

CALCULATION OF THE EQUATION OF MOTION FOR THE VELOCITY OPERATOR

Equation (2.10) is calculated in this appendix. When the Hamiltonian of Eq. (2.7) is inserted into the Heisenberg equation of motion for one component of the velocity operator,

$$\frac{\partial \hat{V}_\ell(\bar{y},t)}{\partial t} = \frac{i}{\hbar}\left[\hat{H}\hat{V}_\ell(\bar{y},t) - \hat{V}_\ell(\bar{y},t)\hat{H}\right], \tag{E1}$$

the result is

$$\frac{\partial \hat{V}_\ell}{\partial t} = \frac{i}{\hbar}\frac{1}{2}\int\left[\hat{\bar{V}}(\bar{x})\cdot\hat{\rho}(\bar{x})\hat{\bar{V}}(\bar{x}), \hat{V}_\ell(\bar{x})\right]d^3x + \frac{i}{\hbar}\left[\int\hat{\rho}(\bar{x})\mathcal{E}(\hat{\rho})\,d^3x, \hat{V}_\ell(\bar{y})\right]. \tag{E2}$$

The first term on the right side of Eq. (E2), denoted by T_1, will be treated first. The second term, denoted by T_2, will then be treated in the two ways described in Section II. When the dot product in T_1 is taken the result is

$$\frac{i}{\hbar}\int\left[\frac{1}{2}\hat{\bar{V}}(\bar{x})\cdot\hat{\rho}(\bar{x})\hat{\bar{V}}(\bar{x}), \hat{V}_\ell(\bar{y})\right]d^3x$$

$$= \frac{i}{2\hbar}\int\left[\left(\hat{V}_k(\bar{x})\hat{\rho}(\bar{x})\hat{V}_k(\bar{x}) + \hat{V}_\ell(\bar{x})\hat{\rho}(\bar{x})\hat{V}_\ell(\bar{x}) + \hat{V}_m(\bar{x})\hat{\rho}(\bar{x})\hat{V}_m(\bar{x})\right), \hat{V}_\ell(\bar{y})\right]d^3x, \tag{E3}$$

where k, l, m, denote components 1, 2, 3, not necessarily in that order.

Expanding the commutator gives

$$T_1 = \frac{i}{2\hbar}\int\Big\{\hat{V}_k(\bar{x})\hat{\rho}(\bar{x})\left[\hat{V}_k(\bar{x}),\hat{V}_\ell(\bar{y})\right] + \left[\hat{V}_k(\bar{x})\hat{\rho}(\bar{x}),\hat{V}_\ell(\bar{y})\right]\hat{V}_k(\bar{x})$$

$$+ \hat{V}_\ell(\bar{x})\hat{\rho}(\bar{x})\left[\hat{V}_\ell(\bar{x}),\hat{V}_\ell(\bar{y})\right] + \left[\hat{V}_\ell(\bar{x})\hat{\rho}(\bar{x}),\hat{V}_\ell(\bar{y})\right]\hat{V}_\ell(\bar{x})$$

$$+ \hat{V}_m(\bar{x})\hat{\rho}(\bar{x})\left[\hat{V}_m(\bar{x}),\hat{V}_\ell(\bar{y})\right] + \left[\hat{V}_m(\bar{x})\hat{\rho}(\bar{x}),\hat{V}_\ell(\bar{y})\right]\hat{V}_m(\bar{x})\Big\}d^3x$$

$$= \frac{i}{2\hbar}\int\Big\{\hat{V}_k(\bar{x})\hat{\rho}(\bar{x})\left[\hat{V}_k(\bar{x}),\hat{V}_\ell(\bar{y})\right] + \hat{V}_k(\bar{x})\left[\hat{\rho}(\bar{x}),\hat{V}_\ell(\bar{y})\right]V_k(\bar{x})$$

$$+ \left[\hat{V}_k(\bar{x}),\hat{V}_\ell(\bar{y})\right]\hat{\rho}(\bar{x})\hat{V}_k(\bar{x}) + \hat{V}_\ell(\bar{x})\left[\hat{\rho}(\bar{x}),\hat{V}_\ell(\bar{y})\right]\hat{V}_\ell(\bar{x})$$

$$+ \left[\hat{V}_\ell(\bar{x}),\hat{V}_\ell(\bar{y})\right]\hat{\rho}(\bar{x})\hat{V}_\ell(\bar{x}) + \hat{V}_m(\bar{x})\hat{\rho}(\bar{x})\left[\hat{V}_m(\bar{x}),\hat{V}_\ell(\bar{y})\right]$$

$$+ \hat{V}_m(\bar{x})\left[\hat{\rho}(\bar{x}),\hat{V}_\ell(\bar{y})\right]\hat{V}_m(\bar{x}) + \left[\hat{V}_m(\bar{x}),\hat{V}_\ell(\bar{y})\right]\hat{\rho}(\bar{x})\hat{V}_m(\bar{x})\Big\}d^3x \ . \qquad \text{(E4)}$$

Inserting the values of the commutators given in Eqs. (2.4c) and (2.4d) into Eq. (E4) gives

$$T_1 = \frac{i}{2\hbar}\int\Big\{\hat{V}_k(\bar{x})\frac{\hbar}{i}\delta(\bar{x}-\bar{y})\left(\frac{\partial\hat{V}_k(\bar{x})}{\partial x_\ell} - \frac{\partial\hat{V}_\ell(\bar{x})}{\partial x_k}\right) + \hat{V}_k(\bar{x})\frac{\hbar}{i}\left[\frac{\partial}{\partial x_\ell}\delta(\bar{x}-\bar{y})\right]\hat{V}_k(\bar{x})$$

$$+\frac{\hbar}{i}\delta(\bar{x}-\bar{y})\hat{\rho}^{-1}(\bar{x})\left(\frac{\partial\hat{V}_\ell(\bar{x})}{\partial x_k}-\frac{\partial\hat{V}_k(\bar{x})}{\partial x_\ell}\right)\hat{\rho}(\bar{x})\hat{V}_k(\bar{x})+\frac{\hbar}{i}\hat{V}_\ell(\bar{x})\frac{\partial\delta(\bar{x}-\bar{y})}{\partial x_\ell}\hat{V}_\ell(\bar{x})$$

$$-\frac{\hbar}{i}\delta(\bar{x}-\bar{y})\hat{\rho}^{-1}(\bar{x})\left(\frac{\partial\hat{V}_\ell(\bar{x})}{\partial x_m}-\frac{\partial\hat{V}_m(\bar{x})}{\partial x_\ell}\right)\hat{\rho}(\bar{x})\hat{V}_m(\bar{x})$$

$$-\hat{V}_m(\bar{x})\frac{\hbar}{i}\delta(\bar{x}-\bar{y})\left(\frac{\partial\hat{V}_\ell(\bar{x})}{\partial x_m}-\frac{\partial\hat{V}_m(\bar{x})}{\partial x_\ell}\right)+\frac{\hbar}{i}\hat{V}_m(\bar{x})\left[\frac{\partial}{\partial x_\ell}\delta(\bar{x}-\bar{y})\right]\hat{V}_m(\bar{x})\Big\}\,d^3x. \qquad \text{(E5)}$$

When the integration is done the result is

$$T_1=-\frac{1}{2}\Big\{\hat{V}_k(\bar{y})\left(\frac{\partial\hat{V}_\ell(\bar{y})}{\partial y_m}-\frac{\partial\hat{V}_k(\bar{y})}{\partial y_\ell}\right)+\hat{V}_k(\bar{y})\frac{\partial V_k(\bar{y})}{\partial y_\ell}$$

$$+\frac{\partial V_k(\bar{y})}{\partial y_\ell}\hat{V}_k(\bar{y})+\left(\frac{\partial\hat{V}_\ell(\bar{y})}{\partial y_k}-\frac{\partial\hat{V}_k(\bar{y})}{\partial y_\ell}\right)\hat{V}_k(\bar{y})+\hat{V}_\ell(\bar{y})\frac{\partial\quad\hat{V}_\ell(\bar{y})}{\partial y_\ell}$$

$$+\frac{\partial\hat{V}_\ell(\bar{y})}{\partial y_\ell}\hat{V}_\ell(\bar{y})+\hat{V}_m(\bar{y})\left(\frac{\partial\hat{V}_\ell(\bar{y})}{\partial y_m}-\frac{\partial\hat{V}_m(\bar{y})}{\partial y_\ell}\right)+\left(\frac{\partial\hat{V}_\ell(\bar{y})}{\partial y_m}-\frac{\partial\hat{V}_m(\bar{y})}{\partial y_\ell}\right)\hat{V}_m(\bar{y})$$

$$\frac{\partial\hat{V}_m(\bar{y})}{\partial y_\ell}\hat{V}_m(\bar{y})+\hat{V}_m(\bar{y})\frac{\partial\hat{V}_m(\bar{y})}{\partial y_\ell}\Big\}\;, \qquad \text{(E6)}$$

where the following equation, which follows simply from Eq. (2.4c), was used to obtain Eq. (E5):

$$\left[\left(\frac{\partial \hat{V}_\ell(\bar{x})}{\partial x_k} - \frac{\partial \hat{V}_k(\bar{x})}{\partial x_\ell}\right), \hat{\rho}(\bar{x})\right] = 0 . \qquad \text{(E7)}$$

When terms are combined in Eq. (E5) the result is

$$T_1 = -\frac{1}{2}\left\{\hat{V}_k(\bar{y})\frac{\partial \hat{V}_\ell(\bar{y})}{\partial y_k} + \frac{\partial \hat{V}_\ell(\bar{y})}{\partial y_k}\hat{V}_k(\bar{y}) + \hat{V}_\ell(\bar{y})\frac{\partial \hat{V}_\ell(\bar{y})}{\partial y_\ell}\right.$$

$$\left. + \frac{\partial \hat{V}_\ell(\bar{y})}{\partial y_\ell}\hat{V}_\ell(\bar{y}) + \hat{V}_m(\bar{y})\frac{\partial \hat{V}_\ell(\bar{y})}{\partial y_m} + \frac{\partial \hat{V}_\ell(\bar{y})}{\partial y_m}\hat{V}_m(\bar{y})\right\}$$

$$= -\frac{1}{2}\left\{\left(\bar{\nabla}\hat{V}_\ell(\bar{y})\right)\cdot\hat{\bar{V}}(\bar{y}) + \hat{\bar{V}}(\bar{y})\cdot\bar{\nabla}\hat{V}_\ell(\bar{y})\right\} , \qquad \text{(E8)}$$

which is the final result for T_1.

The value of the term T_2 in Eq. (E2) will now be calculated. In this first calculation of T_2, the term $\hat{\rho}(\bar{x})\mathcal{E}(\hat{\rho})$ is treated as a function of $\hat{\rho}(\bar{x})$

$$T_2 = \frac{i}{\hbar}\int\left[\hat{\rho}(\bar{x})\mathcal{E}\left(\hat{\rho}(\bar{x})\right), \hat{V}_\ell(\bar{y})\right] d^3x. \qquad \text{(E9)}$$

Expanding $\mathcal{E}(\hat{\rho})$ as a power series in $\hat{\rho}(\bar{x})$ gives

$$\hat{\mathcal{E}}\left(\hat{\rho}(\bar{x})\right) = \sum_{j=0}^{\infty} C_j \, \hat{\rho}^{\,j}(\bar{x}) \,. \tag{E10}$$

Then the following expressions may be written:

$$\hat{\rho}(\bar{x})\hat{\mathcal{E}}\left(\hat{\rho}(\bar{x})\right) = \sum_{j=0}^{\infty} C_j \, \hat{\rho}^{\,j+1}(\bar{x}) \quad , \tag{E11}$$

and

$$\left[\hat{\rho}(\bar{x})\hat{\mathcal{E}}\left(\hat{\rho}(\bar{x})\right), \hat{V}_\ell(\bar{y})\right] = \sum_{j=0}^{\infty} C_j \left[\hat{\rho}^{\,j+1}(\bar{x}) \, , \hat{V}_\ell(\bar{y})\right] \quad . \tag{E12}$$

Since $\hat{\rho}$ commutes with the commutator $\left[\hat{\rho}, \hat{V}\right]$, the relation

$$\left[\hat{\rho}^{\,n}, \hat{V}\right] = n\hat{\rho}^{\,n-1} \left[\hat{\rho}, \hat{V}\right]$$

may be used to obtain

$$\left[\hat{\rho}(\bar{x})\hat{\mathcal{E}}\left(\hat{\rho}(\bar{x})\right), \; \hat{V}_\ell(\bar{y})\right] = \sum_{j=0}^{\infty} C_j \, (j+1) \, \hat{\rho}^{\,j}(\bar{x}) \left[\hat{\rho}(\bar{x}), \hat{V}_\ell(\bar{y})\right]$$

$$= -\sum_{j=0}^{\infty} C_j \, (j+1) \, \hat{\rho}^{\,j}(\bar{x}) \, \frac{\hbar}{i} \left[\frac{\partial}{\partial y_\ell} \delta(\bar{x}-\bar{y})\right] \quad . \tag{E13}$$

Substituting the above expression back into the integral gives

$$T_2 = \frac{i}{\hbar} \int -\sum_{j=0}^{\infty} C_j \, (j+1) \, \hat{\rho}^{\,j}(\bar{x}) \, \frac{\hbar}{i} \left[\frac{\partial}{\partial y_\ell} \delta(\bar{x}-\bar{y})\right] d^3x$$

$$= -\sum_{j=0}^{\infty} C_j \, (j+1) \left[\frac{\partial}{\partial y_\ell} \int \hat{\rho}^{\,j}(\bar{x}) \, \delta(\bar{x}-\bar{y}) \, d^3x\right]$$

$$= -\sum_{j=0}^{\infty} C_j \, (j+1) \left[\frac{\partial}{\partial y_\ell} \rho^{j}(\bar{y})\right] = -\sum C_j \, (j+1) \, j \, \hat{\rho}^{\,j-1}(\bar{y}) \left[\frac{\partial}{\partial y_\ell} \hat{\rho}(\bar{y})\right] \quad . \tag{E14}$$

Equation (E14) may be put in the form

$$T_1 = -\frac{\partial}{\partial y_\ell} \sum_{j=0}^{\infty} C_j\,(j+1)\,\hat{\rho}^{\,j}(\bar{y})$$

$$= -\frac{\partial}{\partial y_\ell}\,\frac{d}{d\hat{\rho}}\left[\hat{\rho}(\bar{y})\,\mathcal{E}\big(\hat{\rho}(\bar{y})\big)\right] . \tag{E15}$$

The final result for the Heisenberg equation of motion of the velocity operator is thus

$$\frac{\partial V_\ell(\bar{y})}{\partial t} = T_1 + T_2$$

$$= -\frac{i}{2}\left\{\left(\bar{\nabla}\hat{V}_\ell(\bar{y})\right)\cdot\hat{\bar{V}}(\bar{y}) + \hat{\bar{V}}(\bar{y})\cdot\bar{\nabla}\hat{V}_\ell(\bar{y})\right\} - \frac{\partial}{\partial y_\ell}\,\frac{d}{d\hat{\rho}}\left[\hat{\rho}(\bar{y})\,\mathcal{E}\,\hat{\rho}(\bar{y})\right] , \tag{E16}$$

which then gives Eq. (2.10).

The value of T_2 will now be calculated by treating the internal energy term in the Hamiltonian of Eq. (2.7) as a functional. That is,

$$\int \hat{\rho}(\bar{x})\,\mathcal{E}\big(\hat{\rho}(\bar{x})\big)\,d^3x \equiv E\left[\hat{\rho}\right] . \tag{E17}$$

The term T_2 may then be written as

$$T_2 = \frac{i}{\hbar}\left[E\left[\hat{\rho}\right],\ \hat{V}_\ell(\bar{y})\right] . \tag{E18}$$

When $E[\hat{\rho}]$ is expanded as

$$E\left[\hat{\rho}\right] = \sum_{n=1}^{\infty}\left[\int f(\bar{x}_1 \ldots \bar{x}_n)\hat{\rho}(\bar{x}_1)\ldots\hat{\rho}(\bar{x}_n)\, d^3x_1 \ldots d^3x_n\right] , \tag{E19}$$

and inserted into Eq. (E18), the result is

$$T_2 = \frac{i}{\hbar}\sum_{n=1}^{\infty}\int f(\bar{x}_1 \ldots \bar{x}_n)\left[\hat{\rho}(\bar{x}_1)\ldots\hat{\rho}(\bar{x}_n), \hat{V}_\ell(\bar{y})\right] d^3x_1 \ldots d^3x_n \ . \tag{E20}$$

Since the density operators commute with each other, it is apparent that $f(\bar{x}_1 \cdots \bar{x}_n)$ must be symmetric with respect to its arguments. Using this fact and Eq. (2.4c) in Eq. (E20) gives

$$T_2 = \frac{i}{\hbar}\sum_{n=1}^{\infty} n\int f(\bar{x}_1, \ldots, \bar{x}_n)\hat{\rho}(\bar{x}_1)\ldots\hat{\rho}(\bar{x}_{n-1})\left[-\frac{\hbar}{i}\frac{\partial}{\partial y_\ell}\delta(\bar{x}-\bar{y})\right] d^3x_1 \ldots d^3x_n$$

$$= -\frac{\partial}{\partial y_\ell}\sum_{n=1}^{\infty} n\int f(\bar{x}_1, \ldots, \bar{x}_n)\hat{\rho}(\bar{x}_1)\ldots\hat{\rho}(\bar{x}_{n-1})\,\delta(\bar{x}_n-\bar{y})\, d^3x_1 \ldots d^3x_n \ . \tag{E21}$$

Doing the integration over $\bar{x}_n$ gives

$$T_2 = -\frac{\partial}{\partial y_\ell}\sum_{n=1}^{\infty} n\int f(\bar{x}_1, \ldots, \bar{x}_{n-1}, \bar{y})\hat{\rho}(\bar{x}_1)\cdots\hat{\rho}(\bar{x}_{n-1})\, d^3x_1 \ldots d^3x_{n-1} \ .$$

$$= -\frac{\partial}{\partial y_\ell}\,\frac{\delta}{\delta\hat{\rho}(\bar{y})}\,E\left[\hat{\rho}\right] \ . \tag{E22}$$

This form of T_2 gives the more general form of the equation of motion for the velocity operator

$$\frac{\partial \hat{V}_\ell(\bar{y})}{\partial t} = -\frac{1}{2}\left\{\left(\bar{\nabla}\hat{V}_\ell(\bar{y})\right)\cdot\hat{\bar{V}}(\bar{y}) + \hat{\bar{V}}(\bar{y})\cdot\bar{\nabla}\hat{V}_\ell(\bar{y})\right\} - \frac{\partial}{\partial y_\ell}\frac{\delta}{\delta\hat{\rho}(\bar{y})} E\left[\hat{\rho}\right] , \qquad \text{(E23)}$$

which is the same as Eq. (2.11).

In the special case where $f(\bar{x}_1 \ldots \bar{x}_n)$ has the form

$$f(\bar{x}_1 \ldots \bar{x}_n) = C_{n-1}\,\delta(\bar{x}_1-\bar{x}_2)\,\delta(\bar{x}_1-\bar{x}_3)\ldots\delta(\bar{x}_1-\bar{x}_n) , \qquad \text{(E24)}$$

T_2 has the form

$$T_2 = -\frac{\partial}{\partial y_\ell}\sum_{n=1}^{\infty} n C_{n-1}\,\hat{\rho}^{\,n-1}(\bar{y}) \quad . \qquad \text{(E25)}$$

Defining j as

$$j = n-1$$

gives the result

$$T_2 = -\frac{\partial}{\partial y_\ell}\sum_{j=0}^{\infty} C_j\,(j+1)\,\hat{\rho}^{\,j}(\bar{y}) , \qquad \text{(E26)}$$

which is the same as Eq. (E16). Equation (2.10) has thus been shown to be a special case of Eq. (2.11).

APPENDIX F

CALCULATION OF THE DENSITY-DENSITY COMMUTATOR IN SECOND QUANTIZATION

The calculation of the density-density commutator in the formalism of second quantization proceeds as follows:

$$
\begin{aligned}
\left[\hat{\rho}(\bar{x}), \hat{\rho}(\bar{y})\right] &= \left[\hat{\psi}^{+}(\bar{x})\, \hat{\psi}(\bar{x}), \hat{\psi}^{+}(\bar{y})\, \hat{\psi}(\bar{y})\right] \\
&= \hat{\psi}^{+}(\bar{x}) \left[\hat{\psi}(\bar{x}), \hat{\psi}^{+}(\bar{y})\, \hat{\psi}(\bar{y})\right] + \left[\hat{\psi}^{+}(\bar{x}), \hat{\psi}^{+}(\bar{y})\, \hat{\psi}(\bar{y})\right] \hat{\psi}(\bar{x}), \\
&= \hat{\psi}^{+}(\bar{x}) \left[\hat{\psi}(\bar{x}), \hat{\psi}^{+}(\bar{y})\right] \hat{\psi}(\bar{y}) + \hat{\psi}^{+}(\bar{x})\, \hat{\psi}^{+}(\bar{y}) \left[\hat{\psi}(\bar{x}), \hat{\psi}(\bar{y})\right] \\
&\quad + \hat{\psi}^{+}(\bar{y}) \left[\hat{\psi}^{+}(\bar{x}), \hat{\psi}(\bar{y})\right] \hat{\psi}(\bar{x}) + \left[\hat{\psi}^{+}(\bar{x}), \hat{\psi}^{+}(\bar{y})\right] \hat{\psi}(\bar{y})\, \hat{\psi}(\bar{x}) .
\end{aligned}
\tag{F1}
$$

Substituting the commutation relations for the field operators, Eq. (3.2) and Eq. (3.3), into Eq. (F2) gives

$$
\left[\hat{\rho}(\bar{x}), \hat{\rho}(\bar{y})\right] = \hat{\psi}^{+}(\bar{x})\, \delta(\bar{x}-\bar{y})\, \hat{\psi}(\bar{y}) - \hat{\psi}^{+}(\bar{y})\, \delta(\bar{y}-\bar{x})\, \hat{\psi}(\bar{x}) = 0 \;, \tag{F2}
$$

which is the same as Eq. (3.5).

