R-MODULES FOR THE ALEXANDER COHOMOLOGY THEORY

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In Chapter I, the algebraic and topological material necessary to construct the theory is listed. This material is found in most graduate level algebra or topology courses.

The definition of the cohomology R-modules is given in Chapter II. Also, it is verified that this construction does give a cohomology theory. The axioms necessary for this are proved as Theorems 1-7.

In Chapter III, additional theorems extend the theory. One of these is the Mayer-Vietoris Theorem. In this theorem, the Mayer-Vietoris sequence is shown to be exact.

Some applications of the theory are given in Chapter IV. One of the applications shows the existence of floors for each element of $H^p(X,A)$ if X is compact Hausdorff, and if A is a closed subset of X. The modules of the n-cell and the n-sphere are also given.

R-MODULES FOR THE ALEXANDER COHOMOLOGY THEORY

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CHAPTER I

INTRODUCTION

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Algebraic Preliminaries

The following preliminaries concerning R-modules will be used.

Preliminary 1. If A and B are submodules of a R-module G, then A + B is a submodule of G and $A \cap B$ is a submodule of G.

Preliminary 2. If G and H are R-modules, and if $f: G \to H$ and $f(ra + tb) = rf(a) + tf(b)$ for $r,t \in R$, $a,b \in G$, then f is an R-homomorphism of G into H. The image of a submodule of G is a submodule of H. If f is also 1-1 and onto, then f is called an isomorphism of G onto H, and we say G and H are isomorphic (denoted $G \cong H$).

Preliminary 3. The kernel of f (denoted ker f) is $\{x \in G \mid f(x) = 0\}$. The set ker f is a submodule of G.

Preliminary 4. Let G be an R-module, and let H be a submodule of G. Sets of the form a + H where $a \in G$ are called cosets of G mod H. The collection $\{a + H \mid a \in G\}$ is an R-module under the operations "$\oplus$," "$\odot$" defined as follows: for $a,b \in G$, $r \in R$, $(a + H) \oplus (b + H) = (a + b) + H$ and $r \odot (a + H) = ra + H$. This collection is called the factor module of G mod H (denoted G/H).

Define $\eta: G \to G/H$ as follows: $\eta(a) = a + H$ for $a \in G$. The function η is an R-homomorphism of G onto G/H. The function η is called the natural map.

<u>Preliminary 5</u>. <u>Fundamental Homomorphism Theorem</u>. If G and H are R-modules, and $f: G \to H$ is an onto homomorphism, then there is a submodule N of G such that $G/N \cong H$.

<u>Preliminary 6</u>. If A,B,C are R-modules, and $A \overset{f}{\to} B \overset{g}{\to} C$ where f and g are R-homomorphisms, and $h = gf$, then (1) h is an R-homomorphism; (2) if h is an isomorphism onto, then

(i) g is onto,

(ii) f is 1-1,

(iii) $B = f(A) + \ker g$.

<u>Preliminary 7</u>. <u>Induced Homomorphism Theorem</u>.

If G,H,M,N are R-modules, α,β,f are R-homomorphisms, α is onto, and $f(\ker\alpha) \subseteq \ker\beta$, then there is a unique R-homomorphism $f^*: G \to M$ such that $f^*\alpha = \beta f^*$.

$$\begin{array}{ccc} G & \overset{f^*}{\dashrightarrow} & M \\ \alpha \uparrow & & \uparrow \beta \\ H & \overset{f}{\longrightarrow} & N \end{array}$$

<u>Definition 1.1</u>. Let $\{M_i\}_{i=n}^{k}$ be a sequence of R-modules, and $\{h_i\}_{i=n}^{k}$ be a sequence of R-homomorphisms such that (1) k is a positive integer or ∞; (2) for every i, $h_i: M_i \to M_{i+1}$; (3) h_n is a 1-1 R-homomorphism into M_{n+1}; (4) for every i, $h_i(M_i) = \ker h_{i+1}$; and (5) if $k \neq \infty$, then h_{k-1} is onto. Then the sequence (M_i, h_i) is exact.

Notation

The following notation will be used.

(1) Unless confusion arises, the natural map (defined above) will be denoted by η;

(2) The empty set will be denoted by "$\square$";

(3) If $A \subseteq X$, then the inclusion map $i: A \to X$ defined by $i(x) = x$ for all $x \in A$, will be denoted by $i: A \subsetneq X$;

(4) If $A \subseteq X$, then the closure of A will be denoted by $\overline{A}$. The interior of A will be denoted by Int (A);

(5) If $f: X \to Y$, then $\text{Im } f = \{f(x) \in Y \mid x \in X\}$.

Topological Preliminaries

The following topological theorems will be used.

<u>Preliminary 8</u>. Every compact Hausdorff space is fully normal.

<u>Preliminary 9</u>. Let X be a compact Hausdorff space. If β is a descending family of closed sets, U is an open subset of X, and $\bigcap\beta \subseteq U$, then there exists $F \in \beta$ such that $F \subseteq U$.

<u>Preliminary 10</u>. A space X is connected if and only if for every open cover β of X, and $a,b \in X$, there exists a finite subcollection $U_1,\ldots,U_n$ of β such that $a \in U_1$, $b \in U_n$, and $U_i \cap U_j \neq \square$ if and only if $|i - j| \leq 1$.

<u>Preliminary 11</u>. Let X,Y be topological spaces. If $f: X \to Y$, X is compact, Y is Hausdorff, f is continuous, f is 1-1, and f is onto, then f is a homeomorphism.

CHAPTER II

THE COHOMOLOGY AXIOMS

Let X be a topological space, and let M be an R-module. For p a non-negative integer, let $C^p(X) = \{f: X^{p+1} \to M\}$. For $\phi_1, \phi_2 \in C^p(X)$, define addition (denoted "+") as follows: $(\phi_1 + \phi_2)(x_0, \ldots, x_p) = \phi_1(x_0, \ldots, x_p) + \phi_2(x_0, \ldots, x_p)$. For $\phi \in C^p(X)$, $r \in R$, define multiplication (denoted "$\cdot$") as follows: $(r \cdot \phi)(x_0, \ldots, x_p) = r\phi(x_0, \ldots, x_p)$. Define $0 \in C^p(X)$ by $0(x_0, \ldots, x_p) = 0_1$ where 0_1 is the additive identity in M. For $\phi \in C^p(X)$, define $-\phi \in C^p(X)$ by $-\phi(x_0, \ldots, x_p) = -(\phi(x_0, \ldots, x_p))$. Then $(C^p(X), +, \cdot)$ _is an R-module and is called the R-module of cochains of X with dimension p._

Define $\delta_p: C^p(X) \to C^{p+1}(X)$ as follows: $\delta_p(\phi)$ is that function in $C^{p+1}(X)$ such that $\delta_p(\phi)(x_0, \ldots, x_{p+1}) = \sum_{i=0}^{p+1}(-1)^i \phi(x_0, \ldots, \hat{x}_i, \ldots, c_{p+1})$. The hat on x_i means to delete x_i.

Note 2.1. The function δ_p is an R-homomorphism.

Proof: Let $r, t \in R$, ϕ_1, $\phi_2 \in C^p(X)$. Then

$$\delta_p(r\phi_1 + t\phi_2)(x_0, \ldots, x_{p+1}) = \sum_{i=1}^{p+1}(-1)^i(r\phi_1 + t\phi_2)(x_0, \ldots \hat{x}_i, \ldots, x_{p+1})$$

$$= \sum_{i=0}^{p+1}(-1)^i(r\phi_1(x_0, \ldots, \hat{x}_i, \ldots, x_{p+1}) + t\phi_2(x_0, \ldots, \hat{x}_i, \ldots, x_{p+1}))$$

$$= r\sum_{i=0}^{p+1}(-1)^i\phi_1(x_0, \ldots, \hat{x}_i, \ldots, x_{p+1}) + t\sum^{p+1}(-1)^i\phi_2(x_0, \ldots, \hat{x}_i, \ldots, x_{p+1})$$

$= r\delta_p\phi_1(x_0,\ldots,x_{p+1}) + t\delta_p\phi_2(x_0,\ldots,x_{p+1})$. Hence, δ_p is an R-homomorphism.

Note 2.2. The composition $\delta_{p+1}\delta_p = 0$.

Proof: Let $\phi \in C^p(X)$, and let $(x_0,\ldots,x_{p+2}) \in X^{p+3}$.

Then $\delta_{p+1}\delta_p\phi(x_0,\ldots,x_{p+2}) = \sum_{i=1}^{p+2}(-1)^i\delta_p\phi(x_0,\ldots,\hat{x}_i,\ldots,x_{p+2})$

$= \sum_{i=1}^{p+2}(-1)^i\sum_{j=0}^{p+1}(-1)^j\phi(h^0{}_i,\ldots,\hat{h}^j{}_i,\ldots,h^{p+1}_i)$, where

$$h^j_i = \begin{cases} x_j \text{ if } j < i \\ x_{j+1} \text{ if } j \geq i. \end{cases}$$

If $j < i$, and $(-1)^{i+j} = 1$, then

$\phi(h^0{}_i,\ldots,\hat{h}^j{}_i,\ldots,h^{p+1}_i) = \phi(x_0,\ldots,\hat{x}_j,\ldots,\hat{x}_i,\ldots,x_{p+2})$

$= \phi(h^0{}_{j-1},\ldots,\hat{h}^i{}_{j-1},\ldots,h^{p+1}_{j-1}) =$

$-(-1)^{(i+j-1)}\phi(h^0{}_{j-1},\ldots,\hat{h}^i{}_{j-1},\ldots,h^{p+1}_{j-1})$.

If $j \geq i$, and $(-1)^{i+j} = 1$, then

$\phi(h^0{}_i,\ldots,\hat{h}^j{}_i,\ldots,h^{p+1}_i) = \phi(x_0,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,x_{p+2})$

$= \phi(h^0{}_{j+1},\ldots,\hat{h}^i{}_{j+1},\ldots,h^{p+1}_{j+1}) =$

$-(-1)^{(i+j+1)}\phi(h^0{}_{j+1},\ldots,\hat{h}^i{}_{j+1},\ldots,h^{p+1}_{j+1})$.

Hence, $\sum_{i=1}^{p+2}(-1)^i\sum_{j=0}^{p+1}(-1)^j\phi(h^0{}_i,\ldots,\hat{h}^j{}_i,\ldots,h^{p+1}_i)$

$= \delta_{p+1}\delta_p\phi(x_0,\ldots,x_{p+2}) = 0$. Hence,

$\delta_{p+1}\delta_p = 0$.

The following notation will be used. If $A \subseteq 2^X$, then $A^m = \{x \in X^m |$ there exists $U \in A$, and $x \in U^m\}$.

Definition 2.1. Let $A \subseteq X$, and let p be a non-negative integer. $C^p(X,A) = \{\phi \in C^p(X) |$ there exists a cover β of A by sets open in X such that $\phi | \beta^{p+1} \cap A^{p+1} = 0\}$.

Note 2.3. $C^p(X, A)$ is a submodule.

Proof: Let $\phi_1, \phi_2 \in C^p(X, A)$. Then there exists open covers β_1, β_2 of A such that $\phi_1 | \beta_1{}^{p+1} \cap A^{p+1} = 0$, and $\phi_2 | \beta_2{}^{p+1} \cap A^{p+1} = 0$. Let $\beta = \{U \cap V | U \in \beta_1, V \in \beta_2$, and $U \cap V \neq \square\}$. Let $x \in A$. Then there exists $U \in \beta_1$ and $V \in \beta_2$ such that $x \in U$ and $x \in V$. Therefore, $x \in U \cap V \neq \square$. Hence, β is an open cover of A.

Let $(x_0,\ldots,x_p) \in \beta^{p+1} \cap A^{p+1}$. Then $x_0,\ldots,x_p \in U$ for some $U \in \beta$. $U = U_1 \cap U_2$ where $U_1 \in \beta_1$, and $U_2 \in \beta_2$. Therefore, $x_0,\ldots,x_p \in U_1$, and $x_0,\ldots,x_p \in U_2$. Therefore, $(x_0,\ldots,x_p) \in \beta_1{}^{p+1} \cap A^{p+1}$, and $(x_0,\ldots,x_p) \in \beta_2{}^{p+1} \cap A^{p+1}$. Therefore, $\phi_1(x_0,\ldots,x_p) = 0$ and $\phi_2(x_0,\ldots,x_p) = 0$. Thus, $(\phi_1 - \phi_2)(x_0,\ldots,x_p) = \phi_1(x_0,\ldots,x_p) - \phi_2(x_0,\ldots,x_p) = 0$. Hence, $\phi_1 - \phi_2 \in C^p(X, A)$. Therefore, $C^p(X, A)$ is an additive subgroup of $C^p(X)$.

Now let $r \in R$, and let $\phi \in C^p(X, A)$. There exists an open cover β of A such that $\phi | \beta^{p+1} \cap A^{p+1} = 0$. Let $(x_0,\ldots,x_p) \in \beta^{p+1} \cap A^{p+1}$. $r\phi(x_0,\ldots,x_p) = r(\phi(x_0,\ldots,x_p)) = r(0) = 0$. Therefore, $r\phi \in C^p(X, A)$. Hence, $C^p(X, A)$ is a submodule of $C^p(X)$.

Lemma 2.1. The set $\delta_p[C^p(X,A)] \subseteq C^{p+1}(X,A)$.

Proof: Let $\delta_p\phi \in \delta_p[C^p(X,A)]$. Then $\phi \in C^p(X,A)$. Thus, there exists an open cover β of A such that $\phi|\beta^{p+1} \cap A^{p+1} = 0$. Let $(x_0,\ldots,x_{p+1}) \in \beta^{p+2} \cap A^{p+2}$. Now $\delta_p\phi(x_0,\ldots,x_p) = \sum_{i=0}^{p+1}(-1)^i\phi(x_0,\ldots,\hat{x}_i,\ldots,x_{p+1})$. Also, $(x_0,\ldots,x_{p+1}) \in \beta^{p+2}$ implies there exists $U \in \beta$ such that $(x_0,\ldots,x_{p+1}) \in U^{p+2}$. Hence, $(x_0,\ldots,\hat{x}_i,\ldots,x_{p+1}) \in U^{p+1}$. Therefore, $(x_0,\ldots,\hat{x}_i,\ldots,x_{p+1}) \in \beta^{p+1} \cap A^{p+1}$ for $i = 0,\ldots,p+1$. Hence, $\phi(x_0,\ldots,\hat{x}_i,\ldots,x_{p+1}) = 0$ for $i = 0,\ldots,;+1$. Thus, $\sum_{i=0}^{p+1}(-1)^i\phi(x_0,\ldots,\hat{x}_i,\ldots,x_{p+1}) = \delta_p\phi(x_0,\ldots,x_{p+1}) = 0$. Therefore, $\delta_p\phi|\beta^{p+2} \cap A^{p+2} = 0$. Therefore, $\delta_p\phi \in C^{p+1}(X,A)$. Therefore, $\delta_p[C^p(X,A)] \subseteq C^{p+1}(X,A)$.

Definition 2.2. Let $Z^p(X,A) = C^p(X,A) \cap \delta_p^{-1}C^p(X,X)$.

Note 2.4. The set $Z^p(X,A)$ is a submodule.

Proof: Let $\phi \in \delta_p^{-1}C^{p+1}(X,X)$, and let $r \in R$. Thus, $\delta_p(\phi) \in C^{p+1}(X,X)$. Since, by Note 2.3, $C^{p+1}(X,X)$ is a submodule, $r\delta_p(\phi) \in C^{p+1}(X,X)$. By Note 2.1, δ_p is an R-homomorphism. Thus, $r\delta_p(\phi) = \delta_p(r\phi)$. Therefore, $r\phi \in \delta_p^{-1}C^{p+1}(X,X)$.

Now let ϕ_1, $\phi_2 \in \delta_p^{-1}C^{p+1}(X,X)$. Then $\delta_p\phi_1$, $\delta_p\phi_2 \in C^{p+1}(X,X)$. Since $C^{p+1}(X,X)$ is a submodule, $\delta_p\phi_1 - \delta_p\phi_2 \in C^{p+1}(X,X)$. Since δ_p is an R-homomorphism, $\delta_p\phi_1 - \delta_p\phi_2 = \delta_p(\phi_1 - \phi_2)$. Therefore, $\phi_1 - \phi_2 \in \delta_p^{-1}C^{p+1}(X,X)$. Hence, $\delta_p^{-1}C^p(X,X)$ is a submodule.

By Note 2.3, $C^p(X,A)$ is also a submodule. Thus, by Preliminary 1, $C^p(X,A) \cap \delta_p^{-1}C^{p+1}(X,X) = Z^p(X,A)$ is a submodule.

$Z^p(X, A)$ is called the submodule of p-cocycles of X mod A.

Definition 2.3. Let

$$B^p(X, A) = \begin{cases} 0 \text{ if } p = 0 \\ \delta_{p-1}C^{p-1}(X, A) + C^p(X, X) \text{ if } p > 0. \end{cases}$$

Note 2.5. $B^p(X, A)$ is a submodule.

Proof: Assume $p > 0$. $C_{p-1}(X, A)$ is a submodule, and δ_{p-1} is an R-homomorphism. Therefore, by Preliminary 2, $\delta_{p-1}C^{p-1}(X, A)$ is a submodule. $C^p(X, X)$ is a submodule. Thus, by Preliminary 1, $\delta_{p-1}C^{p-1}(X, A) + C^p(X, X) = B^p(X, A)$ is a submodule. If $p = 0$, then $B^p(X, A) = \{0\}$ which is a submodule.

$B^p(X, A)$ is called the submodule of cobounding cocycles of dimension p.

Lemma 2.2. If $B \subseteq A \subseteq X$, then $C^p(X, A)$ is a subset of $C^p(X, B)$.

Proof: Let $\phi \in C^p(X, A)$. Then there exists an open cover β of A such that $\phi | \beta^{p+1} \cap A^{p+1} = 0$. Since $B \subseteq A$, β is an open cover of B. Let $(x_0,\ldots,x_p) \in \beta^{p+1} \cap B^{p+1}$. Then $(x_0,\ldots,x_p) \in A^{p+1}$. Therefore, $\phi(x_0,\ldots,x_p) = 0$. Hence, $\phi | \beta^{p+1} \cap B^{p+1} = 0$. Thus, $\phi \in C^p(X, B)$. Therefore, $C^p(X, A)$ is a subset of $C^p(X, B)$.

Note 2.6. $B^p(X, A) \subseteq Z^p(X, A)$.

Proof: Let $\phi \in B^p(X, A)$. Then $\phi = \delta_{p-1}\phi_1 + \phi_2$ where $\phi_1 \in C^{p-1}(X, A)$, and $\phi_2 \in C^p(X, X)$. Now $\delta_p(\phi) = \delta_p(\delta_{p-1}\phi_1 + \phi_2)$. Since δ_p is an R-homomorphism,

$\delta_p(\delta_{p-1}\phi_1 + \phi_2) = \delta_p\delta_{p-1}\phi_1 + \delta_p\phi_2$. By Note 2.2, $\delta_p\delta_{p-1}\phi_1 = 0$. Therefore, $\delta_p\phi = 0 + \delta_p\phi_2 = \delta_p\phi_2$. Thus, $\delta_p\phi_2 \in \delta_p C^p(X, X)$. By Lemma 2.1, $\delta_p C^p(X, X) \subseteq C^{p+1}(X, X)$. Therefore, $\delta_p\phi_2 = \delta_p\phi \in C^{p+1}(X, X)$. Hence, $\phi \in \delta_p^{-1}C^{p+1}(X, X)$. Now $\delta_{p-1}\phi_1 \in {}_{p-1}C^{p-1}(X, A)$ which is a subset of $C^p(X, A)$ by Lemma 2.1. Thus, $\delta_{p-1}\phi_1 \in C^p(X, A)$. By Lemma 2.2, $C^p(X, X) \subseteq C^p(X, A)$. Thus, $\phi_2 \in C^p(X, A)$. Therefore, $\delta_{p-1}\phi_1 + \phi_2 = \phi \in C^p(X, A)$. Hence, $\phi \in C^p(X, A) \cap \delta_p^{-1}C^{p+1}(X,X)$ $= Z^p(X, A)$. Therefore, $B^p(X, A) \subseteq Z^p(X, A)$.

<u>Definition 2.3</u>. For $p \geq 0$, $H^p(X, A) = \dfrac{Z^p(X, A)}{B^p(X, A)}$, and <u>$H^p(X, A)$ is called the pth cohomology module of X mod A</u>.

<u>Definition 2.4</u>. Let X, Y be topological spaces, and let $f\colon X \to Y$. Define $f^{\#}\colon C^p(Y) \to C^p(X)$ by $f^{\#}\phi$ is that function in $C^p(X)$ defined as follows:

$f^{\#}\phi(x_0,\ldots,x_p) = \phi(f(x_0),\ldots,f(x_p))$.

The following notation will be used. $f\colon (X, A) \to (Y, B)$ means that $f\colon X \to Y$, $A \subseteq X$, $B \subseteq Y$, and $f(A) \subseteq B$.

<u>Note 2.7</u>. If $f\colon (X, A) \to (Y, B)$, then

(1) $f^{\#}$ is an R-homomorphism;

(2) $f^{\#}\delta_p = \delta_p f^{\#}$;

(3) if f is continuous, then $f^{\#}[C^p(Y, B)] \subseteq C^p(X, A)$;

(4) if f is continuous, then $f^{\#}[Z^p(Y, B)] \subseteq Z^p(X, A)$;

(5) if f is continuous, then $f^{\#}[B^p(Y, B)] \subseteq B^p(X, A)$.

Proof of (1): Let $\phi_1, \phi_2 \in C^p(Y)$, and let $r, t \in R$. $f^{\#}(r\phi_1 + t\phi_2)(x_0,\ldots,x_p) = (r\phi_1 + t\phi_2)(f(x_0),\ldots,f(x_p)) =$ $r\phi_1(f(x_0),\ldots,f(x_p)) + t\phi_2(f(x_0),\ldots,f(x_p)) =$ $rf^{\#}\phi_1(x_0,\ldots,x_p) + tf^{\#}\phi_2(x_0,\ldots,x_p)$. Hence,

$f^{\#}(r\phi_1 + t\phi_2) = rf^{\#}\phi_1 + tf^{\#}\phi_2$. Hence, $f^{\#}$ is an R-homomorphism.

Proof of (2): Let $\phi \in C^p(Y, B)$.
$f^{\#}\delta_p\phi(x_0,\ldots,x_{p+1}) = f^{\#}[\sum_{i=o}^{p+1}(-1)^i\phi(x_0,\ldots,\hat{x}_i,\ldots,x_{p+1})] = \sum_{i=0}^{p+1}(-1)^i\phi(f(x_0),\ldots,f(\dot{x}_i),\ldots,f(x_{p+1})) =$
$\delta_p f^{\#}\phi(x_0,\ldots,x_{p+1})$. Therefore, $f^{\#}\delta_p = \delta_p f^{\#}$.

Proof of (3): Let $f^{\#}\phi \in f^{\#}[C^p(Y, B)]$ where $\phi \in C^p(Y, B)$. Then there is an open cover β of B such that $\phi|\beta^{p+1} \cap B^{p+1} = 0$. Since f is continuous, $f^{-1}(\beta)$ is an open cover of A. Let $(x_0,\ldots,x_p) \in f^{-1}(\beta)^{p+1} \cap A^{p+1}$. $f^{\#}\phi(x_0,\ldots,x_p) = \phi(f(x_0),\ldots,f(x_p))$. Since $f(A) \subseteq B$, $(f(x_0),\ldots,f(x_p)) \in f(f^{-1}(\beta))^{p+1} \cap B^{p+1} = \beta^{p+1} \cap B^{p+1}$. Therefore, $\phi(f(x_0),\ldots,f(x_p)) = 0$. Therefore, $f^{\#}\phi(x_0,\ldots,x_p) = 0$. Thus, $f^{\#}\phi \in C^p(X, A)$. Hence, $f^{\#}[C^p(Y, B)] \subseteq C^p(X, A)$.

