A*-ALGEBRAS AND MINIMAL IDEALS

IN TOPOLOGICAL RINGS

APPROVED:

B. D. Malviya
Major Professor

George Copp
Minor Professor

Frank Conner
Director of the Department of Mathematics

Robert B. Toulouse
Dean of the Graduate School

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The present thesis mainly concerns B*-algebras, A*-algebras, and minimal ideals in topological rings. It is divided into three chapters and an appendix:

Chapter I Preliminaries

Chapter II B*-algebras and A*-algebras

Chapter III Minimal ideals in topological rings.

Appendix

In Chapter I we reproduce all the basic important definitions which are used in this thesis. For the most part, these definitions have been adapted from the books by Dunford, Jacobson, Naimark and Rickart, listed in the Bibliography.

Chapter II is devoted to the study of B*-algebras and A*-algebras. This chapter has been divided into seven sections. In Section 1, for our later use, we prove two theorems about Banach *-algebras with the condition $||x||^2 \leq k||x^*x||$, with k being a positive constant. In Section 2, we consider the uniqueness problem of an auxiliary norm of an A*-algebra. In Section 3, we characterize a dual B*-algebra as a weakly completely continuous B*-algebra. In Section 4, we show that a c.c. Banach *-algebra under certain conditions becomes an A*-algebra and that a c.c. A*-algebra under certain conditions becomes dual. For a c.c. dual A*-algebra A, we show that every closed right ideal of

A is the intersection of regular maximal right ideals of A containing it. In Section 5, we consider the conditions under which a dual A*-algebra becomes completely continuous. In Section 6, we study weakly completely continuous A*-algebra and obtain a necessary and sufficient condition for a dual A*-algebra to be of the first kind. In Section 7, we study certain properties of dual A*-algebras.

Chapter III deals with minimal ideals in a topological ring. In Section 1, we study some minimal ideal in topological rings. In Section 2, we work on some purely algebraic developments concerning minimal ideals. Finally we list some fundamental theorems used in the text of the present thesis in the form of an appendix.

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CHAPTER I

PRELIMINARIES

All algebras unless otherwise stated are over the field of complex numbers.

Let A be any algebra and a mapping $x \to x^*$ of A onto itself is called an INVOLUTION * provided the following conditions are satisfied:

(1) $(x^*)^* = x$

(2) $(x+y)^* = x^* + y^*$

(3) $(xy)^* = y^* x^*$

(4) $(\alpha x)^* = \bar{\alpha} x^*$, where α is a complex number.

An algebra with an involution is called a *-ALGEBRA.

A complete normed algebra is called a BANACH ALGEBRA. A Banach algebra with an involution * is called a BANACH *-ALGEBRA.

Let A be normed *-algebra and let k be a positive constant. If for every x in A the condition $||x||^2 \leq k||x^*x||$ is satisfied, we say that the normed *-algebra has WEAKENED B*-PROPERTY. If a normed *-algebra satisfies the condition $||x||^2 = ||x^*x||$ for every x, we say that the normed *-algebra has B*-PROPERTY. Any Banach *-algebra with B*-condition is called a B*-algebra. Two B*-algebras A and B are said to be EQUIVALENT if (1) there exists a *-isomorphism ϕ of A onto B, and (2) if ϕ, ϕ^{-1} are continuous.

Let A be a Banach *-algebra. A is called C*-ALGEBRA if (1) A is a B*-algebra, and (2) $-x^*x$ has a quasi-inverse for every x in A.

Let A be any Banach *-algebra. A is called an A*-ALGEBRA if the following conditions are satisfied: (1) there exists an auxiliary norm $|x|$ such that $|xy| \leq |x||y|, x, y \in A$, and (2) $|x|^2 = |x^*x|$.

A*-algebras in general do not have a unique auxiliary norm. The completeness of an A*-algebra with respect to $|\cdot|$ is not necessary. A*-algebras always include B*-algebras but A*-algebras are not necessarily B*-algebras. Two norms $|x|$ and $|x|_1$ in a normed space X are EQUIVALENT if $|x_n| \to 0$ if and only if $|x_n|_1 \to 0$, $x_n \in X$.

A mapping $f: x \to Y$ is called CLOSED if whenever $x_n \to x$ and $f(x_n) \to y$, it follows that $y = f(x)$. If X,Y are Banach space, and T is a closed linear operator from X to Y with the domain of T equal to X, then (1) there are positive constants r,M such that $||Tx|| \leq M$ whenever $||x|| \leq r$, and (2) T belongs to the set of bounded linear operators from X to Y (CLOSED GRAPH THEOREM).

The sum of all minimal left (right) ideals in an algebra is called the LEFT (RIGHT) SOCLE of the algebra. If the left and right socles exist and are equal, then the resulting two-sided ideal is called simply the SOCLE of an algebra. See Rickart (14, p. 46) for details. Let A be any algebra and S be the socle of A. If $S^a = L(S) = \{x \in A:$

$xS=(0)\}=R(S)$, then S^a is called the ANTI-SOCLE of A. The left (right) socle exists if and only if the algebra contains minimal left (right) ideals.

The QUOTIENT of a left ideal L in an algebra A is the two-sided ideal L:A consisting of all $a \in A$ such that $aA \subseteq L$. A two-sided ideal is called PRIMITIVE if it is the quotient of a maximal modular left ideal. In a Banach algebra, every maximal modular ideal is closed, hence the primitive ideal is also closed.

Let A be an algebra and let e be in A such that $e \neq 0, 1$ and $e^2=e$, then e is called a PROPER IDEMPOTENT in A. Two idempotents e_1 and e_2 are said to be ORTHOGONAL if $e_1e_2=e_2e_1=0$. An idempotent is said to be PRIMITIVE if it is not the sum of two mutually orthogonal idempotents. An idempotent e such that eAe is a division algebra is called a MINIMAL IDEMPOTENT.

An element x belonging to an algebra A is called SELF-ADJOINT if $x=x^*$. A *-algebra A is said to be SYMMETRIC if $-x^*x$ has a quasi-inverse for every $x \in A$. If every closed commutative self-adjoint subalgebra of a *-algebra is symmetric, then it is called a C-SYMMETRIC ALGEBRA. A symmetric algebra is always C-symmetric.

Let A and B denote two *-algebras, then a homomorphism $a \to a^t$ of A into B is called a *-HOMOMORPHISM if $(a^*)^t=(a^t)^*$ for every $a \in A$. Let A and B be two *-algebras, then an isomorphism $a \to a^t$ of A onto B is called a *-ISOMORPHISM if $(a^*)^t=(a^t)^*$ for every $a \in A$.

Let $\{A_i\}$ be a set of B*-algebras. Let $x=\{x_i\}$, $x_i \epsilon A_i$ with $||x_i||$ bounded and define $||x||=\sup||x_i||$, then the resulting algebra A is called the B*-SUM of $\{A_i\}$. The algebraic operations in A are defined in an obvious way. The set of all sequences $\{x_i\}$ with the property that for any positive ϵ, $||x_i||<\epsilon$ for all but a finite number of x_i's. We call this subalgebra the B*(∞)-SUM.

A set in a topological space X is said to be RELATIVELY COMPACT if its closure is compact in X. An operator T in a Banach space X is said to be COMPLETELY CONTINUOUS if it maps every bounded set into a relatively compact set. Let T belong to the set of bounded linear operators from X into Y. Let S be the closed unit sphere in X. The operator T is said to be WEAKLY COMPLETELY CONTINUOUS if the weak closure of T(s) is compact in weak topology of Y. A Banach algebra A is said to be COMPLETELY CONTINUOUS (WEAKLY COMPLETELY CONTINUOUS) provided that right- and left-multiplication by any element of A are completely continuous (weakly completely continuous) operators on A.

A set R of elements (x,y,z,...) is called a TOPOLOGICAL RING if

(1) R is a ring,

(2) R is a locally convex topological linear space,

(3) The product xy is a continuous in x(y) for each fixed y(x).

Let A be any algebra and I be any subset of A. The set $R(I)=\{x : x\in A, Ix=(0)\}$ ($L(I)=\{x\in x\in A, Ix=(0)\}$), then $R(I)$ ($L(I)$) is called the RIGHT (LEFT) ANNIHILATOR of I.
A left (right) ideal M(N) is called a LEFT (RIGHT) ANNIHILATOR IDEAL if it has the form $M=L(I)$ ($N=R(I)$) for some set $I\subseteq A$ where A is an algebra. A topological algebra A is called an ANNIHILATOR ALGEBRA if, for any closed left ideal I_ℓ and any closed right ideal I_r in A, the following conditions are satisfied:

(1) $R(I_\ell)=(0)$ if and only if $I_\ell=A$, and

(2) $L(I_r)=(0)$ if and only if $I_r=A$.

Let A be a topological algebra. For every closed right ideal I_r we have $R[L(I_r)]=I_r$ and $L(I_r)=(0)$ if and only if $I_r=A$. And for every closed left ideal I_ℓ we have $L[R(I_\ell)]=I_\ell$ and $R(I_\ell)=(0)$ if and only if $I_\ell=A$. Then A is called a DUAL ALGEBRA.