APPENDIX G

CALCULATION OF THE DENSITY-CURRENT COMMUTATOR IN SECOND QUANTIZATION

The l-th component of the current operator is given by

$$\hat{J}_l(\bar{y}) = \frac{\hbar}{2i}\left(\hat{\psi}^+(\bar{y})\,\frac{\partial\hat{\psi}(\bar{y})}{\partial y_l} - \frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_l}\,\hat{\psi}(\bar{y})\right) . \tag{G1}$$

Upon substituting the expressions for $\hat{\rho}(\bar{x})$ and $\hat{J}_l(\bar{y})$ into the density-current commutator, one obtains

$$\left[\hat{\rho}(\bar{x}), \hat{J}_l(\bar{y})\right] = \left[\hat{\psi}^+(\bar{x})\hat{\psi}(\bar{x}), \frac{\hbar}{2i}\left(\hat{\psi}^+(\bar{y})\frac{\partial\hat{\psi}(\bar{y})}{\partial y_l} - \frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_l}\,\hat{\psi}(\bar{y})\right)\right]$$

$$= \hat{\psi}^+(\bar{x})\left[\hat{\psi}(\bar{x}), \frac{\hbar}{2i}\,\hat{\psi}^+(\bar{y})\,\frac{\partial\hat{\psi}(\bar{y})}{\partial y_l}\right] - \hat{\psi}^+(\bar{x})\left[\hat{\psi}(\bar{x}), \frac{\hbar}{2i}\,\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_l}\,\hat{\psi}(\bar{y})\right]$$

$$+ \left[\hat{\psi}^+(\bar{x}), \frac{\hbar}{2i}\,\hat{\psi}^+(\bar{y})\,\frac{\partial\hat{\psi}(\bar{y})}{\partial y_l}\right]\hat{\psi}(\bar{x}) - \left[\hat{\psi}^+(\bar{x}), \frac{\hbar}{2i}\,\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_l}\,\hat{\psi}(\bar{y})\right]\hat{\psi}(\bar{x})$$

$$= \hat{\psi}^+(\bar{x})\,\hat{\psi}^+(\bar{y})\left[\hat{\psi}(\bar{x}), \frac{\hbar}{2i}\,\frac{\partial\psi(\bar{y})}{\partial y_l}\right] + \frac{\hbar}{2i}\,\hat{\psi}^+(\bar{x})\left[\hat{\psi}(\bar{x}), \hat{\psi}^+(\bar{y})\right]\frac{\partial\hat{\psi}(\bar{y})}{\partial y_l}$$

$$- \frac{\hbar}{2i}\,\hat{\psi}^+(\bar{x})\,\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_l}\left[\hat{\psi}(\bar{x}), \hat{\psi}(\bar{y})\right] - \frac{\hbar}{2i}\,\hat{\psi}^+(\bar{x})\left[\hat{\psi}(\bar{x}), \frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_l}\right]\hat{\psi}(\bar{y})$$

$$+\frac{\hbar}{2i}\hat{\psi}^+(\bar{y})\left[\hat{\psi}^+(\bar{x}),\frac{\partial\hat{\psi}(\bar{y})}{\partial y_\ell}\right]\hat{\psi}(\bar{x})+\frac{\hbar}{2i}\left[\hat{\psi}^+(\bar{x}),\hat{\psi}^+(\bar{y})\right]\frac{\partial\hat{\psi}(\bar{y})}{\partial y_\ell}\hat{\psi}(\bar{x})$$

$$-\frac{\hbar}{2i}\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_\ell}\left[\hat{\psi}^+(\bar{x}),\hat{\psi}(\bar{y})\right]\hat{\psi}(\bar{x})\ \frac{\hbar}{2i}\left[\hat{\psi}^+(\bar{x}),\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_\ell}\right]\hat{\psi}(\bar{y})\hat{\psi}(\bar{x})\ . \tag{G2}$$

Equations (3.2) and (3.3) may be differentiated with respect to y_1 to obtain

$$\frac{\partial}{\partial y_\ell}\left[\hat{\psi}^+(\bar{x}),\hat{\psi}^+(\bar{y})\right]=\frac{\partial}{\partial y_\ell}\left[\hat{\psi}(\bar{x}),\hat{\psi}(\bar{y})\right]=\left[\hat{\psi}^+(\bar{x}),\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_\ell}\right]=\left[\hat{\psi}(\bar{x}),\frac{\partial\hat{\psi}(\bar{y})}{\partial y_\ell}\right]=0, \tag{G3}$$

and

$$\frac{\partial}{\partial y_\ell}\left[\hat{\psi}(\bar{x}),\hat{\psi}^+(\bar{y})\right]=\left[\hat{\psi}(\bar{x}),\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_\ell}\right]=\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y_\ell}\ . \tag{G4}$$

Substituting Eqs. (G4) and (G3) into Eq. (G2) gives

$$\left[\hat{\rho}(\bar{x}),\hat{J}_\ell(\bar{y})\right]=\frac{\hbar}{2i}\hat{\psi}^+(\bar{x})\delta(\bar{x}-\bar{y})\frac{\partial\hat{\psi}(\bar{y})}{\partial y_\ell}-\frac{\hbar}{2i}\hat{\psi}^+(\bar{x})\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y_\ell}\hat{\psi}(\bar{y})$$

$$-\frac{\hbar}{2i}\hat{\psi}^+(\bar{y})\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y_\ell}\hat{\psi}(\bar{x})+\frac{\hbar}{2i}\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_\ell}\delta(\bar{y}-\bar{x})\hat{\psi}(\bar{x})\ . \tag{G5}$$

Applying the identity in Eq. (3.8) to Eq. (G5) gives

$$\left[\hat{\rho}(\bar{x}), \hat{J}_\ell(\bar{y})\right] = \frac{\hbar}{2i}\,\delta(\bar{x}-\bar{y})\,\hat{\psi}^+(\bar{x})\,\frac{\partial\hat{\psi}(\bar{y})}{\partial y_\ell} - \frac{\hbar}{2i}\,\hat{\psi}^+(\bar{x})\,\hat{\psi}(\bar{x})\left(\frac{\partial}{\partial y_\ell}\,\delta(\bar{x}-\bar{y})\right)$$

$$+\frac{\hbar}{2i}\,\hat{\psi}^+(\bar{x})\,\frac{\partial\hat{\psi}(\bar{x})}{\partial x_\ell}\,\delta(\bar{x}-\bar{y}) - \frac{\hbar}{2i}\,\hat{\psi}^+(\bar{x})\,\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y_\ell}\,\hat{\psi}(\bar{x})$$

$$+\frac{\hbar}{2i}\,\delta(\bar{x}-\bar{y})\left(\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_\ell}\right)\hat{\psi}(\bar{x}) + \frac{\hbar}{2i}\,\delta(\bar{x}-\bar{y})\,\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_\ell}\,\hat{\psi}(\bar{x}),$$

$$= -i\hbar\,\delta(\bar{x}-\bar{y})\,\hat{\psi}^+(\bar{x})\frac{\partial\hat{\psi}(\bar{x})}{\partial x_\ell} - i\hbar\,\delta(\bar{x}-\bar{y})\,\frac{\partial\hat{\psi}^+(\bar{x})}{\partial x_\ell}\,\hat{\psi}(\bar{x}) - i\hbar\frac{\partial\delta(\bar{x}-\bar{y})}{\partial x_\ell}\hat{\psi}^+(\bar{x})\hat{\psi}(\bar{x}),$$

$$= -i\hbar\frac{\partial}{\partial x_\ell}\left[\delta(\bar{x}-\bar{y})\;\hat{\rho}(\bar{x})\right], \tag{G6}$$

which is the same as Eq. (3.6).

APPENDIX H

CALCULATION OF THE CURRENT-CURRENT COMMUTATOR IN SECOND QUANTIZATION

The value of the commutator of the components of the mass current density operator is calculated in this Appendix. The lth component of the operator is

$$\hat{J}_\ell(\bar{x}) = \frac{\hbar}{2i}\left(\hat{\Psi}^+(\bar{x})\frac{\partial\hat{\Psi}(\bar{x})}{\partial x_\ell} - \frac{\partial\hat{\Psi}^+(\bar{x})}{\partial x_\ell}\hat{\Psi}(\bar{x})\right) . \tag{H1}$$

When the components of the operator are inserted into the commutator, the result is

$$\left[\hat{J}_k(\bar{x}),\hat{J}_\ell(\bar{y})\right] = \left[\frac{\hbar}{2i}\left(\hat{\Psi}^+(\bar{x})\frac{\partial\hat{\Psi}(\bar{x})}{\partial x_k} - \frac{\partial\hat{\Psi}^+(\bar{x})}{\partial x_k}\hat{\Psi}(\bar{x})\right), \frac{\hbar}{2i}\left(\hat{\Psi}^+(\bar{y})\frac{\partial\hat{\Psi}(\bar{y})}{\partial y_\ell} - \frac{\partial\hat{\Psi}^+(\bar{y})}{\partial y_\ell}\hat{\Psi}(\bar{y})\right)\right] . \tag{H2}$$

Expanding the commutator gives

$$\left[\hat{J}_k(\bar{x}),\hat{J}_\ell(\bar{y})\right] = -\frac{\hbar^2}{4}\left[\hat{\Psi}^+(\bar{x})\frac{\partial\hat{\Psi}^+(\bar{x})}{\partial x_k}, \hat{\Psi}^+(\bar{y})\frac{\partial\hat{\Psi}(\bar{y})}{\partial y_\ell}\right]$$

$$+\frac{\hbar^2}{4}\left[\frac{\partial\hat{\Psi}^+(\bar{x})}{\partial x_k}\hat{\Psi}(\bar{x}), \hat{\Psi}^+(\bar{y})\frac{\partial\hat{\Psi}(\bar{y})}{\partial y_\ell}\right]$$

$$+\frac{\hbar^2}{4}\left[\hat{\psi}^+(\bar{x})\frac{\partial\hat{\psi}(\bar{x})}{\partial x_k}, \frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_\ell}\hat{\psi}(\bar{y})\right]$$

$$-\frac{\hbar^2}{4}\left[\frac{\partial\hat{\psi}^+(\bar{x})}{\partial x_k}\hat{\psi}(\bar{x}), \frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_\ell}\hat{\psi}(\bar{y})\right] ,$$

$$= -\frac{\hbar^2}{4}\hat{\psi}^+(\bar{x})\left[\frac{\partial\hat{\psi}(\bar{x})}{\partial x_k}, \hat{\psi}^+(\bar{y})\frac{\partial\hat{\psi}(\bar{y})}{\partial y_\ell}\right] - \frac{\hbar^2}{4}\left[\hat{\psi}^+(\bar{x}), \hat{\psi}^+(\bar{y})\frac{\partial\hat{\psi}(\bar{y})}{\partial y_\ell}\right]\frac{\partial\hat{\psi}(\bar{x})}{\partial x_k}$$

$$+\frac{\hbar^2}{4}\frac{\partial\hat{\psi}^+(\bar{x})}{\partial x_k}\left[\hat{\psi}(\bar{x}), \hat{\psi}^+(\bar{y})\frac{\partial\hat{\psi}(\bar{y})}{\partial y_\ell}\right] + \frac{\hbar^2}{4}\left[\frac{\partial\hat{\psi}^+(\bar{x})}{\partial x_k}, \hat{\psi}^+(\bar{y})\frac{\partial\hat{\psi}(\bar{y})}{\partial y_\ell}\right]\hat{\psi}(\bar{x})$$

$$+\frac{\hbar^2}{4}\hat{\psi}^+(\bar{x})\left[\frac{\partial\hat{\psi}(\bar{x})}{\partial x_k}, \frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_\ell}\hat{\psi}(\bar{y})\right] + \frac{\hbar^2}{4}\left[\hat{\psi}^+(\bar{x}), \frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_\ell}\hat{\psi}(\bar{y})\right]\frac{\partial\hat{\psi}(\bar{x})}{\partial x_k}$$

$$-\frac{\hbar^2}{4}\frac{\partial\hat{\psi}^+(\bar{x})}{\partial x_k}\left[\hat{\psi}(\bar{x}), \frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_\ell}\hat{\psi}(\bar{y})\right] - \frac{\hbar^2}{4}\left[\frac{\partial\hat{\psi}^+(\bar{x})}{\partial x_k}, \frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_\ell}\hat{\psi}(\bar{y})\right]\hat{\psi}(\bar{x}). \tag{H3}$$

Expanding the commutators further gives

$$\left[\hat{J}_k(\bar{x}), \hat{J}_\ell(\bar{y})\right] = -\frac{\hbar^2}{4}\hat{\psi}^+(\bar{x})\psi^+(\bar{y})\left[\frac{\partial\hat{\psi}(\bar{x})}{\partial x_k}, \frac{\partial\hat{\psi}(\bar{y})}{\partial y_\ell}\right]$$

$$-\frac{\hbar^2}{4}\hat{\psi}^+(\bar{x})\left[\frac{\partial\hat{\psi}(\bar{x})}{\partial x_k}, \hat{\psi}^+(\bar{y})\right]\frac{\partial\hat{\psi}(\bar{y})}{\partial y_\ell} - \frac{\hbar^2}{4}\hat{\psi}^+(\bar{y})\left[\hat{\psi}^+(\bar{x}), \frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_\ell}\right]\frac{\partial\hat{\psi}(\bar{x})}{\partial x_k}$$

$$-\frac{\hbar^2}{4}\left[\hat{\psi}^+(\bar{x}),\hat{\psi}^+(\bar{y})\right]\frac{\partial\hat{\psi}(\bar{y})}{\partial y_l}\frac{\partial\hat{\psi}(\bar{x})}{\partial x_k}+\frac{\hbar^2}{4}\frac{\partial\hat{\psi}^+(\bar{x})}{\partial x_k}\hat{\psi}^+(\bar{y})\left[\hat{\psi}^+(\bar{x}),\frac{\partial\hat{\psi}(\bar{y})}{\partial y_l}\right]$$

$$+\frac{\hbar^2}{4}\frac{\partial\hat{\psi}^+(\bar{x})}{\partial x_k}\left[\hat{\psi}(\bar{x}),\hat{\psi}^+(\bar{y})\right]\frac{\partial\hat{\psi}(\bar{y})}{\partial y_l}+\frac{\hbar^2}{4}\hat{\psi}^+(\bar{y})\left[\frac{\partial\hat{\psi}^+(\bar{x})}{\partial x_k},\frac{\partial\hat{\psi}(\bar{y})}{\partial y_l}\right]\hat{\psi}(\bar{x})$$

$$+\frac{\hbar^2}{4}\left[\frac{\partial\hat{\psi}^+(\bar{x})}{\partial x_k},\hat{\psi}^+(\bar{y})\right]\frac{\partial\hat{\psi}(\bar{y})}{\partial y_l}\hat{\psi}(\bar{x})+\frac{\hbar^2}{4}\hat{\psi}^+(\bar{x})\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_l}\left[\frac{\partial\hat{\psi}(\bar{x})}{\partial x_k},\hat{\psi}(\bar{y})\right]$$

$$+\frac{\hbar^2}{4}\hat{\psi}^+(\bar{x})\left[\frac{\partial\hat{\psi}(\bar{x})}{\partial x_k},\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_l}\right]\hat{\psi}(\bar{y})+\frac{\hbar^2}{4}\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_l}\left[\hat{\psi}^+(\bar{x}),\hat{\psi}(\bar{y})\right]\frac{\partial\hat{\psi}(\bar{x})}{\partial x_k}$$

$$+\frac{\hbar^2}{4}\left[\hat{\psi}^+(\bar{x}),\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_l}\right]\hat{\psi}(\bar{y})\frac{\partial\hat{\psi}(\bar{x})}{\partial x_k}-\frac{\hbar^2}{4}\frac{\partial\hat{\psi}^+(\bar{x})}{\partial x_k}\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_l}\left[\hat{\psi}(\bar{x}),\hat{\psi}(\bar{y})\right]$$

$$-\frac{\hbar^2}{4}\frac{\partial\hat{\psi}^+(\bar{x})}{\partial x_k}\left[\hat{\psi}(\bar{x}),\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_l}\right]\hat{\psi}(\bar{y})-\frac{\hbar^2}{4}\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_l}\left[\frac{\partial\hat{\psi}^+(\bar{x})}{\partial x_k},\hat{\psi}(\bar{y})\right]\hat{\psi}(\bar{x})$$

$$-\frac{\hbar^2}{4}\left[\frac{\partial\hat{\psi}^+(\bar{x})}{\partial x_k},\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_l}\right]\hat{\psi}(\bar{y}),\hat{\psi}(\bar{x})\,. \tag{H4}$$

Inserting the commutation relations, Eq. (3.2) and (3.3), and their derivatives, into Eq. (H4) gives

$$\left[\hat{J}_k(\bar{x}),\hat{J}_\ell(\bar{y})\right] = -\frac{\hbar^2}{4}\frac{\partial\delta(\bar{x}-\bar{y})}{\partial x_k}\hat{\psi}^+(\bar{x})\frac{\partial\hat{\psi}(\bar{y})}{\partial y_\ell}$$

$$+\frac{\hbar^2}{4}\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y_\ell}\hat{\psi}^+(\bar{y})\frac{\partial\hat{\psi}(\bar{x})}{\partial x_k} + \frac{\hbar^2}{4}\delta(\bar{x}-\bar{y})\frac{\partial\hat{\psi}^+(\bar{x})}{\partial x_k},\frac{\partial\hat{\psi}(\bar{y})}{\partial y_\ell}$$

$$-\frac{\hbar^2}{4}\frac{\partial^2\delta(\bar{x}-\bar{y})}{\partial x_k\partial y_\ell}\hat{\psi}^+(\bar{y})\hat{\psi}(\bar{x}) + \frac{\hbar^2}{4}\frac{\partial^2\delta(\bar{x}-\bar{y})}{\partial x_k\partial y_\ell}\hat{\psi}^+(\bar{x})\hat{\psi}(\bar{y})$$

$$-\frac{\hbar^2}{4}\delta(\bar{x}-\bar{y})\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_\ell}\frac{\partial\hat{\psi}(\bar{x})}{\partial x_k} - \frac{\hbar^2}{4}\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y_\ell}\frac{\partial\hat{\psi}^+(\bar{x})}{\partial x_k}\hat{\psi}(\bar{y})$$

$$-\frac{\hbar^2}{4}\frac{\partial\delta(\bar{x}-\bar{y})}{\partial x_k}\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_\ell}\hat{\psi}(\bar{x}) \quad . \tag{H5}$$