Proof of (4): Let $f^{\#}\phi \in f^{\#}[Z^p(Y, B)]$ where $\phi \in Z^p(Y, B)$. Then $\phi \in C^p(Y, B)$, and $\phi \in \delta_p^{-1}C^{p+1}(Y, Y)$. By part (3) of this note, $f^{\#}\phi \in C^p(X, A)$, and $f^{\#}\delta_p\phi \in C^{p+1}(X, X)$. By part (2), $f^{\#}\delta_p\phi = \delta_p f^{\#}\phi$. Therefore, $f^{\#}\phi \in \delta_p^{-1}C^{p+1}(X, X)$. Hence, $f^{\#}\phi \in C^p(X, A) \cap \delta_p^{-1}C^{p+1}(X, X) = Z^p(X, A)$. Therefore, $f^{\#}Z^p(Y, B) \subseteq Z^p(X, A)$.

Proof of (5): Let $\phi \in B^p(Y, B)$. If $p = 0$, then $\phi = 0$. Since $f^{\#}$ is an R-homomorphism, $f^{\#}\phi = 0$. Therefore, $f^{\#}\phi \in B^p(X, A)$. If $p > 0$, then $\phi = \delta_{p-1}\phi_1 + \phi_2$ where $\phi_1 \in C^{p-1}(Y, B)$, and $\phi_2 \in C^p(Y, Y)$. $f^{\#}\delta_{p-1}\phi_1 = \delta_{p-1}f^{\#}\phi_1$.

Since $f^{\#}[C^{p-1}(Y, B)] \subseteq C^{p-1}(X, A)$, $f^{\#}\phi_1 \in C^{p-1}(X, A)$. Therefore, $\delta_{p-1}f^{\#}\phi_1 \in \delta_{p-1}C^{p-1}(X, A)$. Since $\phi_2 \in C^p(Y, Y)$, $f^{\#}\phi_2 \in C^p(X, X)$. Therefore, $f^{\#}\delta_{p-1}\phi_1 + f^{\#}\phi_2 \in \delta_{p-1}C^{p-1}(X, A) + C^p(X, X)$. By part (1), $f^{\#}\delta_{p-1}\phi_1 + f^{\#}\phi_2 = f^{\#}(\delta_{p-1}\phi_1 + \phi_2) = f^{\#}\phi$. Therefore, $f^{\#}\phi \in B^p(X, A)$. Hence, $f^{\#}[B^p(Y, B)] \subseteq B^p(X, A)$.

Applying the Induced Homomorphism Theorem to Note 2.7 shows that $f^{\#}$ induces a unique R-homomorphism f^*: $H^p(X, A) \to H^p(Y, B)$. Also, $f^*\eta = \eta f^{\#}$.

$$\begin{array}{ccc} H^p(X, A) & \xrightarrow{f^*} & H^p(Y, B) \\ \eta \uparrow & & \uparrow \eta \\ Z^p(X, A) & \xrightarrow{f^{\#}} & Z^p(Y, B) \end{array}$$

<u>Theorem 1</u>. Let $f: (X, A) \to (X, A)$ be the identity map. Then $f^*: H^p(X, A) \to H^p(X, A)$ is the identity isomorphism.

Proof: $f^{\#}: C^p(X, A) \to C^p(X, A)$. For $\phi \in C^p(X, A)$, $f^{\#}\phi(x_0,\ldots,x_p) = \phi(f(x_0),\ldots,f(x_p)) = \phi(x_0,\ldots,x_p)$. Hence, $f^{\#}\phi = \phi$. Thus, $f^{\#}$ is the identity map. Let $\phi \in H^p(X, A)$. Since η is onto, $\phi = \eta\phi_1$ for some $\phi_1 \in Z^p(X, A)$. Thus, $f^*\phi = f^*\eta\phi_1 = \eta f^{\#}\phi_1 = \eta\phi_1 = \phi$. Hence, f^* is the identity isomorphism.

<u>Theorem 2</u>. If $(X, A) \xrightarrow{f} (Y, B) \xrightarrow{g} (Z, C)$, and if f,g are continuous, then $f^*g^* = (gf)^*$.

Proof: First, it will be shown that $f^{\#}g^{\#} = (gf)^{\#}$. Let $\phi \in Z^p(Z, C)$. For $(x_0,\ldots,x_p) \in A^p$,

$$f^{\#}g^{\#}\phi(x_0,\ldots,x_p) = g^{\#}\phi(f(x_0),\ldots,f(x_p))$$
$$= \phi(gf(x_0),\ldots,gf(x_p))$$
$$= (gf)^{\#}\phi(x_0,\ldots,x_p).$$

Hence, $f^{\#}g^{\#} = (gf)^{\#}$.

Now to prove the theorem, let $h \in H^p(Z,C)$. Then $h = \eta\phi$ for some $\phi \in Z^p(Z,C)$. Thus, $f^*g^*h = f^*g^*\eta\phi = f^*\eta g^{\#}\phi = \eta f^{\#}g^{\#}\phi = \eta(gf)^{\#}\phi = (gf)^*\eta\phi = (gf)^*h$. Hence, $f^*g^* = (gf)^*$.

<u>Corollary</u> <u>2.1</u>. If $f\colon (X,A) \to (Y,B)$ is an onto homeomorphism, then $f^*\colon H^p(Y,B) \to H^p(X,A)$ is an onto isomorphism.

Proof: Since f is a homeomorphism, $f\cdot f^{-1}\colon (Y,B) \to (Y,B)$ is the identity map. Therefore, by Theorem 1, $(f\cdot f^{-1})^*$ is the identity isomorphism. By Theorem 2, $(f\cdot f^{-1})^* = f^{-1*}f^*$ since f and f^{-1} are continuous. Now $f^{-1*}f^*$ is an onto isomorphism so by Preliminary 6, f^* is 1-1. Similarly, $(f^{-1}\ f)^* = f^*f^{-1*}$ is the identity isomorphism. Thus, by Preliminary 6, f^* is onto. Hence, $f^*\colon H^p(Y,B) \to H^p(X,A)$ is an onto isomorphism.

Let $B \subseteq A \subseteq X$. Let $i: A \subsetneqq X$. Then $i^{\#}: C^p(X) \to C^p(A)$.
Consider the following diagram.

$$\begin{array}{ccc} H^p(A,B) & & H^{p+1}(X,A) \\ \eta\uparrow & & \uparrow\eta \\ Z^p(A,B) & & Z^{p+1}(X,A) \\ \cap| & & \cap| \\ C^p(A) & & C^{p+1}(X) \\ i^{\#}\uparrow & \nearrow^{\delta_p} & \\ C^p(X) & & \end{array}$$

Note 2.8. (1) The function $i^{\#}$ is onto. (2) If $h \in H^p(A,B)$, and $\phi \in i^{\#-1}\eta^{-1}(h)$, then $\delta_p\phi \in Z^{p+1}(X,A)$.

Proof of (1): Let $\phi \in C^p(A)$. Define $\phi' \in C^p(X)$ as follows:

$$\phi'(x_0,\ldots,x_p) = \begin{cases} \phi(x_0,\ldots x_p) \text{ if } (x_0,\ldots,x_p) \in A^{p+1} \\ 0 \text{ otherwise.} \end{cases}$$

$i^{\#}\phi' \in C^p(A)$. Let $(x_0,\ldots,x_p) \in A^{p+1}$. Then $i^{\#}\phi'(x_0,\ldots,x_p) = \phi'(i(x_0),\ldots,i(x_p)) = \phi'(x_0,\ldots,x_p) = \phi(x_0,\ldots,x_p)$. Therefore, $i^{\#}\phi' = \phi$. Hence, $i^{\#}$ is onto.

Proof of (2): Let $h \in H^p(A,B)$, and let $\phi \in i^{\#-1}\eta^{-1}(h)$. Then $i^{\#}\phi \in Z^p(A,B)$. Therefore, $i^{\#}\phi \in \delta_p^{-1}C^{p+1}(A,A)$. Therefore, $\delta_p i^{\#}\phi \in C^{p+1}(A,A)$. Hence, there exists a cover β of A by sets open in A such that $\delta_p i^{\#}\phi | \beta^{p+2} \cap A^{p+2} = 0$.

Let $\beta' = \{U \subseteq X | \ U \cap A \in \beta, U \text{ is open in } X\}$. β' is a cover of A by sets open in X. Let $(x_0,\ldots,x_{p+1}) \in \beta'^{p+2} \cap A^{p+2}$. Then there exists

$U \in \beta'$ such that $(x_0,\ldots,x_{p+1}) \in U^{p+1}$. Thus, $(x_0,\ldots,x_{p+1}) \in (U \cap A)^{p+2}$. $U \cap A \in \beta$. Therefore, $(x_0,\ldots,x_{p+1}) \in \beta^{p+2} \cap A^{p+2}$; $\delta_p\phi(x_0,\ldots,x_{p+1}) = \delta_p\phi(i(x_0),\ldots,i(x_{p+1})) = i^{\#}\delta_p\phi(x_0,\ldots,x_{p+1}) = \delta_p i^{\#}\phi(x_0,\ldots,x_{p+1}) = 0$. Hence, $\delta_p\phi|\beta'^{p+2} \cap A^{p+2} = 0$. Therefore, $\delta_p\phi \in C^{p+1}(X,A)$.

By Note 2.2, $\delta_{p+1}\delta_p\phi = 0$. Therefore, $\delta_{p+1}\delta_p\phi \in C^{p+2}(X, X)$. Thus, $\delta_p\phi \in \delta_{p+1}^{-1}C^{p+2}(X,X)$. Therefore, $\delta_p\phi \in C^{p+1}(X,A) \cap \delta_{p+1}^{-1}C^{p+2}(X,X) = Z^{p+1}(X,A)$.

Define δ: $H^p(A,B) \to H^{p+1}(X,A)$ by $\delta(h) = \eta\delta_p\phi$ where $\phi \in i^{\#-1}\eta^{-1}(h)$.

<u>Note</u> <u>2.9</u>. The function δ is well-defined.

Proof: Let $h_1,h_2 \in H^p(A,B)$ such that $h_1 = h_2$. Since η and $i^{\#}$ are onto, there exists $\phi_1,\phi_2 \in C^p(X)$ such that $\eta i^{\#}\phi_1 = h_1$, and $\eta i^{\#}\phi_2 = h_2$. Now $\delta(h_1) = \eta\delta_p\phi_1$, and $\delta(h_2) = \eta\delta_p\phi_2$. Since $\eta i^{\#}\phi_1 = \eta i^{\#}\phi_2$, $i^{\#}\phi_1 - i^{\#}\phi_2 \in B^p(A,B)$. Two cases will be considered.

Case I: $p = 0$. Then $i^{\#}\phi_1 - i^{\#}\phi_2 = i^{\#}(\phi_1 - \phi_2) = 0$. Now $i^{\#}(\phi_1 - \phi_2) \in C^0(A)$. X is an open cover of A. Let $(x_0) \in X \cap A$. Then $(\phi_1 - \phi_2)(x_0) = (\phi_1 - \phi_2)(i(x_0)) = i^{\#}(\phi_1 - \phi_2)(x_0) = 0$. Therefore, $\phi_1 - \phi_2|X \cap A = 0$. Therefore, $\phi_1 - \phi_2 \in C^0(X,A)$. Hence, $\delta_0(\phi_1 - \phi_2) \in \delta_0C^0(X,A)$. Also, $\delta_0(\phi_1 - \phi_2) = \delta_0\phi_1 - \delta_0\phi_2 \in B^1(X,A)$. Thus, $\eta\delta_0\phi_1 = \eta\delta_0\phi_2$. Hence, $\delta(h_1) = \delta(h_2)$.

Case II: $p > 0$. Since $i^{\#}(\phi_1 - \phi_2) \in B^p(A,B)$, $i^{\#}(\phi_1 - \phi_2) = \delta_{p-1}\alpha_1 + \alpha_2$ where $\alpha_1 \in C^{p-1}(A,B)$, and

$\alpha_2 \in C^p(A,A)$. Since $C^{p-1}(A,B) \subseteq C^{p-1}(A)$, $\alpha_1 \in C^{p-1}(A)$. Therefore, $\alpha_1 = i^{\#}\alpha_1'$ where $\alpha_1' \in C^{p-1}(X)$. Thus, $\delta_{p-1}\alpha_1 = \delta_{p-1}i^{\#}\alpha_1' = i^{\#}\delta_{p-1}\alpha_1'$. Therefore, $i^{\#}(\phi_1 - \phi_2) = i^{\#}\delta_{p-1}\alpha_1' + \alpha_2$. So, $i^{\#}(\phi_1 - \phi_2) - i^{\#}\delta_{p-1}\alpha_1' = i^{\#}(\phi_1 - \phi_2 - \delta_{p-1}\alpha_1') = \alpha_2 \in C^p(A,A)$. Therefore, there exists a cover β of A by sets open in A such that $i^{\#}(\phi_1 - \phi_2 - \delta_{p-1}\alpha_1')|\beta^{p+1} \cap A^{p+1} = 0$. Let $\beta' = \{U \subseteq X|\ U \cap A \in \beta\}$; β' is a cover of A by sets open in X. Let $(x_0,\ldots,x_p) \in \beta'^{p+1} \cap A^{p+1}$. Then $(x_0,\ldots,x_p) \in U^{p+1}$ for some $U \in \beta'$. Since $(x_0,\ldots,x_p) \in A^{p+1}$, $(x_0,\ldots,x_p) \in (U \cap A)^{p+1}$. Now $U \cap A \in \beta$. Therefore, $(x_0,\ldots,x_p) \in \beta^{p+1} \cap A^{p+1}$. Therefore, $(\phi_1 - \phi_2 - \delta_{p-1}\alpha_1')(x_0,\ldots,x_p) = (\phi_1 - \phi_2 - \delta_{p-1}\alpha_1')(i(x_0),\ldots,i(x_p)) = i^{\#}(\phi_1 - \phi_2 - \delta_{p-1}\alpha_1')(x_0,\ldots,x_p) = 0$. Thus, $(\phi_1 - \phi_2 - \delta_{p-1}\alpha_1')|\beta'^{p+1} \cap A^{p+1} = 0$. Hence, $\phi_1 - \phi_2 - \delta_{p-1}\alpha_1' \in C^p(X,A)$. So, $\delta_p(\phi_1 - \phi_2 - \delta_{p-1}\alpha_1') \in \delta_p C^p(X,A)$. Since $\delta_p(\phi_1 - \phi_2 - \delta_{p-1}\alpha_1') = \delta_p\phi_1 - \delta_p\phi_2 - \delta_p\delta_{p-1}\alpha_1' = \delta_p\phi_1 - \delta_p\phi_2 + 0$, $\delta_p\phi_1 - \delta_p\phi_2 + 0 \in \delta_p C^p(X,A) + C^{p+1}(X,X) = B^{p+1}(X,A)$. Therefore, $\delta_p\phi_1 = \delta_p\phi_2$. Hence, $\delta(h_1) = \delta(h_2)$.

In any case, $\delta(h_1) = \delta(h_2)$. Hence, δ is well-defined.

<u>Note 2.10</u>. The function δ is an R-homomorphism.

Proof: Let $h_1,h_2 \in H^p(A,B)$, and let $r_1,r_2 \in R$. By definition, $\delta(h_1) = \eta\delta_p\phi_1$ where $\phi_1 \in i^{\#-1}\eta^{-1}h_1$, and $\delta(h_2) = \eta\delta_p\phi_2$ where $\phi_2 \in i^{\#-1}\eta^{-1}h_2$. Then $r_1\delta h_1 + r_2\delta h_2 = r_1\eta\delta_p\phi_1 + r_2\eta\delta_p\phi_2$. Since η and δ_p are R-homomorphisms,

$r_1\eta\delta_p\phi_1 + r_2\eta\delta_p\phi_2 = \eta(r_1\delta_p\phi_1 + r_2\delta_p\phi_2) =$
$\eta\delta_p(r_1\phi_1 + r_2\phi_2)$. Since $i^{\#}$ is also an R-homomorphism,
$\eta i^{\#}(r_1\phi_1 + r_2\phi_2) = \eta(r_1 i^{\#}\phi_1 + r_2 i^{\#}\phi_2) =$
$r_1\eta i^{\#}\phi_1 + r_2\eta i^{\#}\phi_2 = r_1h_1 + r_2h_2$. Therefore,
$r_1\phi_1 + r_2\phi_2 \in i^{\#-1}\eta^{-1}(r_1h_1 + r_2h_2)$. Therefore,
$\delta(r_1h_1 + r_2h_2) = \eta\delta_p(r_1\phi_1 + r_2\phi_2) = r_1\delta h_1 + r_2\delta h_2$.
Hence, δ is an R-homomorphism.

The following notation will be used in Theorem 3. Let $f: X \xrightarrow{\text{continuous}} X'$. Let $B \subseteq A \subseteq X$, $f(A) \subseteq A'$, and $f(B) \subseteq B'$. Then $f: (X,A,B) \to (X',A',B')$. Let $f_1 = f: (X,A) \to (X',A')$, and let $f_3 = f: (A,B) \to (A',B')$.

Theorem 3. In the following diagram, $\delta f_3{}^* = f_1{}^*\delta$.

$$\begin{array}{ccc} H^p(A,B) & \xrightarrow{\delta} & H^{p+1}(X,A) \\ f_3{}^*\uparrow & & \uparrow f_1{}^* \\ H^p(A',B') & \xrightarrow{\delta} & H^{p+1}(X',A') \end{array}$$

Proof: Let $i_1: A \subsetneq X$, and let $i_2: A' \subsetneq X'$. Then $i_1{}^*: C^p(X) \to C^p(A)$, $i_2{}^*: C^p(X') \to C^p(A')$. Let $h \in H^p(A',B')$. Then $h = \eta i_2{}^{\#}\phi$ for some $\phi \in C^p(X')$. Now $\delta(h) = \eta\delta_p\phi$. So, $f_1{}^*\delta(h) = f_1{}^*\eta\delta_p\phi = \eta f_1{}^*\delta_p\phi$. Let $(x_0,\ldots,x_p) \in C^p(A)$. Then $i_1{}^{\#}f_1{}^{\#}\phi(x_0,\ldots,x_p) =$
$f_1{}^{\#}\phi(i_1(x_0),\ldots,f(x_p))$. Since $x_i \in A$ for $0 \leq i \leq p$, $f_1(x_i) = f_3(x_i)$ for $0 \leq i \leq p$. Therefore,
$\phi(f_1(x_0),\ldots,f_1(x_p)) = \phi(f_3(x_0),\ldots,f_3(x_p))$. Since $f_3(A) \subseteq A'$, and $i_2: A' \to X'$,
$\phi(f_3(x_0),\ldots,f_3(x_p)) = \phi(i_2f_3(x_0),\ldots,i_2f_3(x_p)) =$
$i^{\#}\phi(f_3(x_0),\ldots,f_3(x_p)) = f_3i^{\#}\phi(x_0,\ldots,x_p)$. Hence,

$i_1{}^{\#}f_1{}^{\#}\phi = f_3{}^{\#}i_2{}^{\#}\phi$. Therefore, $f_3{}^{*}h = f_3\eta i_2{}^{\#}\phi = \eta f_3{}^{\#}i_2{}^{\#}\phi = \eta i_1{}^{\#}f_1{}^{\#}\phi$. Therefore, $\delta f_3{}^{*}h = \delta\eta i_1{}^{\#}f_1{}^{\#}\phi$. Since $f_1{}^{\#}\phi \in i_1{}^{\#-1}\eta^{-1}(f_3h)$, $\delta(f_3{}^{*}h) = \eta\delta_p f_1{}^{\#}\phi = \eta f_1{}^{\#}\delta_p\phi = f_1{}^{*}\delta(h)$. Hence, $\delta f_3{}^{*} = f_1{}^{*}\delta$.

Lemma 2.3. Let $B \subseteq A \subseteq X$, and let $j: (X,B) \subsetneq (X,A)$. Then $j^{\#}: C^p(X,A) \to C^p(X,B)$ is 1-1 for all p.

Proof: Let $j^{\#}\phi_1 = j^{\#}\phi_2$ for $\phi_1,\phi_2 \in C^p(X,A)$. Let $(x_0,\ldots,x_p) \in X^{p+1}$. Then $\phi_1(x_0,\ldots,x_p) = \phi_1(j(x_0),\ldots,j(x_p)) = j^{\#}\phi_1(x_0,\ldots,x_p) = j^{\#}\phi_2(x_0,\ldots,x_p) = \phi_2(j(x_0),\ldots,j(x_p)) = \phi_2(x_0,\ldots,x_p)$. Therefore, $\phi_1 = \phi_2$. Hence, $j^{\#}$ is 1-1.

Theorem 4. Let $A \subseteq X$, and let $j: (X,\square) \subsetneq (X,A)$, and let $i: A \subsetneq X$. The sequence

$$H^0(X,A) \xrightarrow{j^*} H^0(X,\square) \xrightarrow{i^*} H^0(A,\square) \xrightarrow{\delta} H^1(X,A) \xrightarrow{j^*} H^1(X,\square) \xrightarrow{i^*} H^1(A,\square) \xrightarrow{\delta} \ldots$$

is exact.

Proof: Let $h \in H^p(X,A)$ for any $p \geq 0$. Then $j^*h \in I_m j^*$. Now $h = \eta\phi$ for some $\phi \in Z^p(X,A)$. Therefore, $\phi \in C^p(X,A)$. Thus, there exists an open cover β of A such that $\phi|\beta^{p+1} \cap A^{p+1} = 0$. Now $i^*j^*h = i^*(j^*\eta\phi) = i^*(\eta j^{\#}\phi) = \eta i^{\#}j^{\#}\phi$. Let $(x_0,\ldots,x_p) \in \beta^{p+1} \cap A^{p+1}$. Then $i^{\#}j^{\#}\phi(x_0,\ldots,x_p) = i^{\#}\phi(j(x_0),\ldots,j(x_p)) = i^{\#}\phi(x_0,\ldots,x_p) = \phi(i(x_0),\ldots,i(x_p)) = \phi(x_0,\ldots,x_p) = 0$. Hence, $i^{\#}j^{\#}\phi|\beta^{p+1} \cap A^{p+1} = 0$. Therefore, $i^{\#}j^{\#}\phi \in C^p(A,A)$. Therefore, $0 + i^{\#}j^{\#}\phi \in \delta_{p-1}C^{p-1}(A,\square) + C^p(A,A) = B^p(A,\square)$. Hence, $\eta(0 + i^{\#}j^{\#}\phi) = \eta i^{\#}j^{\#}\phi = i^*\eta j^{\#}\phi = i^*j^*\eta\phi = i^*j^*(h) = 0$. Therefore, $j^*h \in \ker i^*$. Hence, $\mathrm{Im} j^* \subseteq \ker i^*$ for any p.

Let $h \in \ker i^*$ for some $h \in H^p(X,\square)$. Then $h = \eta\phi$ for some $\phi \in Z^p(X,\square)$, and $i^*(h) = 0$. Two cases will be considered.

Case I: $p = 0$. So, $\phi \in C^0(X,\square)$. X is an open cover of A. Let $(x_0) \in X \cap A$. Then $\phi(x_0) = \phi(i(x_0)) = i^{\#}\phi(x_0)$. Since $i^*(h) = 0$, $i^{\#}\phi \in B^0(X,\square)$. Thus, $i^{\#}\phi = 0$. Hence, $i^{\#}\phi(x_0) = \phi(x_0) = 0$. Therefore, $\phi | X \cap A = 0$. Hence, $\phi \in C^0(X,A)$. Since $\phi \in Z^0(X,\square)$, $\delta_0\phi \in C^1(X,X)$. Therefore, $\phi \in C^0(X,A) \cap \delta_0{}^{-1}C^1(X,X) = Z^0(X,A)$. Thus, $\eta\phi \in H^0(X,A)$. Now $j^*(\eta\phi) = \eta j^{\#}\phi = \eta\phi = h$. Therefore, $h \in \operatorname{Im} j^*$.