A ring A with no nilpotent one-sided ideals$\neq(0)$ is called a SEMI-PRIME ring. An element x of a ring such that $x^n=(0)$ for some positive integer n is called a NILPOTENT element.

Let A be a semi-prime ring and B be a subset of A. If $L(M)\neq(0)$ ($R(M)\neq(0)$) for every modular maximal right (left) ideal M of A, then A is called a LEFT (RIGHT) MODULAR ANNIHILATOR RING.

Let A be a commutative Banach algebra, then $\hat{A}$ is said to be a GELFAND REPRESENTATION of A if $\hat{A}$ is an algebra of complex-valued continuous functions vanishing at ∞ on two locally compact Hausdorff space Ω which is the set of modular maximal ideals of A. See Rickart (14, pp. 118-120). A is called REGULAR provided, for every closed set $F \subset \Omega$ and $p_o \in \Omega - F$, there exists $x \in A$ such that $\hat{x}(f)=0$ and $\hat{x}(p_o)=1$, where Ω is a locally compact Hausdorff space and $\hat{x} \in \hat{A}$ its Gelfand representation.

The space of bounded linear functionals on a normed space X is called the DUAL (or CONJUGATE) and is denoted by X*. And X** is dual (or conjugate) of X*. Let X be a normed linear space, and X** be the conjugate of the Banach space X*. The mapping $\phi : x \to \hat{x}$ of X onto X**, defined by $\hat{x}\, x^* = x^* x$ for every $x^* \in X^*$, then X is said to be REFLEXIVE.

Let U be a Banach algebra. Let π_U denote the set of all primitive ideals. Let A be a subset of U and F a subset of π_U. Then the set L(A) of all $B \in \pi_U$ which contain A is called the HULL OF A in π_U and the intersection k(F) of all the ideals in F is called the KERNEL of F in U. If F= h(k(F)) then F is called a HULL and, if A=k(h(A)), then A is called a KERNEL. The topology deformined in π_U by a closure operation $E \to \overline{E}$, where E is any subset of π_U, (if E is empty, define $E=\overline{E}$, otherwise define $\overline{E}=h(k(E))$) is called the HULL-KERNEL TOPOLOGY. The space π_U under its hull-kernel topology is called the STRUCTURE SPACE of U.

Let A be a dual A*-algebra with an auxiliary norm $|x|$. Let U be the completion of A by $|x|$. If A is an ideal of

U, we say that A IS OF THE FIRST KIND. If A is not an ideal of U, we say that A IS OF THE SECOND KIND.

Let U be a ring and e be an idempotent element in U, then every $a \in U$ can be written in the form $a=ea+(a-ea)$. Evidently, the set $\{ea : a \in U\}=eU$ and $(1-e)U=\{a-ea : a \in U\}$ since $eb=b$ for all $b \in U$ and $eb=0$ for all $b \in (1-e)U$, $eU \cap (1-e)U=0$. Thus we have

$U=eU \oplus (1-e)U$ ...(1).

Likewise $U=Ue \oplus U(1-e)$ ...(2).

Moreover $U=eUe \oplus eU(1-e) \oplus (1-e)Ue \oplus (1-e)U(1-e)$ ...(3).

The decompositions (1), (2) and (3) of the additive group $(U,+)$ are called, respectively, THE RIGHT, LEFT AND TWO-SIDED PEIRCE DECOMPOSITIONS OF U relative to the idempotent element e.

Let A be any algebra over a field $\mathcal{F}$. Let X be a linear vector space over the same field $\mathcal{F}$ and denote by L(X) the algebra of all linear transformations of X into itself. Then any homomorphism of A into the algebra L(X) is called a REPRESENTATION of A in L(X) or on X. A linear subspace M of X is said to be INVARIANT with respect to B if $T(M) \subseteq M$ for every $T \in B$, where B is a subalgebra of the algebra of all linear operators on the linear space X. And B is said to be STRICTLY IRREDUCIBLE provided (0) and X are the only invariant subspaces. In the case of a normed vector space X, (0) and X are the only closed invariant subspaces.

The RADICAL R of an algebra A is the intersection of the kernels of all (strictly) irreducible representations of A. If the radical R of A is (0), then A is said to be SEMI-SIMPLE. If the radical R of A coincides with A, then A is called a RADICAL ALGEBRA. The radical is a two-sided ideal in A.

The STRONGLY RADICAL R_s of an algebra A is the intersection of all maximal modular two-sided ideals of A unless there are no such ideals, in which case $R_s=A$. If $R_s=(0)$, the A is said to be STRONGLY SEMI-SIMPLE.

A Banach *-algebra A whose underlying Banach space is a Hilbert space with inner product (x,y) such that $(xy,z)=(y,x^*z)=(x,zy^*)$ for all $x,y,z\epsilon a$, then A is called an H*-ALGEBRA. An H*-algebra is said to be PROPER if $xA=(0)$ implies $x=0$.

Let U be a normed algebra. A collection $\{e_\Lambda:\Lambda\epsilon\Lambda\}$ of elements of U, where the index set Λ is a directed set, is called an APPROXIMATE IDENTITY for U if the following two conditions are satisfied:

(1) $||e_\Lambda||\leq 1$ for each Λ, and

(2) $\lim_\lambda e_\Lambda x=x=\lim_\lambda xe_\Lambda$ for every $x\epsilon U$.

Let Λ be an index set of arbitrary cardinality, finite or infinite. Denote by M_Λ the collection of all doubly index sets $\{\xi_{\lambda\mu}\}$ of complex numbers $\xi_{\lambda\mu}(\lambda,\mu\epsilon\Lambda)$ such that $\sum_\lambda\sum_\mu|\xi_{\lambda\mu}|^2<\infty$.

Let $x=\{\xi_{\lambda\mu}\}$ and $y=\{\eta_{\lambda\mu}\}$ be any two elements of M_Λ and define

$$\alpha x=\{\alpha\xi_{\lambda\mu}\}$$

$$x+y=\{\xi_{\lambda\mu}+\eta_{\lambda\mu}\}$$
$$xy=\{\sum_{\nu}\xi_{\lambda\nu}\eta_{\nu\mu}\}$$
$$(x,y)=\sum_{\lambda,\mu}\xi_{\lambda\mu}\eta_{\lambda\mu}\ .$$

Under these operations and inner product (x,y), M_Λ is called a FULL MATRIX ALGEBRA OF ORDER Λ over the complex field.

Let U be the Banach algebra of all real or complex continuous functions vanishing at ∞ on a locally compact Hausdorff space X. Then any other norm on U (whether complete or not) is at least as large as the function norm. Therefore we say that the usual (function) norm of U has a MINIMAL CHARACTER.

The CENTER of an algebra A is the set of these elements $a\epsilon A$ which commute with all the elements of A. The set of idempotents of an algebra A is called the CENTRAL IDEMPOTENT if it is the center of A.

Let X be a linear topological space. A generalized sequence $\{x_\alpha\}$ in X is said to be WEAKLY CONVERGENT if there is an x in X with $x^*x=\lim_\alpha x^*x_\alpha$ for every $x^*\epsilon X^*$ where X^* is a conjugate space of X. The point x is called a WEAK LIMIT of the generalized sequence, and the generalized sequence $\{x_\alpha\}$ is said to CONVERGE WEAKLY to x. Every sequence $\{x_n\}$ such that $\{x^*x_n\}$ is a Cauchy sequence of scalars for each $x^*\epsilon X^*$ is called a WEAK CAUCHY SEQUENCE. The space X is said to be WEAKLY COMPLETE if every weak Cauchy sequence has a weak limit.

A series $\{x_n\}$ in a normed linear space is said to be SUMMABLE to a sum s if s is in the space and the sequence of partial sums of the series converges to s, i.e., $||s-\sum_{i=1}^{\infty} x_i||\to 0$. If this is the case, we write $s=\sum_{i=1}^{\infty} x_i$.

Let $\{I_\lambda : \lambda\in\Lambda\}$ be a family of (left, right) ideals in an algebra A. Then the smallest (left, right) ideal in A which contains every I_λ is called the SUM of the ideal I_λ. If the intersection of each I_λ with the sum of the remaining ideals contains only zero, then the sum is called a DIRECT SUM. If A is a topological algebra, then the closure of the sum of ideals I_λ is called their TOPOLOGICAL SUM. If each I_λ is closed and intersects the topological sum of the remaining ideals in the zero element, then the topological sum is called a DIRECT TOPOLOGICAL SUM.

Let $\{R_i : i\in S\}$ be a collection of rings by a set S and consider the product set of the sets R_i. The elements of the product set are functions a(i), $i\in S$, $a(i)\in R_i$, we define addition and multiplication in this set by $(a+b)(i)=a(i)+b(i)$, $(ab)(i)=a(i)b(i)$. It is easy to verify that the result is a ring. And $\{R_i\}$ is called a COMPLETE DIRECT SUM. A subring B of a complete direct sum is called a SUBDIRECT SUM if and only if $b\to b(i)$ is an onto for every i.

CHAPTER II

B*-ALGEBRAS AND A*-ALGEBRAS

Throughout this chapter we let w.c.c. and c.c. stand for WEAKLY COMPLETELY CONTINUOUS and COMPLETELY CONTINUOUS, respectively.