Applying the identity in Eq. (3.8) to Eq. (H5) gives

$$\left[\hat{J}_k(\bar{x}),\hat{J}_\ell(\bar{y})\right] = -\frac{\hbar^2}{4}\frac{\partial\delta(\bar{x}-\bar{y})}{\partial x_k}\hat{\psi}^+(\bar{y})\frac{\partial\hat{\psi}(\bar{y})}{\partial y_\ell} + \frac{\hbar^2}{4}\delta(\bar{x}-\bar{y})\frac{\partial\hat{\psi}^+(\bar{x})}{\partial x_k}\frac{\partial\hat{\psi}(\bar{y})}{\partial y_\ell}$$

$$+\frac{\hbar^2}{4}\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y_\ell}\hat{\psi}^+(\bar{x})\frac{\partial\hat{\psi}(\bar{x})}{\partial x_k} - \frac{\hbar^2}{4}\delta(\bar{x}-\bar{y})\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_\ell}\frac{\partial\hat{\psi}(\bar{x})}{\partial x_k}$$

$$+\frac{\hbar^2}{4}\delta(\bar{x}-\bar{y})\frac{\partial\hat{\psi}^+(\bar{x})}{\partial x_k}\frac{\partial\hat{\psi}(\bar{x})}{\partial y_\ell} - \frac{\hbar^2}{4}\frac{\partial^2\delta(\bar{x}-\bar{y})}{\partial x_k\,\partial y_\ell}\hat{\psi}^+(\bar{x})\hat{\psi}(\bar{x})$$

$$+\frac{\hbar^2}{4}\frac{\partial\delta(\bar{x}-\bar{y})}{\partial x_k}\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_\ell}\hat{\psi}(\bar{x}) - \frac{\hbar^2}{4}\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y_\ell}\frac{\partial\hat{\psi}^+(\bar{x})}{\partial x_k}\hat{\psi}(\bar{x})$$

$$+\frac{\hbar^2}{4}\frac{\partial^2\delta(\bar{x}-\bar{y})}{\partial x_k\,\partial y_\ell}\hat{\psi}^+(\bar{x})\hat{\psi}(\bar{x}) + \frac{\hbar^2}{4}\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y_\ell}\hat{\psi}^+(\bar{x})\frac{\partial\hat{\psi}(\bar{x})}{\partial x_k}$$

$$-\frac{\hbar^2}{4}\frac{\partial\delta(\bar{x}-\bar{y})}{\partial x_k}\hat{\psi}^+(\bar{x})\frac{\partial\hat{\psi}(\bar{y})}{\partial y_\ell} - \frac{\hbar^2}{4}\delta(\bar{x}-\bar{y})\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_\ell}\frac{\partial\hat{\psi}(\bar{x})}{\partial x_k}$$

$$+\frac{\hbar^2}{4}\frac{\partial\delta(\bar{x}-\bar{y})}{\partial x_k}\frac{\partial\hat{\psi}^+(\bar{x})}{\partial x_\ell}\hat{\psi}(\bar{x}) + \frac{\hbar^2}{4}\delta(\bar{x}-\bar{y})\frac{\partial\hat{\psi}^+(\bar{x})}{\partial x_k}\frac{\partial\hat{\psi}(\bar{x})}{\partial x_\ell}$$

$$-\frac{\hbar^2}{4}\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y_\ell}\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_\ell}\hat{\psi}(\bar{y}) - \frac{\hbar^2}{4}\delta(\bar{x}-\bar{y})\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_\ell}\frac{\partial\hat{\psi}(\bar{y})}{\partial y_k}\,. \tag{H6}$$

Applying the identity in Eq. (3.9) and again applying Eq. (3.8) to Eq. (H6) and rearranging terms gives

$$\left[\hat{J}_k(\bar{x}),\hat{J}_\ell(\bar{y})\right] = \frac{\hbar^2}{2}\delta(\bar{x}-\bar{y})\frac{\partial\hat{\psi}^+(\bar{x})}{\partial x_k}\frac{\partial\hat{\psi}(\bar{x})}{\partial x_\ell} - \frac{\hbar^2}{2}\delta(\bar{x}-\bar{y})\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_\ell}\frac{\partial\hat{\psi}(\bar{y})}{\partial y_k}$$

$$-\frac{\hbar^2}{2}\frac{\partial\delta(\bar{x}-\bar{y})}{\partial x_\ell}\hat{\psi}^+(\bar{x})\frac{\partial\hat{\psi}(\bar{x})}{\partial x_\kappa}+\frac{\hbar^2}{4}\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y_\kappa}\hat{\psi}^+(\bar{y})\frac{\partial\hat{\psi}(\bar{y})}{\partial y_\ell}$$

$$+\frac{\hbar^2}{2}\frac{\partial\delta(\bar{x}-\bar{y})}{\partial x_\ell}\frac{\partial\hat{\psi}^+(\bar{x})}{\partial x_\kappa}\hat{\psi}(\bar{x})-\frac{\hbar^2}{4}\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y_\kappa}\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_\ell}\hat{\psi}(\bar{y})$$

$$+\frac{\hbar^2}{4}\delta(\bar{x}-\bar{y})\frac{\partial\hat{\psi}^+(\bar{x})}{\partial x_\kappa}\frac{\partial\hat{\psi}(\bar{x})}{\partial x_\ell}-\frac{\hbar^2}{4}\delta(\bar{x}-\bar{y})\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_\ell}\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_\kappa}$$

$$-\frac{\hbar^2}{4}\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y_\kappa}\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_\ell}\hat{\psi}(\bar{y})-\frac{\hbar^2}{4}\delta(\bar{x}-\bar{y})\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_\ell}\frac{\partial\hat{\psi}(\bar{y})}{\partial y_\kappa}$$

$$+\frac{\hbar^2}{4}\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y_\kappa}\hat{\psi}^+(\bar{y})\frac{\partial\hat{\psi}(\bar{y})}{\partial y_\ell}+\frac{\hbar^2}{4}\delta(\bar{x}-\bar{y})\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_\kappa}\frac{\partial\hat{\psi}(\bar{y})}{\partial y_\ell}\,. \qquad \text{(H7)}$$

Upon rearranging terms in Eq. (H7) and adding zero to it in the form

$$0=-\frac{\hbar^2}{2}\delta(\bar{x}-\bar{y})\left\{\hat{\psi}^+(\bar{x})\frac{\partial^2\hat{\psi}(\bar{x})}{\partial x_\ell\partial x_\kappa}-\frac{\partial^2\hat{\psi}^+(\bar{x})}{\partial x_\ell\partial x_\kappa}\hat{\psi}(\bar{x})\right\}$$

$$+\frac{\hbar^2}{2}\delta(\bar{x}-\bar{y})\left\{\hat{\psi}^+(\bar{y})\frac{\partial^2\hat{\psi}(\bar{y})}{\partial y_\kappa\partial y_\ell}-\frac{\partial^2\hat{\psi}^+(\bar{y})}{\partial y_\kappa\partial y_\ell}\hat{\psi}(\bar{y})\right\}, \qquad \text{(H8)}$$

the result is

$$\left[\hat{J}_k(\bar{x}), \hat{J}_\ell(\bar{y})\right] = -\frac{\hbar^2}{2}\left\{\frac{\partial\delta(\bar{x}-\bar{y})}{\partial x_\ell}\left(\hat{\psi}^+(\bar{x})\frac{\partial\hat{\psi}(\bar{x})}{\partial x_k} - \frac{\partial\hat{\psi}^+(\bar{x})}{\partial x_k}\hat{\psi}(\bar{x})\right)\right.$$

$$+ \delta(\bar{x}-\bar{y})\left(\frac{\partial\hat{\psi}^+(\bar{x})}{\partial x_\ell}\frac{\partial\hat{\psi}(\bar{x})}{\partial x_k} - \frac{\partial^2\hat{\psi}^+(\bar{x})}{\partial x_\ell\partial x_k}\hat{\psi}(\bar{x})\right)$$

$$\left. + \delta(\bar{x}-\bar{y})\left(\hat{\psi}^+(\bar{x})\frac{\partial^2\hat{\psi}(\bar{x})}{\partial x_\ell\partial x_k} - \frac{\partial\hat{\psi}^+(\bar{x})}{\partial x_k}\frac{\partial\hat{\psi}(\bar{x})}{\partial x_\ell}\right)\right\}$$

$$+\frac{\hbar^2}{2}\left\{\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y_k}\left(\hat{\psi}^+(\bar{y})\frac{\partial\hat{\psi}(\bar{y})}{\partial y_\ell} - \frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_\ell}\hat{\psi}(\bar{y})\right)\right.$$

$$+ \delta(\bar{x}-\bar{y})\left(\frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_k}\frac{\partial\hat{\psi}(\bar{y})}{\partial y_\ell} - \frac{\partial^2\hat{\psi}(\bar{y})}{\partial y_k\partial y_\ell}\hat{\psi}(\bar{y})\right)$$

$$\left. + \delta(\bar{x}-\bar{y})\left(\hat{\psi}^+(\bar{y})\frac{\partial^2\hat{\psi}(\bar{y})}{\partial y_k\partial y_\ell} - \frac{\partial\hat{\psi}^+(\bar{y})}{\partial y_\ell}\frac{\partial\hat{\psi}(\bar{y})}{\partial y_k}\right)\right\},$$

$$= -i\hbar\left\{\frac{\partial\delta(\bar{x}-\bar{y})}{\partial x_\ell}\hat{J}_k(\bar{x}) + \delta(\bar{x}-\bar{y})\frac{\partial\hat{J}_k(\bar{x})}{\partial x_\ell}\right\}$$

$$+ i\hbar\left\{\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y_k}\hat{J}_\ell(\bar{y}) + \delta(\bar{x}-\bar{y})\frac{\partial\hat{J}_\ell(\bar{y})}{\partial y_k}\right\},$$

$$= -i\hbar \frac{\partial}{\partial x_\ell}\left[\delta(\bar{x}-\bar{y})\,\hat{J}_k(\bar{x})\right] + i\hbar \frac{\partial}{\partial y_k}\left[\delta(\bar{x}-\bar{y})\,\hat{J}_\ell(\bar{y})\right] , \tag{H9}$$

which is the same as Eq. (3.7).

APPENDIX I

CALCULATION OF THE HAMILTONIAN OPERATOR IN TERMS OF THE CURRENT AND DENSITY OPERATORS

The Hamiltonian operator is written in the formalism of second quantization as

$$\hat{H} = \hat{T} + \hat{U} + \hat{V} \quad , \tag{I1}$$

where the kinetic energy is

$$\hat{T} \equiv \int \hat{\psi}^{+}(\bar{x}) \left(-\frac{\hbar^2}{2m} \bar{\nabla} \hat{\psi}(\bar{x}) \right) d^3x \, , \tag{I2a}$$

the external potential is

$$\hat{U} \equiv \int \hat{\psi}^{+}(\bar{x}) \hat{\psi}(\bar{x}) U(|x|) \, d^3x \quad , \tag{I2b}$$

and the two body potential is

$$\hat{V} \equiv \frac{1}{2} \int \hat{\psi}^{+}(\bar{x}) \hat{\psi}^{+}(\bar{y}) V(|\bar{x}-\bar{y}|) \hat{\psi}(\bar{x}) \hat{\psi}(\bar{y}) \, d^3x \, d^3y \, . \tag{I2c}$$

$U(|\bar{x}|)$ is the external potential and $V(|\bar{x}-\bar{y}|)$ is the two-body internal potential.

Upon substituting Eq. (3.1) into Eq. (12b), the result is

$$\hat{U}' = \frac{1}{m} \int \hat{\rho}(\bar{x}) U(|x|) \, d^3x = U \, , \tag{I3}$$

which is the same as Eq. (3.14b).

Upon rearranging Eq. (I2c) and applying the commutation relations, Eq. (3.2) and Eq. (3.3), the result is

$$\hat{V} = \frac{1}{2}\int V(|\bar{x}-\bar{y}|)\,\hat{\psi}^{+}(\bar{x})\left[\hat{\psi}(\bar{x})\hat{\psi}^{+}(\bar{y}) - \delta(\bar{x}-\bar{y})\right]\hat{\psi}(\bar{y})\,d^3x\,d^3y\ . \qquad (I4)$$

Applying the commutation relations for $\hat{\psi}$ and $\hat{\psi}^{+}$ and integrating the second term over $\bar{X}$ gives

$$\hat{V} = \frac{1}{2m^2}\int V(|\bar{x}-\bar{y}|)\,\hat{\rho}(\bar{x})\,\hat{\rho}(\bar{y})\,d^3x\,d^3y + \frac{1}{2}\int V(0)\,\hat{\rho}(\bar{y})\,d^3y = \hat{V}'\ , \qquad (I5)$$

which is the same as Eq. (3.14c).

To cast Eq. (I2a) in the form of Eq. (3.3a), it must first be rewritten as

$$\hat{T}' = -\int \hat{\psi}^{+}(\bar{x})\left(\frac{-\hbar^2}{i\sqrt{2m}}\right)\bar{\nabla}\cdot\ \left(\frac{-\hbar}{i\sqrt{2m}}\right)\left(\bar{\nabla}\hat{\psi}(\bar{x})\right)d^3x\ . \qquad (I6)$$

The hermiticity of $\frac{\bar{\nabla}}{i}$ may be used in Eq. (I6) to write

$$\hat{T}' = \frac{\hbar^2}{2m}\int\left(\frac{\bar{\nabla}}{i}\hat{\psi}^{+}(\bar{x})\right)\cdot\left(-\frac{\bar{\nabla}}{i}\hat{\psi}(\bar{x})\right)d^3x$$

$$= \frac{\hbar^2}{2m}\int\left(\bar{\nabla}\hat{\psi}^{+}(\bar{x})\right)\cdot\left(\bar{\nabla}\hat{\psi}(\bar{x})\right)d^3x\ . \qquad (I7)$$

Now if one writes the gradient of the density operator as

$$\bar{\nabla}\hat{\rho}(\bar{x}) = m\bar{\nabla}\left(\hat{\psi}^{+}(\bar{x})\,\hat{\psi}(\bar{x})\right) = m\left(\bar{\nabla}\hat{\psi}^{+}(\bar{x})\right)\hat{\psi}(\bar{x}) + m\,\hat{\psi}^{+}(\bar{x})\left(\bar{\nabla}\hat{\psi}(\bar{x})\right), \qquad (I8)$$

and notices the following identities:

$$\frac{1}{m}\bar{\nabla}\hat{\rho}(\bar{x}) + \frac{2i}{\hbar}\hat{\bar{\bar{J}}}(\bar{x}) = 2\hat{\psi}^{+}(\bar{x})\,\bar{\nabla}\hat{\psi}(\bar{x}) \,, \tag{I9}$$

$$\frac{1}{m}\bar{\nabla}\hat{\rho}(\bar{x}) - \frac{2i}{\hbar}\hat{\bar{\bar{J}}}(\bar{x}) = 2\left(\bar{\nabla}\hat{\psi}^{+}(\bar{x})\right)\hat{\psi}(\bar{x}) \,, \tag{I10}$$

then one may write

$$\left(\hat{\psi}^{+}(x)\right)^{-1}\left[\frac{1}{m}\bar{\nabla}\hat{\rho}(\bar{x}) + \frac{2i}{\hbar}\hat{\bar{\bar{J}}}(\bar{x})\right] = \left(\hat{\psi}^{+}(\bar{x})\right)^{-1} 2\hat{\psi}^{+}(\bar{x})\left(\bar{\bar{\nabla}}\hat{\psi}(\bar{x})\right)$$

$$= 2\bar{\nabla}\hat{\psi}(\bar{x}) \,, \tag{I11}$$

and

$$\left[\frac{1}{m}\bar{\nabla}\hat{\rho}(\bar{x}) - \frac{2i}{\hbar}\hat{\bar{\bar{J}}}\right]\left(\hat{\psi}(\bar{x})\right)^{-1} = 2\left(\bar{\bar{\nabla}}\hat{\psi}^{+}(\bar{x})\right)\hat{\psi}(\bar{x})\left(\hat{\psi}(\bar{x})\right)^{-1}$$

$$= 2\bar{\bar{\nabla}}\hat{\psi}^{+}(\bar{x}) \,. \tag{I12}$$

The use of both inverse field operators and the inverse density operator in Eqs. (I11) and (I12) and in Eq. (I13), which follows, should be noted. Since it is shown in Section V that these operators do not exist, the derivation of Eq. (I15) must be regarded as merely formal.

Since

$$\hat{\rho}^{-1}(\bar{x}) = \hat{\psi}^{-1}(\bar{x})\left(\hat{\psi}^{+}(\bar{x})\right)^{-1} , \tag{I13}$$

Eq. (I11) and Eq. (I12) may be combined with Eq. (I13) to give

$$\hat{\psi}\int \bar{\nabla}\hat{\psi}^{+}(\bar{x})\cdot\bar{\nabla}\hat{\psi}(\bar{x})\,d^3x = \int\left[\frac{1}{m}\bar{\nabla}\hat{\rho}(\bar{x}) - \frac{2i}{\hbar}\hat{\bar{\bar{J}}}(\bar{x})\right]\cdot\hat{\rho}^{-1}(\bar{x})\left[\frac{1}{m}\bar{\nabla}\hat{\rho}(\bar{x}) + \frac{2i}{\hbar}\hat{\bar{\bar{J}}}(\bar{x})\right]d^3x, \tag{I14}$$

or

$$\hat{T}' = \frac{\hbar^2}{8m}\int\left[\frac{1}{m}\bar{\nabla}\hat{\rho}(\bar{x}) - \frac{2i}{\hbar}\hat{\bar{\bar{J}}}(\bar{x})\right]\cdot\hat{\rho}^{-1}(\bar{x})\left[\frac{1}{m}\bar{\nabla}\hat{\rho}(\bar{x}) + \frac{2i}{\hbar}\hat{\bar{\bar{J}}}(\bar{x})\right]d^3x = \hat{T}, \tag{I15}$$

which is the same as Eq. (3.14a). Thus the Hamiltonian written in terms of currents and densities, Eq. (3.13), has been derived from the Hamiltonian written in the formalism of second quantization.

APPENDIX J

VERIFICATION OF THE OPERATION OF $\hat{\rho}(\bar{x})$ AND $\hat{J}_\ell(\bar{x})$ ON THE WAVE FUNCTIONAL $\Psi(\rho)$

Since the operation of $\hat{\rho}(\bar{x})$ on $\Psi(\rho)$ is just multiplication of $\Psi(\rho)$ by the function $\hat{\rho}(\bar{x})$, it is easy to see that the functional realizations given in Eq. (3.17a) and (3.17b) satisfy Eq. (3.5). Since

$$\left[\hat{\rho}(\bar{x}), \hat{\rho}(\bar{x})\right] = 0, \tag{J1}$$

Eq. (3.5) may be realized in the functional representation by

$$\left[\hat{\rho}(\bar{x}), \hat{\rho}(\bar{y})\right]\Psi(\rho) \longrightarrow \left[\rho(\bar{x}), \rho(\bar{y})\right]\Psi(\rho) = 0 . \tag{J2}$$

That Eq. (3.6) is satisfied by the functional representation may be seen by writing the commutator with $\hat{\rho}(\bar{x})$ and $\hat{J}_k(\bar{x})$ replaced by their functional representations as follows:

$$\left[\hat{\rho}(\bar{x}), \hat{J}_\ell(\bar{y})\right]\Psi(\rho) \rightarrow \left[\rho(\bar{x}), -i\hbar\rho(\bar{y})\frac{\partial}{\partial y_\ell}\frac{\delta}{\delta\rho(\bar{y})}\right]\Psi(\rho)$$

$$= -i\hbar\rho(\bar{x})\rho(\bar{y})\frac{\partial}{\partial y_\ell}\frac{\delta}{\delta\rho(\bar{y})}\Psi(\rho) + i\hbar\rho(\bar{y})\frac{\partial}{\partial y_\ell}\frac{\delta}{\delta\rho(\bar{y})}\left[\rho(\bar{x})\Psi(\rho)\right]$$

$$= -i\hbar\rho(\bar{x})\rho(\bar{y})\frac{\partial}{\partial y_\ell}\frac{\delta}{\delta\rho(\bar{y})}\Psi(\rho) + i\hbar\rho(\bar{y})\frac{\partial}{\partial y_\ell}\left[\delta(\bar{x}-\bar{y})\Psi(\rho)\right]$$

$$+ i\hbar\rho(\bar{y})\frac{\partial}{\partial y_\ell}\left[\rho(\bar{x})\frac{\delta\Psi(\rho)}{\delta\rho(\bar{y})}\right]$$

$$= -i\hbar\rho(\bar{x})\rho(\bar{y})\frac{\partial}{\partial y_\ell}\frac{\delta\Psi(\rho)}{\delta\rho(\bar{y})} + i\hbar\rho(\bar{y})\rho(\bar{x})\frac{\partial}{\partial y_\ell}\frac{\delta\Psi(\rho)}{\delta\rho(\bar{y})} + i\hbar\rho(\bar{y})\frac{\partial}{\partial y_\ell}\left[\delta(\bar{x}-\bar{y})\Psi(\rho)\right]$$

$$= i\hbar\rho(\bar{y})\left[\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y_\ell}\right]\Psi(\rho) + i\hbar\rho(\bar{y})\delta(\bar{x}-\bar{y})\left[\frac{\partial}{\partial y_\ell}\Psi(\rho)\right]$$

$$= i\rho(\bar{y})\left[\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y_\ell}\right]\Psi(\rho)\ , \tag{J3}$$

where the identities

$$\frac{\delta f(\bar{y})}{\delta f(\bar{x})} = \delta(\bar{x}-\bar{y}) \tag{J4}$$

and

$$\frac{\partial}{\partial y_\ell}\Psi(\rho) = 0 \tag{J5}$$

were used. Applying the identity, Eq. (3.8), to Eq. (J3) gives

$$\left[\hat{\rho}(\bar{x}), \hat{J}_\ell(\bar{y})\right]\Psi(\rho) \longrightarrow -i\hbar\rho(\bar{x})\left[\frac{\partial\delta(\bar{x}-\bar{y})}{\partial x_\ell}\right]\Psi(\rho) - i\hbar\delta(\bar{x}-\bar{y})\left[\frac{\partial\rho(\bar{x})}{\partial x_\ell}\right]\Psi(\rho)$$

$$= -i\hbar \frac{\partial}{\partial x_\ell} \left[\delta(\bar{x}-\bar{y}) \rho(\bar{x}) \right] \Psi(\rho) , \tag{J6}$$

which is the same as Eq. (3.6) in the functional representation.