Case II: $p > 0$. Since $i^*(h) = 0$, $i^{\#}\phi \in B^p(A,\square)$. Therefore, $i^{\#}\phi = \delta_{p-1}\alpha_1 + \alpha_2$ where $\alpha_1 \in C^{p-1}(A,\square)$, and $\alpha_2 \in C^p(A,A)$. Since $i^{\#}$ is onto, there exists $\overline{\alpha_1} \in C^{p-1}(X,\square)$ such that $i^{\#}\overline{\alpha_1} = \alpha_1$. Therefore, $i^{\#}\phi = \delta_{p-1}\alpha_1 + \alpha_2 = \delta_{p-1}i^{\#}\overline{\alpha_1} + \alpha_2 = i^{\#}\delta_{p-1}\overline{\alpha_1} + \alpha_2$. Therefore, $\alpha_2 = i^{\#}\phi - i^{\#}\delta_{p-1}\overline{\alpha_1} = i^{\#}(\phi - \delta_{p-1}\overline{\alpha_1}) \in C^p(A,A)$. Hence, there exists a cover β of A by sets open in A such that $i^{\#}(\phi - \delta_{p-1}\overline{\alpha_1}) | \beta^{p+1} \cap A^{p+1} = 0$. Let $\beta_0 = \{U \subseteq X | U \cap A \in \beta\}$. β_0 is a cover of A by sets open in X. Let $(x_0,\ldots,x_p) \in \beta_0{}^{p+1} \cap A^{p+1}$. Then for some $U \in \beta_0$, $(x_0,\ldots,x_p) \in U^{p+1}$. Since $U \cap A \in \beta$, and $(x_0,\ldots,x_p) \in (U \cap A)^{p+1} \cap A^{p+1}$, $(x_0,\ldots,x_p) \in \beta^{p+1} \cap A^{p+1}$. Therefore, $(\phi - \delta_{p-1}\overline{\alpha_1})(x_0,\ldots,x_p) = (\phi - \delta_{p-1}\overline{\alpha_1})(i(x_0),\ldots,i(x_p))$ $= i^{\#}(\phi - \delta_{p-1}\overline{\alpha_1})(x_0,\ldots,x_p) = 0$. Therefore, $\phi - \delta_{p-1}\overline{\alpha_1} | \beta_0{}^{p+1} \cap A^{p+1} = 0$. Hence, $\phi - \delta_{p-1}\overline{\alpha_1} \in C^p(X,A)$. Now $\delta_p(\phi - \delta_{p-1}\overline{\alpha_1}) = \delta_p\phi - \delta_p\delta_{p-1}\overline{\alpha_1} = \delta_p\phi$. Since

$\phi \in Z^p(X,\square)$, $\delta_p\phi \in C^{p+1}(X,X)$. Therefore, $\phi - \delta_{p-1}\overline{\alpha_1} \in \delta_p{}^{-1}C^{p+1}(X,X)$. Hence, $\phi - \delta_{p-1}\overline{\alpha_1} \in Z^p(X,A)$. Since $\overline{\alpha_1} \in C^{p-1}(X,\square)$, $j^{\#}\overline{\alpha_1} \in C^{p-1}(X,\square)$ and $\delta_{p-1}j^{\#}\overline{\alpha_1} \in \delta_{p-1}C^{p-1}(X,\square)$. Therefore, $\delta_{p-1}j^{\#}\overline{\alpha_1} + 0 \in B^p(X,\square)$. Thus, $\eta\delta_{p-1}j^{\#}\overline{\alpha_1} = 0$. Now $j^*[\eta(\phi - \delta_{p-1}\overline{\alpha_1})] = \eta j^{\#}(\phi - \delta_{p-1}\overline{\alpha_1}) = \eta j^{\#}\phi - \eta j^{\#}\delta_{p-1}\overline{\alpha_1} = \eta j^{\#}\phi - 0 = \eta\phi = h$. Therefore, $h \in \operatorname{Im} j^*$.

In any case $h \in \operatorname{Im} j^*$. Therefore, $\ker i^* \subseteq \operatorname{Im} j^*$. Hence, $\ker i^* = \operatorname{Im} j^*$ for all p.

Let $i^*h \in \operatorname{Im} i^*$ for $h \in H^p(X, \square)$. Then $h = \eta\phi$ for some $\phi \in Z^p(X, \square)$. Therefore, $\delta i^*h = \delta i^*\eta\phi = \delta\eta i^{\#}\phi = \eta\delta_p\phi$. Since $\phi \in Z^p(X,\square)$, $\delta_p\phi \in C^{p+1}(X,X)$. Therefore, $0 + \delta_p\phi \in B^{p+1}(X,A)$. So, $\eta(0 + \delta_p\phi) = \eta\delta_p\phi = \delta i^*h = 0$. Therefore, $i^*h \in \ker\delta$. Hence, $\operatorname{Im} i^* \subseteq \ker\delta$ for all $p \geq 0$.

Let $h \in \ker\delta$ for some $h \in H^p(A, \square)$. Then $\delta h = \eta\delta_p\phi = 0$ for some $\phi \in i^{\#-1}\eta^{-1}h$. Therefore, $\delta_p\phi \in B^{p+1}(X,A)$. Thus, $\delta_p\phi = \delta_p\alpha_1 + \alpha_2$ where $\alpha_1 \in C^p(X,A)$, and $\alpha_2 \in C^{p+1}(X,X)$. Therefore, $\delta_p\phi - \delta_p\alpha_1 = \delta_p(\phi - \alpha_1) = \alpha_2 \in C^{p+1}(X,X)$. So, $\phi - \alpha_1 \in \delta_p{}^{-1}C^{p+1}(X,X)$. Since $\phi \in i^{\#-1}\eta^{-1}h$, $\phi \in C^p(X,\square)$. Since $\alpha_1 \in C^p(X,A)$, $\alpha_1 \in C^p(X,\square)$. Therefore, $\phi - \alpha_1 \in C^p(X,\square)$. Hence, $\phi - \alpha_1 \in Z^p(X,\square)$. Thus, $\eta(\phi - \alpha_1) \in H^p(X,\square)$.

Consider $\eta i^{\#}\alpha_1$. Since $\alpha_1 \in C^p(X,A)$, there is an open cover β of A such that $\alpha_1|\, \beta^{p+1} \cap A^{p+1} = 0$. If $(x_0,\ldots,x_p) \in \beta^{p+1} \cap A^{p+1}$, then

$i^{\#}\alpha_1(x_0,\ldots,x_p) = \alpha_1(i(x_0),\ldots,i(x_p)) = \alpha_1(x_0,\ldots,x_p) = 0$.
Therefore, $i^{\#}\alpha_1 | \beta^{p+1} \cap A^{p+1} = 0$. Hence, $i^{\#}\alpha_1 \in C^p(A,A)$.
Therefore, $0 + i^{\#}\alpha_1 \in B^p(A,\square)$. Thus,
$\eta(0 + i^{\#}\alpha_1) = \eta i^{\#}\alpha_1 = 0$.

Now $i^*[\eta(\phi - \alpha_1)] = \eta i^{\#}(\phi - \alpha_1) = \eta i^{\#}\phi - \eta i^{\#}\alpha_1 = \eta i^{\#}\phi - 0 = h - 0 = h$. Hence, $h \in \operatorname{Im} i^*$. Therefore, $\ker \delta \subseteq \operatorname{Im} i^*$. Therefore, $\ker \delta = \operatorname{Im} i^*$ for all $p \geq 0$.

Let $\delta h \in \operatorname{Im} \delta$ for some $h \in H^p(A,\square)$. Then $\delta h = \eta\delta_p\phi$ for some $\phi \in i^{\#-1}\delta^{-1}h$. Then $j^*\delta h = j^*\eta\delta_p\phi = \eta j^{\#}\delta_p\phi = \eta\delta_p j^{\#}\phi$. Since $\phi \in C^p(X)$, $j^{\#}\phi \in C^p(X,\square)$. Therefore, $\delta_p j^{\#}\phi \in \delta_p C^{p+1}(X,\square)$. Therefore, $\delta_p j^{\#}\phi + 0 \in B^{p+1}(X,\square)$. Hence, $\eta(\delta_p j^{\#}\phi + 0) = \eta\delta_p j^{\#}\phi = j^*\delta h = 0$. Thus, $\delta h \in \ker j^*$. Hence, $\operatorname{Im} \delta \subseteq \ker j^*$.

Let $h \in \ker j^*$ for some $h \in H^p(X,A)$. Then $h = \eta\phi$ for some $\phi \in Z^p(X,A)$, and $j^*h = j^*\eta\phi = \eta j^{\#}\phi = 0$. Therefore, $j^{\#}\phi \in B^p(X,\square)$. Thus, $j^{\#}\phi = \delta_{p-1}\alpha_1 + \alpha_2$ where $\alpha_1 \in C^{p-1}(X,\square)$, and $\alpha_2 \in C^p(X,X)$. Let $(x_0,\ldots,x_p) \in X^{p+1}$. Then
$\delta_{p-1}\alpha_1(x_0,\ldots,x_p) = \delta_{p-1}\alpha_1(j(x_0),\ldots,j(x_p)) = j^{\#}\delta_{p-1}\alpha_1(x_0,\ldots,x_p)$,
and $\alpha_2(x_0,\ldots,x_p) = \alpha_2(j(x_0),\ldots,j(x_p)) = j^{\#}\alpha_2(x_0,\ldots,x_p)$.
Thus, $j^{\#}\delta_{p-1}\alpha_1 = \delta_{p-1}\alpha_1$, and $j^{\#}\alpha_2 = \alpha_2$. Therefore, $j^{\#}\phi = j^{\#}\delta_{p-1}\alpha_1 + j^{\#}\alpha_2 = j^{\#}(\delta_{p-1}\alpha_1 + \alpha_2)$. By Lemma 2.3, $j^{\#}$ is 1-1. Therefore, $\phi = \delta_{p-1}\alpha_1 + \alpha_2$. Thus, $h = \eta\phi = \eta(\delta_{p-1}\alpha_1 + \alpha_2) = \eta\delta_{p-1}\alpha_1 + \eta\alpha_2$. Since $\alpha_2 \in C^p(X,X)$, $\alpha_2 \in B^p(X,A)$. Therefore, $\eta\alpha_2 = 0$. Thus, $h = \eta\delta_{p-1}\alpha_1 = \delta(\eta i^{\#}\alpha_1)$. Therefore, $h \in \operatorname{Im} \delta$. So, $\ker j^* \subseteq \operatorname{Im} \alpha$. Hence, $\ker j^* = \operatorname{Im} \delta$, for all $p \geq 0$.

To show $j^*: H^0(X,A) \to H^0(X,\square)$ is 1-1, let $j^*(h) = 0$ for some $h \in H^0(X,A)$. Then $h = \eta\phi$ for some $\phi \in Z^0(X,A)$. Now $j^*(h) = j^*\eta\phi = \eta j^{\#}\phi = 0$. Therefore, $j^{\#}\phi \in B^0(X,\square) = 0$. Thus, $j^{\#}\phi = 0$. By Lemma 2.3, $j^{\#}$ is 1-1. Hence, $\phi = 0$. Therefore, $\eta\phi = h = 0$. Hence, $j^*: H^0(X,A) \quad H^0(X,\square)$ is 1-1. Therefore, by Definition 1.1, the sequence is exact.

Corollary 2.2. If $H^p(X,A) = 0$ for all $p \geq 0$, then i^* is an isomorphism of $H^p(X,\square)$ onto $H^p(A,\square)$.

Proof: Since $H^p(X,A) = 0$, Im $j^* = 0$. By Theorem 4, Im $j^* =$ ker i^*. Therefore, ker $i^* = 0$. Hence, i^* is 1-1. Let $\phi \in H^p(A,\square)$. Then $\delta\phi \in H^{p+1}(X,A)$. By hypothesis, $H^{p+1}(X,A) = 0$. Thus, $\delta\phi = 0$. Hence, $\phi \in$ ker $\delta =$ Im i^*. Hence, k^* is onto. Therefore, i^* is an isomorphism.

Theorem 4A. Let $B \subseteq A \subseteq X$. Let $m: (A,B) \subsetneq (X,B)$, $n: (X,B) \subsetneq (X,A)$, $j: (A,\square) \subsetneq (A,B)$, and $i: A \subsetneq X$. Let $\tilde{\delta} = \delta j^*$. Then the sequence $\ldots H^{p-1}(X,B) \xrightarrow{m^*} H^{p-1}(A,B) \xrightarrow{\tilde{\delta}} H^p(X,A) \xrightarrow{n^*} H^p(X,B) \xrightarrow{m^*} H^p(A,B) \xrightarrow{m^*} \ldots$ is exact.

Proof: Let $m^*h \in$ Im m^* for some $h \in H^{p-1}(X,B)$. Then $h = \eta\phi$ for some $\phi \in Z^{p-1}(X,B)$. Thus, $\tilde{\delta}m^*(h) = \delta j^*(m^*\eta\phi) = \delta j^*\eta m^{\#}\phi = \delta\eta j^{\#}m^{\#}\phi$. Since $i^{\#} = j^{\#}m^{\#}$, $\delta\eta j^{\#}m^{\#}\phi = \delta\eta i^{\#}\phi = \eta\delta_{p-1}\phi$. Since $\phi \in Z^{p-1}(X,B)$, $\delta_{p-1}\phi \in C^p(X,X)$. Therefore, $0 + \delta_{p-1}\phi \in B^p(X,A)$. Therefore, $\eta(0 + \delta_{p-1}\phi \in B^p(X,A)$. Therefore, $\eta(0 + \delta_{p-1}\phi) = \eta\delta_{p-1}\phi = \tilde{\delta}m^*h = 0$. Hence, $m^*h \in$ ker $\tilde{\delta}$. Therefore, Im $m^* \subseteq$ ker $\tilde{\delta}$.

Let $h \in \ker \tilde{\delta}$ for some $h \in H^p(A,B)$. Then $h = \eta\phi$ for some $\phi \in Z^p(A,B)$, and $\tilde{\delta}h = 0$. By Note 2.8, $m^{\#}$ is onto. So, $\phi = m^{\#}\phi'$ for some $\phi' \in Z^p(X,B)$. Now, $m^*(\eta\phi') = \eta m^{\#}\phi' = \eta\phi = h$. Thus, $h \in \operatorname{Im} m^*$. Therefore, $\ker \tilde{\delta} \subseteq \operatorname{Im} m^*$. Hence, $\operatorname{Im} m^* = \ker \tilde{\delta}$.

Let $h \in \ker n^*$ for some $h \in H^p(X,A)$. Then $h = \eta\phi$ for some $\phi \in Z^p(X,A)$, and $n^*h = n^*\eta\phi = \eta n^{\#}\phi = 0$. Therefore, $n^{\#}\phi \in B^p(X,B)$. Thus, $n^{\#}\phi = \delta_{p-1}\alpha_1 + \alpha_2$ where $\alpha_1 \in C^{p-1}(X,B)$, and $\alpha_2 \in C^p(X,X)$. Since $n^{\#}\phi = \phi \in C^p(X,B)$, $\phi = \delta_{p-1}\alpha_1 + \alpha_2$. Therefore, $h = \eta\phi = \eta(\delta_{p-1}\alpha_1 + \alpha_2) = \eta\delta_{p-1}\alpha_1 + \alpha_2$. Since $\alpha_2 \in C^p(X,X)$, $0 + \alpha_2 \in B^p(X,A)$. Therefore, $\eta\alpha_2 = 0$. Thus, $h = \eta\delta_{p-1}\alpha_1$. Since $\phi \in C^p(X,A)$, and $\alpha_2 \in C^p(X,X)$, then $\phi \in C^p(A,A)$ and $\alpha_2 \in C^p(A,A)$. Therefore, $\phi - \alpha_2 = \delta_{p-1}\alpha_1 \in C^p(A,A)$. Thus, $\alpha_1 \in \delta_{p-1}^{-1}C^p(A,A)$. Also, $\alpha_1 \in C^{p-1}(X,B)$ implies $\alpha_1 \in C^{p-1}(A,B)$. Hence, $\alpha_1 \in Z^{p-1}(A,B)$. Now $\tilde{\delta}(\eta\alpha_1) = \delta j^*(\eta\alpha_1) = \delta(\eta j^{\#}\alpha_1) = \eta\delta_{p-1}\alpha_1 = h$. Therefore, $h \in \operatorname{Im} \tilde{\delta}$. Hence, $\ker n^* \subseteq \operatorname{Im} \tilde{\delta}$.

Let $\tilde{\delta}h \in \operatorname{Im} \tilde{\delta}$ for some $h \in H^{p-1}(A,B)$. Let $k: B \subsetneq A$. By Theorem 4, $\ldots H^{p-1}(A,B) \xrightarrow{j^*} H^{p-1}(A) \xrightarrow{k^*} H^{p-1}(B) \xrightarrow{\delta} H^p(A,B) \rightarrow \ldots$ is an exact sequence. Therefore, $\operatorname{Im} j^* = \ker k^*$. Now $n^*(\tilde{\delta}h) = n^*\delta j^*h$. By Theorem 3, $n^*\delta = \delta k^*$. Therefore, $n^*\delta j^*h = \delta k^*j^*h$. Since $\operatorname{Im} j^* = \ker k^*$, $k^*(j^*h) = 0$. Thus, $\delta k^*j^*h = n^*\tilde{\delta}h = 0$. Therefore, $\tilde{\delta}h \in \ker n^*$. Therefore, $\operatorname{Im} \tilde{\delta} \subseteq \ker n^*$. Hence, $\operatorname{Im} \tilde{\delta} = \ker n^*$.

Let $n^*h \in \operatorname{Im} n^*$ for some $n \in H^p(X,A)$. Then $h = \eta\phi$ for some $\phi \in Z^p(X,A)$. Thus, $m^*n^*h = m^*n^*\eta\phi = m^*\eta n^{\#}\phi = \eta m^{\#}n^{\#}\phi$. Since $\phi \in Z^p(X,A)$, $\phi \in C^p(X,A)$. Therefore, there is an open cover β of A such that $\phi|\beta^{p+1} \cap A^{p+1} = 0$. Let $(x_0,\ldots,x_p) \in \beta^{p+1} \cap A^{p+1}$. Then $m^{\#}n^{\#}\phi(x_0,\ldots,x_p) = m^{\#}\phi(n(x_0),\ldots,n(x_p)) = m^{\#}\phi(x_0,\ldots,x_p) = \phi(m(x_0),\ldots,m(x_p)) = \phi(x_0,\ldots,x_p) = 0$. Hence, $m^{\#}n^{\#}\phi \in C^p(A,A)$. Therefore, $0 + m^{\#}n^{\#}\phi \in B^p(A,B)$. Therefore, $\eta m^{\#}n^{\#}\phi = m^*n^*h = 0$. Thus, $n^*h \in \ker m^*$. Hence, $\operatorname{Im} n^* \subseteq \ker m^*$.

Let $h \in \ker m^*$ for some $h \in H^p(X,B)$. Then $h = \eta\phi$ for some $\phi \in Z^p(X,B)$, and $m^*h = m^*\eta\phi = \eta m^{\#}\phi = 0$. Therefore, $m^{\#}h \in B^p(A,B)$. Thus, $m^{\#}\phi = \delta_{p-1}\alpha_1 + \alpha_2$ where $\alpha_1 \in C^{p-1}(A,B)$, and $\alpha_2 \in C^p(A,A)$. Now $\alpha_2 = m^{\#}\phi - \delta_{p-1}\alpha_1 \in C^p(A,A)$. Hence, there exists a cover β of A by sets open in A such that $m^{\#}\phi - \delta_{p-1}\alpha_1|\beta^{p+1} \cap A^{p+1} = 0$. Let $\beta_1 = \{U \subseteq X|\ U \cap A \in \beta\}$. Then β_1 is a cover of A by sets open in X. Let $(x_0,\ldots,x_p) \in \beta_1{}^{p+1} \cap A^{p+1}$. Then $(\phi - \delta_{p-1}\alpha_1)(x_0,\ldots,x_p) = \phi(x_0,\ldots,x_p) - \delta_{p-1}\alpha_1(x_0,\ldots,x_p) = \phi(m(x_0),\ldots,m(x_p) - \delta_{p-1}\alpha_1(x_0,\ldots,x_p) = m^{\#}\phi(x_0,\ldots,x_p) - \delta_{p-1}\alpha_1(x_0,\ldots,x_p) = (m^{\#}\phi - \delta_{p-1}\alpha_1)(x_0,\ldots,x_p)$. Since $(x_0,\ldots,x_p) \in \beta_1{}^{p+1}$, there exists $U \in \beta_1$ such that $(x_0,\ldots,x_p) \in U^{p+1}$. Now $U \cap A \in \beta$. Since $(x_0,\ldots,x_p) \in A^{p+1}$, $(x_0,\ldots,x_p) \in \beta^{p+1}$. Hence, $(m^{\#}\phi - \delta_{p-1}\alpha_1)(x_0,\ldots,x_p) = (\phi - \delta_{p-1}\alpha_1)(x_0,\ldots,x_p) = 0$. Therefore, $\phi - \delta_{p-1}\alpha_1 \in C^p(X,A)$. Since $\phi \in C^p(X,B)$, $\delta_p\phi \in C^{p+1}(X,X)$. Now

$\delta_p(\phi - \delta_{p-1}\alpha_1) = \delta_p\phi - \delta_p\delta_{p-1}\alpha_1 = \delta_p\phi - 0 = \delta_p\phi \in C^{p+1}(X,X)$. Therefore, $\phi - \delta_{p-1}\alpha_1 \in \delta_p^{-1}C^{p+1}(X,X)$. Hence, $\phi - \delta_{p-1}\alpha_1 \in Z^p(X,A)$. Since $\alpha_1 \in C^{p-1}(A,B)$, $n^{\#}\alpha_1 \in C^{p-1}(X,B)$. Therefore, $\delta_{p-1}n^{\#}\alpha_1 \in \delta_{p-1}C^p(X,B)$. Thus, $\delta_{p-1}n^{\#}\alpha_1 + 0 = \delta_{p-1}n^{\#}\alpha_1 \in B^p(X,B)$. Therefore, $\eta\delta_{p-1}n^{\#}\alpha_1 = 0$. Hence, $n^*(\eta(\phi - \delta_{p-1}\alpha_1)) = \eta n^{\#}(\phi - \delta_{p-1}\alpha_1) = \eta n^{\#}\phi - \eta n^{\#}\delta_{p-1}\alpha_1 = \eta n^{\#}\phi - \eta\delta_{p-1}n^{\#}\alpha_1 = \eta n^{\#}\phi - 0 = \eta\phi = h$. Therefore, $h \in \operatorname{Im} n^*$. Therefore, $\ker m^* \subseteq \operatorname{Im} n^*$. Hence, $\operatorname{Im} n^* = \ker m^*$.

To show $n^*: H^0(X,A) \to H^0(X,B)$ is 1-1, let $n^*(h) = 0$ for some $h \in H^0(X,A)$. Then $h = \eta\phi$ for some $\phi \in Z^p(X,A)$. Now $n^*(h) = n^*\eta\phi = \eta n^{\#}\phi = 0$. Therefore, $n^{\#}\phi \in B^0(X,B)$. Thus, $n^{\#}\phi = 0$. By Lemma 2.3, $n^{\#}$ is 1-1. Hence, $\phi = 0$. Therefore, $\eta\phi = h = 0$. Hence, $n^*: H^0(X,A) \to H^0(X,B)$ is 1-1. Therefore, by Definition 1.1, the sequence is exact.

Definition 2.5. Let $f: X \to Y$, and let $g: X \to Y$. Define $D: C^p(X) \to C^{p-1}(X)$ as follows: for $\phi \in C^p(X)$, $D\phi$ is that function in $C^{p-1}(X)$ such that

$$D\phi(x_0,\ldots,x_{p-1}) = \sum_{i=0}^{p-1}(-1)^i\phi(g(x_0),\ldots,g(x_i),f(x_i),\ldots,f(x_{p-1})).$$

The function *$D\phi$ is called the deformation cochain of ϕ with respect to f and g.*

Lemma 2.4. If $f: X \to Y$, and $g: X \to Y$, then $D\delta_0\phi = f^{\#}\phi - g^{\#}\phi$, and if $p > 0$, $\delta_{p-1}D\phi + D\delta_p\phi = f^{\#}\phi - g^{\#}\phi$.