In Section 1 we prove two theorems for our later use. In Section 2 we prove that under certain conditions an A*-algebra has a unique auxiliary norm (Theorem 2.2.5). In Section 3 we characterize a dual B*-algebra as a w.c.c. B*-algebra (Theorem 2.3.2). In Section 4 we show that a c.c. Banach *-algebra under certain conditions becomes an A*-algebra (Theorem 2.4.1) and a c.c. A*-algebra under certain conditions becomes dual (Theorem 2.4.3). For a c.c. dual A*-algebra A, we show that every closed right ideal of A is the intersection of regular maximal right ideals of A containing it (Theorem 2.4.4). In Section 5 we consider the conditions under which a dual A*-algebra becomes c.c (Theorem 2.5.4). In Section 6 we study w.c.c. A*-algebra and obtain a necessary and sufficient condition for a dual A*-algebra to be of the first kind (Theorem 2.6.6). In Section 7 we study certain properties of dual A*-algebras.

1. Two Basic Theorems

Theorem 2.1.1: Let A be a Banach *-algebra in which $||x||^2 \leq k||x^*x||$, k being a positive constant, holds. Let I be a closed ideal of A, then I is self-adjoint and $^A/I$ is a Banach *-algebra satisfying $||x||^2 \leq \gamma||x^*x||$ for a positive constant γ, which may be chosen to depend only on k continuously and reduces to 1 for k=1.

Proof: Let $x \in A$ and c be a positive constant. Write $y=[c(x^*x)^2]'x+x$ then $y^*=\{[c(x^*x)^2]'\}^*x^*+x^*=[c(x^*x)^2]'x^*+x^*$. So that $yy^*=\{[c(x^*x)^2]'\}^2x^*x+2[c(x^*x)^2]'x^*x+x^*x=\{[c(x^*x)^2]'+1\}^2x^*x=[(cz^2)'+1]^2Z$, where $Z=x^*x$.

Since the function $f(t)=\dfrac{t}{(ct^2+1)^2}$ has its maximum for positive t and $ct^2=\frac{1}{3}$. We have that $||y^*y||=||[(cz^2)'+1]^2Z||$ and the maximum value of the function f(t) is $\dfrac{9(\frac{1}{3c})^{\frac{1}{2}}}{16}$ for $ct^2=\frac{1}{3}$. Hence $[ct^2+1]^2=\frac{9}{16}$. Therefore, $||y^*y||\leq\frac{9}{16}\cdot\dfrac{1}{(3c)^{\frac{1}{2}}}$.

This implies that $||y||^2\leq k'\cdot\frac{9}{16}\cdot\dfrac{1}{(3c)^{\frac{1}{2}}}$ for some constant k'.

Take $k'=1$, then we can write $||y||^2\leq\frac{9}{16}\cdot\dfrac{1}{(3c)^{\frac{1}{2}}}$, i.e.,

$$||y||\leq\frac{3}{4}\cdot\frac{1}{(3c)^{\frac{1}{4}}}=\frac{3^{\frac{3}{4}}}{4}\cdot\frac{1}{c^{\frac{1}{4}}}.$$ Let $c=d^4$, so that $||y||\leq\dfrac{3^{\frac{3}{4}}}{4d}$.

Let I be any closed ideal in A. If $x^*\in I$, then $z=x^*x\in I$ and $(cz^2)'\in I$ since $IA\subset I$. As $c,d\to\infty$ implies $||y||=0$, i.e., $x\in I$, this implies $I^*\subset I$. But $(I^*)^*=I\subset I^*$, therefore $I^*=I$.

By Theorem 4.9.2 [14, p. 249], we know that $^{A}/I$ is a Banach *-algebra satisfying $||x||^2 \leq \gamma ||x^*x||$, for a positive constant γ. Q.E.D.

Theorem 2.1.2: Let A,B be Banach *-algebras satisfying $||x||^2 \leq k||x^*x||$ and $||x||^2 \leq k'||x^*x||$, respectively, where k, k' are positive constants. If there exists an algebraic *-isomorphism ϕ of A onto B and A,B are equivalent, more precisely $\frac{1}{\alpha\beta}||x|| \leq ||\phi(x)|| \leq \alpha\beta||x||$ where α, β are constants.

Proof: The proof follows from Theorem 3 [11, p. 311].

Q.E.D.

2. Auxiliary Norms of A*-algebra

Lemma 2.2.1: Let A,B be Banach *-algebras satisfying $||x||^2 \leq k||x^*x||$ and $||x||^2 \leq k'||x^*x||$, respectively, where k and k' are positive constants. Let ϕ denote a *-homomorphism of A into B, which is isomorphic on a dense *-subalgebra A' of A. Then ϕ maps A *-isomorphically into B if any of the following conditions is satisfied:

(1) A' is an ideal of A,

(2) if I is any closed ideal of A with $I \cap A' = 0$, then $Ia' = 0$.

Proof: First we will show that (1) implies (2). Assume that A' is an ideal of A. Let I be any closed ideal of A with $I \cap A' = 0$. Since $AA' \subset A'$ and $I \subset A$, then $IA' \subset A$. Since $IA \subset I$ and $A' \subset A$, then $IA' \subset I$. Therefore $IA' \subset I \cap A' = 0$. It follows that $IA' = 0$.

Suppose that (2) holds. Let I denote the kernel of ϕ, i.e., $I=\{x\in A:\phi(x)=0\}$ and let R(I) be the right annihilator of I. Since $I\cap A'=0$ implies $IA'=0$, hence $A'\subset R(I)$. By hypothesis A' is dense in A, hence $\overline{A'}\subset\overline{R(I)}$. In a normed algebra, R(I) is closed ideal. This implies that $\overline{A'}\subset R(I)$, i.e., $A\subset R(I)$ and clearly $R(R)\subset A$. Therefore $R(I)=A$. Since $R(I)=A$, then $LR(I)=L(A)=(0)$, because A is semi-simple. But $I\subset LR(I)=(0)$, i.e., $I=(0)$. In order to prove the isomorphism we must prove that the kernel of the homomorphism is zero and this is precisely the case as $I=(0)$. Q.E.D.

Lemma 2.2.2: Let A,B be Banach *-algebras satisfying $||x||^2\leq k||x^*x||$ and $||x||^2\leq k'||x^*x||$, respectively, where k,k' are positive constants. Let A',B' be dense *-subalgebras of A,B respectively. Assume that A' satisfies any of the conditions (1) and (2) of Lemma 2.2.1.

If ϕ is a continuous *-isomorphism of A' onto B', then ϕ is uniquely extensible to a *-isomorphism of A onto B, and A,B are equivalent.

Proof: Since ϕ is continuous *-isomorphism of A' onto B' and A' is dense *-subalgebra of A, by using the principle of extension by continuity [4, p. 29], ϕ can be uniquely extended to a *-homomorphism ϕ' of A into B.

By Lemma 2.2.1, we have that ϕ' is a *-isomorphism of A onto B. By Theorem 2.1.2, ϕ is a *-isomorphism of A onto B; also A and B are equivalent. Q.E.D.

Lemma 2.2.3: Let A be an A*-algebra with an auxiliary norm $|\cdot|$ in which $||ax||\leq c||a||\ |x|$ for every $a, x\in A$, c being a constant. Let U denote the completion of A by $|x|$, then A is an ideal dense in U.

Proof: Consider the linear mapping $x\rightarrow ax$ from a *-subalgebra A of U to A with its own norm $||x||$. Since $||ax||\leq c\ ||a||\ |x|$ for every $a, x\in A$ so that the linear mapping $x\rightarrow ax$ is uniformly continuous, and therefore is uniquely extensible to a linear mapping $z\rightarrow az$ from U to A with $||az||\leq c||a||\ |z|$ [4, p. 29]. This shows that A is a right ideal in U. Also since U is the completion of A with respect to $|\cdot|$, then $\bar{A}=U$, i.e., A is dense in U.

On the other hand, since the involution in A*-algebra is continuous [14, p. 187, Theorem 4.1.15], so the involution is continuous in both A and its completion U with respect to $|\cdot|$. This implies that $||xa||\leq||x||\ ||a||\leq c'|x|\ ||a||$ for every $x, a\in A$ and c' being a constant [16, p. 626, Theorem 5.4]. In like manner A is a left ideal dense in U. This completes the proof. Q.E.D.

Lemma 2.2.4: Let A' be a dense ideal of an A*-algebra A with norm $||x||$. If A' is a Banach algebra with norm $||x||_1$, then

(1) $||ax||_1\leq c||a||_1||x||$, and

(2) $||xa||_1\leq c||x||\ ||a||_1$

for every $a\in A'$ and $x\in A$, c being a constant.

Proof: In view of the closed graph theorem, it suffices to show that the mapping $x \to ax$ from A to A′ is closed.

Let $||x_n - x|| \to 0$ and $||ax_n - y||_1 \to 0$. Since the mapping $a \to a$ from A′ to A is continuous [14, p. 187, Theorem 4.1.15], then $||ax_n - y|| \to 0$, i.e., $ax_n = y$. Therefore $ax = y$, and hence the mapping $x \to ax$ from A to A′ is closed. Therefore we have $||ax||_1 \leq c||a||_1 ||x||$ for every $a \in A'$, $x \in A$ and c being a constant.