That Eq. (3.7) is satisfied by the functional representation may be seen by substituting the functional representation for $\hat{J}_k(\bar{x})$ into the commutator as follows:

$$\left[\hat{J}_k(\bar{x}), \hat{J}_\ell(\bar{y}) \right] \Psi(\rho) = \left[-i\hbar \rho(\bar{x}) \frac{\partial}{\partial x_k} \frac{\delta}{\delta \rho(\bar{x})} , -i\hbar \rho(\bar{y}) \frac{\partial}{\partial y_\ell} \frac{\delta}{\delta \rho(\bar{y})} \right] \Psi(\rho) ,$$

$$= -i\hbar \rho \frac{\partial}{\partial x_k} \frac{\delta}{\delta \rho(\bar{x})} \left[-i\hbar \rho(\bar{y}) \frac{\partial}{\partial y_\ell} \left[\frac{\delta \Psi(\rho)}{\delta \rho(\bar{y})} \right] \right]$$

$$+ i\hbar \rho(\bar{y}) \frac{\partial}{\partial \rho(\bar{y})} \frac{\delta}{\delta \rho(\bar{y})} \left[-i\hbar \rho(\bar{x}) \frac{\partial}{\partial x_k} \left[\frac{\delta \Psi(\rho)}{\delta \rho(\bar{x})} \right] \right]$$

$$= -i\hbar \rho(\bar{x}) \frac{\partial}{\partial x_k} \left[-i\hbar \delta(\bar{x}-\bar{y}) \frac{\partial}{\partial y_\ell} \left[\frac{\delta \Psi(\rho)}{\delta \rho(\bar{y})} \right] \right]$$

$$- i\hbar \rho(\bar{x}) \frac{\partial}{\partial x_k} \left[-i\hbar \rho(\bar{y}) \frac{\partial}{\partial y_\ell} \left[\frac{\delta^2 \Psi(\rho)}{\delta \rho(\bar{x}) \delta \rho(\bar{y})} \right] \right]$$

$$+ i\hbar \rho(\bar{y}) \frac{\partial}{\partial y_\ell} \left[-i\hbar \delta(\bar{x}-\bar{y}) \frac{\partial}{\partial x_k} \left[\frac{\delta \Psi(\rho)}{\delta \rho(\bar{x})} \right] \right]$$

$$+ i\hbar\rho(\bar{y})\frac{\partial}{\partial y_\ell}\left[-i\hbar\rho(\bar{x})\frac{\partial}{\partial x_k}\left[\frac{\delta^2\Psi(\rho)}{\delta\rho(\bar{y})\,\delta\rho(\bar{x})}\right]\right]$$

$$= - i\hbar^2\rho(\bar{x})\frac{\partial\delta(\bar{x}-\bar{y})}{\partial x_k}\frac{\partial}{\partial y_\ell}\left[\frac{\delta\Psi(\rho)}{\delta\rho(\bar{y})}\right] - \hbar^2\rho(\bar{x})\,\delta(\bar{x}-\bar{y})\frac{\partial^2}{\partial x_k\partial y_\ell}\left[\frac{\delta\Psi(\rho)}{\delta\rho(\bar{y})}\right]$$

$$- \hbar^2\rho(\bar{x})\rho(\bar{y})\frac{\partial^2}{\partial x_k\partial y_\ell}\left[\frac{\delta^2\Psi(\rho)}{\delta\rho(\bar{x})\delta\rho(\bar{y})}\right] + \hbar^2\rho(\bar{y})\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y_\ell}\frac{\partial}{\partial x_k}\left[\frac{\delta\Psi(\rho)}{\delta\rho(\bar{x})}\right]$$

$$+ \hbar^2\rho(\bar{y})\,\delta(\bar{x}-\bar{y})\frac{\partial^2}{\partial y_\ell\partial x_k}\left[\frac{\delta\Psi(\rho)}{\delta\rho(\bar{x})}\right] + \hbar^2\rho(\bar{y})\rho(\bar{x})\frac{\partial^2}{\partial y_\ell\partial x_k}\left[\frac{\delta^2\Psi(\rho)}{\delta\rho(\bar{x})\,\delta\rho(\bar{y})}\right]$$

$$= - \hbar^2\rho(\bar{x})\frac{\partial\delta(\bar{x}-\bar{y})}{\partial x_k}\frac{\partial}{\partial y_\ell}\left[\frac{\delta\Psi(\rho)}{\delta\rho(\bar{y})}\right] + \hbar^2\rho(\bar{y})\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y_\ell}\frac{\partial}{\partial x_k}\left[\frac{\delta\Psi(\rho)}{\delta\rho(\bar{x})}\right]$$

$$- \hbar^2\rho(\bar{x})\,\delta(\bar{x}-\bar{y})\frac{\partial}{\partial x_k\partial y_\ell}\left[\frac{\delta\Psi(\rho)}{\delta\rho(\bar{y})}\right]$$

$$+ \hbar^2\rho(\bar{y})\,\delta(\bar{x}-\bar{y})\frac{\partial^2}{\partial y_\ell\partial x_k}\left[\frac{\delta\Psi(\rho)}{\delta\rho(\bar{x})}\right] , \qquad (J7)$$

where Eq. (J4) was used in the third step. Applying the identity in Eq. (3.8) to Eq. (J7) gives

$$\left[\hat{J}_k(\bar{x}), \hat{J}_l(\bar{y})\right]\Psi(\rho) \longrightarrow -\hbar\rho(\bar{y})\frac{\partial\delta(\bar{x}-\bar{y})}{\partial x_k}\frac{\partial}{\partial y_l}\left[\frac{\delta\Psi(\rho)}{\delta\rho(\bar{y})}\right]$$

$$+\hbar^2\frac{\partial\rho(\bar{x})}{\partial x_k}\delta(\bar{x}-\bar{y})\frac{\partial}{\partial y_l}\left[\frac{\delta\Psi(\rho)}{\delta\rho(\bar{y})}\right]$$

$$+\hbar^2\rho(\bar{y})\delta(\bar{x}-\bar{y})\frac{\partial^2}{\partial y_l\partial x_k}\left[\frac{\delta\Psi(\rho)}{\delta\rho(\bar{x})}\right] + \hbar^2\rho(\bar{x})\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y_l}\frac{\partial}{\partial x_k}\left[\frac{\delta\Psi(\rho)}{\delta\rho(\bar{x})}\right]$$

$$-\hbar^2\delta(\bar{x}-\bar{y})\frac{\partial\rho(\bar{y})}{\partial y_l}\frac{\partial}{\partial x_k}\left[\frac{\delta\Psi(\rho)}{\delta\rho(\bar{x})}\right] - \hbar^2\rho(\bar{x})\delta(\bar{x}-\bar{y})\frac{\partial^2}{\partial x_k\partial y_l}\left[\frac{\delta\Psi(\rho)}{\delta\rho(\bar{y})}\right] . \qquad \text{(J8)}$$

Applying the property of the delta function, Eq. (3.9), to Eq. (J8) gives

$$\left[\hat{J}_k(\bar{x}), \hat{J}_l(\bar{y})\right]\Psi(\rho) \longrightarrow \hbar^2\left(\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y_k}\rho(\bar{y})\frac{\partial}{\partial y_l}\frac{\delta}{\delta\rho(\bar{y})}\right.$$

$$\left.+\delta(\bar{x}-\bar{y})\frac{\partial\rho(\bar{x})}{\partial x_k}\frac{\partial}{\partial y_l}\frac{\delta}{\delta\rho(\bar{y})} + \delta(\bar{x}-\bar{y})\rho(\bar{y})\frac{\partial^2}{\partial y_k\partial y_l}\frac{\delta}{\delta\rho(\bar{y})}\right)\Psi(\rho)$$

$$-\hbar^2\left(\frac{\partial\delta(\bar{x}-\bar{y})}{\partial x_l}\rho(\bar{x})\frac{\partial}{\partial x_k}\frac{\delta}{\delta\rho(\bar{x})} + \delta(\bar{x}-\bar{y})\frac{\partial\rho(\bar{x})}{\partial x_l}\frac{\partial}{\partial x_k}\frac{\delta}{\delta\rho(\bar{x})}\right.$$

$$+\delta(\bar{x}-\bar{y})\rho(\bar{x})\frac{\partial^2}{\partial x_\ell \partial x_k}\frac{\delta}{\delta\rho(\bar{x})}\Bigg)\Psi(\rho)$$

$$= -i\hbar\frac{\partial}{\partial x_\ell}\left[\delta(\bar{x}-\bar{y})\hat{J}_k(\bar{x})\right]\Psi(\rho)$$

$$+ i\hbar\frac{\partial}{\partial y_k}\left[\delta(\bar{x}-\bar{y})\ \hat{J}_\ell(\bar{y})\right]\Psi(\rho)\ , \qquad \text{(J9)}$$

which is the same as Eq. (317) in the functional representation.

APPENDIX K

CALCULATION OF THE DENSITY-VELOCITY COMMUTATOR IN SECOND QUANTIZATION FORMULATION

In this appendix the density-velocity commutator is calculated rigorously in the formulation of second quantization. The implicit definition of the velocity operator given by Eq. (4.2) is used, and no recourse is taken to inverse operators. The result in Eq. (4.3) is obtained by using the known value of the density-current commutator.

When the form of the current operator given in Eq. (4.2) is substituted into the density-current commutator, the result is

$$\left[\hat{\rho}(\bar{x}), \hat{J}_\ell(\bar{y})\right] = \left[\hat{\rho}(\bar{x}), \frac{1}{2}\left(\hat{\rho}(\bar{y})\hat{\bar{V}}_\ell(\bar{y}) + \hat{\bar{V}}_\ell(\bar{y})\hat{\rho}(\bar{y})\right)\right]$$

$$= \frac{1}{2}\left[\hat{\rho}(\bar{x}), \hat{\rho}(\bar{y})\hat{\bar{V}}_\ell(\bar{y})\right] + \frac{1}{2}\left[\hat{\rho}(\bar{x}), \hat{\bar{V}}_\ell(\bar{y})\hat{\rho}(\bar{y})\right]$$

$$= \frac{1}{2}\hat{\rho}(\bar{y})\left[\hat{\rho}(\bar{x}), \hat{\bar{V}}_\ell(\bar{y})\right] + \frac{1}{2}\left[\hat{\rho}(\bar{x}), \hat{\rho}(\bar{y})\right]\hat{\bar{V}}_\ell(\bar{y})$$

$$+ \frac{1}{2}\hat{\bar{V}}_\ell(\bar{y})\left[\hat{\rho}(\bar{x}), \hat{\rho}(\bar{y})\right] + \frac{1}{2}\left[\hat{\rho}(\bar{x}), \hat{\bar{V}}_\ell(\bar{y})\right]\hat{\rho}(\bar{y}) . \qquad \text{(K1)}$$

Applying Eq. (3.5) for the density-density commutator to Eq. (K1) gives

$$\left[\hat{\rho}(\bar{x}), \hat{\bar{\bar{J}}}_\ell(\bar{y})\right] = \frac{1}{2}\hat{\rho}(\bar{y})\left[\hat{\rho}(\bar{x}), \hat{\bar{V}}_\ell(\bar{y})\right] + \frac{1}{2}\left[\hat{\rho}(\bar{x}), \hat{\bar{V}}_\ell(\bar{y})\right]\rho(\bar{y}). \tag{K2}$$

Substituting the value of the density-current commutator given in Eq. (3.6) into Eq. (K2) gives

$$-i\hbar\frac{\partial}{\partial x_\ell}\left[\hat{\rho}(\bar{x})\,\delta(\bar{x}-\bar{y})\right] = \frac{1}{2}\hat{\rho}(\bar{y})\left[\hat{\rho}(\bar{x}), \hat{V}_\ell(\bar{y})\right]$$

$$+\frac{1}{2}\left[\hat{\rho}(\bar{x}), \hat{V}_\ell(\bar{y})\right]\hat{\rho}(\bar{y}). \tag{K3}$$

Working with the left side of Eq. (K3) gives

$$-i\hbar\left[\frac{\partial}{\partial x_\ell}\rho(\bar{x})\right]\delta(\bar{x}-\bar{y}) - i\hbar\hat{\rho}(\bar{x})\frac{\partial}{\partial x_\ell}\delta(\bar{x}-\bar{y}) = -i\hbar\frac{\partial}{\partial x_\ell}\left[\hat{\rho}(\bar{x})\delta(\bar{x}-\bar{y})\right]. \tag{K4}$$

Applying the identity Eq. (3.8) to Eq. (K4) gives

$$-i\hbar\frac{\partial}{\partial x_\ell}\left[\hat{\rho}(\bar{x})\,\delta(\bar{x}-\bar{y})\right] = -i\hbar\hat{\rho}(\bar{y})\left[\frac{\partial}{\partial x_\ell}\delta(\bar{x}-\bar{y})\right]. \tag{K5}$$

Substituting Eq. (K5) into Eq. (K3) gives

$$\frac{1}{2}\hat{\rho}(\bar{y})\left[\hat{\rho}(\bar{x}), \hat{V}_\ell(\bar{y})\right] + \frac{1}{2}\left[\hat{\rho}(\bar{x}), \hat{V}_\ell(\bar{y})\right]\hat{\rho}(\bar{y}) = -i\hbar\hat{\rho}(\bar{y})\frac{\partial}{\partial x_\ell}\delta(\bar{x}-\bar{y}), \tag{K6}$$

which is satisfied if

$$\left[\hat{\rho}(\bar{x}), \hat{V}_\ell(\bar{y})\right] = -i\hbar\left[\frac{\partial}{\partial x_\ell}\delta(\bar{x}-\bar{y})\right] = i\hbar\left[\frac{\partial}{\partial y_\ell}\delta(\bar{x}-\bar{y})\right], \tag{K7}$$

which is the same as Eq. (4.3).

APPENDIX L

CALCULATION OF THE VELOCITY-VELOCITY COMMUTATOR IN THE FORMALISM OF SECOND QUANTIZATION

In this appendix the value of the velocity-velocity commutator is calculated rigorously in the formalism of second quantization. Equation (4.4) is calculated using the form of the density operator given in Eq. (4.2) and the known values of the density-density, current-current, and density-velocity commutators. When Eq. (4.2) is substituted into the current-current commutator, the result is

$$\left[\hat{J}_k(\bar{x}), \hat{J}_\ell(\bar{y})\right] = \left[\frac{1}{2}\left(\hat{\rho}(\bar{x})\hat{V}_k(\bar{x}) + \hat{V}_k(\bar{x})\hat{\rho}(\bar{x})\right), \frac{1}{2}\left(\hat{\rho}(\bar{y})\hat{V}_\ell(\bar{y}) + \hat{V}_\ell(\bar{y})\hat{\rho}(\bar{y})\right)\right]$$

$$= \frac{1}{4}\left[\hat{\rho}(\bar{x})\hat{V}_k(\bar{x}), \hat{\rho}(\bar{y})\hat{V}_\ell(\bar{y})\right] + \frac{1}{4}\left[\hat{\rho}(\bar{x})\hat{V}_k(\bar{x}), \hat{V}_\ell(\bar{y})\hat{\rho}(\bar{y})\right]$$

$$+ \frac{1}{4}\left[\hat{V}_k(\bar{x})\hat{\rho}(\bar{x}), \hat{\rho}(\bar{y})\hat{V}_\ell(\bar{y})\right] + \frac{1}{4}\left[\hat{V}_k(\bar{x})\hat{\rho}(\bar{x}), \hat{V}_\ell(\bar{y})\hat{\rho}(\bar{y})\right]$$

$$= \frac{1}{4}\hat{\rho}(\bar{x})\left[\hat{V}_k(\bar{x}), \hat{\rho}(\bar{y})\hat{V}_\ell(\bar{y})\right] + \frac{1}{4}\left[\hat{\rho}(\bar{x}), \hat{\rho}(\bar{y})\hat{V}_\ell(\bar{y})\right]\hat{V}_k(\bar{x})$$

$$+\frac{1}{4}\hat{\rho}(\bar{x})\left[\hat{V}_k(\bar{x}),\hat{V}_\ell(\bar{y})\hat{\rho}(\bar{y})\right]+\frac{1}{4}\left[\hat{\rho}(\bar{x}),\hat{V}_\ell(\bar{y})\hat{\rho}(\bar{y})\right]\hat{V}_k(\bar{x})$$

$$+\frac{1}{4}\hat{V}_k(\bar{x})\left[\hat{\rho}(\bar{x}),\hat{\rho}(\bar{y})\hat{V}_\ell(\bar{y})\right]+\frac{1}{4}\left[\hat{V}_k(\bar{x}),\hat{\rho}(\bar{y})\hat{V}_\ell(\bar{y})\right]\hat{\rho}(\bar{x})$$

$$+\frac{1}{4}\hat{V}_k(\bar{x})\left[\hat{\rho}(\bar{x}),\hat{V}_\ell(\bar{y})\hat{\rho}(\bar{y})\right]+\frac{1}{4}\left[\hat{V}_k(\bar{x}),\hat{V}_\ell(\bar{y})\hat{\rho}(\bar{y})\right]\hat{\rho}(\bar{x})\,. \tag{L1}$$

Expanding the commutators further gives

$$\left[\hat{J}_k(\bar{x}),\hat{J}_\ell(\bar{y})\right]=\frac{1}{4}\hat{\rho}(\bar{x})\hat{\rho}(\bar{y})\left[\hat{V}_k(\bar{x}),\hat{V}_\ell(\bar{y})\right]+\frac{1}{4}\hat{\rho}(\bar{x})\left[\hat{V}_k(\bar{x}),\hat{\rho}(\bar{y})\right]\hat{V}_\ell(\bar{y})$$

$$+\frac{1}{4}\hat{\rho}(\bar{y})\left[\hat{\rho}(\bar{x}),\hat{V}_\ell(\bar{y})\right]\hat{V}_k(\bar{x})+\frac{1}{4}\left[\hat{\rho}(\bar{x}),\hat{\rho}(\bar{y})\right]\hat{V}_\ell(\bar{y})\hat{V}_k(\bar{x})$$

$$+\frac{1}{4}\hat{\rho}(\bar{x})\hat{V}_\ell(\bar{y})\left[\hat{V}_k(\bar{x}),\hat{\rho}(\bar{y})\right]+\frac{1}{4}\hat{\rho}(\bar{x})\left[\hat{V}_k(\bar{x}),\hat{V}_\ell(\bar{y})\right]\hat{\rho}(\bar{y})$$

$$+\frac{1}{4}\hat{V}_\ell(\bar{y})\left[\hat{\rho}(\bar{x}),\hat{\rho}(\bar{y})\right]\hat{V}_k(\bar{x})+\frac{1}{4}\left[\hat{\rho}(\bar{x}),\hat{V}_\ell(\bar{y})\right]\hat{\rho}(\bar{y})\hat{V}_k(\bar{x})$$

$$+\frac{1}{4}\hat{V}_k(\bar{x})\hat{\rho}(\bar{y})\left[\hat{\rho}(\bar{x}),\hat{V}_\ell(\bar{y})\right]+\frac{1}{4}\hat{V}_k(\bar{x})\left[\hat{\rho}(\bar{x}),\hat{\rho}(\bar{y})\right]\hat{V}_\ell(\bar{y})$$

$$+\frac{1}{4}\hat{\rho}(\bar{y})\left[\hat{V}_k(\bar{x}),\hat{V}_\ell(\bar{y})\right]\hat{\rho}(\bar{x})+\frac{1}{4}\left[\hat{V}_k(\bar{x}),\hat{\rho}(\bar{y})\right]\hat{V}_\ell(\bar{y})\hat{\rho}(\bar{x})$$

$$+\frac{1}{4}\hat{V}_k(\bar{x})\hat{V}_\ell(\bar{y})\left[\hat{\rho}(\bar{x}),\hat{\rho}(\bar{y})\right]+\frac{1}{4}\hat{V}_k(\bar{x})\left[\hat{\rho}(\bar{x}),\hat{V}_\ell(\bar{y})\right]\hat{\rho}(\bar{y})$$

$$+\frac{1}{4}\hat{V}_\ell(\bar{y})\left[\hat{V}_k(\bar{x}),\hat{\rho}(\bar{y})\right]\hat{\rho}(\bar{x})+\frac{1}{4}\left[\hat{V}_k(\bar{x}),\hat{V}_\ell(\bar{y})\right]\hat{\rho}(\bar{y})\hat{\rho}(\bar{x})\,. \tag{L2}$$