Proof: Assume $p = 0$. Let $x_0 \in X$. Then $D\delta_0\phi(x_0) = \delta_p\phi(g(x_0), f(x_0)) = \phi(f(x_0)) - \phi(g(x_0)) = f^{\#}\phi(x_0) - g^{\#}\phi(x_0)$. Therefore, $D\delta_p\phi = f^{\#}\phi - g^{\#}\phi$ for $p = 0$.

Assume $p > 0$. Let $(x_0,\ldots,x_p) \in X^{p+1}$. Then

$D\delta_p\phi(x_0,\ldots,x_p) = \sum_{i=0}^{p}(-1)^i[\sum_{j=0}^{p+1}(-1)^j\phi(h_0^i,\ldots,\hat{h}_j^i,\ldots,h_{p+1}^i)]$

where $h_j^i = \begin{cases} g(x_j) \text{ if } j \leq i \\ f(x_{j-1}) \text{ if } j > i. \end{cases}$

Also, $\delta_{p-1}D\phi(x_0,\ldots,x_p) =$

$\sum_{j=0}^{p}(-1)^j[\sum_{i=0}^{p-1}(-1)^i\phi(g(k_D^j),\ldots,g(k_i^j), f(k_i^j),\ldots,f(k_{p-1}^j))]$

where $k_j^i = \begin{cases} x_i \text{ if } i < j \\ x_{i+1} \text{ if } i \geq j. \end{cases}$

Thus, $D\delta_p\phi(x_0,\ldots,x_p) = \sum_{i=0}^{p}(-1)^i[\sum_{j=1}^{p}(-1)^j\phi(h_0^i,\ldots,\hat{h}_j^i,\ldots,h_{p+1}^i)] +$
$\phi(\hat{h}_0^0,\ldots,h_{p+1}^0) + (-1)^p(-1)^{p+1}\phi(h_0^p,\ldots,\hat{h}_{p+1}^p) =$
$\sum_{i=0}^{p}(-1)^i[\sum_{j=1}^{p}(-1)^j\phi(h_0^i,\ldots,\hat{h}_j^i,\ldots,h_{p+1}^i)] + f^{\#}\phi(x_0,\ldots,x_p) -$
$g^{\#}\phi(x_0,\ldots,x_p)$. For $i = j \neq 0$,

$(-1)^{ij}\phi(h_0^i,\ldots,\hat{h}_j^i,\ldots,h_{p+1}^i) =$
$(-1)^{ij}\phi(g(x_0),\ldots,g(x_{j-1}),f(x_{j-1}),\ldots,f(x_p)) =$
$-((-1)^{j(i-1)})\phi(g(k_0^j),\ldots,g(k_{i-1}^j),f(k_{i-1}^j),\ldots,f(k_{p-1}^j)$.

For $i < j$, $(-1)^{ij}\phi(h_0^i,\ldots,\hat{h}_j^i,\ldots,h_{p+1}^i) =$
$(-1)^{ij}\phi(g(x_0),\ldots,g(x_i),f(x_i),\ldots,f(x_{j-2}),f(x_j),\ldots,f(x_p)) =$
$-((-1)^{j(i-1)}\phi(g(k_0^j),\ldots,g(k_{i-1}^j),f(k_{i-1}^j),\ldots,f(k_{p-1}^j))$.

For $j < i$, $(-1)^{ij}\phi(h_0^i,\ldots,\hat{h}_j^i,\ldots,h_{p+1}^i) =$
$(-1)^{ij}\phi(g(x_0),\ldots,g(x_{j-1}),g(x_{j+1}),\ldots g(x_i),f(x_i),\ldots,f(x_p))$
$= -((-1)^{j(i-1)})\phi(g(k_0^j),\ldots,g(k_{i-1}^j),f(k_{i-1}^j),\ldots,f(k_{p-1}^j))$.

Therefore, $\sum_{i=0}^{p}(-1)^i[\sum_{j=1}^{p}(-1)^j\phi(h_o^i,\ldots,\hat{h}_j^i,\ldots,h_{p+1}^i)] =$
$-[\sum_{j=0}^{p}(-1)^j[\sum_{i=0}^{p-1}(-1)^i\phi(g(k_0^j),\ldots,g(k_i^j),f(k_i^j),\ldots,f(k_{p-1}^j))]] =$
$-[\delta_{p-1}D\phi(x_0,\ldots,x_p)]$. Hence,

$\delta_{p-1}D\phi(x_0,\ldots,x_p) + D\delta_p\phi(x_0,\ldots,x_p) =$

$\delta_{p-1}D\phi(x_0,\ldots,x_p) + \sum_{i=0}^{p}(-1)^i[\sum_{j=1}^{p}(-1)^j\phi(h_0^i,\ldots,\hat{h}_j^i,\ldots,h_{p+1}^i)] +$

$f^{\#}\phi(x_0,\ldots,x_p) - g^{\#}\phi(x_0,\ldots,x_p) =$

$\delta_{p-1}D\phi(x_0,\ldots,x_p) - \delta_{p-1}D\phi(x_0,\ldots,x_p) + f^{\#}\phi(x_0,\ldots,x_p) -$

$g^{\#}\phi(x_0,\ldots,x_p) = f^{\#}\phi(x_0,\ldots,x_p) - g^{\#}\phi(x_0,\ldots,x_p)$. Thus, for $p > 0$, $\delta_{p-1}D\phi + D\delta_p\phi = f^{\#}\phi - g^{\#}\phi$.

The Fundamental Lemma of Spanier.

(1) Let $f,g\colon (X,A) \to (Y,B)$ (not necessarily continuous).

(2) Let $\phi \in Z^p(Y,B)$.

(3) There exists an open cover β of Y such that $\delta_p\phi|\beta^{p+2} = 0$ and $\phi|\beta^{p+1} \cap B^{p+1} = 0$.

(4) There exists an open cover γ of X such that $U \in \gamma$ implies there exists $V \in \beta$ such that $f(U) \cup g(U) \subseteq V$.

If (1)-(4) are satisfied, then

(a) $f^{\#}\phi$, $g^{\#}\phi \in Z^p(X,A)$, and (b) $f^{\#}\phi - g^{\#}\phi \in B^p(X,A)$.

Proof: Let $(x_0,\ldots,x_p) \in \gamma^{p+1} \cap A^{p+1}$. Then $(f(x_0),\ldots,f(x_p)) \in B^{p+1}$, and $(g(x_0),\ldots,g(x_p)) \in B^{p+1}$. Also, $(x_0,\ldots,x_p) \in U^{p+1}$ for some $U \in \gamma$. By (4), there is a $V \in \beta$ such that $f(U) \cup g(U) \subseteq V$. Thus, $(f(x_0),\ldots,f(x_p)) \in V^{p+1}$, and $(g(x_0),\ldots,g(x_p)) \in V^{p+1}$. Hence, $(f(x_0),\ldots,f(x_p))$, $(g(x_0),\ldots,g(x_p)) \in \beta^{p+1} \cap B^{p+1}$. Therefore, by (3), $\phi(f(x_0),\ldots,f(x_p)) = f^{\#}\phi(x_0,\ldots,x_p) = 0$, and $\phi(g(x_0),\ldots,g(x_p)) = g^{\#}\phi(x_0,\ldots,x_p) = 0$. Hence, γ is an open cover of A, and $f^{\#}\phi|\gamma^{p+1} \cap A^{p+1} = 0$, and $g^{\#}\phi|\gamma^{p+1} \cap A^{p+1} = 0$. Therefore, $f^{\#}\phi, g^{\#}\phi \in C^p(X,A)$.

Let $(x_0,\ldots,x_{p+1}) \in \gamma^{p+2}$. Then for some $U \in \gamma$, $(x_0,\ldots,x_{p+1}) \in U^{p+2}$. By (4), there exists $V \in \beta$ such that $(f(x_0),\ldots,f(x_{p+1})) \in V^{p+2}$, and $g(x_0),\ldots,g(x_{p+1})) \in V^{p+2}$. Therefore, $(f(x_0),\ldots,f(x_{p+1})),g(x_0),\ldots,g(x_{p+1})) \in \beta^{p+2}$ Hence, $\delta_p\phi(f(x_0),\ldots,f(x_{p+1})) = f^{\#}\delta_p\phi(x_0,\ldots,x_{p+1}) = \delta_p f^{\#}\phi(x_0,\ldots,x_{p+1}) = 0$, and $\delta_p\phi(g(x_0),\ldots,g(x_{p+1})) = g^{\#}\delta_p\phi(x_0,\ldots,x_{p+1}) = \delta_p g^{\#}\phi(x_0,\ldots,x_p) = 0$. Therefore, γ is an open cover of X, and $\delta_p f^{\#}\phi|\gamma^{p+2} = 0$, and $\delta_p g^{\#}\phi|\gamma^{p+2} = 0$. Thus, $\delta_p f^{\#}\phi \in C^{p+1}(X,X)$, and $\delta_p g^{\#}\phi \in C^{p+1}(X,X)$. Therefore, $f^{\#}\phi, g^{\#}\phi \in \delta_p^{-1}C^{p+1}(X,X)$. Hence, $f^{\#}\phi$, $g^{\#}\phi \in Z^p(X,A)$.

To prove (b), it will be shown that $D\phi \in C^{p-1}(X,A)$. Let $(x_0,\ldots,x_{p-1}) \in \gamma^p \cap A^p$. Then $(x_0,\ldots,x_{p-1}) \in U^p$ for some $U \in \gamma$. By (4), there exists $V \in \beta$ such that $f(U) \cup g(U) \subseteq V$. Let $K = \{(g(x_0),\ldots,g(x_i),f(x_i),\ldots,f(x_{p-1}))| 0 \le i \le p-1\}$. Thus, K is a subset of V^{p+1}. Therefore, every element in K is an element of β^{p+1}. Since $x_i \in A$ for $0 \le i \le p-1$, K is a subset of B^{p+1}. Therefore, if $(g(x_0),\ldots,g(x_i),f(x_i),\ldots,f(x_{p-1})) \in K$, then $(g(x_0),\ldots,g(x_i),f(x_i),\ldots,f(x_{p-1})) \in \beta^{p+1} \cap B^{p+1}$. Hence, by (3), $D\phi(x_0,\ldots,x_{p-1}) = \sum_{i=0}^{p-1}(-1)^i\phi(g(x_0),\ldots,g(x_i),f(x_i),\ldots,f(x_{p-1})) = 0$. Therefore, γ is an open cover of A, and $D\phi|\gamma^p \cap A^p = 0$. Thus, $D\phi \in C^{p-1}(X,A)$.

Now it will be shown that $D\delta_p\phi \in C^p(X,X)$. Let $(x_0,\ldots,x_p) \in \gamma^{p+1} \cap X^{p+1}$. Then $(x_0,\ldots x_p) \in U$ for some

$U \in \gamma$. By (4), there is a $V \in \beta$ such that $f(U) \cup g(U) \subseteq V$. Let $K = \{(g(x_0),\ldots,g(x_i),f(x_i),\ldots,f(x_p)) | 0 \leq i \leq p\}$. Then K is a subset of V^{p+1}. Therefore, if $(g(x_0),\ldots,g(x_i),f(x_i),\ldots,f(x_p)) \in K$, then $(g(x_0),\ldots,g(x_i),f(x_i),\ldots,f(x_p)) \in \beta^{p+1}$. By (3), $\delta_p\phi|\beta^{p+1} = 0$. Hence, $D\delta_p\phi(x_0,\ldots,x_p) = \sum_{i=0}^{p}(-1)^i\delta_p\phi(g(x_0),\ldots,g(x_i),f(x_i),\ldots,f(x_p)) = 0$. Therefore, γ is an open cover of X, and $D\delta_p\phi|\gamma^{p+1} = 0$. Hence, $D\delta_p\phi \in C^p(X,X)$.

If $p = 0$, then by Lemma 2.4, $f^{\#}\phi - g^{\#}\phi = D\delta_0\phi \in C^0(X,X)$. Thus, $0 + f^{\#}\phi - g^{\#}\phi = f^{\#}\phi - g^{\#}\phi \in B^0(X,A)$. If $p > 0$, then by Lemma 2.4, $f^{\#}\phi - g^{\#}\phi = \delta_{p-1}D\phi - D\delta_p\phi$. Since $D\phi \in C^{p-1}(X,A)$, then $\delta_{p-1}D\phi \in \delta_{p-1}C^{p-1}(X,A)$. Therefore, $\delta_{p-1}D\phi + D\delta_p\phi \in \delta_{p-1}C^{p-1}(X,A) + C^p(X,X) = B^p(X,A)$. Thus, $f^{\#}\phi - g^{\#}\phi \in B^p(X,A)$. In any case, $f^{\#}\phi - g^{\#}\phi \in B^p(X,A)$.

The following notation will be used. For $\pi: (X,A) \times T \xrightarrow{\text{continuous}} (Y,B)$, fix $t \in T$. Define $\hat{t}: (X,A) \to (Y,B)$ by $\hat{t}(x) = \pi(t,x)$ for each $x \in X$. Then $\hat{t}$ induces $\hat{t}^*: H^p(Y,B) \to H^p(X,A)$.

Lemma 2.5. Let X be compact. Let B be a closed subset of Y. Let $\pi: (X,A) \times T \to (Y,B)$. Let $h \in H^p(Y,B)$, and $t \in T$. There exists a set O open in T such that $t \in O$, and if $s \in O$, then $\hat{s}^*(h) = \hat{t}^*(h)$. Further, if T is connected, then for each $s,t \in T$, $\hat{s}^* = \hat{t}^*$.

Proof: Since $h \in H^p(Y,B)$, $h = \eta\phi$ for some $\phi \in Z^p(Y,B)$. Therefore, there exists an open cover β_1 of B such that $\phi|\beta_1^{p+1} \cap B^{p+1} = 0$. Also, there exists an open cover β_2 of Y such that $\delta_p\phi|\beta_2^{p+2} = 0$. Then $\beta_1 \cup (Y\setminus B)$ is an open cover of Y. Let $\beta = \{V_1 \cap V_2 | V_1 \in \beta_1 \cup (Y\setminus B), V_2 \in \beta_2, V_1 \cap V_2 \neq \square\}$. Then β is an open cover of Y, and $\phi|\beta^{p+1} \cap B^{p+1} = 0$, and $\delta_p\phi|\beta^{p+2} = 0$. Since π is continuous, $\pi^{-1}(\beta)$ is an open cover of $X \times T$. For each $x \in X$, $(x,t) \in X \times T$. Therefore, $(x,t) \in \pi^{-1}(V)$ for some $V \in \beta$. Since $\pi^{-1}(V)$ is open in $X \times T$, there exist sets G_{xv}, k_{xv} such that G_{xv} is open in X, K_{xv} is open in T, and $(x,t) \in G_{xv} \times K_{xv} \subseteq \pi^{-1}(V)$. Let $\beta' = \{G_{xv} | x \in X\}$. Then β' is an open cover of X. Therefore, there is a finite subcollection $\beta'' = \{G_{v1},\ldots,G_{vn}\}$ of β' that covers X. Now $t \in K_{vi}$ for $i = 1,\ldots,n$. Therefore, $t \in \bigcap_{i=1}^{n} K_{vi}$. Let $O = \bigcap_{i=1}^{n} K_{vi}$. Then O is open in T and $t \in O$. Let $s \in O$. Let $G_{vi} \in \beta''$. If $x \in G_{vi}$, then $\hat{t}(x) = \pi(x,t)$. Since $t \in K_{vi}$, and $G_{vi} \times K_{vi} \subseteq \pi^{-1}(V_i)$, then $\pi(x,t) \in V_i$. Also, $\hat{s}(x) = \pi(x,s) \in V_i$.

The above establishes the following: $\hat{s},\hat{t}: (X,A) \to (Y,B)$; $\phi \in Z^p(Y,B)$; there is an open cover β of Y such that $\delta_p\phi|\beta^{p+2} = 0$, and $\phi|\beta^{p+1} \cap B^{p+1} = 0$; there is an open cover β'' of X such that $G \in \beta''$ implies there exists $V \in \beta$ such that $\hat{s}(G) \cup \hat{t}(G) \subseteq V$. Hence, the hypothesis of the Fundamental Lemma of Spanier is satisfied.

Thus, $\hat{s}^{\#}\phi - \hat{t}^{\#}\phi \in B^p(X,A)$. Therefore, $\eta\hat{s}^{\#}\phi = \eta\hat{t}^{\#}\phi$. Hence, $\hat{s}^*(h) = \hat{t}^*(h)$.

Now assume also that T is connected. Suppose there exists $t,s \in T$ such that $\hat{t}^*(h) \neq \hat{s}^*(h)$ for some $h \in H^p(Y,B)$. From the first part of this proof, there is an open set O such that $\hat{r}^*(h) = \hat{t}^*(h)$ for all $r \in O$. Let $M = \bigcup\{O \mid \text{for every } r \in O, \hat{r}^*(h) = \hat{t}^*(h)\}$. Then $s \in T\setminus M$, and $t \in M$. M is open in T. Thus, $T\setminus M$ is closed in T. Therefore, $M \cap (\overline{T\setminus M}) = M \cap (T\setminus M) = \square$. Now suppose $\overline{M} \cap (T\setminus M) \neq \square$. Let $x \in \overline{M} \cap (T\setminus M)$. Then every open set about x intersects M. There exists an open set G about x such that for every $y \in G$, $\hat{y}^*(h) = \hat{x}^*(h)$. Since $G \cap M \neq \square$, there exists $y \in G \cap M$ such that $\hat{y}^*(h) = \hat{x}^*(h)$. Therefore, $\hat{t}^*(h) = \hat{y}^*(h) = \hat{x}^*(h)$. Therefore, $x \in M$. This contradicts $x \in T\setminus M$. Thus, $\overline{M} \cap (T\setminus M) = \square$. Therefore, $\overline{M} \cap (T\setminus M) = M \cap (\overline{T\setminus M}) = \square$. Since $T = M \cup (T\setminus M)$, this implies T is not connected. This is a contradiction. Hence, for each $t,s \in T$, $\hat{t}^* = \hat{s}^*$.

<u>Theorem 5</u>. (The Generalized Homotopy Theorem). Let T be connected, and $f,g\colon (X,A) \to (Y,B)$ where X is compact, B is closed, and f and g are continuous. If there exists a continuous function $h\colon (X,A) \times T \to (Y,B)$, and if there exists $s,t \in T$ such that $h(x,s) = f(x)$, and $h(x,t) = g(x)$ for all $x \in X$, then $f^* = g^*$.

Proof: Define $\hat{s},\hat{t}$: $(X,A) \to (Y,B)$ by $\hat{s}(x) = h(x,s)$, and $\hat{t}(x) = h(x,t)$ for all $x \in X$. Then $\hat{s} = f$, and $\hat{t} = g$. By Lemma 2.5, $\hat{s}^* = \hat{t}^*$. Therefore, $f^* = g^*$.

Definition 2.6. Let I be the unit interval. Let f,g: $X \to Y$. Then f is homotopic to g (denoted $f \simeq g$) if there exists h: $X \times I \to Y$ such that h is continuous, and $h(x,1) = f(x)$ for all $x \in X$, and $h(x,0) = g(x)$ for all $x \in X$.

Corollary 2.3. If X is compact, and B is closed, and f,g: $(X,A) \to (Y,B)$, and f and g are continuous, then $f \simeq g$ implies $f^* = g^*$.

Proof: Since $f \simeq g$, there exists a continuous function h: $X \times I \to Y$ such that $h(x,1) = f(x)$, and $h(x,0) = g(x)$ for all $x \in X$. Since I is connected, the hypothesis of Theorem 5 is satisfied. Hence, $f^* = g^*$.

Definition 2.7. If f is a continuous function from X into Y, then f is null-homotopic if f is homotopic to a constant map.

Definition 2.8. A space X is contractible if the identity map from X to X is null-homotopic.

Definition 2.9. Let $A \subseteq X$. Then A is a retract of X if there exists r: $X \to A$ such that r is continuous, and $r \circ j = i$ where i is the identity map on A, and j: $A \hookrightarrow X$.

Definition 2.10. Let $A \subseteq X$. Then A is a deformation retract of X if there exists r: $X \to A$ such that r is a retraction of X onto A, and $j \circ r \simeq i$ where i is the identity map on X, and j: $A \hookrightarrow X$.

Corollary 2.4. If A is a deformation retract of a compact space X, then $H^p(X) \cong H^p(A)$ for all p.

Proof: Let i be the identity map from X to X. Since A is a deformation retract of X, there exists $r: X \to A$ such that r is onto and $jr \simeq i$ where $j: A \subsetneqq X$. Now $r^*: H^p(A) \to H^p(X)$. It will be shown that r^* is the desired isomorphism. By Corollary 2.3, $(jr)^* = i^*$. By Theorem 2, $(jr)^* = r^*j^* = i^*$. Now r^* is an R-homomorphism. Let $h \in H^p(X)$. Then $j^*(h) \in H^p(A)$. $r^*(j^*(h)) = i^*(h) = h$. Thus, r^* is onto. Let $r^*(h) = 0$ for some $h \in H^p(A)$. Then $h = \eta\phi$ for some $\phi \in Z^p(A)$. By Note 2.8, $j^{\#}$ is onto. Thus, $\phi = j^{\#}\phi'$ for some $\phi' \in Z^p(X)$. Therefore, $\eta\phi' \in H^p(X)$, and $j^*\eta\phi' = \eta j^{\#}\phi' = \eta\phi = h$. Hence, $r^*(h) = r^*(j^*\eta\phi') = i^*(\eta\phi') = 0$. By Theorem 1, i^* is an isomorphism. Hence, $i^*(\eta\phi') = 0$ implies $\eta\phi' = 0$. Therefore, $\phi' \in B^p(X)$. This implies $j^{\#}\phi' \in B^p(A)$. Thus, $\eta j^{\#}\phi' = h = 0$. Therefore, $r^*(h) = 0$ implies $h = 0$. Hence, r^* is 1-1. Hence, $r^*: H^p(A) \to H^p(X)$ is an isomorphism for all p.

Theorem 6. (Weak Excision Theorem).

Let $A \subseteq X$. Let U be open such that $\overline{U} \subseteq \text{Int}(A)$. Let $j: (X\setminus U, A\setminus U) \subsetneqq (X,A)$. Then $j^*: H^p(X,A) \to H^p(X\setminus U, A\setminus U)$ is an onto isomorphism.

Proof: To show j^* is onto, let $h \in H^p(X\setminus U, A\setminus U)$. Then $h = \eta\phi$ for some $\phi \in Z^p(X\setminus U, A\setminus U)$. Define $\phi' \in C^p(X)$ as follows:

$$\phi'(x_0,\ldots,x_p) = \begin{cases} \phi(x_0,\ldots,x_p) \text{ if } (x_0,\ldots,x_p) \in (X\setminus U)^{p+1} \\ 0 \text{ if } (x_0,\ldots,x_p) \notin (X\setminus U)^{p+1}. \end{cases}$$

Since $\phi \in Z^p(X\setminus U,\ A\setminus U)$, $\phi \in C^p(X\setminus U,\ A\setminus U)$. Thus, there exists a covering β of $A\setminus U$ by sets open in $X\setminus U$ such that $\phi|\beta^{p+1} \cap (A\setminus U)^{p+1} = 0$. Let $\beta_1 = \{U\} \cup \{G|\ G \text{ is open in } X,\ G \cap (X\setminus U) \in \beta\}$. Then β_1 is a covering of A by sets open in X. Let $(x_0,\ldots,x_p) \in \beta_1^{p+1} \cap A^{p+1}$. If $x_i \in U$ for any $0 \le i \le p$, then by the definition of ϕ', $\phi'(x_0,\ldots,x_p) = 0$. Otherwise, $(x_0,\ldots,x_p) \in [G \cap (X\setminus U)]^{p+1}$ for some $G \cap (X\setminus U) \in \beta$. Therefore, $(x_0,\ldots,x_p) \in \beta^{p+1} \cap (A\setminus U)^{p+1}$. Hence, $\phi'(x_0,\ldots,x_p) = \phi(x_0,\ldots,x_p) = 0$. In any case, $\phi'(x_0,\ldots,x_p) = 0$. Hence, $\phi' \in C^p(X,A)$.