In the same manner, we can prove that $a \to ax$ is closed and then we also have $||xa||_1 \leq c||x|| \; ||a||_1$ for every $a \in A'$, $x \in A$ and c being a constant. Q.E.D.

Theorem 2.2.5: Let A be an A*-algebra with an auxiliary norm $|x|$. A has a unique auxiliary norm if any of the following conditions is satisfied:

(1) $||ax|| \leq \alpha ||a|| \; |x|$ for every $a, x \in A$ and α being a constant, and

(2) the socle of A is dense in A.

Proof: Let $|x|_1$ be any other auxiliary norm of A. Set $|x|_2 = |x| + |x|_1$, then A is clearly a normed algebra with respect to $|x|_2$.

By the weak B*-property, we let $|x|^2 \leq k|x^*x|$ and $|x|_1^2 \leq k_1|x^*x|$ where k, k_1 are positive constants. Then $|x^*x|_2 = |x^*x| + |x^*x|_1 \geq \frac{1}{k}|x|^2 + \frac{1}{k_1}|x|_1^2 \geq \frac{1}{k_2}(|x| + |x|_1)^2 = \frac{1}{k_2}|x|_2^2$ where $k_2 = 2 \max(k, k_1)$. This shows that $|x|_2$ is also an auxiliary norm of A. Let U, U_1, U_2 denote the completions of A by $|x|$, $|x|_1$, $|x|_2$ respectively.

Case (1): Assume that (1) is satisfied. Lemma 2.2.3 shows that A is a dense ideal in both U and U_2. In fact, $||ax||\leq\beta||a||\ |x|_1$ and $||ax||\leq\alpha||a||\ |x|$, therefore $2||ax||\leq ||a||(\alpha|x|+\beta|x|_1)$, i.e., $||ax||\leq\gamma||a||(|x|+|x|_1)$, where $\phi=\max(\alpha,\beta)$ and γ is some constant. This implies that $||ax||\leq \gamma||a||\ |x|_2$ for every x, $a\epsilon A$. Therefore, by Lemma 2.2.3, A is a dense ideal in U_2. From Lemma 2.2.2, U, U_2 and U_1, U_2 are equivalent, i.e., U and U_1 are equivalent. This implies that A has a unique auxiliary norm.

Case (2): By weakened B*-property $x^*x=0$ implies $x=0$. Since every minimal right ideal R of A is generated by a uniquely determined self-adjoint primitive idempotent, hence by Lemma 2.1 [15, p. 29], every minimal right ideal R can be written as R=eA, where e is a minimal self-adjoint idempotent. Let I be any closed ideal of U_2 such that $I\cap A=(0)$. We want to show that IA=(0). Suppose that $IA\neq(0)$, then there exists an R in A such that $IR\neq 0$ and therefore $Ie\neq(0)$, since R is generated by e. Let $z\epsilon I$ with $ze\neq 0$. Since eAe=(the complex field)e [5, p. 13, Proof of Theorem 5], $ez^*ze\epsilon I$. Therefore, $ez^*ze=\lambda e(\lambda\neq 0)$, hence $e\epsilon I$. But $e\epsilon A$, i.e., $I\cap A\neq(0)$, a contradiction. Therefore IA=(0). Again, by Lemma 2.2.2, we have that U, U_2 and U_1, U_2 are equivalent respectively, and therefore U, U_1 are equivalent, i.e., the norms $|x|$, $|x|_1$ are equivalent. Q.E.D.

Theorem 2.2.6: Let A be an A*-algebra with an auxiliary norm $|x|$ satisfying (1) or (2) of the preceding Theorem 2.2.5. Let B be a Banach *-algebra with weakened B*-property and ϕ be an algebraic *-homomorphism of A into B. Then ϕ can be uniquely extended to a continuous *-homomorphism ϕ' of the completion U of A by $|x|$ into B and $\phi'(U)$ is a closed *-subalgebra of B. If ϕ is *-isomorphism of A into B, then ϕ' is also *-isomorphism of U into B and U, $\phi(U)$ are equivalent.

Proof: Let $|x|_1=|x|+||\phi(x)||$. By Theorem 2.2.5, $|x|$ is an auxiliary norm of A. A has a unique auxiliary norm, hence $\alpha|x|_1\leq|x|$. Therefore $\alpha(|x|+||\phi(x)||)\leq|x|$. This implies $\alpha|x|_1\leq|x|$ and $||\phi(x)||\leq\frac{1}{\alpha}|x|$ for some constant α. Hence we have that $||\phi(x)||\leq c|x|$, for some constant $c=\frac{1}{\alpha}$. Therefore ϕ can be uniquely extended to a continuous *-homomorphism ϕ' of U into B. By the proof of Theorem 2.1.2, $\phi'(U)$ is a closed *-subalgebra of B, since ϕ is a *-homomorphism of A into B.

Next suppose that ϕ is a *-isomorphism of A into B. Let I be any closed ideal of U with $I\cap A=(0)$. From the proof of Theorem 2.2.5, we see that $IA=(0)$. Then using Lemma 2.2.1, we know that ϕ' is a *-isomorphism and from Lemma 2.2.2, U and $\phi'(U)$ are equivalent. Q.E.D.

Theorem 2.2.7: Let A be an A*-algebra in which every maximal commutative *-subalgebra is regular. Then A has a unique auxiliary norm.

Proof: Let B denote any maximal commutative *-subalgebra of A. Let $|x|$, $|x|_1$ be two auxiliary norms satisfying $||x||^2\leq k||x^*x||$ and $||x||^2\leq k_1||x^*x||$, respectively, where k, k_1 are positive constants.

Let U, U_1 be the completions of B by $|x|$, $|x|_1$, respectively. Since any *-subalgebra of an A*-algebra is semi-simple [14, p. 188, Theorem 4.1.19] and also B is regular, hence by a theorem of Rickart [17, p. 193, Theorem 1], we may assume that B, U, U_1 have the same representation space Ω, where Ω is a locally compact Hausdorff space. Let U, U_1 be the algebra of continuous functions on Ω. Using Theorem 2.1.2, U, U_1 are equivalent, i.e., $\frac{1}{kk}|x|\leq kk_1|x|$ for every $x\in B$.

Now consider any $x\in A$ and let B be a maximal commutative *-subalgebra of A which contains x^*x. Then $\frac{1}{kk}|x^*x|\leq|x^*x|_1\leq kk_1|x^*x|$ which in turn implies $\frac{1}{kk_1}|x|^2\leq|x|_1{}^2\leq kk_1|x|^2$, i.e., $\frac{1}{p}|x|\leq|x|_1\leq p|x|$ for any constant p. Therefore A has a unique auxiliary norm. Q.E.D.

<u>Remark 2.2.8</u>: Let A be a proper H*-algebra with norm $||x||=(x,x)^{\frac{1}{2}}$, then $|x|^2=|x^*x|$ and $||xy||\leq|x|\ ||y||$.

<u>Proof</u>: Consider an auxiliary norm $|x|$ defined by $|x|=\underset{||y||=1}{\text{l.u.b.}}\ ||xy||$. Since $||x||=(x,x)^{\frac{1}{2}}$,

$$|x|^2=(\underset{||y||=1}{\text{l.u.b.}}\ ||xy||)^2=\underset{||y||=1}{\text{l.u.b.}}\ [xy,xy)^{\frac{1}{2}}]^2$$

$$=\underset{||y||=1}{\text{l.u.b.}}\ (xy,xy)=\underset{||y||=1}{\text{l.u.b.}}\ (x^*xy,y)$$

$$=\underset{||y||=1}{\text{l.u.b.}}\ ||x^*xy||\ ||y||\leq|x^*x|$$

Also, $$|x^*|^2=(\underset{||y||=1}{\text{l.u.b.}}\ ||x^*y||)^2$$

$$=\underset{||y||=1}{\text{l.u.b.}}\ (x^*y,x^*y)=\underset{||y||=1}{\text{l.u.b.}}\ ||xx^*y||\ ||y||$$

$$=\text{l.u.b.}\,|xx^*|\leq|x|\ |x^*|$$

That is, $|x^*| < |x|$. Therefore we have that $|x^*| \cdot |x| < |x|^2$, i.e., $|x^*x| < |x|^2$.

Now, since $|x| = \underset{||y||=1}{\text{l.u.b.}} ||xy||$ then we have that $||xy| \leq |x|$, and $||x \cdot \frac{y'}{||y'||}|| \leq |x|$ where $y = \frac{y'}{||y'||}$, $||xy'|| \leq |x|\ ||y'||$.

Q.E.D.

3. w.c.c. B*-algebras

Lemma 2.3.1: Let Ω be a locally compact Hausdorff space and $C(\Omega)$ be the Banach algebra of complex-valued continuous functions vanishing at ∞ on Ω. Then $C(\Omega)$ is w.c.c. if and only if Ω is discrete.

Proof: Let G be any relatively compact open subset of Ω and let $C(G)$ denote the subalgebra of $C(\Omega)$ consisting of the functions vanishing outside G, i.e., $C(G) = \{f : f(x) = 0$ as every $x \notin G\} \subseteq C(\Omega)$. Hence $C(G)$ is clearly a closed subalgebra of $C(\Omega)$.