Substituting Eqs. (4.3), (3.5) and (3.7) into Eq. (L2) gives

$$\left[\hat{J}_k(\bar{x}), \hat{J}_\ell(\bar{y})\right] = \frac{1}{4}\hat{\rho}(\bar{x})\hat{\rho}(\bar{y})\left[\hat{V}_k(\bar{x}), \hat{V}_\ell(\bar{y})\right] - \frac{1}{4} i\hbar\left[\frac{\partial\delta(\bar{x}-\bar{y})}{\partial x_k}\right]\hat{\rho}(\bar{x})\hat{V}_\ell(\bar{y})$$

$$+\frac{1}{4} i\hbar\left[\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y_\ell}\right]\hat{\rho}(\bar{y})\hat{V}_k(\bar{x}) - \frac{1}{4} i\hbar\left[\frac{\partial\delta(\bar{x}-\bar{y})}{\partial x_k}\right]\hat{\rho}(\bar{x})\hat{V}_\ell(\bar{y})$$

$$+\frac{1}{4}\hat{\rho}(\bar{x})\left[\hat{V}_k(\bar{x}), \hat{V}_\ell(\bar{y})\right]\hat{\rho}(\bar{y}) + \frac{1}{4} i\hbar\left[\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y_\ell}\right]\hat{\rho}(\bar{y})\hat{V}_k(\bar{x})$$

$$+\frac{1}{4} i\hbar\left[\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y_\ell}\right]\hat{V}_k(\bar{x})\hat{\rho}(\bar{y}) + \frac{1}{4}\hat{\rho}(\bar{y})\left[\hat{V}_k(\bar{x}), \hat{V}_\ell(\bar{y})\right]\hat{\rho}(\bar{x})$$

$$-\frac{1}{4} i\hbar\left[\frac{\partial\delta(\bar{y}-\bar{x})}{\partial x_k}\right]\hat{V}_\ell(\bar{y})\hat{\rho}(\bar{x}) + \frac{1}{4} i\hbar\left[\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y_\ell}\right]\hat{V}_k(\bar{x})\hat{\rho}(\bar{y})$$

$$-\frac{1}{4} i\hbar\left[\frac{\partial\delta(\bar{y}-\bar{x})}{\partial x_k}\right]\hat{V}_\ell(\bar{y})\hat{\rho}(\bar{x}) + \frac{1}{4}\left[\hat{V}_k(\bar{x}), \hat{V}_\ell(\bar{y})\right]\hat{\rho}(\bar{y})\hat{\rho}(\bar{x})$$

$$= i\hbar\frac{\partial}{\partial x_\ell}\left[\delta(\bar{x}-\bar{y})\hat{J}_k(\bar{x})\right] + i\hbar\frac{\partial}{\partial y_k}\left[\delta(\bar{x}-\bar{y})\hat{J}_\ell(\bar{y})\right] \quad . \tag{L3}$$

Expanding the last two terms in Eq. (L3) gives

$$\left[\hat{J}_k(\bar{x}), \hat{J}_\ell(\bar{y})\right] = -\frac{i\hbar}{2}\left[\frac{\partial\delta(\bar{x}-\bar{y})}{\partial x_\ell}\right]\hat{\rho}(\bar{x})\ \hat{V}_k(\bar{x})$$

$$-\frac{i\hbar}{2}\left[\frac{\partial\delta(\bar{x}-\bar{y})}{\partial x_\ell}\right]\hat{V}_k(\bar{x})\hat{\rho}(\bar{x}) - \frac{i\hbar}{2}\delta(\bar{x}-\bar{y})\left[\frac{\partial\left(\hat{\rho}(\bar{x})\ \hat{V}_k(\bar{x})\right)}{\partial x_\ell}\right]$$

$$-\frac{i\hbar}{2}\delta(\bar{x}-\bar{y})\left[\frac{\partial\left(\hat{V}_k(\bar{x})\hat{\rho}(\bar{x})\right)}{\partial x_\ell}\right] + \frac{i\hbar}{2}\left[\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y_k}\right]\hat{\rho}(\bar{y})\ \hat{V}_\ell(\bar{y})$$

$$+\frac{i\hbar}{2}\left[\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y_k}\right]\hat{V}_\ell(\bar{y})\ \hat{\rho}(\bar{y}) + \frac{i\hbar}{2}\delta(\bar{x}-\bar{y})\left[\frac{\partial}{\partial y_k}\left(\hat{\rho}(\bar{y})\ \hat{V}_\ell(\bar{y})\right)\right]$$

$$+\frac{i\hbar}{2}\delta(\bar{x}-\bar{y})\left[\frac{\partial}{\partial y_k}\left(\hat{V}_\ell(\bar{y})\hat{\rho}(\bar{y})\right)\right]. \qquad \text{(L4)}$$

Applying the identity, Eq. (3.8), to Eq. (L4) gives

$$\left[\hat{J}_k(\bar{x}), \hat{J}_\ell(\bar{y})\right] = \frac{i\hbar}{2}\left[\frac{\partial\delta(\bar{x}-\bar{y})}{\partial y_\ell}\right]\hat{\rho}(\bar{y})\ \hat{V}_k(\bar{x}) + \frac{i\hbar}{2}\left[\frac{\partial\delta(\bar{x}-y)}{\partial y_\ell}\right]\hat{V}_k(\bar{x})\hat{\rho}(\bar{y})$$

$$-\frac{i\hbar}{2}\delta(\bar{x}-\bar{y})\left[\hat{\rho}(\bar{x})\frac{\partial\hat{V}_k(\bar{x})}{\partial x_\ell} + \frac{\partial\hat{V}_k(\bar{x})}{\partial x_\ell}\rho(\bar{x})\right]$$

$$-\frac{i\hbar}{2}\left[\frac{\partial\delta(\bar{x}-\bar{y})}{\partial x_k}\right]\hat{V}_\ell(\bar{y})\ \hat{\rho}(\bar{x}) - \frac{i\hbar}{2}\left[\frac{\partial\delta(\bar{x}-\bar{y})}{\partial x_k}\right]\rho(\bar{x})\ \hat{V}_\ell(\bar{y})$$

$$+\frac{i\hbar}{2}\,\delta(\bar{x}-\bar{y})\left[\hat{\rho}(\bar{y})\,\frac{\partial\hat{V}_{\ell}(\bar{y})}{\partial y_{k}}+\frac{\partial\hat{V}_{\ell}(\bar{y})}{\partial y_{k}}\,\hat{\rho}(\bar{y})\right]\;. \tag{L5}$$

When Eq. (L5) is combined with Eq. (L3), the result is

$$\frac{1}{4}\hat{\rho}(\bar{x})\hat{\rho}(\bar{y})\left[\hat{V}_{k}(\bar{x}),\hat{V}_{\ell}(\bar{y})\right]+\frac{1}{4}\left[\hat{V}_{k}(\bar{x}),\hat{V}_{\ell}(\bar{y})\right]\hat{\rho}(\bar{y})\hat{\rho}(\bar{x})$$

$$+\frac{1}{4}\hat{\rho}(\bar{x})\left[\hat{V}_{k}(\bar{x}),\hat{V}_{\ell}(\bar{y})\right]\hat{\rho}(\bar{y})+\frac{1}{4}\hat{\rho}(\bar{y})\left[\hat{V}_{k}(\bar{x}),\hat{V}_{\ell}(\bar{y})\right]\hat{\rho}(\bar{x})$$

$$=-\frac{i\hbar}{2}\,\delta(\bar{x}-\bar{y})\left[\hat{\rho}(\bar{x})\,\frac{\partial\hat{V}_{k}(\bar{x})}{\partial x_{\ell}}+\frac{\partial\hat{V}_{k}(\bar{x})}{\partial x_{\ell}}\,\hat{\rho}(\bar{x})\right]$$

$$+\frac{i\hbar}{2}\,\delta(\bar{x}-\bar{y})\left[\hat{\rho}(\bar{y})\,\frac{\partial\hat{V}_{\ell}(\bar{y})}{\partial y_{k}}+\frac{\partial\hat{V}_{\ell}(\bar{y})}{\partial y_{k}}\,\hat{\rho}(\bar{y})\right]\;. \tag{L6}$$

The right side of Eq. (L6) may be manipulated to give

$$-i\hbar\,\delta(\bar{x}-\bar{y})\left[\hat{\rho}(\bar{y})\,\frac{\partial\hat{V}_{k}(\bar{x})}{\partial x_{\ell}}-\hat{\rho}(\bar{y})\,\frac{\partial\hat{V}_{\ell}(\bar{x})}{\partial x_{k}}\right]$$

$$+\frac{i\hbar}{2}\,\delta(\bar{x}-\bar{y})\left[\frac{\partial^{2}}{\partial x_{k}\partial x_{\ell}}\,\delta(\bar{x}-\bar{y})\right]-\frac{i\hbar}{2}\,\delta(\bar{x}-\bar{y})\left[\frac{\partial^{2}\delta(\bar{x}-\bar{y})}{\partial x_{\ell}\,\partial x_{k}}\right]\;, \tag{L7}$$

where Eq. (4.3) was used to obtain Eq. (L7). If the density operator commutes with the velocity-velocity commutator, Eqs. (L6) and (L7) may be combined to give

$$\hat{\rho}(\bar{x})\left[\hat{V}_k(\bar{x}),\ \hat{V}_\ell(\bar{y})\right] = i\hbar\,\delta(\bar{x}-\bar{y})\left[\frac{\partial \hat{V}_\ell(\bar{x})}{\partial x_k} - \frac{\partial \hat{V}_k(\bar{x})}{\partial x_\ell}\right], \tag{L8}$$

which is consistent with the condition of the density operator commuting with the velocity-velocity commutator, and is the same as Eq. (4.4).

APPENDIX M

DERIVATION OF THE KINETIC ENERGY TERM IN LANDAU'S HAMILTONIAN FROM THE SECOND QUANTIZATION HAMILTONIAN

In this appendix Eq. (4.5b) for the kinetic energy term in the Hamiltonian of Eq. (4.6) is derived from Eq. (3.14a). When Eq. (4.2) for the current operator is substituted into Eq. (3.14a), the result is

$$\hat{T}' = \frac{\hbar^2}{8m}\int\left[\frac{1}{m}\bar{\nabla}\hat{\rho}(\bar{x}) - \frac{i}{\hbar}\left(\hat{\rho}(\bar{x})\hat{\bar{V}}(\bar{x}) + \hat{\bar{V}}(\bar{x})\hat{\rho}(\bar{x})\right)\right]$$

$$\cdot\, m\hat{\rho}^{-1}(\bar{x})\left[\frac{1}{m}\bar{\nabla}\hat{\rho}(\bar{x}) + \frac{i}{\hbar}\left(\hat{\rho}(\bar{x})\hat{\bar{V}}(\bar{x}) + \hat{\bar{V}}(\bar{x})\hat{\rho}(\bar{x})\right)\right]d^3x$$

$$= \frac{\hbar^2}{8m}\int\left[\left(\bar{\nabla}\hat{\rho}(\bar{x})\right)\hat{\rho}^{-1}(\bar{x}) - \frac{im}{\hbar}\hat{\rho}(\bar{x})\hat{\bar{V}}(\bar{x})\hat{\rho}^{-1}(\bar{x}) - \frac{im}{\hbar}\hat{\bar{V}}(\bar{x})\right]$$

$$\cdot\left[\frac{1}{m}\bar{\nabla}\hat{\rho}(\bar{x}) + \frac{i}{\hbar}\left(\hat{\rho}(\bar{x})\hat{\bar{V}}(\bar{x}) + \hat{\bar{V}}(\bar{x})\hat{\rho}(\bar{x})\right)\right]d^3x$$

$$= \frac{\hbar^2}{8m}\int\left[\frac{1}{m}\left(\bar{\nabla}\hat{\rho}(\bar{x})\right)\cdot\hat{\rho}^{-1}(\bar{x})\left(\bar{\nabla}\hat{\rho}(\bar{x})\right) - \frac{i}{\hbar}\hat{\bar{V}}(\bar{x})\cdot\bar{\nabla}\hat{\rho}(\bar{x})\right.$$

$$-\frac{i}{\hbar}\hat{\rho}(\bar{x})\hat{\bar{V}}(\bar{x})\hat{\rho}^{-1}(\bar{x})\cdot\left(\bar{\nabla}\hat{\rho}(\bar{x})\right)$$

$$+\frac{i}{\hbar}\left(\bar{\nabla}\hat{\rho}(\bar{x})\right)\cdot\hat{\bar{V}}(\bar{x}) + \frac{m}{\hbar^2}\hat{\rho}(\bar{x})\hat{\bar{V}}(\bar{x})\cdot\hat{\bar{V}}(\bar{x}) + \frac{m}{\hbar^2}\hat{\bar{V}}(\bar{x})\cdot\hat{\rho}(\bar{x})\hat{\bar{V}}(\bar{x})$$

$$+\frac{i}{\hbar}\bar{\nabla}\hat{\rho}(\bar{x})\hat{\rho}^{-1}(\bar{x})\cdot\hat{\bar{V}}(\bar{x})\hat{\rho}(\bar{x}) + \frac{m}{\hbar^2}\hat{\rho}(\bar{x})\hat{\bar{V}}(\bar{x})\hat{\rho}^{-1}(\bar{x})\cdot\hat{\bar{V}}(\bar{x})\hat{\rho}(\bar{x})$$

$$+\frac{m}{\hbar^2}\hat{\bar{V}}(\bar{x})\cdot\hat{\bar{V}}(\bar{x})\hat{\rho}(\bar{x})\Big]\,d^3x\,. \qquad \text{(M1)}$$

Rearranging terms allows $\hat{T}'$ to be written as

$$\hat{T}' = \frac{\hbar^2}{8m}\int\Big[\frac{1}{m}\left(\bar{\nabla}\hat{\rho}(\bar{x})\right)\cdot\hat{\rho}^{-1}(\bar{x})\left(\bar{\nabla}\hat{\rho}(\bar{x})\right) - \frac{i}{\hbar}\hat{\bar{V}}(\bar{x})\cdot\bar{\nabla}\hat{\rho}(\bar{x}) - \frac{i}{\hbar}\hat{\bar{V}}(\bar{x})\cdot\bar{\nabla}\hat{\rho}(\bar{x})$$

$$-\frac{i}{\hbar}\left[\hat{\rho}(\bar{x}),\hat{\bar{V}}(\bar{x})\right]\cdot\hat{\rho}^{-1}(\bar{x})\bar{\nabla}\hat{\rho}(\bar{x}) + \frac{i}{\hbar}\left(\bar{\nabla}\hat{\rho}(\bar{x})\right)\cdot\hat{\bar{V}}(\bar{x})$$

$$+\frac{i}{\hbar}\left(\bar{\nabla}\hat{\rho}(\bar{x})\right)\cdot\hat{\bar{V}}(\bar{x}) + \frac{i}{\hbar}\left(\bar{\nabla}\rho(\bar{x})\right)\cdot\hat{\rho}^{-1}(\bar{x})\left[\hat{\bar{V}}(\bar{x}),\hat{\rho}(\bar{x})\right]$$

$$+\frac{m}{\hbar^2}\hat{\bar{V}}(\bar{x})\cdot\hat{\rho}(\bar{x})\hat{\bar{V}}(\bar{x}) + \frac{m}{\hbar^2}\left[\hat{\rho}(\bar{x}),\hat{\bar{V}}(\bar{x})\right]\cdot\hat{\bar{V}}(\bar{x})$$

$$+\frac{m}{\hbar^2}\hat{\bar{V}}(\bar{x})\cdot\hat{\rho}(\bar{x})\hat{\bar{V}}(\bar{x})+\frac{m}{\hbar^2}\hat{\bar{V}}(\bar{x})\cdot\hat{\rho}(\bar{x})\hat{\bar{V}}(\bar{x})$$

$$+\frac{m}{\hbar^2}\left[\hat{\rho}(\bar{x}),\hat{\bar{V}}(\bar{x})\right]\hat{\rho}^{-1}(\bar{x})\cdot\hat{\bar{V}}(\bar{x})\hat{\rho}(\bar{x})+\frac{m}{\hbar^2}\hat{\bar{V}}(\bar{x})\left[\hat{\bar{V}}(\bar{x}),\hat{\rho}(\bar{x})\right]$$

$$+\frac{m}{\hbar^2}\hat{\bar{V}}(\bar{x})\cdot\hat{\rho}(\bar{x})\hat{\bar{V}}(\bar{x})+\frac{m}{\hbar^2}\hat{\bar{V}}(\bar{x})\cdot\left[\hat{\bar{V}}(\bar{x}),\hat{\rho}(\bar{x})\right]\Bigg]d^3x. \tag{M2}$$

Since the density-velocity commutator is the gradient of a delta function, it will commute with the velocity operator. Using this fact and further rearranging terms gives

$$\hat{T}'=\frac{\hbar^2}{8m}\int\Bigg\{\frac{4}{\hbar^2}\hat{\bar{V}}(\bar{x})\cdot\hat{\rho}(\bar{x})\hat{\bar{V}}(\bar{x})+\frac{1}{m}\left(\bar{\nabla}\hat{\rho}(\bar{x})\right)\cdot\hat{\rho}^{-1}(\bar{x})\left(\bar{\nabla}\hat{\rho}(\bar{x})\right)$$

$$+\frac{2i}{\hbar}\left[\bar{\nabla}\hat{\rho}(\bar{x});\hat{\bar{V}}(\bar{x})\right]+\frac{2i}{\hbar}\left(\bar{\nabla}\hat{\rho}(\bar{x})\right)\cdot\hat{\rho}^{-1}(\bar{x})\left[\hat{\bar{V}}(\bar{x}),\hat{\rho}(\bar{x})\right]$$

$$+\frac{m}{\hbar^2}\left[\hat{\rho}(\bar{x}),\hat{\bar{V}}(\bar{x})\right]\hat{\rho}^{-1}(\bar{x})\cdot\left[\hat{\bar{V}}(\bar{x}),\hat{\rho}(\bar{x})\right]\Bigg\}d^3x. \tag{M3}$$

The third term in Eq. (M3) is an infinite constant which may be neglected without altering the equations of motion. The result of evaluating the density-velocity commutator of Eq. (4.3) is

$$\left[\hat{\rho}(\bar{y}),\hat{\bar{V}}(\bar{x})\right]=-i\hbar\bar{\nabla}_x\,\delta(\bar{x}-\bar{y})\Big|_{\bar{x}=\bar{y}}=0\ , \tag{M4}$$

since the derivative of the delta function is an odd function. This result may be substituted into Eq. (M3) to obtain

$$\hat{T}' = \int \left[\hat{\bar{V}}(\bar{x}) \cdot \frac{\rho(\bar{x})}{2} \hat{\bar{V}}(\bar{x}) + \frac{\hbar^2}{8m^2} \hat{\rho}^{-1}(\bar{x}) \left(\bar{\nabla}\hat{\rho}(\bar{x}) \right) \cdot \left(\bar{\nabla}\hat{\rho}(\bar{x}) \right) \right] d^3x , \qquad \text{(M5)}$$

which is the same as Eq. (4.5b).

APPENDIX N

DERIVATION OF QUANTUM CORRECTION TERM IN VELOCITY OPERATOR EQUATION OF MOTION

Equation (4.9) is derived in this appendix as follows:

$$\frac{i}{\hbar}\left[\hat{H}_Q, \hat{V}_\ell(\bar{x})\right] = \frac{i}{\hbar}\frac{\hbar^2}{8m^2}\int\left[\hat{\rho}^{-1}(\bar{y})\left(\bar{\nabla}\hat{\rho}(\bar{y})\right)\cdot\left(\bar{\nabla}\hat{\rho}(\bar{y})\right), \hat{V}_\ell(\bar{x})\right] d^3y$$

$$= \frac{i\hbar}{8m^2}\int\left\{\hat{\rho}^{-1}(\bar{y})\left(\bar{\nabla}\hat{\rho}(\bar{y})\right)\cdot\bar{\nabla}_y\left[\hat{\rho}(\bar{y}), \hat{V}_\ell(\bar{x})\right] + \left[\hat{\rho}^{-1}(\bar{y})\left(\bar{\nabla}\rho(\bar{y})\right), \hat{V}_\ell(\bar{x})\right]\cdot\left(\bar{\nabla}\hat{\rho}(\bar{y})\right)\right\} d^3y$$

$$= \frac{i\hbar}{8m^2}\int\left\{\hat{\rho}^{-1}(\bar{y})\left(\bar{\nabla}\hat{\rho}(\bar{y})\right)\cdot\bar{\nabla}_y\left(i\hbar\frac{\partial}{\partial x_\ell}\left[\delta(\bar{y}-\bar{x})\right]\right) + \hat{\rho}^{-1}(\bar{y})\,\bar{\nabla}_y\left[\hat{\rho}(\bar{y}), V_\ell(\bar{x})\right]\cdot\bar{\nabla}\hat{\rho}(\bar{y})\right.$$

$$\left. + \left(\bar{\nabla}\hat{\rho}(\bar{y})\right)\cdot\left[\hat{\rho}^{-1}(\bar{y}), \hat{V}_\ell(\bar{x})\right]\left(\bar{\nabla}\hat{\rho}(\bar{y})\right)\right\} d^3y$$

$$= \frac{\hbar^2}{4m^2}\frac{\partial}{\partial x_\ell}\left[\bar{\nabla}\cdot\left(\hat{\rho}^{-1}(\bar{x})\,\bar{\nabla}\hat{\rho}(\bar{x})\right)\right]$$

$$+ \frac{i\hbar}{8m^2}\int\left(\bar{\nabla}\hat{\rho}(\bar{y})\right)\cdot\hat{\rho}^{-1}\left[\hat{V}_\ell(\bar{x}), \hat{\rho}(\bar{y})\right]\hat{\rho}^{-1}(\bar{y})\left(\bar{\nabla}\hat{\rho}(\bar{y})\right) d^3y$$

$$= \frac{\hbar^2}{4m^2} \frac{\partial}{\partial x_\ell} \bar{\nabla} \cdot \left(\hat{\rho}^{-1}(\bar{x}) \bar{\nabla} \hat{\rho}(\bar{x}) \right) + \frac{\hbar^2}{8m^2} \frac{\partial}{\partial x_\ell} \left[\left(\bar{\nabla} \hat{\rho}(\bar{x}) \right) \cdot \hat{\rho}^{-1}(\bar{x}) \hat{\rho}^{-1}(\bar{x}) \bar{\nabla} \hat{\rho}(\bar{x}) \right], \quad \text{(N1)}$$

which is the same as Eq. (4.9).