It should be noted that the above is true for any $\phi \in C^p(X\setminus U,\ A\setminus U)$. By the definition of ϕ', $j^{\#}\phi' = \phi$. Hence, $j^{\#}$ is onto $C^p(X\setminus U,\ A\setminus U)$.

Now since $\phi \in Z^p(X\setminus U,\ A\setminus U)$, $\delta_p\phi \in C^{p+1}(X\setminus U,\ X\setminus U)$. Therefore, there is a covering γ of $X\setminus U$ by sets open in $X\setminus U$ such that $\delta_p\phi|\gamma^{p+2} \cap (X\ U)^{p+2} = 0$. Since $\phi' \in C^p(X,A)$, $\delta_p\phi' \in C^{p+1}(X,A)$. Therefore, there is a covering γ_1 of A by sets open in X such that $\delta_p\phi'|\gamma_1^{p+2} \cap A^{p+2} = 0$. Let $\gamma_2 = \{G|\ G \text{ is open in } X,\ (X\setminus U) \cap G \in \gamma\}$. Let $\gamma_3 = \{G \cap (X\setminus \bar{U})|\ G \in \gamma_2\}$, and let $\gamma_4 = \{V \cap \mathrm{Int}(A)|\ V \in \gamma_1\}$. Let $\Gamma = \gamma_3 \cup \gamma_4$. It will be shown that Γ is an open cover of X. Let $x \in X$. If $x \in \bar{U}$, then $x \in \mathrm{Int}(A)$. Therefore, $x \in V$ for some $V \in \gamma_1$. Thus, $x \in V \cap \mathrm{Int}(A) \in \Gamma$. If $x \in X\setminus\bar{U}$, then $x \in X\setminus U$. Therefore,

$x \in G$ for some $G \in \gamma$. Then $G = (X\setminus U) \cap G'$ where G' is open in X. Therefore, $x \in G' \cap (X\setminus\overline{U}) \in \Gamma$. In any case, x is covered by a set in Γ. Every set in γ_3 is open, and every set in γ_4 is open. Hence, Γ is an open cover of X. Let $(x_0,\ldots,x_{p+1}) \in \Gamma^{p+2}$. If $(x_0,\ldots,x_{p+1}) \in \gamma_3^{p+2}$, then $(x_0,\ldots,x_{p+1}) \in [G \cap (X\setminus\overline{U})]^{p+2} \subseteq [G \cap (X\setminus U)]^{p+2}$. Since $G \cap (X\setminus U) \in \gamma$, $(x_0,\ldots,x_{p+1}) \in \gamma^{p+2}$. Therefore, $\delta_p\phi(x_0,\ldots,x_p) = 0$. For each i, $0 \leq i \leq p + 1$, $(x_0,\ldots,\hat{x}_i,\ldots,x_{p+1}) \in (X\setminus U)^{p+1}$. Thus, $\phi'(x_0, \ldots,\hat{x}_i,\ldots,x_{p+1}) = \phi(x_0,\ldots,x_i,\ldots,x_{p+1})$. Therefore, $\delta_p\phi(x_0,\ldots,x_{p+1}) = \sum_{i=0}^{p+1}(-1)^i\phi(x_0,\ldots,\hat{x}_i,\ldots,x_{p+1}) = \sum_{i=0}^{p+1}(-1)^i\phi'(x_0,\ldots,\hat{x}_i,\ldots,x_{p+1}) = \delta_p\phi'(x_0,\ldots,x_{p+1}) = 0$. If $(x_0,\ldots,x_{p+1}) \in \gamma_4^{p+2}$, then $(x_0,\ldots,x_{p+1}) \in (V \cap \text{Int}(A))^{p+2}$ for some $V \in \gamma_1$. Thus, $(x_0,\ldots,x_{p+1}) \in \gamma_1^{p+2} \cap A^{p+2}$. Therefore, $\delta_p\phi'(x_0,\ldots,x_{p+1}) = 0$. In any case, $\delta_p\phi' = 0$. Hence, $\delta_p\phi'|\Gamma^{p+2} = 0$. Therefore, $\delta_p\phi' \in C^{p+2}(X,X)$. Therefore, $\phi' \in \delta_p^{-1}C^{p+1}(X,X)$. By the definition of ϕ', $j^{\#}\phi' = \phi$. Thus, $j^*(\eta\phi') = \eta j^{\#}\phi' = \eta\phi = h$. Hence, j^* is onto.

To show j^* is 1-1, let $j^*(h) = 0$. Then $h = \eta\phi$ for some $\phi \in Z^p(X,A)$. Since $j^*(h) = j^*(\eta\phi) = \eta j^{\#}\phi = 0$, $j^{\#}\phi \in B^p(X\setminus U, A\setminus U)$. Therefore, $j^{\#}\phi = \delta_{p-1}\alpha_1 + \alpha_2$ where $\alpha_1 \in C^{p-1}(X\setminus U, A\setminus U)$, and $\alpha_2 \in C^p(X\setminus U, X\setminus U)$. It was noted earlier in this proof that $j^{\#}$ is onto $C^{p-1}(X\setminus U, A\setminus U)$. Thus, $\alpha_1 = j^{\#}\overline{\alpha}_1$ for some $\overline{\alpha}_1 \in C^{p-1}(X,A)$. Therefore,

$\alpha_2 = j^{\#}\phi - \delta_{p-1}\alpha_1 = j^{\#}\phi - \delta_{p-1}j^{\#}\bar{\alpha}_1 = j^{\#}\phi - j^{\#}\delta_{p-1}\bar{\alpha}_1 =$
$j^{\#}(\phi - \delta_{p-1}\bar{\alpha}_1) \in C^p(X\backslash U, X\backslash U)$. Hence, there is a covering β of $X\backslash U$ by sets open in $X\backslash U$ such that
$j^{\#}(\phi - \delta_{p-1}\bar{\alpha}_1)|\beta^{p+1} = 0$. Now $\phi - \delta_{p-1}\bar{\alpha}_1 \in C^p(X,A)$. Thus, there is a covering γ of A by sets open in X such that
$\phi - \delta_{p-1}\bar{\alpha}_1 |\ \gamma^{p+1} \cap A^{p+1} = 0$. Let
$\beta_1 = \{G|\ G$ is open in X, $(X\backslash U) \cap G \in \beta\}$. Let
$\beta_2 = \{G \cap (X\backslash \bar{U})|\ G \in \beta_1\}$. Let
$\gamma_1 = \{G \cap \mathrm{Int}(A)|\ G \in \gamma\}$. Let $\Gamma = \beta_2 \cup \gamma_1$. Similar to an earlier part of this proof, Γ is an open cover of X. Let $(x_0,\ldots,x_p) \in \Gamma^{p+1}$. If $(x_0,\ldots,x_p) \in \beta_2^{p+1}$, then $(x_0,\ldots,x_p) \in [G \cap (X\backslash\bar{U})]^{p+1}$. Thus, $G \cap (X\backslash U) \in \beta$, and $(x_0,\ldots,x_p) \in \beta^{p+1}$. Hence, $j^{\#}(\phi - \delta_{p-1}\bar{\alpha}_1)(x_0,\ldots,x_p) = 0$. Since $x_i \in X\backslash U$ for $0 \leq i \leq p$, $j^{\#}(\phi - \delta_{p-1}\bar{\alpha}_1)(x_0,\ldots,x_p) =$
$(\phi - \delta_{p-1}\bar{\alpha}_1)(j(x_0),\ldots,j(x_p)) = (\phi - \delta_{p-1}\bar{\alpha}_1)(x_0,\ldots,x_p) = 0$.
If $(x_0,\ldots,x_p) \in \gamma_1^{p+1}$, then $(x_0,\ldots,x_p) \in (G \cap \mathrm{Int}(A)^{p+1})$ for some $G \in \gamma$. Hence, $(x_0,\ldots,x_p) \in \gamma^{p+1} \cap A^{p+1}$. Therefore, $(\phi - \delta_{p-1}\bar{\alpha}_1)(x_0,\ldots,x_p) = 0$. In any case, $(\phi - \delta_{p-1}\bar{\alpha}_1)(x_0,\ldots,x_p) = 0$. Therefore, Γ is an open cover of X and $\phi - \delta_{p-1}\bar{\alpha}_1|\Gamma^{p+1} = 0$. Hence,
$\phi - \delta_{p-1}\bar{\alpha}_1 \in C^p(X,X)$. Now $\delta_{p-1}\bar{\alpha}_1 \in \delta_{p-1}C^{p-1}(X,A)$. Thus,
$\phi = \delta_{p-1}\bar{\alpha}_1 + (\phi - \delta_{p-1}\bar{\alpha}_1) \in \delta_{p-1}C^{p-1}(X,A) + C^p(X,X) = B^p(X,A)$.
Therefore, $\eta\phi = h = 0$. Hence, j^* is 1-1. Hence, j^* is an onto isomorphism.

<u>Theorem</u> <u>6A</u>. (Excision Theorem).

If $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$ where A and B are closed subsets of X, and if $j: (B, A \cap B) \subsetneq (X,A)$, then

$j^*: H^p(X,A) \to H^p(B, A \cap B)$ is an onto isomorphism.

Proof: Let $U = X \setminus B$. Let $x \in \bar{U}$. If $x \notin Int(A)$, then $x \in Int(B)$. Therefore, since $x \in \bar{U}$, $Int(B) \cap U \neq \square$. But $Int(B) \cap X \setminus B = \square$. This is a contradiction. Thus, $x \in Int(A)$. Hence, $\bar{U} \subseteq Int(A)$. Since $B = X \setminus (X \setminus B)$, and $A \cap B = A \setminus (X \setminus B)$, $j: (X \setminus U, A \setminus U) \to (X,A)$. Thus, the hypothesis of Theorem 6 is satisfied. Hence, j^* is an onto isomorphism.

Theorem 7. (Dimension Theorem).

If the cardinality of X is 1, then

$$H^p(X) \cong \begin{cases} G \text{ if } p = 0 \\ 0 \text{ if } p > 0 \end{cases}$$

where G is the coefficient R-module.

Proof: Consider the case $p = 0$. Let $\phi \in C^0(X)$. Then $\delta_0\phi \in C^1(X)$. Let $(x,x) \in X^2$. $\delta_0\phi(x,x) = \sum_{i=0}^{1}(-1)^i\phi(x) = \phi(x) - \phi(x) = 0$. Therefore, $\delta_0\phi \in C^1(X,X)$. Thus, $\phi \in \delta_0{}^{-1}C^1(X,X)$. Hence, $\delta_0{}^{-1}C^1(X,X) = C^0(X)$. This implies that $Z^0(X) = C^0(X)$. Now $B^0(X) = 0$. Hence, $H^0(X) = Z^0(X)/B^0(X) = C^0(X)/0 = C^0(X)$. For each $\phi \in C^0(X)$, $\phi(x) = g$ for some $g \in G$. Define $F: C^0(X) \to G$ by $F(\phi) = g$. To show F is an R-homomorphism, let $\phi_1, \phi_2 \in C^0(X)$, and let $r,t \in G$. Then $\phi_1(x) = g_1$, and $\phi_2(x) = g_2$ for some $g_1, g_2 \in G$. Then $rF(\phi_1) + tF(\phi_2) = rg_1 + tg_2 = r\phi_1(x) + t\phi_2(x) = (r\phi_1 + t\phi_2)(x) = F(r\phi_1 + t\phi_2)$. Hence, F is an R-homomorphism. F is onto since for each $g \in G$, there is a function $\phi \in C^0(X)$ such that $\phi(x) = g$. If $F(\phi) = 0$, then $\phi(x) = 0$. Hence, by definition, $\phi = 0$. Thus, F is 1-1. Hence $F: H^0(X) \to G$ is an isomorphism.

Now assume that $p > 0$. Assume also that p is odd. For the remainder of this proof, let x_n denote the p-tuple from X^n. Let $\phi \in C^p(X)$. Since p is odd, $\delta_p\phi(x_{p+2}) = \sum_{i=0}^{p+2}(-1)^i\phi(x_{p+1}) = 0$. Therefore, $\delta_p\phi \in C^{p+1}(X,X)$. Therefore, $\phi \in \delta_p{}^{-1}C^{p+1}(X,X)$. Thus, $C^p(X) = \delta_p{}^{-1}C^{p+1}(X,X)$. Hence, $Z^p(X) = C^p(X)$. Now it will be shown that $C^p(X) = \delta_{p-1}C^{p-1}(X)$. For $\phi \in C^p(X)$, $\phi(x_{p+1}) = g$ for some $g \in G$. Define $\phi' \in C^{p-1}(X)$ by $\phi'(x_p) = g$. Then since p is odd, $\delta_{p-1}\phi'(x_{p+1}) = \sum_{i=0}^{p+1}(-1)^i\phi'(x_p) = \phi'(x_p) + \sum_{i=1}^{p+1}(-1)^i\phi'(x_p)$ $= g + 0 = g$. Therefore, $\delta_{p-1}\phi' = \phi$. Hence, $\phi \in \delta_{p-1}C^{p-1}(X)$. Thus, $\delta_{p-1}C^{p-1}(X) = C^p(X)$. Now for any p (even or odd), $C^p(X,X) = \{\phi \in C^p(X) \mid \phi(x_{p+1}) = 0\} = \{0\}$. Thus, $B^p(X) = \delta_{p-1}C^{p-1}(X) + C^p(X,X) = C^p(X) + 0 = C^p(X)$. Hence, $H^p(X) = Z^p(X)/B^p(x) = C^p(X)/C^p(X) \cong 0$.

Consider now the case that $p > 0$ and p is even. Let $\phi \in \delta_p{}^{-1}C^{p+1}(X,X)$. Then, $\delta_p\phi \in C^{p+1}(X,X) = \{0\}$. Now $\delta_p\phi(x_{p+2}) = \sum_{i=0}^{p+2}(-1)^i\phi(x_{p+1}) = \phi(x_{p+1}) + \sum_{i=1}^{p+2}(-1)^i\phi(x_{p+1}) = 0$. Since p is even, $\sum_{i=1}^{p+2}(-1)^i\phi(x_{p+1}) = 0$. Hence, $\phi(x_{p+1}) = 0$. Therefore, $\phi = 0$. Thus, $\delta_p{}^{-1}C^{p+1}(X,X) = \{0\}$. So, $Z^p(X) = C^p(X) \cap \delta_p{}^{-1}C^{p+1}(X,X) = \{0\}$. Since $B^p(X) \subseteq Z^p(X)$, $B^p(X) = \{0\}$. Hence, $H^p(X) = Z^p(X)/B^p(X) = \{0\}/\{0\} \cong 0$. In any case, if $p > 0$, then $H^p(X) \cong 0$.

Corollary 2.5. If X is compact and contractible, then

$$H^p(X) \approx \begin{cases} G & \text{if } p = 0 \\ 0 & \text{if } p > 0. \end{cases}$$

Proof: If X is compact and contractible, then the identity map $i: X \to X$ is homotopic to a constant map $f: X \to X$. Since f is constant, there exists $k \in X$ such that $f(x) = k$ for every $x \in X$. Now f is a retraction of X onto $\{k\}$, $f: X \to \{k\}$, and $f \simeq i$. Thus, $\{k\}$ is a deformation retract of S. Therefore, by Corollary 2.4, $H^p(X) \approx H^p(\{k\})$ for all p. By Theorem 7, $H^p(\{k\}) \approx \begin{cases} G & \text{if } p = 0 \\ 0 & \text{if } p > 0. \end{cases}$

Hence, $H^p(X) \approx \begin{cases} G & \text{if } p = 0 \\ 0 & \text{if } p > 0. \end{cases}$

CHAPTER III

EXTENDING THE THEORY

Some Topological Theorems

Definition 3.1. Let X be a topological space. Let β,γ be collections of non-empty subsets of X. Then β refines γ (denoted $\beta < \gamma$) if for each $V \in \beta$, there exists $U \in \gamma$ such that $V \subseteq U$.

The following notation will be used. If β is an open cover of X, $U_0 \in \beta$, let $U_0^* = \bigcup\{U \in \gamma \mid U \cap U_0 \neq \square\}$. Let $\beta^* = \{U^* \mid U \in \beta\}$.

Definition 3.2. The space X is fully normal if for each open cover β of X, there exists an open cover γ of X such that $\gamma^* < \beta$.

Theorem 8. A closed subset of a fully normal space is fully normal.

Proof: Let X be fully normal, and let K be a closed subset of X. Let β be a covering of K by sets open in K. Let $\beta' = \{U \subseteq X \mid U \text{ is open in } X, U \cap K \in \beta\}$. Then $\beta' \cup (X \setminus K)$ is an open cover of X. Hence, there exists an open cover γ of X such that $\gamma^* < \beta' \cup (X\setminus K)$. Let $\gamma' = \{V \cap K \mid V \in \gamma, V \cap K \neq \square\}$. Then γ' is a cover of K by sets open in K. Let $V^* \in \gamma'^*$. Then $V = V_1 \cap K$ for some $V_1 \in \gamma$. Furthermore, $V^* = \bigcup\{H \mid H \cap V \neq \square, H \in \gamma'\} = \bigcup\{H \cap K \mid (H \cap K) \cap V_1 \neq \square, H \in \gamma\}$. Also,

$V_1{}^* = \{H \mid H \cap V_1 \neq \square, H \in \gamma\}$. Therefore, $V^* \subseteq V_1{}^*$. Since $\gamma' < \beta' \cup (X\ K)$, $V_1{}^* \subseteq U$ for some $U \in \beta' \cup (X \setminus K)$. If $U = X \setminus K$, then $V \subseteq V^* \subseteq V_1{}^* \subseteq X \setminus K$. This is a contradiction since $V \subseteq K$. Therefore, $U \in \beta'$. Thus, $U \cap K \in \beta$. Now $V^* \subseteq K$, so $V^* \subseteq U \cap K$. Therefore, for each $V^* \in \gamma'^*$, $V^* \subseteq U$ for some $U \in \beta$. Hence, K is fully normal.

<u>Theorem</u> <u>9</u>. A fully normal space is normal.

Proof: Let X be fully normal. Let A, B be closed disjoint nonempty subsets of X. Let $\beta = \{X \setminus A, X \setminus B\}$. Then β is an open cover of X. Hence, there exists an open cover γ of X such that $\gamma^* < \beta$. Therefore, for every $U \in \gamma$, $U^* \subseteq X \setminus A$ or $U^* \subseteq X \setminus B$. Let $M = \{U \in \gamma \mid A \cap U \neq \square\}$, and let $N = \{U \in \gamma \mid B \cap U \neq \square\}$. Then since γ covers X, M and N are nonempty. If $U \in M$, then $U \subseteq U^* \subseteq X \setminus A$ implies $U \subseteq X \setminus A$. Thus, $A \cap U = \square$, and $U \notin M$. Hence, $U \in M$, implies $U \subseteq X \setminus B$. Likewise, $U \in N$, implies $U \subseteq X \setminus A$. Suppose $U \in M$, and $V \in N$, and $U \cap V \neq \square$. Then $U \subseteq X \setminus B$. Since $V \cap U \neq \square$, $V \subseteq U^* \subseteq X \setminus B$. Therefore, $V \cap B = \square$. This contradicts $V \in N$. Therefore, if $U \in M$, $V \in N$, then $U \cap V = \square$. Hence, $\bigcup M \cap \bigcup N = \square$. Thus, $\bigcup M$ is an open set containing A, and $\bigcup N$ is an open set containing B, and $\bigcup M \cap \bigcup N = \square$. Therefore, X is normal.

Reduced Modules

Let $A \subseteq X$. Let Q denote a one point space. Define the following:

$f: (X,A) \to (Q,Q)$

$g: X \to Q$

$h: A \to Q$

$m: (Q,\square) \to (Q,Q)$

$q: Q \to Q.$

Let $\tilde{H}^0(X,A) = H^0(X,A)/f^*H^0(Q,Q)$, $\tilde{H}^0(X) = H^0(X)/g^*H^0(Q)$, $\tilde{H}^0(A) = H^0(A)/h^*H^0(A)$, and $\tilde{H}^1(X,A) = H^1(X,A)/f^*H^1(Q,Q)$. Consider the following diagram.

$$\begin{array}{ccccccccc}
H^0(Q,Q) & \xrightarrow{m^*} & H^0(Q) & \xrightarrow{q^*} & H^0(Q) & \xrightarrow{\delta} & H^1(Q,Q) & & \\
f^*\downarrow & & g^*\downarrow & & h^*\downarrow & & f^*\downarrow & & \\
H^0(X,A) & \xrightarrow{j^*} & H^0(X) & \xrightarrow{i^*} & H^0(A) & \xrightarrow{\delta} & H^1(X,A) & \xrightarrow{i^*} & H^1(X) \\
\eta_1\downarrow & & \eta_2\downarrow & & \eta_3\downarrow & & \eta_4\downarrow & & \eta_5\downarrow \\
\tilde{H}^0(X,A) & \xrightarrow{\tilde{j}^*} & \tilde{H}^0(X) & \xrightarrow{\tilde{i}^*} & \tilde{H}^0(A) & \xrightarrow{\tilde{\delta}} & \tilde{H}^1(X,A) & \xrightarrow{j^*} & H^1(X)
\end{array}$$

Fig. 1--Reduced sequence

In the above diagram, $j: (X,\square) \overset{\subsetneq}{\to} (X,A)$, $i: (X,\square) \to (A,\square)$. Also, η_1, η_2, η_3, η_4, and η_5 are natural maps.

<u>Lemma 3.1</u>.

In Figure 1, $g^*m^* = j^*f^*$, $h^*q^* = i^*g^*$, and $f^*\delta = \delta h^*$.

Proof: Let $e \in H^0(Q,Q)$. Then $e = \eta\phi$ for some $\phi \in Z^0(Q,Q)$. Thus, $g^*m^*(e) = g^*m^*(\eta\phi) = g^*\eta m^\#\phi = \eta g^\# m^\#\phi$. Similarly, $j^*f^*(e) = \eta j^\# f^\#\phi$. Let $x \in X$. Then $g^\# m^\#\phi(x) = m^\#\phi(g(x)) = m^\#\phi(Q) = \phi(m(Q)) = \phi(Q)$. Also, $j^\# f^\#\phi(x) = f^\#\phi(j(x)) = f^\#\phi(x) = \phi(f(x)) = \phi(Q)$. Therefore, $g^\# m^\#\phi = j^\# f^\#\phi$. Hence, $\eta g^\# m^\#\phi = \eta j^\# f^\#\phi$. Therefore, $g^*m^*(e) = j^*f^*(e)$. Hence, $g^*m^* = j^*f^*$.

In a similar manner, $h^*q^* = i^*g^*$. By Theorem 3, $f^*\delta = \delta h^*$.

Lemma 3.2. In Figure 1, j, i, and δ induce maps $\tilde{j}^*$, $\tilde{i}^*$, and $\tilde{\delta}$.

Proof: All the natural maps are onto. Let $a \in \ker \eta_1$. Therefore, $a \in f^*H^0(Q,Q)$. So, $a = f^*(a')$ for some $a' \in H^0(Q,Q)$. By Lemma 3.1, $j^*(a) = j^*(f^*a') = g^*m^*a'$. Therefore, $j^*a \in g^*H^0(Q)$. Hence, $j^*a \in \ker \eta_2$. Therefore, $j^*\ker \eta_1 \subseteq \ker \eta_2$. Hence, by Preliminary 7, there exists an R-homomorphism $\tilde{j}^*: \tilde{H}^0(X,A) \to \tilde{H}^0(X)$.

Now let $a \in \ker \eta_2$. Then $a \in g^*H^0(Q)$. Thus, $a = g^*a'$ for some $a' \in H^0(Q)$. By Lemma 3.1, $i^*(a) = i^*(g^*a') = h^*q^*a'$. Therefore, $i^*(a) \in h^*H^0(Q) = \ker \eta_3$. Therefore, $i^*\ker \eta_2 \subseteq \ker \eta_3$. Hence, by Preliminary 7, there exists an R-homomorphism $\tilde{i}^*: \tilde{H}^0(X) \to \tilde{H}^0(A)$.