Now consider $g \in C(\Omega)$ such that $g=1$ on G, i.e., $gf=f$ for every $f \in C(G)$. Since $gf(x)=g(x)f(x)$ for every x in G and $gf(x)=f(x)$, therefore $gf(x)=f(x)$ for every x in G, i.e., $gf=f$.

Assume that $C(\Omega)$ is w.c.c., then $C(G)$ is weakly compact [4, p. 482, Definition]. This implies that $C(G)$ is locally weakly compact, therefore G is finite [12, p. 87]. Hence Ω is discrete. Suppose that Ω is discrete, then $C(\Omega)$ becomes c.c. and hence w.c.c.

Q.E.D.

Theorem 2.3.2: The following statements are equivalent for a B*-algebra A:

(1) A is w.c.c.,

(2) A is a $B^*(\infty)$-sum of C*-algebras, each of which consists of the set of all completely continuous operators on a Hilbert space.

Proof: Let Ω be a locally compact Hausdorff space and let $C(\Omega)$ be the Banach algebra of complex-valued continuous functions vanishing at ∞ on Ω. Let B be a maximal commutative *-subalgebra of A. B is necessarily closed subalgebra of A [14, p. 182, Theorem 4.1.3]. And B is isomorphic with $C(\Omega)$ considered in Lemma 2.3.1. Since B is closed in the w.c.c. algebra A, B is w.c.c. as well. By Lemma 2.3.1, Ω is discrete. Now let e_α be the elements of B corresponding to the characteristic functions of the points $\alpha\in\Omega$. Since $f_\alpha^*(x)=\overline{f_\alpha(x)}$ and $e_\alpha^*(x)=f_\alpha^*(x)=\overline{f_\alpha(x)}=\overline{1}=1=e_\alpha(x)$, or $e_\alpha(x)=f_\alpha^*(x)=\overline{f_\alpha(x)}=\overline{0}=0=e_\alpha(x)$. Hence $e_\alpha^*=e_\alpha$, i.e., e_α is self-adjoint. Thus $\{e_\alpha\}$ is an orthogonal family of self-adjoint primitive idempotents of B such that, for every $x\in B$, we can write $x=\Sigma\lambda_\alpha e_\alpha$, where the right hand series is summable in the norm, i.e., for any given positive number ϵ the number of α's such that $|\lambda_\alpha|\geq\epsilon$ is finite. Conversely if $\Sigma\lambda_\alpha e_\alpha$ is summable, it clearly represents an element of B. Now we show that $e_\alpha A$ is a minimal right ideal of A. Let a be any self-adjoint element of A. Then $e_\alpha ae_\alpha=e_\alpha^*(e_\alpha a)^*=e_\alpha a^*e_\alpha^*=e_\alpha ae_\alpha=(e_\alpha ae_\alpha)^*$, i.e., $e_\alpha ae_\alpha$ is self-adjoint and commutative with every e_β, therefore $e_\alpha a e_\alpha \epsilon B$. Hence $e_\alpha a e_\alpha = \lambda e_\alpha$. Since $(e_\alpha a e_\alpha)^* = \bar{\lambda} e_\alpha = e_\alpha a e_\alpha = \lambda e_\alpha$, then $\bar{\lambda} = \lambda$, i.e., λ being a real. Therefore $e_\alpha A e_\alpha$ is a division algebra, so $e_\alpha A e_\alpha$=(the complex field)$\times e_\alpha$. Since A is semi-simple, i.e., A contains no nil-potent ideals, hence $e_\alpha A$ is a minimal right ideal.

If $e_\alpha z=0$, for any given $z \epsilon A$, then $z=0$, $e_\alpha z z^* = 0$. Hence $zz^* e_\alpha = 0$. Let us consider the directed set of finite sums $e_{\alpha_1} + e_{\alpha_2} + \ldots + e_{\alpha_n}$ of mutually orthogonal e_α. Let z be any element of A. Since A is w.c.c., then the unit sphere of A is transformed by right multiplication by z into a relatively weakly compact subset of A. Let z' be any limiting point (in the weak topology) of a directed set $\{(e_{\alpha_1} + \ldots + e_{\alpha_n})z\}$. Since $e_\alpha z = e_\alpha z'$, hence $z \epsilon z'$. This implies that $\{(e_{\alpha_1} + e_{\alpha_2} + \ldots + e_{\alpha_n})z\}$ converges weakly to z. In the same manner we can show that if, in this discussion, e_α is confined to any sub-family, $\{(e_{\alpha_1} + \ldots + e_{\alpha_n})z\}$ converges weakly to an element of A. Therefore $\Sigma e_\alpha z$ is summable to z by a theorem of Orlicz's [2, p. 240]. That is, $e_\alpha z \neq 0$ for only a countable number of e_α's, and $\{(e_{\alpha_1} + \ldots + e_{\alpha_n})z\}$ converges to z in the norm.

For any given closed ideal I of A, $e_\alpha A \cap I = (0)$ or $e_\alpha A \subset I$. Indeed, $e_\alpha A \cap I \neq (0)$ implies $(0) \neq e_\alpha a \epsilon I$ for some $a \epsilon A$ and therefore $e_\alpha a a^* e_\alpha = \lambda e_\alpha (\lambda \neq 0)$ implies $e_\alpha \epsilon I$ so that $e_\alpha A \subset I$. Let $\{e_{\alpha'}\}$ be the set of e_α's with $e_{\alpha'} A \subset I$ and $\{e_{\alpha''}\}$ the rest of e_α's. Denote by H', H'' the closed subspaces of A spanned by $\{e_{\alpha'} A\}$, $\{e_{\alpha''} A\}$, respectively. Then A is a direct sum of H' and H'', i.e., $A = H' \oplus H''$. Evidently $H' \subset I$. We show that $H' = I$ and $H'' =$

$L(I)$. Let $z \in I$. Since $z = \Sigma e_\alpha z$ where $e_\alpha z \neq 0$, then $e_\alpha z \in I$. This implies $I \subset H'$. Hence $H' = I$. Since $e_\alpha''AI \subset e_\alpha''A \cap I = (0)$, so $e_\alpha''AI = (0)$. Therefore $e_\alpha''A \subset L(I)$, i.e., $H'' \subset L(I)$. Conversely let $z \in L(I)$, then $z^* \in L(I)$ since $L(z)$ is a closed ideal of A and therefore $L(I) = [L(I)]^*$, i.e., it is self-adjoint. Hence $z^* e_\alpha = 0$ so that $(z^* e_\alpha)^* = 0^* = 0 = e_\alpha z$. Therefore $z = 0$, i.e., $z \in H''$. Hence $H'' = L(I)$. Thus $A = I \oplus L(I)$. Let I be any primitive ideal of A. By [6, p. 65], $L(I)$ is then a primitive algebra with minimal right ideals since $L(I)$ is isomorphic with A/I. Let $e_\alpha, e_\beta \in L(I)$. Since $Ae_\alpha A$ and $Ae_\beta A$ are non-zero ideals of $L(I)$, then $(Ae_\alpha A)(Ae_\beta A) \neq (0)$. Therefore $e_\alpha a e_\beta \neq 0$ for $a \in A$. Some $e_\beta A \neq (0)$, then $e_\beta a^* \neq 0$ for $a^* \in A$. Therefore $e_\beta a^* e_\alpha a e_\beta \neq 0$. Hence $e_\beta a^* e_\alpha a e_\beta = \lambda e_\beta (\lambda \neq 0)$. This implies $e_\beta \in Ae_\alpha A$ so that $Ae_\alpha A$ is the socle of $L(I)$ and $L(I)$ is the closure of $Ae_\alpha A$. Conversely for any e_α, the closure J_α of $Ae_\alpha A$ is $L(I)$, where I is a primitive ideal. Consider the maximal set $\{J_\alpha\}$ such that $J_\alpha \cap J_\beta = (0)$ for $\alpha \neq \beta$. The direct sum of J_α's is dense in A. Since J_α is *-isomorphic with a C*-algebra of all completely continuous operators on a Hilbert space [9, p. 408, Theorem 6.3]. Therefore A is *-isomorphic with a $B^*(\infty)$-sum of C*-algebras, each of which consists of the set of all completely continuous operators on a Hilbert space.

(2) $\Rightarrow$ (1). This is evident from the fact that a C*-algebra of completely continuous operators on a Hilbert space is w.c.c. [13, p. 362, Theorem 4]. Q.E.D.

Corollary 2.3.4: Let A be a w.c.c. B*-algebra. Let e be a self-adjoint idempotent of A. Then eAe is *-isomorphic with a direct sum of full matrix algebras of finite orders over the complex field.

Proof: Since A is w.c.c. and e is a self-adjoint idempotent of A, then eAe is a w.c.c. B*-algebra with a unit e. This implies eAe is weakly complete. By Theorem 2 [13, p. 361], eAe is finite-dimensional. Again, by Theorem 1 [13, p. 360], eAe is *-isomorphic with a direct sum of full matrix algebras of finite orders over the complex field. Q.E.D.

Theorem 2.3.5: Let A be a w.c.c. B*-algebra. A is c.c. if and only if A is a B*(∞)-sum of finite-dimensional B*-algebras.

Proof: By Theorem 8.2 [9, p. 412], a complex completely continuous B*-algebra is the B*(∞)-sum of finite-dimensional B*-algebras.