APPENDIX O

DERIVATION OF THE VELOCITY OPERATOR IN TERMS OF THE FIELD OPERATORS OF SECOND QUANTIZATION

In this appendix the form of the velocity operator given by Eq. (4.11) is shown to be inconsistent with the definition of the velocity operator in terms of density and current operators given by Eq. (2.3), when the density and current operators are written in terms of the field operators in Eqs. (3.1) and (3.4). If the inverse density operator is written as

$$\hat{\rho}^{-1}(\bar{x}) = \frac{1}{m} \hat{\psi}^{-1}(\bar{x}) \left(\hat{\psi}^{+}(\bar{x})\right)^{-1} , \tag{O1}$$

and inserted into Eq. (2.3) along with the form of the current operator given in Eq. (3.4), the result is

$$\begin{aligned}\hat{\bar{V}}(\bar{x}) &= \frac{1}{2}\left\{\frac{1}{m}\hat{\psi}^{-1}(\bar{x})\left(\hat{\psi}^{+}(\bar{x})\right)^{-1}\left[\frac{\hbar}{2i}\hat{\psi}^{+}(\bar{x})\left(\bar{\nabla}\hat{\psi}(\bar{x})\right)-\left(\bar{\nabla}\hat{\psi}^{+}(\bar{x})\right)\hat{\psi}(\bar{x})\right]\right. \\ &\left. + \frac{\hbar}{2i}\hat{\psi}^{+}(\bar{x})\left(\bar{\nabla}\hat{\psi}(\bar{x})\right)-\left(\bar{\nabla}\hat{\psi}^{+}(\bar{x})\hat{\psi}(\bar{x})\right)\frac{1}{m}\hat{\psi}^{-1}(\bar{x})\left(\hat{\psi}^{+}(\bar{x})\right)^{-1}\right\}, \\ &= \frac{\hbar}{4mi}\left\{\hat{\psi}^{-1}(\bar{x})\left(\left(\bar{\nabla}\hat{\psi}(\bar{x})\right)-\left(\bar{\nabla}\hat{\psi}^{+}(\bar{x})\right)\left(\hat{\psi}^{+}(\bar{x})\right)\right)^{-1}-\hat{\psi}^{-1}(\bar{x})\left(\hat{\psi}^{+}(\bar{x})\right)^{-1}\left(\bar{\nabla}\hat{\psi}^{+}(\bar{x})\right)\hat{\psi}(\bar{x})\right. \\ &\left. + \hat{\psi}^{+}(\bar{x})\left(\bar{\nabla}\hat{\psi}(\bar{x})\right)\hat{\psi}^{-1}(\bar{x})\left(\hat{\psi}^{-1}(\bar{x})\right)^{-1}\right\} .\end{aligned} \tag{O2}$$

In the succeeding manipulations of the second two terms in Eq. (02), it is assumed that the annihilation and creation operators commute with their inverses as follows:

$$\left[\hat{\Psi}(\bar{x}), \hat{\Psi}^{-1}(\bar{x})\right] = \left[\hat{\Psi}^{+}(\bar{x}), \left(\hat{\Psi}^{+}(\bar{y})\right)^{-1}\right] = 0, \tag{03a}$$

and

$$\bar{\nabla}_x \left[\hat{\Psi}(\bar{x}), \hat{\Psi}^{-1}(\bar{y})\right] = \left[\bar{\nabla}_x \hat{\Psi}(\bar{x}), \hat{\Psi}^{-1}(\bar{y})\right] = 0 . \tag{03b}$$

These are merely formal equations since the inverse operators do not exist, as is shown in Section V. If the velocity operator is now written

$$\hat{\bar{V}}(\bar{x}) = \frac{\hbar}{4mi}\left(\hat{\bar{V}}_1(\bar{x}) + \hat{\bar{V}}_2(\bar{x})\right), \tag{04}$$

where

$$\hat{\bar{\mathcal{U}}}_1(\bar{x}) = \hat{\Psi}^{-1}(\bar{x})\left(\bar{\nabla}\hat{\Psi}(\bar{x})\right) - \left(\bar{\nabla}\hat{\Psi}^{+}(\bar{x})\right)\left(\hat{\Psi}^{+}(\bar{x})\right)^{-1}, \tag{05}$$

and

$$\hat{\bar{V}}_2(\bar{x}) \equiv \hat{\Psi}^{+}(\bar{x})\left(\bar{\nabla}\hat{\Psi}(\bar{x})\right)\hat{\Psi}^{-1}(\bar{x})\left(\hat{\Psi}^{+}(\bar{x})\right)^{-1} - \hat{\Psi}^{-1}(\bar{x})\left(\hat{\Psi}^{+}(\bar{x})\right)^{-1}\left(\bar{\nabla}\hat{\Psi}^{+}(\bar{x})\right)\hat{\Psi}(\bar{x}), \tag{06}$$

then the terms included in $\hat{\bar{V}}_2(\bar{x})$ may be manipulated separately. Commuting terms in $\hat{\bar{V}}_2(\bar{x})$ gives

$$\hat{\bar{V}}_2(\bar{x}) = \hat{\Psi}^{+}(\bar{x})\hat{\Psi}^{-1}(\bar{x})\left(\bar{\nabla}\hat{\Psi}(\bar{x})\right)\left(\hat{\Psi}^{+}(\bar{x})\right)^{-1} - \hat{\Psi}^{-1}(\bar{x})\left(\bar{\nabla}\hat{\Psi}^{+}(\bar{x})\right)\left(\hat{\Psi}^{+}(\bar{x})\right)^{-1}\hat{\Psi}(\bar{x}),$$

$$= \hat{\Psi}^{-1}(\bar{x})\left(\bar{\nabla}\hat{\Psi}(\bar{x})\right) + \left[\hat{\Psi}^{+}(\bar{x}), \hat{\Psi}^{-1}(\bar{x})\left(\bar{\nabla}\hat{\Psi}(\bar{x})\right)\right]\left(\hat{\Psi}^{+}(\bar{x})\right)^{-1}$$

$$-\left(\bar{\nabla}\hat{\Psi}^{+}(\bar{x})\right)\hat{\Psi}^{+}(\bar{x}) - \hat{\Psi}^{-1}(\bar{x})\left[\left(\bar{\nabla}\hat{\Psi}^{+}(\bar{x})\right)\left(\hat{\Psi}^{+}(\bar{x})\right)^{-1}, \hat{\Psi}(\bar{x})\right]. \tag{07}$$

thus

$$\hat{\bar{V}}(\bar{x}) = \frac{\hbar}{4mi}\left(\hat{\bar{V}}_1(\bar{x}) + \hat{\bar{V}}_2(\bar{x})\right) ,$$

$$= \frac{\hbar}{2mi}\left[\hat{\psi}^{-1}(\bar{x})\left(\bar{\nabla}\hat{\psi}(\bar{x})\right) - \left(\bar{\nabla}\hat{\psi}^{+}(\bar{x})\right)\left(\hat{\psi}^{+}(\bar{x})\right)^{-1}\right] + \frac{\hbar}{4mi}\ \hat{\bar{R}}(\bar{x}) , \tag{08}$$

where

$$\hat{\bar{R}}(\bar{x}) \equiv \left[\hat{\psi}^{+}(\bar{x}), \hat{\psi}(\bar{x})\left(\bar{\nabla}\hat{\psi}(\bar{x})\right)\right]\left(\hat{\psi}^{+}(\bar{x})\right)^{-1}$$

$$- \hat{\psi}^{-1}(\bar{x})\left[\left(\bar{\nabla}\hat{\psi}^{+}(\bar{x})\right)\left(\hat{\psi}^{+}(\bar{x})\right)^{-1}, \hat{\psi}(\bar{x})\right] . \tag{09}$$

If $\hat{\bar{R}}(\bar{x})$ is zero, then Eq. (08) is equivalent to Eq. (4.11). This is not the case, however, as is shown below.

Expanding the commutators in $\hat{\bar{R}}(\bar{x})$ gives

$$\hat{\bar{R}}(\bar{x}) = \hat{\psi}^{-1}(\bar{x})\left[\hat{\psi}^{+}(\bar{x}), \left(\bar{\nabla}\hat{\psi}(\bar{x})\right)\right]\left(\hat{\psi}^{+}(\bar{x})\right)^{-1} + \left[\hat{\psi}^{+}(\bar{x}), \hat{\psi}^{-1}(\bar{x})\right]\left(\bar{\nabla}\hat{\psi}(\bar{x})\right)\left(\hat{\psi}^{+}(\bar{x})\right)^{-1}$$

$$- \hat{\psi}^{-1}(\bar{x})\left(\bar{\nabla}\hat{\psi}^{+}(\bar{x})\right)\left[\left(\hat{\psi}^{+}(\bar{x})\right)^{-1}, \hat{\psi}(\bar{x})\right] - \hat{\psi}^{-1}(\bar{x})\left[\bar{\nabla}\hat{\psi}^{+}(\bar{x}), \hat{\psi}(\bar{x})\right]\left(\hat{\psi}^{+}(\bar{x})\right)^{-1} . \tag{010}$$

The commutators

$$\left[\hat{\psi}^{+}(x), \bar{\nabla}\hat{\psi}(\bar{x})\right]$$

and

$$\left[\bar{\nabla}\hat{\psi}^{+}(\bar{x}) , \hat{\psi}(\bar{x})\right]$$

may be written as

$$\bar{\nabla}_y \left[\hat{\psi}^+(\bar{x}),\, \hat{\psi}(\bar{y})\right]\Big|_{\bar{x}=\bar{y}} = + \left[\bar{\nabla}_y\, \delta(\bar{x}-\bar{y})\right]_{\bar{x}=\bar{y}} \tag{011}$$

and

$$\bar{\nabla}_y \left[\hat{\psi}^+(\bar{y}),\, \hat{\psi}(\bar{x})\right]\Big|_{\bar{x}=\bar{y}} = - \left[\bar{\nabla}_y\, \delta(\bar{x}-\bar{y})\right]_{\bar{x}=\bar{y}}, \quad \text{respectively,} \tag{012}$$

where the two variables are set equal after the differentiation has been done. Thus

$$\left[\hat{\psi}^+(\bar{x}),\, \bar{\nabla}\hat{\psi}(\bar{x})\right] = - \left[\bar{\nabla}\hat{\psi}^+(\bar{x}),\, \hat{\psi}(\bar{x})\right], \tag{013}$$

and

$$\hat{\bar{R}}(\bar{x}) = \left[\hat{\psi}^+(\bar{x}),\, \hat{\psi}^{-1}(\bar{x})\right]\left(\bar{\nabla}\hat{\psi}(\bar{x})\right)\left(\hat{\psi}^+(\bar{x})\right)^{-1} - \hat{\psi}^{-1}(\bar{x})\left(\bar{\nabla}\hat{\psi}^+(\bar{x})\right)\left[\left(\hat{\psi}^+(\bar{x})\right)^{-1},\, \hat{\psi}(\bar{x})\right]$$

$$- 2\hat{\psi}^{-1}(\bar{x})\left(\hat{\psi}^+(\bar{x})\right)^{-1}\left[\bar{\nabla}_x\, \delta(\bar{x}-\bar{y})\right]_{\bar{x}=\bar{y}} \quad . \tag{014}$$

If it is now noted that

$$\left[\hat{\psi}^+(\bar{y}),\, \hat{\psi}^{-1}(\bar{x})\right] = \hat{\psi}^{-1}(\bar{x})\left[\hat{\psi}(\bar{x}),\, \hat{\psi}^+(\bar{y})\right]\hat{\psi}^{-1}(\bar{x}) = \delta(\bar{x}-\bar{y})\,\hat{\psi}^{-1}(\bar{x})\,\hat{\psi}^{-1}(\bar{x}), \tag{015}$$

and

$$\left[\left(\hat{\psi}^+(\bar{x})\right)^{-1},\, \hat{\psi}(\bar{y})\right] = \left(\hat{\psi}^+(\bar{x})\right)^{-1}\left[\hat{\psi}(\bar{y}),\, \hat{\psi}^+(\bar{x})\right]\left(\hat{\psi}^+(\bar{x})\right)^{-1} = \delta(\bar{x}-\bar{y})\left(\hat{\psi}^+(\bar{x})\right)^{-1}\left(\hat{\psi}^+(\bar{x})\right)^{-1}, \tag{016}$$

then $\hat{\bar{R}}(\bar{x})$ may be written as

$$\hat{\bar{R}}(\bar{x}) = \delta(\bar{x}-\bar{y})\,\hat{\psi}^{-1}(\bar{x})\,\hat{\psi}^{-1}(\bar{x})\left(\bar{\nabla}\hat{\psi}(\bar{x})\right)\left(\hat{\psi}^+(\bar{x})\right)^{-1}\Big|_{\bar{x}=\bar{y}}$$

$$- 2\hat{\psi}^{-1}(\bar{x})\left(\hat{\psi}^+(\bar{x})\right)^{-1}\left[\bar{\nabla}_x\, \delta(\bar{x}-\bar{y})\right]_{\bar{x}=\bar{y}}$$

$$-\hat{\Psi}^{-1}(\bar{x})\left(\bar{\nabla}\hat{\Psi}^{+}(\bar{x})\right)\left(\hat{\Psi}^{+}(\bar{x})\right)^{-1}\left(\hat{\Psi}^{+}(\bar{x})\right)^{-1} \; \delta(\bar{x}-\bar{y})\Big|_{\bar{x}=\bar{y}} \quad , \tag{017}$$

where the delta functions are evaluated at the point $\bar{x}=\bar{y}$. $\hat{\hat{R}}(\bar{x})$ may thus be written as

$$\hat{\hat{R}}(\bar{x})=\hat{\Psi}^{-1}(\bar{x})\,\hat{\hat{V}}_1(\bar{x})\left(\hat{\Psi}^{+}(\bar{x})\right)^{-1}\delta(\bar{x}-\bar{y})\Big|_{\bar{x}=\bar{y}} - 2\hat{\Psi}^{-1}(\bar{x})\left[\bar{\nabla}_x\,\delta(\bar{x}-\bar{y})\right]\left(\hat{\Psi}^{+}(\bar{x})\right)^{-1}\Big|_{\bar{x}=\bar{y}} , \tag{018}$$

which is not zero unless $\hat{\hat{V}}_1(\bar{x})$ is zero, in which case the velocity operator would be zero and any theory using the velocity operator would have little value. It has thus been shown that the form of the velocity operator given in Eq. (4.11) is inconsistent with the definition of the velocity operator in terms of the density and current operators.

APPENDIX P

CALCULATION OF THE RESOLUTION OF AN ARBITRARY OPERATOR IN THE COHERENT STATES REPRESENTATION

In this appendix the identity, Eq. (6.10), is derived by applying the unit operator given by Eq. (6.8) twice to the operator $\hat{\Theta}$. The result is

$$\hat{\Theta} = \hat{I}\hat{\Theta}\hat{I} = \int |\alpha\rangle\langle\alpha| \,\Theta\, |\beta\rangle\langle\beta| \, D(\alpha)\, D(\beta)$$

$$= \int |\alpha\rangle\langle\beta| \,\Theta(\alpha^*, \beta) \exp\left(-\frac{1}{2}\|\alpha\|^2 - \frac{1}{2}\|\beta\|^2\right) D(\alpha)\, D(\beta), \qquad \text{(P1)}$$

where $\Theta(\alpha^*, \beta)$ is defined as in Eq. (6.11). This equation is the same as Eq. (6.10).

APPENDIX Q

DERIVATION OF EQ. (6.14)

If the operator $\hat{\Theta}$ of Eq. (6.13) is inserted into Eq. (6.11), which defines $\Theta(\alpha^*,\beta)$, the result is

$$\Theta(\alpha^*,\beta) = \langle\alpha|\hat{\Theta}_1\hat{\Theta}_2|\beta\rangle \exp\left(\frac{1}{2}\|\alpha\|^2 + \frac{1}{2}\|\beta\|^2\right). \qquad (Q1)$$

Applying the unit operator in Eq. (6.7) to Eq. (Q1) gives

$$\Theta(\alpha^*,\beta) = \langle\alpha|\hat{\Theta}_1\hat{I}\Theta_2|\beta\rangle \exp\left(\frac{1}{2}\|\alpha\|^2 + \frac{1}{2}\|\beta\|^2\right)$$

$$= \int \langle\alpha|\hat{\Theta}_1|\gamma\rangle\langle\gamma|\hat{\Theta}_2|\beta\rangle \exp\left(\frac{1}{2}\|\alpha\|^2 + \frac{1}{2}\|\beta\|^2\right) D(\gamma). \qquad (Q2)$$

Equation (6.11) may be written as

$$\langle\alpha|\hat{\Theta}|\beta\rangle = \Theta(\alpha^*,\beta) \exp\left(-\frac{1}{2}\|\alpha\|^2 - \frac{1}{2}\|\beta\|^2\right). \qquad (Q3)$$

Using Eq. (Q4) to substitute for the matrix elements in Eq. (Q3) gives

$$\Theta(\alpha^*,\beta)$$

$$= \int \Theta_1(\alpha^*,\beta) \exp\left(-\frac{1}{2}\|\alpha\|^2 - \frac{1}{2}\|\gamma\|^2\right) \Theta_2(\gamma^*,\beta) \times$$

$$\exp\left(-\frac{1}{2}\|\gamma\|^2 - \frac{1}{2}\|\beta\|^2\right) \exp\left(\frac{1}{2}\|\alpha\|^2 + \frac{1}{2}\|\beta\|^2\right) D(\gamma)$$

$$= \int \Theta_1(\alpha^*, \beta)\, \Theta_2(\gamma^*, \beta) \exp\left(-\|\gamma\|^2\right) D(\gamma), \tag{Q4}$$

which is the same as Eq. (6.14).

APPENDIX R

DERIVATION OF EQ. (6.18)

In order to derive Eq. (6.18), it is necessary to insert the form of $\hat{\bar{J}}(\bar{x})$ given by Eq. (6.17) into the defining equation for $\bar{\mathcal{J}}(\alpha^*,\beta;\bar{x})$, Eq. (6.11). The result is

$$\bar{\mathcal{J}}(\alpha^*,\beta;\bar{x}) = \langle\alpha\,|\,\lim_{\epsilon\to 0}\Big[\frac{1}{2}\hat{\rho}(\bar{x})\hat{\bar{\nabla}}(\bar{x}-\bar{\epsilon})$$

$$+\frac{1}{2}\hat{\bar{\nabla}}(\bar{x}+\bar{\epsilon})\hat{\rho}(\bar{x})\Big]|\beta\rangle\exp\left(\frac{1}{2}\|\alpha\|^2+\frac{1}{2}\|\beta\|^2\right)$$

$$=\lim_{\epsilon\to 0}\langle\alpha\,\Big|\Big[\frac{1}{2}\hat{\rho}(\bar{x})\hat{\bar{\nabla}}(\bar{x}+\bar{\epsilon})+\frac{1}{2}\hat{\bar{\nabla}}(\bar{x}+\bar{\epsilon})\hat{\rho}(\bar{x})\Big]\,|\beta\rangle\exp\left(\frac{1}{2}\|\alpha\|^2+\frac{1}{2}\|\beta\|^2\right). \quad \text{(R1)}$$

Let

$$\bar{y}\equiv\bar{x}+\bar{\epsilon}\,.$$

Commuting $\hat{\bar{\nabla}}(\bar{x}+\bar{\epsilon})\hat{\rho}(\bar{x})$ according to Eq. (4.7) gives

$$\hat{\mathcal{J}}(\alpha^*,\beta;\bar{x}) = \lim_{\bar{y}\to\bar{x}}\langle\alpha\,\Big|\Big(\hat{\rho}(\bar{x})\hat{\bar{\nabla}}(\bar{x}+\bar{\epsilon})-\frac{\hbar}{2i}\bar{\nabla}_x\,\delta(\bar{x}-\bar{y})\Big)\Big|\beta\rangle\ \times$$

$$\exp\left(\frac{1}{2}\|\alpha\|^2+\frac{1}{2}\|\beta\|^2\right)$$

$$=\lim_{\bar{y}\to\bar{x}}\langle\alpha\,|\,\hat{\rho}(\bar{x})\hat{\bar{\nabla}}(\bar{x}-\bar{y})\,|\,\beta\rangle\exp\left(\frac{1}{2}\|\alpha\|^2+\frac{1}{2}\|\beta\|^2\right)$$

$$+\lim_{\bar{y}\to\bar{x}}\frac{\hbar}{2i}\,\bar{\nabla}_{\bar{y}}\,\delta(\bar{x}-\bar{y})\langle\alpha\,|\,\beta\rangle\exp\left(\frac{1}{2}\|\alpha\|^2+\frac{1}{2}\|\beta\|^2\right). \quad \text{(R2)}$$

Applying the identity operator given by Eq. (6.7) to Eq. (R2) gives

$$\bar{\mathcal{J}}(\alpha^*, \beta; \bar{x}) = \lim_{\epsilon \to 0} \int \mathcal{R}(\alpha^*, \beta; \bar{x})\, \mathcal{V}(\gamma^*, \beta; \bar{x}) \exp(-\|\gamma\|^2) D(\gamma)$$

$$+ \frac{\hbar}{2i} \exp\left[(\alpha|\beta)\right] \lim_{\epsilon \to 0} \bar{\nabla}_{\bar{\epsilon}}\, \delta(\bar{\epsilon}), \tag{R3}$$

which is the same as Eq. (6.18).