Now let $a \in \ker \eta_3$. Then $a \in h^*H^0(Q)$. Thus, $a = h^*a'$ for some $a' \in H^1(Q,Q)$. By Lemma 3.1, $\delta(a) = \delta(h^*a') = f^*\delta a'$. Therefore, $\delta a \in f^*H^1(Q,Q) = \ker \eta_4$. Thus, $\delta\ker \eta_3 \subseteq \ker \eta_4$. Hence, by Preliminary 7, there exists an R-homomorphism $\tilde{\delta}: \tilde{H}^0(A) \to \tilde{H}^1(X,A)$.

Theorem 10. The bottom line of Figure 1 is an exact sequence.

Proof: By Theorem 4, the middle line of Figure 1 is an exact sequence. To show $\text{Im}\, \tilde{j}^* = \ker \tilde{i}^*$, let $j^*(n) \in \tilde{H}^0(X)$ where $n \in \tilde{H}^0(X,A)$. Then $n = \eta_1 n'$ for some $n' \in H^0(X,A)$. Thus, $\tilde{i}^*\tilde{j}^*(n) = \tilde{i}^*\tilde{j}^*\eta_1 n' = \tilde{i}^*\eta_2 j^*n' = \eta_3 i^*j^*n'$. Since

$j^*n' \in \text{Im } j^*$, $i^*j^*n' = 0$. Therefore, $\eta_3 i^*n' = \tilde{i}^*\tilde{j}^*n = 0$. Hence, $\tilde{j}^*n \in \ker \tilde{i}^*$. Therefore, $\text{Im } \tilde{j}^* \subseteq \ker \tilde{i}^*$. Now let $e \in \ker \tilde{i}^*$ where $e \in \tilde{H}^0(X)$. Then $e = \eta_2 e_1$ for some $e_1 \in H^0(X)$, and $\tilde{i}^*(e) = 0$. Thus, $\tilde{i}^*e = \tilde{i}^*\eta_2 e_1 = \eta_3 i^*e_1 = 0$. Therefore, $i^*(e_1) \in \ker \eta_3 = h^*H^0(Q)$. Hence, $i^*(e_1) = h^*(e_2)$ for some $e_2 \in H^0(Q)$. By Theorem 2, $h^* = i^*g^*$. Thus, $h^*(e_2) = i^*g^*(e_2) = i^*(e_1)$. Therefore, $i^*(e_1 - g^*e_2) = 0$. Hence, $e_1 - g^*e_2 \in \ker i^* = \text{Im } j^*$. So $e_1 - g^*e_2 = j^*(a)$ for some $a \in H^0(X,A)$. Then, $\eta_1 a \in \tilde{H}^0(X,A)$. Further, $\tilde{j}^*(\eta_1 a) = \eta_2 j^*(a) = \eta_2(e_1 - g^*e_2) = \eta_2 e_1 - \eta_2 g^*e_2 = \eta_2(e_1) - 0 = \eta_2(e_1) = e$. Therefore, $e \in \text{Im } \tilde{j}^*$. Therefore, $\ker \tilde{i}^* \subseteq \text{Im } \tilde{j}^*$. Hence, $\text{Im } \tilde{j}^* = \ker \tilde{i}^*$.

To show $\text{Im } \tilde{i}^* = \ker \tilde{\delta}$, let $\tilde{i}^*n \in \tilde{H}^0(A)$ where $n \in \tilde{H}^0(X)$. Then $n = \eta_2 n'$ for some $n' \in H^0(X)$. Thus, $\tilde{\delta}\tilde{i}^*(n) = \tilde{\delta}\tilde{i}^*\eta_2 n' = \tilde{\delta}\eta_3 i^*n' = \eta_3 \delta i^*n'$. Since $i^*n' \in \text{Im } H^0(X)$, $\delta i^*n' = 0$. Therefore, $\eta_3\delta i^*n' = \tilde{\delta}\tilde{i}^*n = 0$. Therefore, $\tilde{i}^*n \in \ker \tilde{\delta}$. Thus, $\text{Im } \tilde{i}^* \subseteq \ker \tilde{\delta}$. Now let $e \in \ker \tilde{\delta}$ where $e \in \tilde{H}^0(A)$. Then $e = \eta_3 e_1$ for some $e_1 \in H^0(A)$. Further, $\tilde{\delta}(e) = \tilde{\delta}(\eta_3 e_1) = \eta_4 \delta e_1$. By Theorem 7, $H^1(Q,Q) = 0$. Therefore, $\tilde{H}^1(X,A) = H^1(X,A)/f^*H^1(Q,Q) = H^1(X,A)/0 = H^1(X,A)$. Thus, $\tilde{\delta}(e) = \eta_4\delta e_1 = \delta(e_1) = 0$. Therefore, $e_1 \in \ker \delta = \text{Im } i^*$. Hence, $e_1 = i^*e_2$ for some $e_2 \in H^0(X)$. Then $\eta_2 e_2 \in \tilde{H}^0(X)$. Further, $\tilde{i}^*(\eta_2 e_2) = \eta_3 i^*e_2 = \eta_3 e_1 = e$. Therefore, $e \in \text{Im } \tilde{i}^*$. Hence, $\ker \tilde{\delta} \subseteq \text{Im } \tilde{i}^*$. Hence, $\text{Im } \tilde{i}^* = \ker \tilde{\delta}$.

To show $\operatorname{Im} \tilde{\delta} = \ker \tilde{j}^*$, let $\tilde{\delta}(n) \in \tilde{H}^1(X,A)$ where $n \in \tilde{H}^0(A)$. Then $n = \eta_3 n'$ for some $n' \in H^0(A)$. Further, $\tilde{j}^*\tilde{\delta}(n) = \tilde{j}^*\tilde{\delta}(\eta_3 n') = \tilde{j}^*\eta_4 \delta n' = \eta_5 j^*\delta n'$. Since $\operatorname{Im} \delta = \ker j^*$, $j^*\delta n' = 0$. Thus, $\eta_5 j^*\delta n' = \tilde{j}^*\tilde{\delta}(n) = \eta_5(0) = 0$. Therefore, $\tilde{\delta}(n) \in \ker \tilde{j}^*$. Thus, $\operatorname{Im} \tilde{\delta} \subseteq \ker \tilde{j}^*$. Now let $e \in \ker \tilde{j}^*$ where $e \in \tilde{H}^1(X,A)$. Then $e = \eta_4 e_1$ for some $e_1 \in H^1(X,A)$. Also, $\tilde{j}^*(e) = \tilde{j}^*(\eta_4 e_1) = \eta_5 j^* e_1 = 0$. Therefore, $j^* e_1 = 0$. Thus, $e_1 \in \ker j^* = \operatorname{Im} \delta$. Therefore, $e_1 = \delta(e_2)$ for some $e_2 \in H^0(A)$. Then $\eta_3 e_2 \in \tilde{H}^0(A)$. Further, $\tilde{\delta}(\eta_3 e_2) = \eta_4 \delta e_2 = \eta_4 e_1 = e$. Therefore, $e \in \operatorname{Im} \tilde{\delta}$. Thus, $\ker \tilde{j}^* \subseteq \operatorname{Im} \tilde{\delta}$. Hence, $\operatorname{Im} \tilde{\delta} = \ker \tilde{j}^*$.

Theorem 11. The following are equivalent:

(1) g^* is onto,

(2) $\tilde{H}^0(X) \cong 0$.

Proof: Assume g^* is onto. Then $g^*H^0(Q) = H^0(X)$. Thus, $\tilde{H}^0(X) = H^0(X)/g^*H^0(Q) = H^0(X)/H^0(X) \cong 0$.

Assume $\tilde{H}^0(X) \cong 0$. Then $H^0(X)/g^*H^0(Q) = 0$. Thus, $g^*H^0(Q) = H^0(X)$. Hence, g^* is onto.

Lemma 3.3. (The Modification Lemma).

Let A and B be closed subsets of a fully normal space X. Let β be an open cover of X. Then there exists a function $f: X \to X$ (not necessarily continuous), an open cover γ of X, and an open set N such that:

(1) $N \supseteq A$,

(2) $f|\ (X \setminus \overline{N}) \cup A$ is the identity,

(3) $f(\overline{N} \cup B) \subseteq A \cap B$,

(4) $f(\overline{N}\setminus A) \subseteq A$, and

(5) if $V \in \gamma$, then there exists $U \in \beta$ such that $f(V) \subseteq V^* \subseteq U$.

Proof: Since X is fully normal, there is an open cover γ_0 of X such that $\gamma_0 < \beta$. Let

$\gamma_1 = \{V \cap (X\setminus A) \mid V \in \gamma_0, V \cap (X\setminus A) \neq \square\}$,

$\gamma_2 = \{V \cap (X\setminus B) \mid V \in \gamma_0, V \cap (X\setminus B) \neq \square\}$. and

$\gamma_3 = \{V \in \gamma_0 \mid V \cap A \cap B \neq \square\}$. Let $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$. Then γ is an open cover of X. Let $V \in \gamma$. If $V \in \gamma_1$, then $V \subseteq V_1$ for some $V_1 \in \gamma_0$. Thus, there exists $U \in \beta$ such that $V_1^* \subseteq U$. Hence, $V^* \subseteq U$. Similarly, if $V \in \gamma_2$, there exists $U \in \beta$ such that $V^* \subseteq U$. If $V \in \gamma_3$, then $V \in \gamma_0$. Thus, there exists $U \in \beta$ such that $V^* \subseteq U$. In any case, $V^* \subseteq U$ for some $U \in \beta$. Hence, $\gamma^* < \beta$. Let $M = \{V \in \gamma \mid V \cap A \neq \square\}$. Then $\bigcup M$ is an open set containing A. By Theorem 9, X is normal. Hence, there exists an open set N such that $A \subseteq N \subseteq \overline{N} \subseteq \bigcup M$. For every $V \in M$, define a_V as follows: if $V \in \gamma_1$ or γ_2, fix $a_V \in V \cap A$; if $V \in \gamma_3$, fix $a_V \in V \cap A \cap B$. Define $f: X \to X$ in the following way. For $x \in (X\setminus\overline{N}) \cup A$, let $f(x) = x$. For $x \in \overline{N}\setminus A$, choose some $V \in M$ such that $x \in V$. Let $f(x) = a_V$. From the definition of f, $f|(X\setminus\overline{N}) \cup A$ is the identity. Also, from the way the a_V's were chosen, $f(\overline{N}\setminus A) \subseteq A$. Let $x \in \overline{N} \cap B$. Since $x \in \overline{N}$, $x \in V$ for some $V \in M$. Therefore, $V \cap A \neq \square$. Thus, $V \notin \gamma_1$. Since $x \in B$, $V \notin \gamma_2$. Hence, $V \in \gamma_3$. Therefore, $f(x) = a_V \in A \cap B$. Hence, $f(\overline{N} \cap B) \subseteq A \cap B$. Thus, (1) - (4) are satisfied.

Now let $V \in \gamma$. Let $x \in V$. If $x \in (X \setminus \overline{N}) \cup A$, then $f(x) = x$. Therefore, $f(x) \in V \subseteq V^*$. If $x \in \overline{N} \setminus A$, then $f(x) \in V_1$ for some $V_1 \in \gamma$ such that $x \in V_1$. Thus, $V \cap V_1 \neq \square$. Hence, $V_1 \subseteq V^*$. Therefore, $f(x) \in V^*$. In any case, $f(x) \in V^*$. Hence, $f(v) \subseteq V^*$. Furthermore, $V \subseteq V_0$ for some $V_0 \in \gamma_0$. Thus, $V^* \subseteq V_0{}^*$. Since $\gamma * < \beta$, there exists $U \in \beta$ such that $V_0{}^* \subseteq U$. Therefore, $V^* \subseteq U$. Hence, $f(V) \subseteq V^* \subseteq U$. Hence, (5) is satisfied. Thus, the proof of the Modification Lemma is complete.

Lemma 3.4. Let A be a closed subset of X. Let $\phi \in Z^p(X,A)$. Then there exists an open cover β of X such that $\phi | \beta^{p+1} \cap A^{p+1} = 0$, and $\delta_p \phi | \beta^{p+2} \cap X^{p+2} = 0$.

Proof: If $\phi \in Z^p(X,A)$, then $\phi \in C^p(X,A)$, and $\delta_p \phi \in C^{p+1}(X,X)$. Thus, there exists an open cover β_1 of A such that $\phi | \beta_1{}^{p+1} \cap A^{p+1} = 0$. Also, there exists an open cover β_2 of X such that $\delta_p \phi | \beta_2{}^{p+2} \cap X^{p+2} = 0$. Since A is closed, $\beta_1 \cup (X \setminus A)$ is an open cover of X. Let $\beta = \{U \cap V | \ U \in \beta_1 \cup (X \setminus A),\ V \in \beta_2,\ U \cap V \neq \square\}$. Let $(x_0, \ldots, x_p) \in \beta^{p+1} \cap A^{p+1}$. Then $(x_0, \ldots, x_p) \in (U \cap V)^{p+1}$ for some $U \cap V \in \beta$. Then $U \in \beta_1 \cup (X \setminus A)$. Since $(x_0, \ldots, x_p) \in A^{p+1}$, $U \neq X \setminus A$. Thus, $U \in \beta_1$. Hence, $(x_0, \ldots, x_p) \in U^{p+1}$. Therefore, $(x_0, \ldots x_p) \in \beta_1{}^{p+1} \cap A^{p+1}$. Thus, $\phi(x_0, \ldots, x_p) = 0$. Now let $(x_0, \ldots, x_p) \in \beta^{p+2} \cap X^{p+2}$. Then $(x_0, \ldots, x_p) \in (U \cap V)^{p+2}$ for some $U \cap V \in \beta$. Thus, $V \in \beta_2$. Also, $(x_0, \ldots, x_p) \in V^{p+2}$. Hence,

$(x_0,\ldots,x_p) \in \beta_2{}^{p+2} \cap X^{p+2}$. Therefore, $\delta_p\phi(x_0,\ldots,x_p) = 0$. Hence, $\phi|\beta^{p+1} \cap A^{p+1} = 0$, and $\delta_p\phi|\beta^{p+2} \cap X^{p+2} = 0$.

Theorem 12. (Extension Theorem).
Let A and B be closed subsets of a fully normal space X. Let $h \in H^p(A, A \cap B)$. Then there exists an open set N, and there exists $e \in H^p(\overline{N}, \overline{N} \cap B)$ such that $A \subseteq N$, and $i^*(e) = h$ where $i: (A, A \cap B) \to (\overline{N}, \overline{N} \cap B)$.

Proof: Since $h \in H^p(A, A \cap B)$, then $h = \eta\phi$ for some $\phi \in Z^p(A, A \cap B)$. Now $A \cap B$ is closed in A. Thus, by Lemma 3.4, there exists a cover β_1 of A by sets open in A such that $\phi|\beta_1{}^{p+1} \cap (A \cap B)^{p+1} = 0$, and $\delta_p\phi|\beta_1{}^{p+2} \cap A^{p+2} = 0$. Let $\beta = \{U \subseteq X|\ U$ is open in X, $U \cap A \in \beta_1\}$. Then $\beta \cup (X\backslash A)$ is an open cover of X. By the Modification Lemma, there exists a function $f: X \to X$, an open cover γ of X, and an open set N such that $A \subseteq N$, $f|(X\backslash\overline{N}) \cup A$ is the identity, $f(\overline{N} \cap B) \subseteq A \cap B$, $f(\overline{N}\backslash A) \subseteq A$, and if $V \in \gamma$, then there exists $U \in \beta \cup (X\backslash A)$ such that $f(V) \subseteq V^* \subseteq U$. Define $\phi_1 \in C^p(\overline{N})$ by $\phi_1(x_0,\ldots,x_p) = \phi(f(x_0),\ldots,f(x_p))$ for all $(x_0,\ldots,x_p) \in \overline{N}^{p+1}$. Let $\gamma_1 = \{V \cap \overline{N}|\ V \in \gamma,\ V \cap \overline{N} \neq \square\}$. Then γ_1 is a cover of $\overline{N} \cap B$ by sets open in $\overline{N}$. Let $(x_0,\ldots,x_p) \in \gamma_1{}^{p+1} \cap (\overline{N} \cap B)^{p+1}$. Then there exists $V \in \gamma$ such that $(x_0,\ldots,x_p) \in V^{p+1}$. By the Modification Lemma, there exists $U \in \beta \cup (X\backslash A)$ such that $f(V) \subseteq U$. Now $U \neq X\backslash A$ since $f(\overline{N} \cap B) \subseteq A \cap B$. Therefore, $(f(x_0),\ldots,f(x_p)) \in \beta^{p+1} \cap (\overline{N} \cap B)^{p+1}$. Since

$f(\overline{N} \cap B) \subseteq A \cap B$, $(f(x_0),\ldots,f(x_p)) \in (A \cap B)^{p+1}$. Therefore, $(f(x_0),\ldots,f(x_p)) \in \beta^{p+1} \cap (A \cap B)^{p+1}$. Therefore, $\phi_1(x_0,\ldots,x_p) = \phi(f(x_0),\ldots,f(x_p)) = 0$. Hence, $\phi_1|\gamma_1{}^{p+1} \cap (\overline{N} \cap B)^{p+1} = 0$. Thus, $\phi_1 \in C^p(\overline{N}, \overline{N} \cap B)$.

Now let $(x_0,\ldots,x_{p+1}) \in \gamma_1{}^{p+2} \cap (\overline{N})^{p+2}$. Then there exists $V \in \gamma$ such that $(x_0,\ldots,x_{p+1}) \in V^{p+2}$. By the Modification Lemma, there exists $U \in \beta \cup (X\setminus A)$ such that $f(V) \subseteq U$. Now $U \neq X\setminus A$ since $f(\overline{N}) \subseteq A$. Also, $(f(x_0),\ldots,f(x_{p+1})) \in \beta^{p+2} \cap \overline{N}^{p+2}$, and $(f(x_0),\ldots,f(x_{p+1})) \in A^{p+2}$. Therefore, $(f(x_0),\ldots,f(x_{p+1})) \in \beta^{p+2} \cap A^{p+2}$. Thus, $\delta_p\phi_1(x_0,\ldots,x_{p+1}) = \delta_p\phi(f(x_0),\ldots,f(x_{p+1})) = 0$. Hence, $\delta_p\phi|\gamma_1{}^{p+2} \cap \overline{N}^{p+2} = 0$. Therefore, $\delta_p\phi_1 \in C^{p+1}(\overline{N},\overline{N})$. Hence, $\phi_1 \in Z^p(\overline{N}, \overline{N} \cap B)$.

Let $e = \eta\phi_1 \in H^p(\overline{N}, \overline{N} \cap B)$. Then $i^*e = \eta i^{\#}\phi_1$. Let $(x_0,\ldots,x_p) \in A^{p+1}$. Then $i^{\#}\phi_1(x_0,\ldots,x_p) = \phi_1(i(x_0),\ldots,i(x_p))$ $= \phi_1(x_0,\ldots,x_p) = \phi(f(x_0),\ldots,f(x_p))$. Since $f|A$ is the identity, $\phi(f(x_0),\ldots,f(x_p)) = \phi(x_0,\ldots,x_p) = \phi_1(x_0,\ldots,x_p)$. Therefore, $i^{\#}\phi_1 = \phi$. Hence, $\eta i^{\#}\phi_1 = \eta\phi = h$. Therefore, $i^*e = h$.

<u>Lemma</u> <u>3.5</u>. Let A and B be closed subsets of X. Let $t\colon (A, A \cap B) \subsetneq (X,B)$. Then $r^{\#}[B^p(X,B)] = B^p(A, A \cap B)$.

Proof: By Note 2.7, $t^{\#}[B^p(X,B)] \subseteq B^p(A, A \cap B)$. Let $\phi \in B^p(A, A \cap B)$. Then $\phi = \delta_{p-1}\alpha_1 + \alpha_2$ where $\alpha_1 \in C^{p-1}(A, A \cap B)$, and $\alpha_2 \in C^p(A,A)$. Then there exists a cover β of $A \cap B$ by sets open in A such that $\alpha_1|\beta^p \cap (A \cap B)^p = 0$. Also, there exists a cover γ of A by sets open in A such that $\alpha_2|\gamma^{p+1} \cap A^{p+1} = 0$. Let $\beta_1 = \{U| U \text{ is open in } X, U \cap A \in \beta\}$, and let $\gamma_1 = \{V| V \text{ is open in } X, V \cap A \in \gamma\}$. Define $\phi_1 \in C^{p-1}(X)$ by $\phi_1(x_0,\ldots,x_{p-1}) =$

$$\begin{cases} \alpha_1(x_0,\ldots,x_{p-1}) \text{ if } (x_0,\ldots,x_{p-1}) \in A^p \\ 0 \text{ otherwise.} \end{cases}$$

Since A is closed, $\beta_1 \cup (X\backslash A)$ is an open cover of B. Let $(x_0,\ldots,x_{p-1}) \in (\beta_1 \cup (X\backslash A))^p \cap B^p$. If $(x_0,\ldots,x_{p-1}) \in (X\backslash A)^p$, then $\phi_1(x_0,\ldots,x_{p-1}) = 0$. If $(x_0,\ldots,x_{p-1}) \in \beta_1{}^p$, then $(x_0,\ldots,x_{p-1}) \in U^p$ for some $U \in \beta_1$. If $x_i \notin A$ for some $0 \leq i \leq p-1$, then $\phi_1(x_0,\ldots,x_{p-1}) = 0$. If $x_i \in A$ for $0 \leq i \leq p-1$, then $(x_0,\ldots,x_{p-1}) \in (U \cap A)^p$. Therefore, $(x_0,\ldots,x_{p-1}) \in \beta^p \cap (A \cap B)^p$. Therefore, $\phi_1(x_0,\ldots,x_{p-1}) = \alpha_1(x_0,\ldots,x_{p-1}) = 0$. In any case, $\phi_1(x_0,\ldots,x_{p-1}) = 0$. Hence, $\phi_1|(\beta_1 \cup (X\backslash A))^p \cap B^p = 0$. Thus, $\phi_1 \in C^{p-1}(X,B)$.

Define $\phi_2 \in C^p(X)$ by $\phi_2(x_0,\ldots,x_p) =$

$$\begin{cases} \alpha_2(x_0,\ldots,x_p) \text{ if } (x_0,\ldots,x_p) \in A^{p+1} \\ 0 \text{ otherwise.} \end{cases}$$

Since A is closed, $\gamma_1 \cup (X\backslash A)$ is an open cover of X. Let $(x_0,\ldots,x_p) \in (\gamma_1 \cup (X\backslash A))^{p+1}$. If $(x_0,\ldots,x_p) \in (X\backslash A)^{p+1}$, then $\phi_2(x_0,\ldots,x_p) = 0$. If $(x_0,\ldots,x_p) \in \gamma^{p+1}$, then $(x_0,\ldots,x_p) \in V^{p+1}$ for some $V \in \gamma_1$. If $x_i \notin A$ for $0 \leq i \leq p$, then $\phi_2(x_0,\ldots,x_p) = 0$. If $x_i \in A$ for $0 \leq i \leq p$, then $(x_0,\ldots,x_p) \in (V \cap A)^{p+1}$. Therefore, $(x_0,\ldots,x_p) \in \gamma^{p+1} \cap A^{p+1}$.

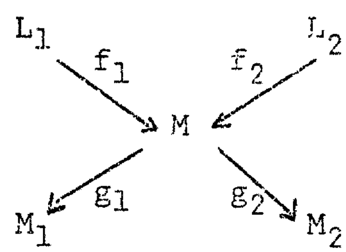
Thus, $\phi_2(x_0,\ldots,x_p) = \alpha_2(x_0,\ldots,x_p) = 0$. In any case, $\phi_2(x_0,\ldots,x_p) = 0$. Hence, $\phi_2|[\gamma_1 \cup (X\setminus A)]^{p+1} = 0$. Therefore, $\phi_2 \in C^p(X,X)$.