Conversely, since the finite-dimensional normed spaces are reflexive, the weak topology and the strong topology coincide. Hence, the proof follows from Theorem 2.3.2.

Q.E.D.

Lemma 2.3.6: Let I be a closed right ideal of a w.c.c. B*-algebra A and $\{e_{\alpha'}\}$ be a maximal family of orthogonal self-adjoint primitive idempotents contained in I. Then for every $z \in I$ we have $z = \Sigma e_{\alpha'} z$.

Proof: Let $z'=\Sigma e_{\alpha'}z$, where the right hand side is summable by Corollary 2.3.3. Now $e_{\alpha'}z'=e_{\alpha'}\Sigma e_{\alpha'}z=e_{\alpha'}z$, since $\{e_{\alpha'}\}$ is a family of orthogonal self-adjoint primitive idempotents. This implies that $e_{\alpha'}(z-z')=0$, and therefore $e_{\alpha'}(z-z')(z-z')^*=0$. The closed subalgebra B generated by $(z-z')(z-z')^*$ and $\{e_{\alpha'}\}$ is a closed commutative *-subalgebra contained in I. B, being closed, is w.c.c. Therefore, by Lemma 2.3.1 and the proof of Theorem 2.3.2, we have that B is generated by self-adjoint idempotents. But $\{e_{\alpha'}\}$ is a maximal family of orthogonal self-adjoint primitive idempotents contained in B so we have that $(z-z')(z-z')^*=0$. By B*-condition, $z-z'=0$, i.e., $z=z'$. Therefore $z'=z=\Sigma e_{\alpha'}z$. Q.E.D.

Theorem 2.3.7: The following statements are equivalent for a Banach *-algebra A satisfying $||x||^2\leq k||x^*x||$, where k is a constant.

(1) A is w.c.c.

(2) A is equivalent to a $B^*(\infty)$-sum of C*-algebra, each of which is the set of all completely continuous operators on a Hilbert space.

(3) A is dual.

Proof: (1)⇔(2). Since B*-algebra is Banach *-algebra satisfying $||x||^2\leq k||x^*x||$, so (1) is equivalent to (2) by Theorem 2.3.2.

(2)⇒(3). Since A is a $B^*(\infty)$-sum of C*-algebra, each of which is the set of all completely continuous operators on a Hilbert space, but the algebra of completely continuous operators on a Hilbert space is dual [11, p. 412], by Lemma 2.4 [10, p. 222], the B*(∞)-sum of dual B*-algebra is dual. This completes our assertion.

(3)⇒(2). By Theorem 8.3 [9, p. 412], a dual B*-algebra is the B*(∞)-sum of algebras, each of which is the algebra of all completely continuous operators on a Hilbert space.

Q.E.D.

Corollary 2.3.8: A closed *-subalgebra of a dual Banach *-algebra A satisfying $||x||^2 \leq k||x^*x||$ is dual.

Proof: Let B be any closed *-subalgebra. By Theorem 2.3.7, A is weakly complete. Hence B is w.c.c. Again using Theorem 2.3.7, we see that B is dual. Q.E.D.

4. c.c. Banach *-algebras

Theorem 2.4.1: The following statements are equivalent for a c.c. Banach *-algebra A.

(1) A is semi-simple and $x^*x=0$ implies $x=0$.

(2) A is a dense subalgebra of a c.c. B*-algebra U.

(3) A is an A*-algebra.

Proof: (1)⇒(2). First we show that A is symmetric, that is, x^*x has a quasi-inverse for every $x \in A$. Suppose that the contrary holds for some $x \in A$. -1 is then a proper value of the c.c. operator defined by the left multiplication by x^*x, and its proper space, being finite-dimensional [2, p. 160], contains a minimal right ideal R. Since A is semi-simple and $x^*x=0$ implies $x=0$, hence R=eA [15, p. 29, Lemma 2.1], where e is a self-adjoint primitive idempotent.

Since $T_{x^*x}e=-e$, then $x^*xe=-e$. Therefore $(x^*xe)^*=(-e)^*$, $e^*(x^*x)^*=-e^*=-e$, $ex^*x=x^*xe$, hence x^*x commutes with e. Since R is minimal then eAe is a division ring [5, p. 13, Theorem 5]. This implies that eAe=[the complex field]x e [14, p. 40, Theorem 1.7.6], and exe, ex^*e are mutually conjugate complex multiples of e. Now let z=ex-exe, then

$$z^*=(ex-exe)^*=(ex)^*-(exe)^*=x^*e^*-e^*(ex)^*$$
$$=x^*e-ex^*e \text{ and } zz^*=(ex-exe)(x^*e-ex^*e)$$
$$=exx^*e-exex^*e-exex^*e+exeex^*e$$
$$=exx^*e-exex^*e-exex^*e+exex^*e$$
$$=exx^*e-exex^*e, \text{ but } xx^*e=-e$$
$$=e(-e)-exex^*e=-e^2-exex^*e=-e-exex^*e.$$

Hence $zz^*=-e-exex^*e=-e-exe^2x^*e^2=-e-exeex^*ee=(1+exe.ex^*e)(-e)$, but exe, ex^*e are conjugate complex, i.e., $zz^*=1+$ real number)$(-e)=-k^2e$ where $k>0$. This implies that $(z+ke)(z+ke)^*=(z+ke)(z^*+ke^*)=(z+ke)(z^*+ke)=zz^*+kez^*+zke+keke=0$. Therefore $z=-ke$ and $zz^*=k^2e$. This is a contradiction. Hence A is symmetric. This implies that A is C-symmetric. Let M be a primitive ideal, then M is self-adjoint [9, p. 403, Theorem 4.4]. Moreover A is a direct sum of M and a full matrix algebra B of finite order over the complex field, which is a self-adjoint simple ideal of A. Let x_B be the B-component of x and its factor space norm by $||x_B||^{A/M}$. By Theorem 4.8.11, [14, p. 244], B is *-isomorphic with a C*-algebra with C*-norm $|x_B|$, the *-homomorphism $x\to x_B$ is continuous from A onto B with norm $|x_B|$ [16, p. 628, Theorem 6.2]. By a result of B. E. Johnson [7, p. 539], a semi-simple Banach algebra has a unique norm topology. This implies that $x \to x^*$ is continuous. So we can suppose that $||x||=||x^*||$. Let $||x_B||_e = \text{l.u.b.}_{||y_B||=1} ||x_B y_B||$, $||x_B||_r = \text{l.u.b.}_{||y_B||=1} ||y_B x_B||$ and $|||x_B||| = ||x_B||_e + ||x_B||_r$, we can show that $|||\cdot|||$ is an algebra norm.

$$
\begin{aligned}
|||x_B w_B||| &= ||x_B w_B||_e + ||x_B w_B||_r \\
&\leq ||x_B||_e ||w_B||_e + ||x_B||_r ||w_B||_r \\
&\leq ||x_B||_e ||w_B||_e + ||x_B||_e ||w_B||_r + ||x_B||_r ||w_B||_e \\
&\quad + ||x_B||_r ||w_B||_r \\
&= (||x_B||_e + ||x_B||_r)(||w_B||_e + ||w_B||_r) \\
&= |||x_B||| \cdot |||w_B|||
\end{aligned}
$$

Since $|||x^*_B||| = ||x^*_B||_e + ||x^*_B||_r = ||x_B||_e + ||x_B||_r = |||x_B|||$ so that $|||x^*_B x_B||| \leq |||x^*_B|||\ |||x_B||| \leq |||x_B|||^2$. Therefore $|||x_B|||^2 \geq |||x^*_B x_B|||$. From the minimal character of the usual norm in an algebra $C(\Omega)$ [9, p. 407, Theorem 6.2], we have that $|||x^*_B x||| \geq |x^*_B x_B| = |x_B|^2$. Therefore $|||x_B|||^2 \geq |||x^*_B x_B||| \geq |x^*_B x_B| = |x_B|^2$. Suppose that there exists an infinite number of B's, such that $|x_B| > \epsilon$, so that $|||x_B||| > \epsilon$, where ϵ is a given positive number. Then we may assume that $||x_{B_n}||_e > \frac{\epsilon}{2}$, $n=1,2,3,\ldots$. Choose $y_n \epsilon B_n$ such that $||xy_n|| = ||x_{B_n} y_n|| > \frac{\epsilon}{2}$, $||y_n||=1$. Since A is c.c., there exists a subsequence $\{xy_{n'}\}$ converging to an element $x' \epsilon A$ and $||x'|| \geq \frac{\epsilon}{2}$.

We know that B's are simple ideals, hence for $B_n \neq B$, $xy_n y_B = 0$. This implies $x' y_B = 0$, since $xy_n \to x'$. Hence $x_B' y_B = 0$, take $x_B'^* = y_B$. Then we have that $x_B' x_B'^* = 0$ and therefore $x_B' = 0$ for every B. But then $x' \epsilon M'$ for every M. This implies $x' = 0$, a contradiction. It follows that $\{x_B\}$ is an element of the $B^*(\infty)$-sum U of B*-algebras B with norm $|x_B|$. A is mapped by $x \to \{x_B\}$ into a dense subalgebra of the B*-algebra U which is c.c. This completes the proof of (1)⇒(2).