APPENDIX S

DERIVATION OF THE IDENTITY EQ. (6.19)

The identity, Eq. (6.19), is most easily proven by first proving the simpler identity given below:

$$\frac{1}{\pi}\int \exp\left(\beta^*\alpha - |\alpha|^2\right)(\alpha^*)^n \, d^2\alpha = (\beta^*)^n , \tag{S1}$$

where the k subscript has been dropped since it is common to all the variables. If α is written in polar form,

$$\alpha = |\alpha| \, e^{i\theta} , \tag{S2}$$

then

$$d^2\alpha \equiv d\,\mathrm{Re}(\alpha)\, d\,\mathrm{Im}(\alpha) = |\alpha| \, d|\alpha| \, d\theta . \tag{S3}$$

The key to the proof lies in expanding the exponential, and noticing that only one term remains after doing the integral over θ. This is shown below:

$$\frac{1}{\pi}\int \exp\left(\beta^*\alpha - |\alpha|\right)^2 (\alpha^*)^n \, d^2\alpha = \frac{1}{\pi}\int_0^\infty\int_0^{2\pi} \exp\left(\beta^*|\alpha| \, e^{i\theta}\right) e^{-|\alpha|^2} |\alpha|^{n+1} e^{-in\theta} \, d\theta \, d|\alpha| ,$$

$$= \frac{1}{\pi}\int_0^\infty\int_0^{2\pi} e^{-|\alpha|^2}\Big[1 + \beta^*|\alpha| \, e^{i\theta} + \frac{(\beta^*)^2}{2!} |\alpha|^2 e^{i2\theta} + \ldots$$

$$+ \frac{(\beta^*)^n}{n!} e^{in\theta} |\alpha|^n + \ldots\Big] |\alpha|^{n+1} e^{-in\theta} \, d\theta \, d|\alpha| \tag{S4}$$

$$= \frac{2\pi}{\pi}\int_0^\infty |\alpha|^{2n+1} \frac{(\beta^*)}{n!} e^{-|\alpha|^2} d|\alpha| = (\beta^*)^n , \tag{S5}$$

which is the same as Eq. (S1). Integration by parts was used.

If $F(\gamma^*)$ is analytic it can be expanded in a power series

$$F(\gamma^*) = \sum_{m=0}^{\infty} C_m (\gamma^*)^m . \tag{S6}$$

Then by using Eq. (S1) the integral identity

$$\frac{1}{\pi}\int \exp\left(\alpha^*\gamma - |\gamma|^2\right) F(\gamma^*)\, d^2\gamma = \sum_m \frac{C_m}{\pi}\int \exp\left(\alpha^*\gamma - |\gamma|^2\right)(\gamma^*)^n\, d^2\gamma$$

$$= \sum_m C_m (\alpha^*)^m = F(\alpha^*) , \tag{S7}$$

is obtained. Differentiating Eq. (S7) with respect to α^* n times gives

$$\left(\frac{\partial}{\partial\alpha^*}\right)^n \frac{1}{\pi}\int \exp\left(\alpha^*\gamma - |\gamma|^2\right) F(\gamma^*)\, d^2\gamma =$$

$$= \frac{1}{\pi}\int \exp\left(\alpha^*\gamma - |\gamma|^2\right) \gamma^n F(\gamma^*)\, d^2\gamma$$

$$= \left(\frac{\partial}{\partial\alpha^*}\right)^n F(\alpha^*) , \tag{S8}$$

which is Eq. (6.19).

APPENDIX T

VERIFICATION OF THE FORM OF THE FUNCTION $\bar{\mathcal{V}}(\alpha^*,\beta;\bar{x})$ GIVEN IN EQ. (6.20) AS A SOLUTION OF EQ. (6.18)

In this appendix it is shown that the function $\bar{\mathcal{V}}(\alpha^*,\beta;\bar{x})$ given by Eq. (6.20) is a solution to Eq. (6.18) by simply substituting this form of $\bar{\mathcal{V}}(\alpha^*,\beta,\bar{x})$ into Eq. (6.18). When the form of $\mathcal{R}(\alpha^*,\beta;\bar{x})$ given in Eq. (6.12a) and the explicit form of $\mathcal{J}(\alpha^*,\beta;\bar{x})$ given in Eq. (6.12b) are substituted into Eq. (6.18), the result is

$$\bar{\mathcal{J}}(\alpha^*,\beta;\bar{x}) = \frac{\hbar}{2i}\alpha^*(\bar{x})\int \gamma(\bar{x})\exp\left[(\alpha|\gamma)+(\gamma|\beta)-\|\gamma\|^2\right]\left[\frac{\bar{\nabla}\beta(\bar{x})}{\beta(\bar{x})}-\frac{\bar{\nabla}\gamma^*(\bar{x})}{\gamma^*(\bar{x})}\right]D(\gamma)$$

$$+\frac{\hbar}{2i}\exp\left[(\alpha|\beta)\right]\lim_{\epsilon\to 0}\bar{\nabla}_{\bar{\epsilon}}\,\delta(\bar{\epsilon})\,. \tag{T1}$$

This may be broken into three terms, I_1, I_2, and I_3, which are

$$I_1 = \frac{\hbar}{2i}\alpha^*(\bar{x})\frac{\bar{\nabla}\beta(\bar{x})}{\beta(\bar{x})}\int \gamma(\bar{x})\exp\left[(\alpha|\gamma)+(\gamma|\beta)-\|\gamma\|^2\right]D(\gamma)\,, \tag{T2}$$

$$I_2 = -\frac{\hbar}{2i}\alpha^*(\bar{x})\int \gamma(\bar{x})\exp\left[(\alpha|\gamma)+(\gamma|\beta)-\|\gamma\|^2\right]\frac{\bar{\nabla}\gamma^*(\bar{x})}{\gamma^*(\bar{x})}D(\gamma) \tag{T3}$$

and

$$I_3 = \frac{\hbar}{2i} \exp\left[(\alpha|\beta)\right] \lim_{\epsilon \to 0} \bar{\nabla}_{\bar{\epsilon}} \delta(\bar{\epsilon}) \,. \tag{T4}$$

The sum of these three terms must be equal to the form of $\bar{\mathcal{G}}(\alpha^*, \beta; \bar{x})$ given by Eq. (6.12b).

The term I_1 is treated first. Expanding $\gamma(\bar{x})$ and the exponential gives

$$I_1 = \frac{\hbar}{2i} \alpha^*(\bar{x}) \frac{\bar{\nabla}\beta(\bar{x})}{\beta(\bar{x})} \int \left(\sum_{\bar{j}} \gamma_j e^{i\bar{j}\cdot\bar{x}}\right)\left(\prod_{\ell} e^{\alpha_\ell^* \gamma_\ell}\right)\left(\prod_m e^{\gamma_m \beta_m}\right)\left(\prod_n e^{-|\gamma_n|^2}\right) \prod_K \frac{d^2\gamma_K}{\pi}$$

$$= \frac{\hbar}{2i} \alpha^*(\bar{x}) \frac{\bar{\nabla}\beta(\bar{x})}{\beta(\bar{x})} \sum_{\bar{j}} e^{i\bar{j}\cdot\bar{x}} \int \gamma_j \prod_K \left[\exp\left(\alpha_K^* \gamma_K + \gamma_K \beta_K - |\gamma|^2\right) \frac{d^2\gamma_K}{\pi}\right] . \tag{T5}$$

If the identity in Eq. (6.19) is applied to Eq. (T5) with

$$F(\gamma_j^*) = e^{\gamma_j^* \beta_j} \,, \quad n = 1 \,,$$

the result is

$$I_1 = \frac{\hbar}{2i} \alpha^*(\bar{x}) \frac{\bar{\nabla}\beta(\bar{x})}{\beta(\bar{x})} \sum_{\bar{j}} \left\{ e^{i\bar{j}\cdot\bar{x}} \frac{\partial e^{\alpha_j^* \beta_j}}{\partial \alpha_j^*} \int \prod_{K \neq j} \left[\exp\left(\alpha_K^* \gamma_K + \gamma_K^* \beta_K - |\gamma|^2\right) \prod_K \frac{d^2\gamma_K}{\pi}\right]\right\} . \tag{T6}$$

Applying the same identity again, with

$$F(\gamma_K^*) = e^{\gamma_K^* \beta_K} \,, \quad n = 0 \,,$$

the result is

$$I_1 = \frac{\hbar}{2i} \alpha^*(\bar{x}) \frac{\bar{\nabla}\beta(\bar{x})}{\beta(\bar{x})} \sum_{\bar{j}} e^{i\bar{j}\cdot\bar{x}} \left[\frac{\partial e^{\alpha^*_j \beta_j}}{\partial \alpha^*_j} \prod_{k\neq j} e^{\alpha^*_k \beta_k} \right]$$

$$= \frac{\hbar}{2i} \alpha^*(\bar{x}) \frac{\bar{\nabla}\beta(\bar{x})}{\beta(\bar{x})} \left(\sum_{\bar{j}} \beta_j e^{i\bar{j}\cdot\bar{x}} \right) \left(\prod_k e^{\alpha^*_k \beta_k} \right)$$

$$= \frac{\hbar}{2i} \alpha^*(\bar{x}) \frac{\bar{\nabla}\beta(\bar{x})}{\beta(\bar{x})} \beta(\bar{x}) e^{(\alpha|\beta)} = \frac{\hbar}{2i} \alpha^*(\bar{x}) \bar{\nabla}\beta(\bar{x}) e^{(\alpha|\beta)} . \tag{T7}$$

Thus the first term on the left side of Eq. (6.12b) has been obtained. The second term is obtained in the same way as follows:

$$I_2 = -\frac{1}{2} \frac{\hbar}{i} \alpha^*(\bar{x}) \int \gamma(\bar{x}) \exp\left[(\alpha|\gamma) + (\gamma|\beta) - \|\gamma\|^2\right] \frac{\bar{\nabla}\gamma^*(\bar{x})}{\gamma^*(\bar{x})} \prod_k \frac{d^2\gamma_k}{\pi}$$

$$= -\frac{\hbar}{2i} \alpha^*(\bar{x}) \int \gamma(\bar{x}) \exp\left[(\alpha|\gamma) + (\gamma\ \beta) - \|\gamma\|^2\right] \frac{\bar{\nabla}\left(\sum_m \gamma^*_m e^{-i\bar{m}\cdot\bar{x}} \right)}{\left(\sum_m \gamma^*_m e^{-i\bar{n}\cdot\bar{x}} \right)} \prod_k \frac{d^2\gamma_k}{\pi}$$

$$= -\frac{1}{2} \frac{\hbar}{i} \alpha^*(\bar{x}) \int \gamma(\bar{x}) \exp\left[(\alpha|\gamma) + (\gamma|\beta) - \|\gamma\|^2\right] \frac{\sum_m \left(-i\bar{m}\gamma_m{}^* e^{-i\bar{m}\cdot\bar{x}} \right)}{\left(\sum_n \gamma^*_n e^{-i\bar{n}\cdot\bar{x}} \right)} \prod_k \frac{d^2\gamma_k}{\pi}$$

$$= -\frac{1}{2} \frac{\hbar}{i} \alpha^*(\bar{x}) \int \left(\sum_{\bar{l}} \gamma_l e^{i\bar{l}\cdot\bar{x}} \right) \frac{\left(\sum_m -i\bar{m}\gamma^*_m e^{-i\bar{m}\cdot\bar{x}} \right)}{\left(\sum_n \gamma^*_n e^{-i\bar{n}\cdot\bar{x}} \right)} \times$$

$$\times \prod_{k} \left\{ \exp\left(\alpha_k^* \gamma_k + \gamma^* \beta_k - |\gamma_k|^2\right) \frac{d^2\gamma_k}{\pi} \right\}$$

$$= -\frac{1}{2}\frac{\hbar}{i}\alpha^*(\bar{x}) \sum_{\ell} e^{i\bar{\ell}\cdot\bar{x}} \int \gamma_\ell \frac{\left(\sum_m -im\gamma_m^* e^{-i\bar{m}\cdot\bar{x}}\right)}{\sum_n \gamma_n^* e^{-i\bar{n}\cdot\bar{x}}}$$

$$\times \prod_{k} \left\{ \exp\left(\alpha_k^* \gamma_k + \gamma_k^* \beta_k - |\gamma_k|^2\right) \frac{d^2\gamma_k}{\pi} \right\} . \qquad \text{(T8)}$$

Applying the identity in Eq. (6.19) to Eq. (T8) gives

$$I_2 = -\frac{1}{2}\frac{\hbar}{i}\alpha^*(\bar{x}) \sum_{\bar{k}} e^{i\bar{k}\cdot\bar{x}} \left\{ \int \frac{\partial}{\partial \alpha_{\bar{k}}^*} \left[\frac{-i\bar{k}\alpha_k^* e^{-i\bar{k}\cdot\bar{x}} e^{\alpha_k^* \beta_k}}{\alpha_k^* e^{-i\bar{k}\cdot\bar{x}} + \sum_{n\neq k} \gamma_n^* e^{-i\bar{n}\cdot\bar{x}}} \right] \times \right.$$

$$\times \prod_{n\neq k} \left[\exp\left(\alpha_n^* \gamma_n + \gamma_n^* \beta_n - \gamma_n{}^2\right) \frac{d^2\gamma_n}{\pi} \right]$$

$$+ \int \left(\sum_{m\neq k} \frac{\partial}{\partial \alpha_k^*} \left[\frac{-im\gamma_m^* e^{-i\bar{m}\cdot\bar{x}} e^{\alpha_k^* \beta_k}}{\alpha_k^* e^{-i\bar{k}\cdot\bar{x}} + \sum_{n\neq k} \gamma_n^* e^{-i\bar{n}\cdot\bar{x}}} \right] \right)$$

$$\left. \times \prod_{m\neq k} \left[\exp\left(\alpha_m^* \gamma_m + \gamma_m \beta_m - \gamma_m{}^2\right) \frac{d^2\gamma_m}{\pi} \right] \right\} . \qquad \text{(T9)}$$

Another application of Eq. (6.19) gives

$$I_2 = -\frac{1}{2}\frac{\hbar}{i}\alpha^*(\bar{x})\sum_k e^{i\bar{k}\cdot\bar{x}}$$

$$\times\left\{\frac{\partial}{\partial\alpha_k^*}\left[\frac{-ik\alpha_k^* e^{-i\bar{k}\cdot\bar{x}}\exp\left(\alpha_k^*\beta_k+\sum_{n\neq k}\alpha_n^*\beta_n\right)}{\alpha_k^* e^{-i\bar{k}\cdot\bar{x}}+\sum_{n\neq k}\alpha_n^* e^{-i\bar{n}\cdot\bar{x}}}\right]\right.$$

$$\left.+\sum_{m\neq k}\left[\frac{\partial}{\partial\alpha_k^*}\left[\frac{-im\alpha_m^* e^{-im\cdot x}\exp\left(\alpha^*_k\beta_k+\sum_{n\neq k}\alpha_n^*\beta_n\right)}{\alpha_k^* e^{-i\bar{k}\cdot\bar{x}}+\sum_{n\neq k}\alpha_n^* e^{-i\bar{n}\cdot\bar{x}}}\right]\right]\right\}. \tag{T10}$$

Carrying out the differentiation leaves

$$I_2 = -\frac{1}{2}\frac{\hbar}{i}\alpha^*(\bar{x})\sum_k e^{i\bar{k}\cdot\bar{x}} \times$$

$$\times\left\{\frac{\left[-ike^{-i\bar{k}\cdot\bar{x}}e^{(\alpha|\beta)}-ik\beta_k\alpha_k^* e^{-i\bar{k}\cdot\bar{x}}e^{(\alpha|\beta)}\right]}{\left(\sum_n\alpha_n^* e^{-i\bar{n}\cdot\bar{x}}\right)^2}\left(\sum_n\alpha_n^* e^{-i\bar{n}\cdot\bar{x}}\right)\right.$$

$$-\frac{e^{-i\bar{k}\cdot\bar{x}}\left(-i\bar{k}\alpha^*_k e^{-i\bar{k}\cdot\bar{x}}e^{(\alpha|\beta)}\right)}{\left(\sum_n\alpha_n^* e^{-i\bar{n}\cdot\bar{x}}\right)^2}$$

$$+\sum_{m\neq k}\left[\frac{-im\alpha_m^* e^{-i\bar{m}\cdot\bar{x}} \quad \beta_k e^{(\alpha|\beta)} \alpha^*(\bar{x})}{\alpha^*(\bar{x})^{\,2}}\right.$$

$$-\frac{(-im)\alpha_m^* e^{-i\bar{m}\cdot\bar{x}} \quad e^{-i\bar{k}\cdot\bar{x}} \quad e^{(\alpha|\beta)}}{\left[\alpha^*(\bar{x})^{\,2}\right]}$$

$$= -\frac{\hbar}{2i}\,\frac{\alpha^*(\bar{x})}{\alpha^*(\bar{x})^2}\left\{\alpha^*(\bar{x})\left[\sum_k\left(-ik\, e^{-ik\cdot\bar{x}} e^{i\bar{k}\cdot\bar{x}} \, e^{(\alpha|\beta)}\right)\right.\right.$$

$$\left. + \sum_k e^{i\bar{k}\cdot\bar{x}}(-ik)\,\beta_k\alpha_k^* \, e^{-i\bar{k}\cdot\bar{x}} \, e^{(\alpha|\beta)}\right]$$

$$+ \alpha^*(\bar{x})\left[\sum_k e^{-i\bar{k}\cdot\bar{x}} \, \beta_k\left(\sum_{m\neq k} -im\,\alpha_m^* \, e^{-i\bar{m}\cdot\bar{x}} \quad e^{(\alpha|\beta)}\right)\right]$$

$$-\sum_{\bar{k}} e^{i\bar{k}\cdot\bar{x}} \quad e^{-i\bar{k}\cdot\bar{x}} \quad (-i\bar{k})\,\alpha_k^* \, e^{-i\bar{k}\cdot\bar{x}} \quad e^{(\alpha|\beta)}$$

$$\left. -\sum_k e^{i\bar{k}\cdot\bar{x}} \, e^{-i\bar{k}\cdot\bar{x}} \sum_{m\neq k} -im\,\alpha_m^* \, e^{-i\bar{m}\cdot\bar{x}} \quad e^{(\alpha|\beta)}\right\}$$

$$= -\frac{1}{2}\,\frac{\hbar}{i}\left\{\sum_k \beta_k \, e^{i\bar{k}\cdot\bar{x}}\,(-ik)\,\alpha_k^* \quad e^{-i\bar{k}\cdot\bar{x}} \quad e^{(\alpha|\beta)}\right.$$

$$+\sum_{k}\beta_k e^{i\bar{k}\cdot\bar{x}}\left(\sum_{m\neq k} -im\alpha_m^* \, e^{-i\bar{m}\cdot\bar{x}} \, e^{(\alpha|\beta)}\right)$$

$$+\sum_{k} -ik\, e^{(\alpha|\beta)} \, \frac{-1}{\alpha^*(\bar{x})}\sum_{k} -ik\alpha_k^* \, e^{-i\bar{k}\cdot\bar{x}} \, e^{(\alpha|\beta)}$$

$$\left.-\frac{1}{\alpha^*(\bar{x})}\sum_{k}\sum_{m\neq k} -im\alpha_m^* \, e^{-i\bar{m}\cdot\bar{x}} \, e^{(\alpha|\beta)}\right\}$$

$$= -\frac{1}{2}\frac{\hbar}{i}\left\{\beta(\bar{x})\bar{\nabla}\alpha^*(\bar{x})\, e^{(\alpha|\beta)} + \sum_{k} -ik\, e^{(\alpha|\beta)}\right.$$

$$\left.-\frac{e^{(\alpha|\beta)}}{\alpha^*(\bar{x})}\sum_{k}\left(\bar{\nabla}\alpha^*(\bar{x}) + ik\alpha_k^* \, e^{-i\bar{k}\cdot\bar{x}}\right)\right\}$$

$$= -\frac{1}{2}\frac{\hbar}{i}\, e^{(\alpha|\beta)}\left\{\beta(\bar{x})\bar{\nabla}\alpha^*(\bar{x}) - \sum_{\bar{k}}(-i\bar{k}) - \frac{1}{\alpha^*(\bar{x})}\left[\bar{\nabla}\alpha^*(\bar{x}) - \bar{\nabla}\alpha^*(\bar{x})\right]\right\}$$

$$= -\frac{1}{2}\frac{\hbar}{i}\, e^{(\alpha|\beta)}\left[\beta(\bar{x})\bar{\nabla}\alpha^*(\bar{x}) - \sum_{\bar{k}}(-i\bar{k})\right]. \tag{T11}$$

Expanding the delta function term in I_3 gives

$$\frac{1}{2}\frac{\hbar}{i}\, e^{(\alpha|\beta)} \lim_{\epsilon\to 0} \bar{\nabla}_\epsilon \delta(\bar{\epsilon}) = \frac{1}{2}\frac{\hbar}{i}\, e^{(\alpha|\beta)} \lim_{\epsilon\to 0} \bar{\nabla}_\epsilon\left(\sum_{\bar{k}} e^{-i\bar{k}\cdot\bar{\epsilon}}\right)$$

$$= \frac{1}{2}\frac{\hbar}{i} e^{(\alpha|\beta)} \lim_{\bar{\epsilon}\to 0} \sum_{\bar{k}} -i\bar{k} e^{-i\bar{k}\cdot\bar{\epsilon}}$$

$$= \frac{1}{2}\frac{\hbar}{i} e^{(\alpha|\beta)} \sum_{\bar{k}} (-i\bar{k}) \; . \tag{T12}$$

where the volume has been set equal to one. I_3 combines with I_2 to give

$$I_2 + \frac{1}{2}\frac{\hbar}{i} e^{(\alpha|\beta)} \lim_{\epsilon\to 0} \bar{\nabla}_\epsilon \delta(\bar{\epsilon}) = -\frac{1}{2}\frac{\hbar}{i} e^{(\alpha|\beta)} \beta(\bar{x}) \bar{\nabla}\alpha^*(\bar{x}) . \tag{T13}$$

The final result is

$$I_1 + I_2 + \frac{1}{2}\frac{\hbar}{i} e^{(\alpha|\beta)} \lim_{\epsilon\to 0} \bar{\nabla}_\epsilon \delta(\bar{\epsilon})$$

$$= \frac{1}{2}\frac{\hbar}{i} e^{(\alpha|\beta)} \left[\alpha^*(\bar{x})\bar{\nabla}\beta(\bar{x}) - \beta(\bar{x})\bar{\nabla}\alpha^*(\bar{x})\right] . \tag{T14}$$

$\gamma(\alpha^*, \beta; \bar{x})$ as given by Eq. (6.20) is therefore a solution of Eq. (6.18).