Now $\delta_{p-1}\phi_1 + \phi_2 \in B^p(X,B)$. Let $(x_0,\ldots,x_p) \in A^p$. Then $t^\#(\delta_{p-1}\phi_1 + \phi_2)(x_0,\ldots,x_p) =$
$(\delta_{p-1}\phi_1 + \phi_2)(t(x_0),\ldots,t(x_p)) =$
$(\delta_{p-1}\phi_1 + \phi_2)(x_0,\ldots,x_p) = \delta_{p-1}\phi_1(x_0,\ldots,x_p) + \phi_2(x_0,\ldots,x_p)$
$= \delta_{p-1}\alpha_1(x_0,\ldots,x_p) + \alpha_2(x_0,\ldots,x_p) = \phi(x_0,\ldots,x_p)$.
Therefore, $t^\#(\delta_{p-1}\phi_1 + \phi_2) = \phi$. Hence, $\phi \in t^\#[B^p(X,B)]$. Therefore, $B^p(A, A \cap B) \subseteq t^\#[B^p(X,B)]$. Hence, $t^\#[B^p(X,B)] = B^p(A, A \cap B)$.

<u>Theorem</u> <u>13</u>. (Reduction Theorem).
Let A and B be closed subsets of a fully normal space X. Let $j: (A, A \cap B) \subsetneq (X,B)$. Let $h \in H^p(X,B)$ such that $j^*(h) = 0$. Then there exists an open set N such that $A \subseteq N$, and $k^*(h) = 0$, where $k: (\overline{N}, \overline{N} \cap B) \subsetneq (X,B)$.
Proof: Since $h \in H^p(X,B)$, $h = \eta\phi$ for some $\phi \in Z^p(X,B)$. Therefore, there exists an open cover β of X such that $\delta_p\phi|\beta^{p+2} = 0$. By the Modification Lemma, there exists an open set N containing A. Let $k: (\overline{N}, \overline{N} \cap B) \subsetneq (X,B)$. Since $j^*(h) = 0$, $j^\#\phi \in B^p(X,B)$. By Lemma 3.5, $B^p(A,A \cap B) = j^\#[B^p(X,B)]$. Therefore, $\phi \in B^p(X,B)$. Also, by Lemma 3.5, $B^p(\overline{N}, \overline{N} \cap B) = k^\#[B^p(X,B)]$. Therefore, $k^\#\phi \in B^p(\overline{N}, \overline{N} \cap B)$. Thus, $\eta k^\#\phi = k^*(h) = 0$.

<u>Lemma</u> <u>3.6</u>. Let M, M_1, M_2, L_1, and L_2 be R-modules. Let f_1, f_2, g_1, and g_2 be R-homomorphisms such that

$\text{Im } f_1 = \ker g_1$, $\text{Im } f_2 = \ker g_2$, and g_2f_1, g_1f_2 are onto isomorphisms.



Then (1) $M = \text{Im } f_1 \oplus \text{Im } f_2 = \ker g_1 \oplus \ker g_2$;

(2) $g_1|\ker g_2$ is an isomorphism, and $g_2|\ker g_1$ is an isomorphism;

(3) $x \in \text{Im } f_1$ implies $x = f_1(g_2f_1)^{-1}g_2(x)$, and $x \in \text{Im } f_2$ implies $x = f_2(g_1f_2)^{-1}g_1(x)$;

(4) for every $x \in M$, $x = f_1(g_2f_1)^{-1}g_2(x) + f_2(g_1f_2)^{-1}g_1(x)$.

Proof: Since g_2f_1 is an isomorphism, Preliminary 6 gives $M = \text{Im } f_1 \oplus \ker g_2 = \text{Im } f_1 \oplus \text{Im } f_2 = \ker g_1 \oplus \ker g_2$. Thus, (1) is satisfied.

By Preliminary 3, $\ker g_2$ is a submodule of M. Therefore, $g_1|\ker g_2$ is an R-homomorphism. Let $a,b \in \ker g_2$. Then $a,b \in \text{Im } f_2$. Thus, $a = f_2(a')$, and $b = f_2(b')$. If $g_1(a) = g_1(b)$, then $g_1f_2(a') = g_1f_2(b')$. Since g_1f_2 is 1-1, $a' = b'$. Thus, $f_1(a') = f_1(b')$. Hence, $a = b$. Therefore, $g_1|\ker g_2$ is 1-1. Let $e \in M_1$. Since g_1f_2 is onto M_1, there exists $e' \in L_2$ such that $g_1f_2(e') = e$. Thus, $f_2e' \in \text{Im } f_2 = \ker g_2$. Therefore, $g_1(f_2e') = e$. Hence, $g_1|\ker g_2$ is onto. Therefore, $g_1|\ker g_2$ is an isomorphism. Similarly, $g_2|\ker g_1$ is an isomorphism. Thus, (2) is satisfied.

Let $x \in \text{Im } f_1$. Then $x \in \ker g_1$ Now since g_2f_1 is an isomorphism, $g_2(x) = g_2(f_1(g_2f_1)^{-1}g_2(x))$. Therefore, since $g_2| \ker g_1$ is 1-1, $x = f_1(g_2f_1)^{-1}g_2(x)$. Similarly, $x \in \text{Im } f_2$ implies $x = f_2(g_1f_2)^{-1}g_1(x)$. Thus, (3) is satisfied.

Let $x \in M$. By (1) of this lemma, $x = f_1(y) + f_2(z)$, where $y \in L_1$, $z \in L_2$. By (3) of this lemma,
$f_1(y) = f_1(g_2f_1)^{-1}g_2(f_1(y))$, and
$f_2(a) = f_2(g_1f_2)^{-1}g_2(f_2(z)$. Since $\text{Im } f_1 = \ker g_1$, and $\text{Im } f_2 = \ker g_2$, $g_2(x) = g_2(f_1(y) + f_2(z)) =$
$g_2f_1(y) + g_2f_2(z) = g_2f_1(y) + 0 = g_2f_1(y)$, and
$g_1(x) = g_1(f_1(y) + f_2(z)) = g_1f_1(y) + g_1f_2(z) =$
$0 + g_1f_2(z) = g_1f_2(z)$. Thus, $x = f_1(y) + f_2(z) =$
$f_1(g_2f_1)^{-1}g_2(f_1(y)) + f_2(g_1f_2)^{-1}g_1(f_2(z)) =$
$f_1(g_2f_1)^{-1}f_1(x) + f_2(g_1f_2)^{-1}g_1(x)$. Thus, (4) is satisfied.

The Mayer-Vietoris Theorem

Consider the following diagram, page 54. In Figure 2, X is a fully normal space, X_1, X_2 are closed subsets of X such that $X = X_1 \cup X_2$, and $A = X_1 \cap X_2$. Also,

$i_2\colon (X_2,A) \subsetneq (X,A)$,

$i\colon A \subsetneq X$,

$j_1\colon (X,A) \subsetneq (X,X_1)$,

$j_2\colon (X,A) \subsetneq (X,X_2)$,

$j\colon X \subsetneq (X,A)$,

$l_1\colon A \subsetneq X_1$,

$l_2\colon A \subsetneq X_2$,

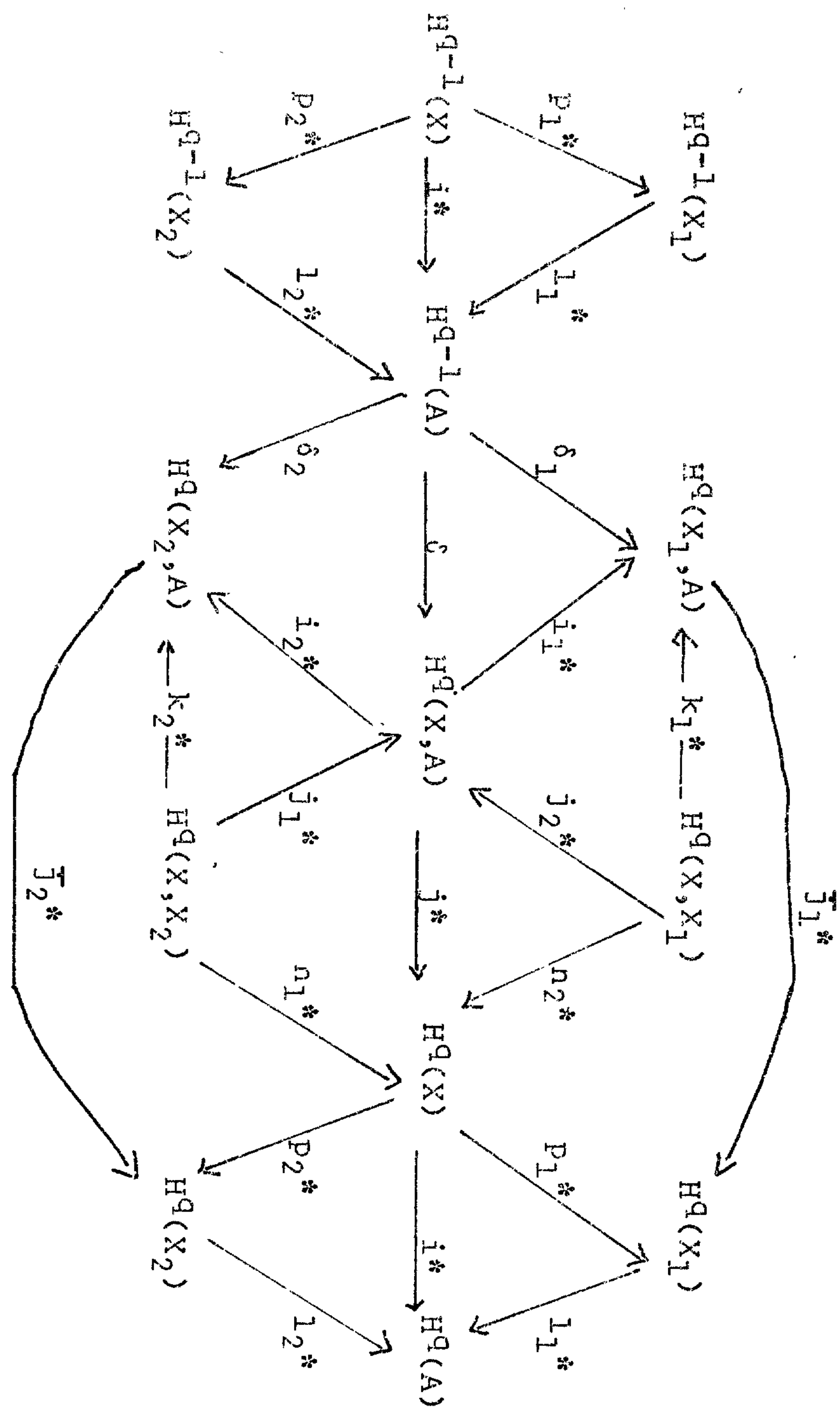


Fig. 2--M-V Sequence

$k_1: (X_1,A) \overset{\subseteq}{\to} (X,X_2)$,

$k_2: (X_2,A) \overset{\subseteq}{\to} (X,X_1)$,

$n_1: X \overset{\subseteq}{\to} (X,X_1)$,

$n_2: X \overset{\subseteq}{\to} (X,X_2)$,

$p_1: X_1 \overset{\subseteq}{\to} X$,

$p_2: X_2 \overset{\subseteq}{\to} X$,

$\bar{j}_1: X_1 \overset{\subseteq}{\to} (X_1,A)$,

$\bar{j}_2: X_2 \overset{\subseteq}{\to} (X_2,A)$.

Using Theorem 4, the following sequences are exact:

(1) $H^q(X,X_1) \overset{n_1^*}{\to} H^q(X) \overset{p_1^*}{\to} H^q(X_1)$,

(2) $H^q(X,X_2) \overset{n_2^*}{\to} H^q(X) \overset{p_2^*}{\to} H^q(X_2)$,

(3) $H^q(X_1) \overset{l_1^*}{\to} H^q(A) \overset{\delta_1}{\to} H^q(X_1,A)$,

(4) $H^q(X_2) \overset{l_2^*}{\to} H^q(A) \overset{\delta_2}{\to} H^q(X_2,A)$,

(5) $H^{q-1}(A) \overset{\delta}{\to} H^q(X,A) \overset{j^*}{\to} H^q(X) \overset{i^*}{\to} H^q(A)$.

Using Theorem 4A, the following sequences are exact:

(6) $H^q(X,X_1) \overset{j_1^*}{\to} H^q(X,A) \overset{i_1^*}{\to} H^q(X_1,A)$,

(7) $H^q(X,X_2) \overset{j_2^*}{\to} H^q(X,A) \overset{i_2^*}{\to} H^q(X_2,A)$,

(8) $H^q(X_1,A) \overset{\bar{j}_1^*}{\to} H^q(X_1) \overset{l_1^*}{\to} H^q(A)$,

(9) $H^q(X_2,A) \overset{\bar{j}_2^*}{\to} H^q(X_2) \overset{l_2^*}{\to} H^q(A)$.

The following are true by Theorem 2:

(10) $j^*j_1^* = n_1^*$,

(11) $j^*j_2^* = n_2^*$,

(12) $l_1^*p_1^* = i^*$,

(13) $l_2^*p_2^* = i^*$,

(14) $i_1^*j_2^* = k_1^*$,

(15) $i_2^*j_1^* = k_2^*$.

The following are true by Theorem 3:

(16) $i_1{*}\delta = \delta_1$,

(17) $i_2{*}\delta = \delta_2$.

Theorem 14. (The Mayer-Vietoris Theorem). Let X be fully normal. Let X_1, X_2 be closed subsets of X such that $X = X_1 \cup X_2$. Let $A = X_1 \cap X_2$. Define the following:

J: $H^q(X) \to H^q(X_1) \oplus H^q(X_2)$ by $J(x) = (p_1{*}(x), p_2{*}(x))$;

I: $H^q(X_1) \oplus H^q(X_2) \to H^q(A)$ by $I(x,y) = i_1{*}(x) - i_2{*}(y)$;

Δ: $H^q(A) \to H^{q+1}(X)$ by $\Delta(x) = -n_1{*}k_2{*}^{-1}\delta_2(x)$.

Then $\ldots \to H^q(X) \overset{J}{\to} H^q(X_1) \oplus H^q(X_2) \overset{I}{\to} H^q(A) \overset{\Delta}{\to} H^{q+1}(X) \overset{J}{\to} \ldots$ is an exact sequence.

Lemma 3.7. For $x \in H^q(A)$, $\Delta(x) = n_2{*}k_1{*}^{-1}\delta_1(x)$.

Proof: By (6), and (7), Im $j_1{*}$ = ker $i_1{*}$, and Im $j_2{*}$ = ker $i_2{*}$. Also, $i_2{*}j_1{*} = k_2{*}$, and $i_1{*}j_2{*} = k_1{*}$ are isomorphisms. Thus, the hypothesis of Lemma 3.6 is satisfied. Let $x \in H^{q-1}(A)$. Then $\delta(x) \in H^q(X,A)$. Therefore, by Lemma 3.6, $\delta(x) = j_1{*}(i_2{*}j_1{*})^{-1}i_2{*}(\delta(x)) + j_2{*}(i_1{*}j_2{*})^{-1}i_1{*}(\delta(x)) = j_1{*}k_2{*}^{-1}i_2{*}\delta(x) + j_2{*}k_1{*}^{-1}i_1{*}\delta(x) = j_1{*}k_2{*}^{-1}\delta_2(x) + j_2{*}k_1{*}\delta_1(x)$. Since Im δ = ker $j{*}$, $j{*}(\delta(x)) = 0$. Hence, $j{*}(j_1{*}k_2{*}^{-1}\delta_2(x) + j_2{*}k_1{*}^{-1}\delta_1(x)) = j{*}j_1{*}k_2{*}^{-1}\delta_2(x) + j{*}j_2{*}k_1{*}^{-1}\delta_1(x) = n_1{*}k_2{*}^{-1}\delta_2(x) + n_2{*}k_1{*}^{-1}\delta_2(x) = 0$. Therefore, $n_2{*}k_1{*}^{-1}\delta_1(x) = -n_1{*}k_2{*}^{-1}\delta_2(x) = \Delta(x)$. Hence, $\Delta(x) = n_2{*}k_1{*}^{-1}\delta_1(x)$.

Proof of Theorem 14: To show J: $H^0(X) \to H^0(X_1) \oplus H^0(X_2)$ is 1-1, let $J(x) = 0$. Then

$(p_1^*(x),\ p_2^*(x) = 0$. Hence, $p_1^*(x) = 0$, and $p_2^*(x) = 0$. Therefore, $x \in \ker p_1^* = \operatorname{Im} n_1^*$, and $x \in \ker p_2^* = \operatorname{Im} n_2^*$. Therefore, $x = n_1^*(a) = n_2^*(b)$ for some $a \in H^q(X,X_1)$, and $b \in H^q(X,X_2)$. Since $n_1^* = j^*j_1^*$, and $n_2^* = j^*j_2^*$, $x = j^*j_1^*(a) = j^*j_2^*(b)$. By (5), $j^*\colon H^0(X,A) \to H^0(X)$ is 1-1. Thus, $j_1^*(a) = j_2^*(b)$. Therefore, $i_1^*j_1^*(a) = i_1^*j_2^*(b)$. Since $\operatorname{Im} j_1^* = \ker i_1^*$, $i_1^*j_1^*(a) = 0$. Therefore, $i_1^*j_2^*(b) = 0$. Since $i_1^*j_2^*$ is an isomorphism, $b = 0$. Hence, $j^*j_2^*(b) = x = 0$. Hence, $J\colon H^0(X) \to H^0(X_1) \oplus H^0(X_2)$ is 1-1.

To show $\operatorname{Im} \Delta \subseteq \ker J$, let $\Delta(x) \in H^q(X)$. Then $J(\Delta(x)) = (p_1^*(\Delta(x)),\ p_2^*(\Delta(x))) = (p_1(-n_1^*k_2^{*-1}\delta_2(x)),\ p_2^*(n_2^*k_1^{*-1}\delta_1(x)))$. Since $\operatorname{Im} n_1^* = \ker p_1^*$, and $\operatorname{Im} n_2^* = \ker p_2^*$, then $p_1^*(-n_1^*k_2^{*-1}\delta_2(x)) = 0$, and $p_2^*n_2^*k_1^{*-1}\delta_1(x) = 0$. Therefore, $J(\Delta(x)) = (0,0)$. Thus, $\Delta(x) \in \ker J$. Hence, $\operatorname{Im} \Delta \subseteq \ker J$.

To show $\operatorname{Im} J \subseteq \ker I$, let $J(x) \in H^q(X_1) \oplus H^q(X_2)$. Then $I(J(x)) = I(p_1^*(x),\ p_2^*(x)) = l_1^*p_1^*(x) - l_2^*p_2^*(x)$. By (12), and (13), $l_1^*p_1^*(x) = i^*(x)$, and $l_2^*p_2^*(x) = i^*(x)$. Thus, $I(J(x)) = i^*(x) - i^*(x) = 0$. Therefore, $J(x) \in \ker I$. Hence, $\operatorname{Im} J \subseteq \ker \Delta$.

To show $\operatorname{Im} I \subseteq \ker \Delta$, let $I(x,y) \in H^q(A)$. Then $\Delta(I(x,y)) = \Delta(l_1^*(x) - l_2^*(y)) = -n_1^*k_2^{*-1}\delta_2(l_1^*(x) - l_2^*(y))$ $= -n_1^*k_2^{*-1}\delta_2 l_1^*(x) + n_1^*k_2^*\delta_2 l_2^*(y) =$ $n_2^*k_1^{*-1}\delta_1 l_1^*(x) + n_1^*k_2^*\delta_2 l_2^*(y)$. By (3) and (4),

$\delta_1 l_1^*(x) = 0$, and $\delta_2 i_2(x) = 0$. Thus, $\Delta(I(x,y)) = 0 + 0 = 0$. Therefore, $I(x,y) \in \ker \Delta$. Hence, $\operatorname{Im} I \subseteq \ker \Delta$.

To show $\ker J \subseteq \operatorname{Im} \Delta$, let $x \in \ker J$, where $x \in H^q(X)$. Then $J(x) = (p_1^*(x), p_2^*(x)) = 0$. Therefore, $p_1^*(x) = 0$, and $p_2^*(x) = 0$. Hence, $x \in \ker p_1^* = \operatorname{Im} n_1^*$, and $x \in \ker p_2^* = \operatorname{Im} n_2^*$. Therefore, $x = n_1^*(x_1) = n_2^*(x_2)$, where $x_1 \in H^q(X,X_1)$, and $x_2 \in H^q(X,X_2)$. By (10) and (11), $x = j^*j_1^*(x_1) = j^*j_2^*(x_2)$. Therefore, $j^*j_1^*(x_1) - j^*j_2^*(x_2) = j^*(j_1^*(x_1) - j_2^*(x_2)) = 0$. Thus, $j_1^*(x_1) - j_2^*(x_2) \in \ker j^* = \operatorname{Im} \delta$. Hence, $j_1^*(x_1) - j_2^*(x_2) = \delta(y)$ for some $y \in H^{q-1}(A)$. By (16), (15), and (7), $\delta_2(y) = i^*_2\delta(y) = i_2^*(j_1^*(x_1) - j_2^*(x_2)) = i_2^*j_1^*(x_1) - i_2^*j_2^*(x_2) = i_2^*j_1^*(x_1) - 0 = i_2^*j_1^*(x_1)$. Thus, $\Delta(-y) = -n_1^*k_2^{*-1}\delta_2^*(-y) = -n_1^*k_2^{*-1}(-k_2^*(x_1)) = n_1^*k_2^{*-1}(k_2^*(x_1)) = n_1^*(x_1) = x$. Therefore, $x \in \operatorname{Im} \Delta$. Hence, $\ker J \subseteq \operatorname{Im} \Delta$.

To show $\ker \Delta \subseteq \operatorname{Im} I$, let $x \in \ker \Delta$, where $x \in H^q(A)$. Then $\Delta(x) = n_2^*k_1^{*-1}\delta_1(x) = j^*j_2^*k_1^{*-1}\delta_1(x) = 0$. Therefore, $j_2^*k_1^{*-1}\delta_1(x) \in \ker j^* = \operatorname{Im} \delta$. Thus, $j_2^*k_1^{*-1}\delta_1(x) = \delta(a)$ for some $a \in H^q(A)$. Hence, $i_1^*(j_2^*k_1^{*-1}\delta_1(x)) = i_1^*\delta(a)$. Since $i_1^*j_2^*k_1^{*-1} = \text{identity}$, $\delta_1(x) = i_1^*\delta(a)$. By (16), $i_1^*\delta(a) = \delta_1(a)$. Thus, $\delta_1(x) - \delta_1(a) = \delta_1(x - a) = 0$. Therefore, $x - a \in \ker \delta_1 = \operatorname{Im} l_1^*$. Thus, $x - a = l_1^*(y)$ for some $y \in H^{q-1}(X_1)$. Also, $i_2^*(j_2^*k_1^{*-1}\delta_1(x)) = i_x^*\delta(a)$. By (17) and (7), $i_2^*\delta(a) = \delta_2(a) = 0$. Therefore, $a \in \ker \delta_2 = \operatorname{Im} l_2^*$. Thus, $a = l_2^*(z)$ for some $z \in H^{q-1}(X_2)$. Now $(y, -z) \in H^{q-1}(X_1) \oplus H^{q-1}(X_2)$, and

$I(y, -z) = l_1^*(y) - l_2^*(-z) = x - a - (-a) = x$. Thus, $x \in \text{Im } I$. Hence, $\ker \Delta \subseteq \text{Im } I$.