(2)⇒(3) is obvious since the B*-norm of U serves as an auxiliary norm of A.

(3)⇒(1) follows from a lemma of Rickart's [16, p. 626, Lemma 5.1]. Q.E.D.

Corollary 2.4.2: A c.c. A*-algebra has a unique auxiliary norm.

Proof: Let A be a c.c. A*-algebra. Let B denote any maximal commutative *-subalgebra of A. By Theorem 5.1 [9, p. 406], the structure space Ω of B is discrete. From the proof of Theorem 2.3.2 that for any point $p_o \epsilon \Omega$ the characteristic function of a single point set $\{p_o\}$ is a Gelfand representation of self-adjoint idempotent of B. Hence B is regular. It follows from Theorem 2.2.5 that A has a unique auxiliary norm. Q.E.D.

Theorem 2.4.3: Let A be a c.c. A*-algebra, A is dual if and only if the socle is dense in A and, for every $x \epsilon A$, the closure of xA contains x.

Proof: Assume that A is dual, then it is annihilator algebra. Also A is A*-algebra and therefore semi-simple. Therefore the socle of A is dense in A [14, p. 100, Corollary 2.8.16]. Since A is dual, we have that $L[R(0)]=(0)=L(A)$, that is, the left annihilator of A is 0. Now to prove that x is in the closure of xA, it will suffice to show that $L(xA)x=(0)$, i.e., $yx=0$ for $y \in L(xA)$. But $yxA=(0)$ implies $yx=0$ since $L(A)=(0)$.

Conversely, suppose that the socle of A is dense in A and $x \in \overline{xA}$ for every $x \in A$. From the proof of Theorem 2.4.1 we have that the socle of A is the direct sum of dual simple algebras B, and therefore dual in relative topology induced by A [8, p. 690, Theorem 2]. Therefore A is dual [8, p. 694, Theorem 7]. Q.E.D.

Theorem 2.4.4: Let A be a c.c. dual A*-algebra. Then every closed right ideal of A is the intersection of regular maximal right ideals of A containing it.

Proof: Let A be a dense subalgebra of a c.c. B*-algebra U as stated in the proof of Theorem 2.4.1. Denote by $\overline{E}$ the closure of a subset $E \subset U$. Let M be any regular maximal right ideal of A. We show that $\overline{M}$ is a regular maximal right ideal of U. Let $L_A(M)=\{x \in A : xM=(0)\}$, the left annihilator of M in A, is a closed minimal left ideal generated by a self-adjoint primitive idempotent $e \in A$ and $M=\{x : ex=(0),\ x \in A\}$. Since $\overline{A}=U$, therefore $\overline{M}=\{z : ez=(0),\ z \in U\}$ and Ue is a minimal left ideal. Therefore $\overline{M}$ is a regular maximal right ideal of U [3, p.157].

Conversely if M_1 is a regular maximal right ideal of U and e is a self-adjoint primitive idempotent generating $L(M_1)$, then by the above proof, $e\epsilon A$ and $M_1 \cap A = \{x : ex = 0, x \epsilon A\}$. Since e is self-adjoint primitive idempotent of A, hence $M_1 \cap A$ is a regular maximal right ideal of A, since $e(x-ex) = ex - e^2x = ex - ex = 0$, $x \epsilon A$. Now let N be any closed right ideal of A, by duality of A, $N = R_A(L_A(N))$. Since $N \subset A$ so $R_A(L_A(N)) = R_A[L(N) \cap A]$, also $L(N) \cap A \subset A$, this gives $R_A[L(N) \cap A] = R[L(N) \cap A] \cap A$. But $L(N) \cap A \subset L(N)$, i.e., $R[L(N) \cap A] \supset RL(N)$. This implies $R[L(N) \cap A] \cap A \supset RL(N) \cap A$. Since $RL(N)$ is a closed ideal of U so $RL(N) = \overline{N}$, this gives $N = R[L(N) \cap A] \cap A \supset RL(N) \cap A$, i.e., $N \supset \overline{N} \cap A$. But $N \subset \overline{N} \cap A$. Therefore $N = \overline{N} \cap A$. We know that every closed right ideal of a dual B*-algebra is the intersection of regular maximal right ideals containing it. Hence $\overline{N}$ is the intersection of regular maximal right ideals M_1 of U containing $\overline{N}$. Therefore $N = (\cap M_1) \cap A = \cap (M_1 \cap A)$, that is, N is the intersection of regular maximal right ideals of A containing N, since $\overline{N} \subset M_1$ implies $N \subset M_1$ and $N \subset M_1 \cap A$. Q.E.D.

<u>Theorem 2.4.5</u>: Let A be a c.c. A*-algebra. A is finite-dimensional if and only if A has a unit.

<u>Proof</u>: Suppose that A is finite-dimensional, we have only a finite number primitive self-adjoint idempotent and A can be expressed as a direct sum of finite number of minimal left ideals. And now, since A is semi-simple, it has an identity.

Conversely, suppose that A has a unit e. A c.c. B*-algebra U in which A is dense has e as a unit, and therefore U is finite-dimensional. This implies that A is finite-dimensional, completing the proof. Q.E.D.

5. Dual A*-algebras

Theorem 2.5.1: Let A be a semi-simple dual Banach *-algebra in which $x^*x=0$ implies $x=0$. If A satisfies any of the following conditions,

(1) every primitive ideal of A is a direct summand,

(2) $x \to x^*$ is continuous,

then A is an A*-algebra and a dense subalgebra of a dual B*-algebra U which is uniquely determined up to *-isomorphism.

Proof: Every minimal right ideal of A is generated by a unique self-adjoint primitive idempotent of A. From the structure of semi-simple dual ring, we see that A is the closure of its socle which is a direct sum of simple dual ideals S_α of form $Ae_\alpha A$, where e_α is a self-adjoint primitive idempotent [8, p. 692, Theorem 5].

Now suppose first that (2) holds. Let $\overline{S}_\alpha$ be the closure of S_α and $M_\alpha=\{x: x\in A,\ x\overline{S}_\alpha=(0)\}$, the left annihilator are closed self-adjoint ideals and M_α is primitive ideal of A since the annihilator of a minimal closed right ideal is a maximal closed left ideal [3, p. 157]. Since $\overline{S}_\alpha+M_\alpha$ is dense in A [14, p. 99, Lemma 2.8.10], the image of S_α is dense in A/M_α, i.e., $\overline{IM(S_\alpha)}=A/M_\alpha$. Let $\dot{x}=x+M_\alpha$ for every $x\in\overline{S}_\alpha$ and $||\dot{x}||$ the factor space norm. Then we can introduce in $(e_\alpha A)^{\cdot}$ an inner product $(\dot{x},\dot{y})$ by the relation $(\dot{x},\dot{y})\dot{e}_\alpha=(e_\alpha xy^*e_\alpha)^{\cdot}$ [14, p. 261, Theorem 4.10.3]. The operator $T_{\dot{z}}:\dot{x}\to(xz)^{\cdot};\ \dot{x}\in(e_\alpha A)^{\cdot}$ is continuous in the norm defined by the above inner product [14, p. 261, Theorem 4.10.3]. Denote by $|\dot{z}|=||T_{\dot{z}}||$ the operator norm. Since $x\to x^*$ is continuous, we may assume $||x||=||x^*||$. And also $|\dot{z}|^2=||T_{\dot{z}}||^2=||T_{\dot{z}}T_{\dot{z}}^*||$, $|\dot{z}\dot{z}^*|=||T_{\dot{z}\dot{z}^*}||=||T_{\dot{z}}T_{\dot{z}^*}||$, that is, $|\dot{z}|^2=|\dot{z}\dot{z}^*|$ so that A/M_α as an A*-algebra with an auxiliary norm $|\dot{x}|$. Therefore $||x||^2\geq||x^*x||\geq||(x^*x)^{\cdot}||\geq|(x^*x)^{\cdot}|=|\dot{x}|^2$. So that we may regard A/M_α as a dense subalgebra of C*-algebra K_α of all completely continuous operators on the Hilbert space obtained by completing $(e_\alpha A)^{\cdot}$. We consider the $B^*(\infty)$-sum U of K_α's. And $x\to\dot{x}$ is continuous, the image of the socle of A by this mapping is dense in U. Since $x\to\dot{x}$ is a *-isomorphic into, we may consider A as a dense subalgebra of U. By Theorem 2.2.5, U is uniquely determined up to *-isomorphism.

Next, we assume that every primitive ideal of A is a direct summand, then $A=\overline{S}_\alpha+M_\alpha$. Since $M_\alpha=\{x\in A: x\overline{S}_\alpha=(0)\}$, let $x\in M_\alpha$ then $x\overline{S}_\alpha=(0)$. Therefore $(x\overline{S}_\alpha)^*=(0)^*=(0)=\overline{S}_\alpha^*x^*=\overline{S}_\alpha x^*=R(S_\alpha)=M_\alpha$, i.e., $M_\alpha=L(S_\alpha)=R(S_\alpha)$. By Lemma 4.10.13 [14, p. 267], S_α being self-adjoint and therefore M_α is self-adjoint. Likewsie $\overline{S}_\alpha$ is self-adjoint. As in the proof of Theorem 2.4.1, $x\to x^*$ is continuous and therefore the proof is completed. Q.E.D.