APPENDIX U

DERIVATION OF EQ. (6.22)

Equation (6.22) is derived in this appendix by simply substituting Eq. (6.21) into Eq. (6.20). The result is given below:

$$\gamma(\alpha^*, \beta, \bar{x}) = \frac{1}{2}\frac{\hbar}{im} e^{(\alpha|\beta)} \left\{ \frac{\bar{\nabla}\left(f_\beta(\bar{x}) e^{i\phi_\beta(\bar{x})}\right)}{f_\beta(\bar{x}) e^{i\phi_\beta(\bar{x})}} - \frac{\bar{\nabla}\left(f_\alpha(\bar{x}) e^{-i\phi_\alpha(\bar{x})}\right)}{f_\alpha(\bar{x}) e^{-i\phi_\alpha(\bar{x})}} \right\}$$

$$= \frac{1}{2}\frac{\hbar}{im} e^{(\alpha|\beta)} \left\{ \frac{e^{i\phi_\beta(\bar{x})}\bar{\nabla} f_\beta(\bar{x}) + i f_\beta(\bar{x}) e^{i\phi_\beta(\bar{x})}\bar{\nabla}\phi_\beta(\bar{x})}{f_\beta(\bar{x}) e^{i\phi_\beta(\bar{x})}} \right.$$

$$\left. - \frac{\left(\bar{\nabla} f_\alpha(\bar{x}) e^{-i\phi_\alpha(\bar{x})} - i f_\alpha(\bar{x}) e^{-i\phi_\alpha(\bar{x})}\bar{\nabla}\phi_\alpha(\bar{x})\right)}{f_\alpha(\bar{x}) e^{-i\phi_\alpha(\bar{x})}} \right\}$$

$$= \frac{1}{2}\frac{\hbar}{im} e^{(\alpha|\beta)} \left\{ \frac{\bar{\nabla} f_\beta(\bar{x})}{f_\beta(\bar{x})} + i\bar{\nabla}\phi_\beta(\bar{x}) - \frac{\bar{\nabla} f_\alpha(\bar{x})}{f_\alpha(\bar{x})} + i\bar{\nabla}\phi_\alpha(\bar{x}) \right\},$$

$$= \frac{1}{2}\frac{\hbar}{im} e^{(\alpha|\beta)} \left\{ \bar{\nabla}\phi_\alpha(\bar{x}) + \bar{\nabla}\phi_\beta(\bar{x}) + i\, \frac{\bar{\nabla} f_\alpha(\bar{x})}{f_\alpha(\bar{x})} - \frac{\bar{\nabla} f_\beta(\bar{x})}{f_\beta(\bar{x})} \right\}$$

$$= \frac{1}{2}\frac{\hbar}{m} e^{(\alpha|\beta)} \left\{ \bar{\nabla}\, \phi_\alpha(\bar{x}) + \phi_\beta(\bar{x}) + i \left(\frac{f_\beta \bar{\nabla} f_\alpha - f_\alpha \bar{\nabla} f_\beta}{f_\alpha f_\beta} \right) \right\}$$

$$= \frac{1}{2}\frac{\hbar}{m} e^{(\alpha|\beta)} \left\{ \bar{\nabla} \left[\phi_\alpha(\bar{x}) + \phi_\beta(\bar{x}) \right] + i \bar{\nabla} \ln \left[f_\alpha(\bar{x}) / f_\beta(\bar{x}) \right] \right\}, \tag{U1}$$

which is the same as Eq. (6.22).

APPENDIX V

DERIVATION OF THE VELOCITY OPERATOR FROM THE FUNCTION $\bar{\gamma}(\alpha^*, \beta; \bar{x})$

In this appendix the form of the velocity operator given in Eq. (6.24) is obtained by substituting the form of $\bar{\gamma}(\alpha^*, \beta; \bar{x})$ given by Eq. (6.20) into Eq. (6.10). The results are shown below:

$$\hat{\bar{V}}(\bar{x}) = \frac{1}{2}\frac{\hbar}{im}\int e^{(\alpha|\beta)} e^{-\frac{1}{2}\|\alpha\|^2 - \frac{1}{2}\|\beta\|^2} \left[\frac{\bar{\nabla}\beta(\bar{x})}{\beta(\bar{x})} - \frac{\bar{\nabla}\alpha^*(\bar{x})}{\alpha^*(\bar{x})}\right] D(\alpha)D(\beta)|\alpha\rangle\langle\beta|$$

$$= \frac{1}{2}\frac{\hbar}{im}\int \langle\alpha|\beta\rangle \left[\bar{\nabla}\ln\beta(\bar{x}) - \bar{\nabla}\ln\alpha^*(\bar{x})\right] |\alpha\rangle\langle\beta| \; D(\alpha)D(\beta)$$

$$= \frac{1}{2}\frac{\hbar}{im}\int \left[\bar{\nabla}\ln\beta(\bar{x}) - \bar{\nabla}\ln\alpha^*(\bar{x})\right] |\alpha\rangle\langle\alpha|\beta\rangle\langle\beta| \; D(\alpha)D(\beta)$$

$$= \frac{1}{2}\frac{\hbar}{im}\int \bar{\nabla}\ln\beta(\bar{x})|\alpha\rangle\langle\alpha|\beta\rangle\langle\beta| D(\alpha)D(\beta)$$

$$- \frac{1}{2}\frac{\hbar}{im}\int \bar{\nabla}\ln\alpha^*(\bar{x})\,|\alpha\rangle\langle\alpha|\beta\rangle\langle\beta| D(\alpha)D(\beta)$$

$$= \frac{1}{2}\frac{\hbar}{im}\int \bar{\nabla} \ln \beta(\bar{x})\, |\beta\rangle\langle\beta|\, D(\beta) - \frac{1}{2}\frac{\hbar}{im}\int \bar{\nabla} \ln \alpha^*(\bar{x})\, |\alpha\rangle\langle\alpha|\, D(\alpha)$$

$$= \frac{1}{2}\frac{\hbar}{im}\int \bar{\nabla} \ln \left(\alpha(\bar{x})/\alpha^*(\bar{x})\right) |\alpha\rangle\langle\alpha|\, D(\alpha). \qquad \text{(V1)}$$

Substituting for $\alpha(\bar{x})$ from Eq. (6.21) into Eq. (V1) gives

$$\hat{V}(\bar{x}) = \frac{1}{2}\frac{\hbar}{im}\int \bar{\nabla} \ln \left[\frac{f_\alpha(\bar{x})\, e^{i\phi_\alpha(\bar{x})}}{f_\alpha(\bar{x})\, e^{-i\phi_\alpha(\bar{x})}}\right] |\alpha\rangle\langle\alpha|\, D(\alpha)$$

$$= \frac{1}{2}\frac{\hbar}{im}\int \bar{\nabla} \ln \left[e^{+2i\phi_\alpha(\bar{x})}\right] |\alpha\rangle\langle\alpha|\, D(\alpha)$$

$$= \frac{1}{2}\frac{\hbar}{im}\int 2i\bar{\nabla}\phi_\alpha(\bar{x})\, |\alpha\rangle\langle\alpha|\, D(\alpha) = \frac{\hbar}{m}\int \bar{\nabla}\phi_\alpha(\bar{x})\, |\alpha\rangle\langle\alpha|\, D(\alpha), \qquad \text{(V2)}$$

which is the same as Eq. (6.2).

APPENDIX W

CALCULATION OF THE HEISENBERG EQUATION OF MOTION FOR THE FIELD ANNIHILATION OPERATOR

Equation (7.14) is calculated by inserting the Hamiltonian of Eq. (7.12) into the Heisenberg equation of motion. The result is

$$\left[\hat{\psi}(\bar{x},t), \hat{H}\right] = \int \left[\hat{\psi}(\bar{x},t), \hat{\psi}^{+}(\bar{y},t) T(\bar{y}) \hat{\psi}(\bar{y},t)\right] d^3y$$

$$+ \frac{1}{2}\int \left[\hat{\psi}(\bar{x},t), \hat{\psi}^{+}(\bar{z},t) \hat{\psi}^{+}(\bar{y},t) V(\bar{z},\bar{y}) \hat{\psi}(\bar{y},t) \hat{\psi}(\bar{z},t)\right] d^3z\, d^3y$$

$$= \int\left\{ \hat{\psi}^{+}(\bar{y},t) \left[\hat{\psi}(\bar{x},t), T(\bar{y}) \hat{\psi}(\bar{y},t)\right] + \left[\hat{\psi}(\bar{x},t), \hat{\psi}^{+}(\bar{y},t) T(\bar{y})\right] \hat{\psi}(\bar{y},t) \right\} d^3y$$

$$= \frac{1}{2}\iint\left\{ \hat{\psi}^{+}(\bar{z},t) \left[\hat{\psi}(\bar{x},t), \hat{\psi}^{+}(\bar{y},t) V(\bar{z},\bar{y}) \hat{\psi}(\bar{y},t) \hat{\psi}(\bar{z},t)\right]\right.$$

$$\left. + \left[\hat{\psi}(\bar{x},t), \hat{\psi}^{+}(\bar{z},t)\right] \hat{\psi}^{+}(\bar{y},t) V(\bar{z},\bar{y}) \hat{\psi}(\bar{y},t) \hat{\psi}(\bar{x},t)\right\} d^3z\, d^3y$$

$$= \int \delta(\bar{x}-y) T(\bar{y}) \hat{\psi}(\bar{y},t)\, d^3y + \frac{1}{2}\int \delta(\bar{x}-\bar{z}) \hat{\psi}^{+}(\bar{y},t) V(\bar{z},\bar{y}) \hat{\psi}(\bar{y},t) \hat{\psi}(\bar{z},t)\, d^3z\, d^3y$$

$$= \frac{1}{2}\iint\left\{ \hat{\psi}^{+}(\bar{z},t) \hat{\psi}^{+}(\bar{y},t) \left[\hat{\psi}(\bar{x},t), V(\bar{z},\bar{y}) \hat{\psi}(\bar{y},t) \hat{\psi}(\bar{z},t)\right]\right.$$

$$\left. + \hat{\psi}^{+}(\bar{z},t) \left[\hat{\psi}(\bar{x},t), \hat{\psi}^{+}(\bar{y},t)\right] V(\bar{z},\bar{y}) \hat{\psi}(\bar{y},t) \hat{\psi}(\bar{z},t)\right\} d^3z\, d^3y$$

$$= T(\bar{x}) \hat{\psi}(\bar{x},t) + \frac{1}{2}\int \hat{\psi}^{+}(\bar{y},t) V(\bar{x},y) \hat{\psi}(\bar{y},t) \hat{\psi}(\bar{x},t)\, d^3y$$

$$+\frac{1}{2}\int \hat{\psi}^{+}(\bar{z},t)\,\delta(\bar{x}-\bar{y})\,V(\bar{z},\bar{y})\,\hat{\psi}(\bar{y},t)\,\hat{\psi}(\bar{z},t)\,d^3z\,d^3y$$

$$= T(\bar{x})\,\hat{\psi}(\bar{x},t)+\frac{1}{2}\int \hat{\psi}^{+}(\bar{y},t)\,V(\bar{x},\bar{y})\,\hat{\psi}(\bar{y},t)\,\hat{\psi}(\bar{x},t)\,d^3y$$

$$+\frac{1}{2}\int \hat{\psi}^{+}(\bar{z},t)\,V(\bar{z},\bar{x})\,\hat{\psi}(\bar{x},t)\,\hat{\psi}(\bar{z},t)\,d^3z\ . \tag{W1}$$

Since $V(\bar{x},\bar{y}) = V(\bar{y},\bar{x})$, the dummy variables of integration in Eq. (W1) may be relabelled to give

$$\left[\hat{\psi}(\bar{x},t),\ \hat{H}\right] = i\hbar\frac{\partial\hat{\psi}}{\partial t}(\bar{x},t) = T(\bar{x})\hat{\psi}(\bar{x},t)+\int\hat{\psi}^{+}(\bar{y},t)V(\bar{x},\bar{y})\hat{\psi}(\bar{y},t)\hat{\psi}(\bar{x},t)d^3y, \tag{W2}$$

which is the same as Eq. (7.14).

APPENDIX X

GROSS-PITAEVSKII EQUATION IN TERMS OF DENSITY AND PHASE

Equation (7.24) is calculated in this Appendix by substituting Eq. (7.19) for the order parameter into the Gross-Pitaevskii equation, Eq. (7.16). Since the order parameter, the density, and the phase are all functions of the position and time, they will be written without their arguments to simplify the notation. That is,

$$\rho(\bar{x},t) \equiv \rho,$$

$$\rho(\bar{x}',t) \equiv \rho'.$$

Making the substitution described above gives

$$\frac{i\hbar}{\sqrt{m}} \frac{\partial}{\partial t}\left(\rho^{1/2} e^{iR}\right) = -\frac{1}{2}\frac{\hbar^2}{m}\nabla^2\left[\left(\frac{\rho}{m}\right)^{1/2} e^{iR}\right] + U\left(\frac{\rho}{m}\right)^{1/2} e^{iR}$$

$$+\frac{1}{m}\int V(\bar{x},\bar{x}')\ \rho' d^3x' \left(\frac{\rho}{m}\right)^{1/2} e^{iR} \quad . \tag{X1}$$

Expanding the derivatives in the above equation results in

$$i\hbar\left[e^{iR}\ \frac{\partial}{\partial t}\left(\frac{\rho}{m}\right)^{1/2} + \left(\frac{\rho}{m}\right)^{1/2} e^{iR}\ \frac{\partial R}{\partial t}\right] = -\frac{1}{2}\frac{\hbar^2}{m}\bar{\nabla}\cdot\left(e^{iR}\ \bar{\nabla}\left(\frac{\rho}{m}\right)^{1/2}\right.$$

$$+\left(\frac{\rho}{m}\right)^{1/2} e^{iR} \; i\bar{\nabla}R\Big) + U\left(\frac{\rho}{m}\right)^{1/2} e^{iR}$$

$$+\left(\frac{\rho}{m}\right)^{1/2} e^{iR}\int V(\bar{x},\bar{x}')\,\rho'\,d^3x' . \tag{X2}$$

On further manipulation, including division by e^{iR} , we obtain

$$i\hbar\,\frac{\partial\left(\frac{\rho}{m}\right)^{1/2}}{\partial t} - \left(\frac{\rho}{m}\right)^{1/2}\hbar\,\frac{\partial R}{\partial t}$$

$$= -\frac{1}{2}\frac{\hbar^2}{m}\Big[\nabla^2\left(\frac{\rho}{m}\right)^{1/2} + i\,(\nabla R)\cdot\nabla\left(\frac{\rho}{m}\right)^{1/2} + i\left(\bar{\nabla}R\right)\cdot\bar{\nabla}\left(\frac{\rho}{m}\right)^{1/2}$$

$$-\left(\frac{\rho}{m}\right)^{1/2}(\nabla R)^2 + i\left(\frac{\rho}{m}\right)^{1/2}\nabla^2 R\Big] + U\left(\frac{\rho}{m}\right)^{1/2}$$

$$+\left(\frac{\rho}{m}\right)^{1/2}\int V(\bar{x},\bar{x}')\,\rho'\,d^3x' , \tag{X3}$$

which is the same as Eq. (7.24).

APPENDIX Y

EQUATION OF CONTINUITY FROM THE GROSS-PITAEVSKII EQUATION

Equation (7.25) is obtained in this appendix by setting the imaginary terms on each side of Eq. (7.24) equal. The result is

$$\hbar \frac{\partial \left(\frac{\rho}{m}\right)^{1/2}}{\partial t} = -\frac{1}{2}\frac{\hbar^2}{m}\left[2(\bar{\nabla}R)\cdot\bar{\nabla}\left(\frac{\rho}{m}\right)^{1/2} + \left(\frac{\rho}{m}\right)^{1/2} \nabla^2 R\right] . \tag{Y1}$$

Noting that

$$\frac{\partial}{\partial t}\left(\frac{\rho}{m}\right)^{1/2} = \frac{1}{2\left(\frac{\rho}{m}\right)^{1/2}} \frac{\partial\left(\frac{\rho}{m}\right)}{\partial t} ,$$

and

$$\bar{\nabla}\left(\frac{\rho}{m}\right)^{1/2} = \frac{1}{2m^{1/2}\rho^{1/2}}\bar{\nabla}\rho ,$$

we see that Eq. (Y1) may be written as

$$\frac{\partial \rho}{\partial t} = -\frac{1}{2}\frac{\hbar}{m}\left[2(\bar{\nabla}R)\cdot(\bar{\nabla}\rho) + 2\nabla^2 R\right]$$

$$= \bar{\nabla}\cdot\left(\rho\frac{\hbar}{m}\bar{\nabla}R\right) = \bar{\nabla}\cdot\bar{j} , \tag{Y2}$$

where Eq. (7.21) was used in the last step. This is the same as Eq. (7.25).

APPENDIX Z

THE BERNOULLI EQUATION FROM THE GROSS-PITAEVSKII EQUATION

Equation (7.26) is derived in this appendix by setting the real terms on each side of Eq. (7.24) equal. The result is

$$-\left(\frac{\rho}{m}\right)^{1/2} \hbar \frac{\partial R}{\partial t} = -\frac{1}{2}\frac{\hbar^2}{m}\left[\nabla^2\left(\frac{\rho}{m}\right)^{1/2} - \left(\frac{\rho}{m}\right)^{1/2} (\nabla R)^2\right] + U\left(\frac{\rho}{m}\right)^{1/2}$$

$$+\left(\frac{\rho}{m}\right)^{1/2} \int V(\bar{x},\bar{x}')\left(\frac{\rho'}{m}\right) d^3x' . \qquad \text{(Z1)}$$

Dividing the above equation by $-\left(\rho/m\right)^{1/2}$ gives

$$\hbar\frac{\partial R}{\partial t} = \frac{1}{2}\frac{\hbar^2}{m}\left[\frac{\nabla^2\left(\frac{\rho}{m}\right)^{1/2}}{\left(\frac{\rho}{m}\right)^{1/2}}\right] - (\bar{\nabla}R)^{1/2} - U - \frac{1}{m}\int V(\bar{x},\bar{x}')\rho' d^3x' . \qquad \text{(Z2)}$$

Identifying $\frac{1}{m}\int V(\bar{x},x')\rho' d^3x$ as the total internal potential V and dividing by m gives the Bernoulli equation

$$\frac{\hbar}{m}\frac{\partial R}{\partial t} = \frac{1}{2}\frac{\hbar^2}{m^2}\left[\frac{\nabla^2\left(\frac{\rho}{m}\right)^{1/2}}{\left(\frac{\rho}{m}\right)^{1/2}} - (\bar{\nabla}R)^2\right] - \frac{U}{m} - \frac{V}{m} . \qquad \text{(Z3)}$$

Taking the gradient of the above equation and using Eq. (7.23) gives

$$\frac{\partial v}{\partial t} + (v \cdot \bar{\nabla}) v - \frac{\hbar^2}{2m} \bar{\nabla} \left[\frac{\nabla^2 (\rho/m)^{1/2}}{(\rho/m)^{1/2}} \right] = -\frac{1}{m} \bar{\nabla} U - \frac{1}{m} \bar{\nabla} V, \tag{24}$$

where

$$\bar{F}_{ext.} = -\bar{\nabla} V$$

and

$$\bar{F}_{int.} = -\bar{\nabla} U$$

which is the same as Eq. (7.16).

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