To show $\ker I \subseteq \text{Im } J$, let $(y,z) \in \ker I$, where $(y,z) \in H^q(X_1) \oplus H^q(X_2)$. Then $I(y,z) = l_1^*(y) - l_2^*(z) = 0$. Thus, $\delta_1(l_1^*(y) - l_2^*(z)) = \delta_1 l_1^*(y) - \delta_1 l_2^*(z) = 0$. By (3), $\delta_1 l_1^*(y) = 0$. Therefore, $\delta_1 l_2^*(z) = 0$. By (16), $\delta_1 l_2^*(z) = i_1^* \delta l_2^*(z)$. Therefore, $\delta l_2^*(z) \in \ker i_1^*$. By (4) and (17), $\delta_2 k_2^*(z) = i_2^* \delta l_2^*(z) = 0$. By Lemma 3.6, $i_2^*|\ker i_1^*$ is an isomorphism. Therefore, $i_2^*(\delta l_2^*(z)) = 0$ implies $\delta l_2^*(z) = 0$. Hence, $l_2^*(z) \in \ker \delta = \text{Im } i^*$. Therefore, $l_2^*(z) = i^*(b)$ for some $b \in H^q(X)$. By (12) and (13), $i^*(b) = l_1^* p_1^*(b) = l_2^*(z)$, and $i^*(b) = l_2^* p_2^*(b) = l_2^*(z)$. Therefore, $l_1^*(y) - l_1^* p_1^*(b) = l_1^*(y - p_1^*(b)) = 0$, and $l_2^* p_2^*(b) - l_2^*(z) = l_2^*(p_2^*(b) - z) = 0$. Therefore, by (8) and (9), $y - p_1^*(b) \in \ker l_1^* = \text{Im } \bar{j}_1^*$, and $p_2^*(b) - z \in \ker l_2^* = \text{Im } \bar{j}_2^*$. Thus, $y - p_1^*(b) = \bar{j}_1^*(g)$ for some $g \in H^q(X_1,A)$, and $p_2^*(b) - z = \bar{j}_2^*(h)$ for some $h \in H^q(X_2,A)$. Since k_1^* and k_2^* are onto isomorphisms, $g = k_1^*(g')$ for some $g' \in H^q(X,X_2)$, and $h = k_2^*(h')$ for some $h' \in H^q(X,X_1)$. Therefore, $\bar{j}_1^*(g) = \bar{j}_1^* k_1^*(g')$, and $\bar{j}_2^*(h) = \bar{j}_2^* k_2^*(h')$. Since $k_1 \bar{j}_1 = n_2 p_1$, $(k_1 \bar{j}_1)^* = (n_2 p_1)^*$. Thus, by Theorem 2, $\bar{j}_1^* k_1^* = p_1^* n_2^*$. Similarly, $\bar{j}_2^* k_2^* = p_2^* n_1^*$. Hence, $\bar{j}_1^* k_1^*(g') = p_1^* n_2^*(g')$, and $\bar{j}_2^* k_2^*(h') = p_2^* n_1^*(h')$. Therefore, $y - p_1^*(b) = p_1^* n_2^*(g')$, and $p_2^*(b) - z = p_2^* n_1^*(h')$. Therefore, $y = p_1 n_2^*(g') + p_1^*(b) = p_1^*(n_2^*(g') + b))$, and

$z = p_2^*n_1^*(h') + p_2^*(b) = p_2^*(n_1^*(h') + b)$. Let $x = n_2^*(g') + b + n_1^*(h') \in H^q(X)$. Then $J(x) = (p_1^*(x), p_2^*(x)) = (p_1^*(n_2^*(g') + b + n_1^*(h')), p_2^*(n_2^*(g') + b + n_1^*(h'))) = (p_1^*(n_2^*(g') + b) + p_1^*(n_1^*(h')), p_2^*(h_1''(h') + b) + p_2^*(n_2^*(g'))) = (y + p_1^*n_1^*(h'), z + p_2^*n_2^*(g'))$. By (1) and (2), $p_1^*n_1^*(h') = 0$, and $p_2^*n_2^*(g') = 0$. Hence, $J(x) = (y + 0, z + 0) = (y,z)$. Therefore, $(y,z) \in \operatorname{Im} J$. Hence, $\ker I \subseteq \operatorname{Im} J$.

Therefore, $\operatorname{Im} J = \ker I$, $\operatorname{Im} I = \ker \Delta$, and $\operatorname{Im} \Delta = \ker J$. Hence, the sequence is exact.

CHAPTER IV

APPLICATIONS

<u>Theorem</u> <u>15</u>. If X is connected, then $H^0(X) \cong G$, where G is the coefficient module.

Proof: Let X be connected. Since $B^0(X) = 0$, $H^0(X) = Z^0(X)/B^0(X) = Z^0(X) = C^0(X) \cap \delta_0^{-1}C^1(X,X)$. Let $\phi \in H^0(X)$. Then $\phi \in \delta_0^{-1}C^1(X,X)$. Thus, there exists an open cover β of X such that $\delta_0\phi|\beta^2 = 0$. Let $a,b \in X$. Then by Preliminary 10, there exists a finite subcollection of β, $U_1,\ldots,U_n$, such that $a \in U_1$, $b \in U_n$, and $U_i \cap U_j \neq \square$ if and only if $|i - j| \leq 1$. Therefore, $U_1 \cap U_2 \neq \square$. Let $k_1 \in U_1 \cap U_2$. Then $(a,k_1) \in \beta^2$. Therefore, $\delta_0\phi(a,k_1) = \phi(a) - \phi(k_1) = 0$. Thus, $\phi(a) = \phi(k_1)$. Similarly, $\phi(k_1) = \phi(k_2)$ for some $k_2 \in U_2 \cap U_3$. Thus, $\phi(a) = \phi(k_1) = \ldots = \phi(k_i) = \ldots = \phi(k_{n-1}) = \phi(b)$, where $k_i \in U_{i-1} \cap U_i$, and $\phi(k_{i-1}) = \phi(k_i)$. Hence, $\phi(a) = \phi(b)$. Since a and b were arbitrary, for any $a,b \in X$, $\phi(a) = \phi(b)$. Therefore, ϕ maps X onto a single point of G. For $x \in G$, define $x\phi \in C^0(X)$ by $x\phi(a) = x$ for each $a \in X$. Now $x\phi \in H^0(X)$, since X is an open cover of X, and $\phi_0 x\phi|X^2 = 0$. Define $F\colon H^0(X) \to G$ by $F(x\phi) = x$ for $x\phi \in H^0(X)$. The above shows that F is well-defined and onto. Let $r,t \in R$, $x\phi, y\phi \in H^0(X)$. Then $rF(x\phi) + tF(y\phi) = rx + ty = F(rx\phi + ty\phi)$. Thus, F is an R-homomorphism. Let

$F(x\phi) = 0$. Then $x = 0$. Thus, $x\phi = 0$. Therefore, F is 1-1. Hence, $F\colon H^0(X) \to G$ is an isomorphism.

Corollary 4.1. If X is connected, then $\tilde{H}^0(X) = 0$.

Proof: Let Q and g be as in Figure 1. By Theorem 15, $H^0(X) \cong G$. By Theorem 7, $H^0(Q) \cong G$. Let $\phi \in H^0(X) = Z^0(X) \cong G$. Then for some $a \in G$, $\phi(x) = a$ for all $x \in X$. Since $H^0(Q) \cong G$, $\phi \in H^0(Q)$. Thus, $\phi(Q) = a$. Now $g^*\phi = g^\#\phi$. Let $x \in X$. Then $g^\#\phi(x) = \phi(g(x)) = \phi(Q) = a = \phi(x)$. Thus, $g^*\phi = \phi$. Hence, g^* is onto. Therefore, by Theorem 11, $\tilde{H}^0(X) = 0$.

Definition 4.1. Let $A \subseteq X$. Let $h \in H^p(X,A)$ such that $h \neq 0$. If there exists a closed subset F of X such that $i^*(h) \neq 0$, where $i\colon (F, F \cap A) \subsetneq (X,A)$, but $i_M{}^*(h) = 0$ for every closed proper subset M of F, where $i_M\colon (M, M \cap A) \subsetneq (X,A)$, then F is called a floor for h.

Theorem 16. Let A be a closed subset of a compact Hausdorff space X. Let $h \in H^p(X,A)$ such that $h \neq 0$. Then there exists a floor for h.

Proof: Let $K = \{F \subseteq X \mid F$ is closed, $i^*(h) \neq 0$, where $i\colon (F, F \cap A) \subsetneq (X,A)\}$. Let $i\colon (A,A) \subsetneq (X,A)$. Now $h = \eta\phi$ for some $\phi \in Z^p(X,A)$. Thus, $i^*(h) = i^*\eta\phi = \eta i^\#\phi$. Let $(x_0,\ldots,x_p) \in A^{p+1}$. Then $i^\#\phi(x_0,\ldots,x_p) = \phi(i(x_0),\ldots,i(x_p)) = \phi(x_0,\ldots,x_p)$. Therefore, $i^\#\phi = \phi$. Since $\eta\phi = h \neq 0$, $\eta i^\#\phi \neq 0$. Therefore, $i^*(h) \neq 0$. Hence, $A \in K$. Thus, $K \neq \square$. Partially order K by set inclusion. Let $\mathcal{P}$ be a linearly ordered subfamily of K. Let $F = \bigcap \mathcal{P}$. Then F is closed. Suppose $i_F^*(h) = 0$, where $i_F: (F, F \cap A) \subsetneq (X,A)$. By the Reduction Theorem, there exists an open set N such that $k^*(h) = 0$, where $k: (\bar{N}, \bar{N} \cap A) \subsetneq (X,A)$, and $F \subseteq N$. Since P is a descending family of closed sets, and N is an open set containing $\cap P$, by Preliminary 8, there exists $Q \in P$ such that $Q \subseteq N$. Let $i_Q: (Q, Q \cap A) \subsetneq (X,A)$. Now $i_Q^*(h) = i_Q^*\eta\phi = \eta i_Q^{\#}\phi$. Let $(x_0,\ldots,x_p) \in Q^{p+1}$. Then $i_Q^{\#}\phi(x_0,\ldots,x_p) = \phi(i_Q(x_0),\ldots,i_Q(x_p)) = \phi(k(x_0),\ldots,k(x_p)) = k^{\#}\phi(x_0,\ldots,x_p)$. Therefore, $i_Q^{\#}\phi = k^{\#}\phi$. Thus, $\eta i_Q^{\#}\phi = \eta k^{\#}\phi$. Therefore, $i_Q^*(h) = k^*(h) = 0$. But since $Q \in K$, $i_Q^*(h) \neq 0$. This is a contradiction. Hence, $i_F^*(h) \neq 0$. Therefore, $F \in K$. Thus, by the Hausdorff Minimal Principle, K has a smallest element, F_1. Clearly, F_1 is a floor for h.

<u>Theorem 17</u>. Let X be fully normal, and $h \in H^p(X)$. Let F be a floor for h. If $p > 0$, then F is connected.

Proof: Since F is a floor, F is closed. Thus, by Theorem 8, F is fully normal. Suppose F is not connected. Then $F = A \cup B$, where A and B are separated. Thus, A and B are closed in F. Let $i_A: A \subsetneq F$, $i_B: B \subsetneq F$, $i: F \subsetneq X$, $j_A: A \subsetneq X$, and $j_B: B \subsetneq X$. Since F is a floor, $i^*(h) \neq 0$. Define $J: H^p(F) \to H^p(A) \oplus H^p(B)$ by $J(x) = (i_A^*(x), i_B^*(x))$. By Theorem 14, $\ker J = \operatorname{Im} \Delta$, where $\Delta: H^{p-1}(A \cap B) \to H^p(F)$. By Theorem 2, $i_A^*i^*(h) = j_A(h)$. Since F is a floor for h, and A is a closed subset of F, $j_A(h) = 0$. Therefore, $i_A^*(i^*(h)) = 0$.

Likewise, $i_B^*(i^*(h)) = j_B^*(h) = 0$. Therefore, $J(i^*(h)) = (i_A^*(i^*(h)), i_B^*(i^*(h)) = (0,0)$. Therefore, $i^*(h) \in \ker J = \text{Im } \Delta$. Since A and B are separated, $A \cap B = \square$. Thus, $H^{p-1}(A \cap B) = H^{p-1}(\square) = 0$. Therefore, Im $\Delta = 0$. Hence, $i^*(h) = 0$. This is a contradiction. Hence, F is connected.

Definition 4.2. Let X be a continuum. X is unicoherent if whenever A and B are subcontinua covering X, $A \cap B$ is a subcontinuum.

Definition 4.3. A continuum X is heriditarily unicoherent if each subcontinua of X is unicoherent.

Note 4.1. If X is a heriditarily unicoherent continuum, the minimal subcontinua between two points is unique.

Proof: Let A and B be the minimal subcontinua between a and b. Then $a,b \in A$, and $a,b \in B$. Hence, $A \cap B \neq \square$. Thus, $A \cup B$ is connected. Also, $A \cup B$ is compact. Therefore, $A \cup B$ is a subcontinuum of X. Now A and B are subcontinua covering $A \cup B$. Therefore, since X is heriditarily unicoherent, $A \cap B$ is a subcontinuum. Since $a,b \in A \cap B$, $A \cap B$ is a subcontinuum between a and b. Since A and B are minimal, $A \subseteq A \cap B$, and $B \subseteq A \cap B$. Hence, $A = A \cap B = B$. Hence, a minimal subcontinuum between two points is unique.

In Corollary 4.1, it was shown that if X is connected, then $\tilde{H}^0(X) = 0$. If it is assumed that $\tilde{H}^0(X) = 0$ implies that X is connected, then the following can be proved: If X is a continuum and $H^1(X) = 0$, then X is unicoherent. An immediate consequence of this is that the n-cell is unicoherent.

Modules of the n-cell

Let S = n-cell = $\{x \in R^n \mid \|x\| \leq 1\}$.

Lemma 4.1. The n-cell is contractible.

Proof: Let $i: S \to S$ by $i(x_1,\ldots,x_n) = (x_1,\ldots,x_n)$. Let $k: S \to S$ by $k(x_1,\ldots,x_n) = (0,\ldots,0)$. Then k is a constant map. Define $h: S \times I \to S$, where I is the unit interval, by $h(x_1,\ldots,x_n, a) = (ax_1,\ldots,ax_n)$. Then
$h(x_1,\ldots,x_n, 1) = (x_1,\ldots,x_n) = i(x_1,\ldots,x_n)$, and
$h(x_1,\ldots,x_n, 0) = (0,\ldots,0) = k(x_1,\ldots,x_n)$, for all
$(x_1,\ldots,x_n) \in S$. Therefore, $i \simeq k$. Hence, S is contractible.

Theorem 18. For G a coefficient module,

$$H^p(S) \cong \begin{cases} G \text{ if } p = 0, \\ 0 \text{ if } p > 0. \end{cases}$$

Proof: By Lemma 4.1, S is contractible. Also, S is compact. Therefore, by Corollary 2.5, $H^p(S) \cong \begin{cases} G \text{ if } p = 0, \\ 0 \text{ if } p > 0. \end{cases}$

Modules of the n-sphere

Let X_n = n-sphere = $\{x \in R^{n+1} \mid \|x\| = 1\}$. Let S_n = n-cell.

Theorem 19. Let G be the coefficient module.

(a) If $p = n = 0$, then $H^p(X_n) \cong G \times G$.

(b) If $p = 0$, and $n > 0$, then $H^p(X_n) \cong G$.

(c) If $p = n$, and $n > 0$, then $H^p(X_n) \cong G$.

Proof of (a): Now $X_0 = \{1, -1\}$. Since X_0 is fully normal, and $X_0 = \{1\} \cup \{-1\}$, by Theorem 14,

$H^0(X_0) \overset{J}{\to} H^0(\{1\}) \oplus H^0(\{-1\}) \overset{I}{\to} H^0(\square)$ is an exact sequence. Therefore, $J: H^0(X_0) \to H^0(\{1\}) \oplus H^0(\{-1\})$ is 1-1. Since $H^0(\square) = 0$, $\ker I = H^0(\{1\}) \oplus H^0(\{-1\})$. Hence, $\operatorname{Im} J = H^0(\{1\}) \oplus H^0(\{-1\})$. Therefore, J is onto. Thus, J is an isomorphism. By the Dimension Theorem, $H^0(\{1\}) \cong G$, and $H^0(\{-1\}) \cong G$. Therefore, $J: H^0(X_0) \to G \times G$ is an isomorphism.

Proof of (b): Now X_n is connected for $n > 0$. Therefore, by Theorem 15, $H^0(X) \cong G$.

<u>Lemma 4.2</u>. For any n, $X_n = A \cup B$, where $A = \{(x_0,\ldots,x_n) \in X_n | \ x_n \geq 0\}$, and $B = \{(x_0,\ldots,x_n) | \ x_n \leq 0\}$. Furthermore, A is homeomorphic to X_n, B is homeomorphic to S_n, and $A \cap B$ is homeomorphic to X_{n-1}.

Proof: Clearly, $X_n = A \cup B$. For $(x_0,\ldots,x_n) \in A$, $x_0^2 + \ldots + x_n^2 = 1$. Thus, $x_0^2 + \ldots + x_{n-1}^2 \leq 1$. Therefore, $x_0,\ldots,x_{n-1} \in S_n$. Define $f: A \to S_n$ by $f(x_0,\ldots,x_n) = (x_0,\ldots,x_{n-1})$. Let $\varepsilon > 0$. Let $N((x_0,\ldots,x_n), \varepsilon) = \{(a_0,\ldots,a_n) \in R^{n+1} | \ (x_0-a_0)^2 + \ldots + (x_n-a_n)^2 < \varepsilon\}$. Let $(x_0,\ldots,x_n) \in A$. Let $0 < \delta < \varepsilon$. Let $(a_0,\ldots,a_n) \in N((x_0,\ldots,x_n), \delta)$. Then $(x_0-a_0)^2 + \ldots + (x_n-a_n)^2 < \delta$. Therefore, $(x_0-a_0)^2 + \ldots + (x_{n-1}-a_{n-1})^2 < \delta - (x_n-a_n)^2 < \delta < \varepsilon$. Therefore, $(a_0,\ldots,a_{n-1}) \in N((x_0,\ldots,x_{n-1}), \varepsilon)$. Hence, f is continuous.

Let $f(x_0,\ldots,x_n) = f(a_0,\ldots,a_n)$, where $(x_0,\ldots,x_n),(a_0,\ldots,a_n) \in A$. Then $x_0^2 + \ldots + x_n^2 = 1 = a_0^2 + \ldots + a_n^2$. Also, $(x_0,\ldots,x_{n-1}) = (a_0,\ldots,a_{n-1})$.

Therefore, $x_n^2 = a_n^2$. Since $(x_0,\ldots,x_n), (a_0,\ldots,a_n) \in A$, $x_n \geq 0$, and $a_n \geq 0$. Hence, $x_n^2 = a_n^2$ implies $x_n = a_n$. Hence, $(x_0,\ldots,x_n) = (a_0,\ldots,a_n)$. Therefore, f is 1-1.

Let $(x_0,\ldots,x_{n-1}) \in S_n$. Then $x_0^2 + \ldots + x_{n-1}^2 \leq 1$. Thus, $1 - (x_0^2 + \ldots + x_{n-1}^2) \geq 0$. Let $x_n = \sqrt{1-(x_0^2+\ldots+x_{n-1}^2)}$. Therefore, $(x_1,\ldots,x_n) \in A$, and $f(x_0,\ldots,x_n) = (x_0,\ldots,x_n)$ Hence, f is onto. Therefore, $f\colon A \to S_n$ is a continuous, 1-1, onto function, and A is compact. Hence, by Preliminary 11, A is homeomorphic to S_n. Similarly, B is homeomorphic to S_n.

Now $A \cap B = \{(x_0,\ldots,x_n) \in X_n \mid x_n = 0\}$. Define $g\colon X_{n-1} \to A \cap B$ by $g(x_0,\ldots,x_{n-1}) = (x_0,\ldots,x_{n-1}, 0)$. Clearly, g is a homeomorphism.

Proof of (c): The proof is by induction. First it will be shown that $H^1(X_1) \cong G$. Let g,h,Q be as in Figure 1. By Theorem 7, $H^0(Q) \cong G$. Thus, there exists an isomorphism $F\colon H^0(Q) \to G$. By (a) of this Theorem, and Lemma 4.2, $H^0(A \cap B) \cong G \times G$, where $A = \{(x,y) \in X_1 \mid y \geq 0\}$, and $B = \{(x,y) \in X_1 \mid y \leq 0\}$. Then $\tilde{H}^0(A \cap B) = H^0(A \cap B)/h^*H^0(Q) \cong G \times G/h^*H^0(Q)$. Let $\phi \in \ker h^*$. Then $h^*\phi = h^\#\phi = 0$. Thus, for $x \in A \cap B$, $h^\#\phi(x) = \phi(h(x)) = \phi(Q) = 0$. Hence, $\phi = 0$. Therefore, h^* is 1-1. Define $f\colon h^*H^0(Q) \to G$ by $f(h^*(a)) = F(a)$ for each $h^*(a) \in h^*H^0(Q)$. Since h^* is 1-1, f is well-defined. If $x \in G$, then $x = F(a)$ for some $a \in H^0(Q)$. Thus, $f(h^*(a)) = F(a) = x$. Therefore, f is onto. If $f(h^*(a)) = 0$, then $F(a) = 0$. Thus, $a = 0$. Therefore,

$h^*(a) = 0$. Hence, f is 1-1. Let $r,t \in R$, $h^*(a), h^*(b) \in h^*H^0(Q)$. Then $rf(h^*(a)) + tf(h^*(b)) = rF(a) + tF(b) = F(ra + tb) = f(rh^*(a) + th^*(b))$. Therefore, f is an R-homomorphism. Hence, $f\colon h^*H^0(Q) \to G$ is an isomorphism. Therefore, $\tilde{H}^0(A \cap B) \cong G \times G/h^*H^0(Q) \cong G \times G/G \cong G$.

Now $\tilde{H}^0(A) = H^0(A)/g^*H^0(Q)$. By Theorem 18, $H^0(A) \cong G$. Similar to the above, $g^*H^0(B) \cong G$. Thus, $H^0(A) \cong G/G \cong 0$. Also, $\tilde{H}^0(B) \cong 0$.

The following sequence is exact:

$$\tilde{H}^0(X_1) \overset{J}{\to} \tilde{H}^0(A) \oplus \tilde{H}^0(B) \overset{I}{\to} \tilde{H}^0(A \cap B) \overset{\Delta}{\to} H^1(X_1) \overset{J}{\to} H^1(A) \oplus H^1(B).$$

By Theorem 18, $H^1(A) \cong 0$, and $H^1(B) \cong 0$. Therefore, $\ker J = H^1(X_1)$. Since $\ker J = \operatorname{Im} \Delta$, Δ is onto. Since $\tilde{H}^0(A) \cong 0$, and $\tilde{H}^0(B) \cong 0$, $\tilde{H}^0(A) \oplus H^0(B) \cong 0$. Therefore, $\operatorname{Im} I = 0$. Since $\operatorname{Im} I = \ker \Delta = 0$, Δ is 1-1. Therefore, Δ is an isomorphism. Since $\tilde{H}^0(A \cap B) \cong G$, $H^1(X_1) \cong G$.

Now assume that $H^{n-1}(X_{n-1}) \cong G$. Let $A = \{(x_0,\ldots,x_n) \in X_n \mid x_n \geq 0\}$, and $B = \{(x_0,\ldots,x_n) \in X_n \mid x_n \leq 0\}$. By Theorem 14, the following sequence is exact:

$$H^{n-1}(A) \oplus H^{n-1}(B) \overset{I}{\to} H^{n-1}(A \cap B) \overset{\Delta}{\to} H_n(X_n) \overset{J}{\to} H^n(A) \oplus H^n(B).$$

By Theorem 18, $H^n(A) \oplus H^n(B) \cong 0$. Therefore, $\ker J = \operatorname{Im} \Delta = H^n(X_n)$. Thus, Δ is onto. Also, by Theorem 18, $H^{n-1}(A) \oplus H^{n-1}(B) \cong 0$. Therefore, $\operatorname{Im} I = \ker \Delta = 0$. Thus, Δ is 1-1. Therefore, $\Delta\colon H^{n-1}(A \cap B) \to H^n(X_n)$ is an isomorphism. By Lemma 4.2 and the induction hypothesis, $H^{n-1}(A \cap B) \cong G$. Hence, $H^n(X_n) \cong G$. Hence, by induction,

BIBLIOGRAPHY

M. R. Hagan, unpublished notes on topology, Department of Mathematics, North Texas State University, 1971-1972.

Y. W. Lau, unpublished notes on algebra, Department of Mathematics, North Texas State University, 1971-1972.