Theorem 2.5.2: Any dual A*-algebra has a unique auxiliary norm and is a dense subalgebra of a dual B*-algebra which is uniquely determined up to *-isomorphism.

Proof: In an A*-algebra any *-subalgebra is semi-simple, $x \to x^*$ is continuous and $x^*x=0$ implies $x=0$ [16, p. 626, Theorem 5.2 and Lemma 5.3]. This together with Theorem 2.5.1 gives the desired result. Q.E.D.

Theorem 2.5.3: Any dual A*-algebra with a unit is finite-dimensional, and is *-isomorphic with a direct sum of full matrix algebra over the complex field.

Proof: The finite-dimensionality of A follows [10A]. We know that A*-algebra is semi-simple and satisfies B*-condition. By Theorem 4.1.19 [14, p. 188], any *-subalgebra of an A*-algebra is semi-simple. Therefore A is *-isomorphic with a direct sum of full matrix algebras over the complex field [13, p. 360, Theorem 1].

6. w.c.c. A*-algebras

Lemma 2.6.1: Let B be a Banach *-algebra which is a dense ideal in a dual A*-algebra A. Let I denote any closed right ideal of B and $\overline{I}$ its closure in A. Then:

(1) $\overline{I} \cap B = R_B(L_B(I))$, the right annihilator of the left annihilator of I in B,

(2) $\overline{I}B \subset I$

(3) B is dual if and only if, for every $x \in B$, the closure of xB in B contains x.

Proof: (1) $L(\overline{I}) = L(I)$ and $I \subset B$ implies $L_B(I) \subset L(I)$. But $L_B(I) \subset B$ and hence $L_B(I) \subset B \cap K(\overline{I})$. Clearly $B \cap L(\overline{I}) \subset L_B(I)$, then

$L_B(I)=B\cap L(\bar{I})$. We know that $BL(\bar{I})\subset B\cap L(\bar{I})$, i.e., $BL(\bar{I})\subset L(\bar{I})$ implies $\overline{BL(\bar{I})}\subset\overline{L(\bar{I})}=L(\bar{I})$. Since $x\in\overline{Ax}$ for every $x\in A$ [8, p. 690, Theorem 1], in particular, $x\in\overline{Ax}$ for $x\in L(\bar{I})$, hence $L(\bar{I})\subset\overline{AL(\bar{I})}$. Since $L(\bar{I})$ is closed left ideal in A, we have that $AL(\bar{I})\subset L(\bar{I})$, so that $\overline{AL(\bar{I})}\subset\overline{L(\bar{I})}=L(\bar{I})$. This implies that $L(\bar{I})=\overline{AL(\bar{I})}$. Again we have $x\in\overline{Ax}$ for every $x\in A$. In particular $x\in\overline{Bx}$ for every $x\in B$. Therefore $AL(I)\subset\overline{BL(I)}$, $AL(\bar{I})\subset\overline{BL(\bar{I})}$, hence $\overline{AL(\bar{I})}\subset\overline{BL(\bar{I})}$. Therefore $\overline{BL(\bar{I})}=\overline{AL(\bar{I})}=L(\bar{I})$, then $L(\bar{I})=\overline{BL(\bar{I})}\subset\overline{L_B(\bar{I})}$, but $L_B(I)\subset L(\bar{I})$, i.e., $\overline{L_B(I)}\subset L(\bar{I})$. Hence $\overline{L_B(I)}=L(\bar{I})$. This implies that $R_B(L_B(I))=R_B\overline{(L(I))}\cap B=R(L(I))\cap B=I\cap B$, since A is dual.

(2) Let z be any element of I. Taking a sequence $\{z_n\}$ from I in such a way that $||z_n-z||_A\to 0$. By Lemma 2.2.4, we have that $||z_nb-zb||_B=||(z_n-z)b||_B\leq C||z_n-z||_A\cdot||b||_B$ for every $b\in B$, where $||\cdot||_A$ and $||\cdot||_B$ stand for norms in A and B respectively. Since $||z_n-z||_A\to 0$ then $||z_nb-zb||_B\to 0$, that is, $zb\in I$, i.e., $IB\subset I$.

(3) If B is dual, then the closure of xB in B contains x [8, p. 690, Theorem 2]. Now suppose that for every $x\in B$, $x\in\overline{xB}$ in B. Let I be any closed right ideal of B. Take any $x\in\bar{I}\cap B$, this implies that $x\in I$ by (2). That is, $\bar{I}\cap B\subset I$. But it is clear that $I\subset\bar{I}\cap B$. Therefore $\bar{I}\cap B=I$. By (1), $I=R_B(L_B)I))$. For every closed left ideal J of B, using the *-involution, we have that $J=L_B(R_B(J))$. Therefore B is dual. Q.E.D.

<u>Lemma 2.6.2</u>: Let B be a Banach *-algebra which is a dense ideal in an A*-algebra A. Then:

(1) A is w.c.c. if B is w.c.c. and A^2 is dense in A.

(2) B is w.c.c. if A is w.c.c. and B^2 is dense in B.

Proof: (1) Assume that B is w.c.c. and A^2 is dense in A, i.e., $\overline{A^2}=A$. Let b,b' be any element of B. Take any $\{x_n\}$ sequence from A such that $||x_n||_A=1$, where $n=1,2,3....$ Since B is a dense ideal in A, by Lemma 2.2.4, $||b'x_n||_B \leq C||x_n||_A||b'||_B$ where C is a constant. That is, $\{b'x_n\}$ is bounded in B. Hence there exists a subsequence $\{X_{n'}\}$ such that $\{bb'x_{n'}\}$ converges weakly to an element in B since B is w.c.c. Also the mapping $x \to x$ from B into A is continuous [16, p. 627, Theorem 6.2], and this makes $\{bb'x_{n'}\}$ converge weakly to an element in A, i.e., bb' is a w.c.c. element of A. And we have that $AB \subset B$ implies $B^2 \subset B$, i.e., $\overline{B^2} \subset \overline{B}=A$. Hence $\overline{B^2} \subset A$. On the other hand, $A \subset \overline{A}=B$, $A \subset B$, i.e., $\overline{A^2} \subset \overline{B^2}$ but $\overline{A^2}=A$ makes $A \subset \overline{B^2}$. Therefore $\overline{B^2}=A$, i.e., B^2 is dense in A. From [13, p. 362], the set of w.c.c. elements is closed, so that A is w.c.c.

(2) Let b,b' be any elements of B. And let $\{x_n\}$ be any bounded sequence from B. Since A*-algebra is semi-simple and this makes $||x|| \leq C||x||$ for every $x \in B$ [16, p. 621, Theorem 2.6]. But $||x||_1$ is bounded, therefore $||x||$ is also bounded, that is, $\{x_n\}$ is also bound in A. By the boundness of A we have that there exists a subsequence $\{x_{n'}\}$ such that $\{b'x_{n'}\}$ converges weakly to an element z of A in A. Let ϕ be any continuous linear functional on B, and put $\psi(x)=\phi(bx)$ for every $x \in A$. Therefore ψ is a continuous

linear functional on A. Since B is an ideal of A, i.e., $BA\subset B$ implies $bz\in B$ where $b\in B$, $z\in A$. Hence $\phi(bb'x_{n'}-bz)=\phi[b(b'x_{n'}-z)]=\psi(b'x_{n'}-z)$. But $\{b'x_{n'}\}$ converges weakly to z, i.e., $\psi(b'x_{n'}-z)\to 0$. This makes $\phi(bb'x_{n'}-bz)\to 0$, so that bb' is a w.c.c. element of B. And B^2 is dense in B, i.e., $\overline{B^2}=B$. Therefore B is w.c.c. Q.E.D.

Lemma 2.6.3: If A is a Banach *-algebra satisfying $||x||^2\leq k||x^*x||$, k being a positive constant, then $A=A^2$.

Proof: Let $x\in A$ be a self-adjoint element with non-negative spectrum. By the Gelfand representation, the involution $x\to x^*$ defined by the equation $x^*(w)=\overline{x(w)}$ for $w\in\Omega$, where Ω is a locally compact Hausdorff space. But $x^*=x$, i.e., $x^*(w)=x(w)=\overline{x(w)}$. This gives us that x becomes a real value, hence we can find $y\in A$ such that $x=y^2$, i.e., $x\in A^2$. Therefore $A\subset A^2$. In a Banach *-algebra, any element z of A has a unique representation $z=x+iy$ where x and y are self-adjoint. So we can conclude that for every $x\in A$, we have $A\subset A^2$. But $AA\subset A$ is clearly $A^2\subset A$, that is, $A=A^2$. Q.E.D.

Theorem 2.6.4: Let A be a w.c.c. A*-algebra with an auxiliary norm $|x|$ such that $||xy||\leq c|x|\ ||y||$ for every x, $y\in A$, where c is a constant. Let one of the following conditions be satisfied:

(1) the closure of xA contains x for every $x\in A$,

(2) A is reflexive.

Then A is dual and is a dense ideal of the completion U of A by $|x|$. U is equivalent to a w.c.c. B*-algebra. For any family of orthogonal self-adjoint idempotents $\{e_\alpha\}$ of A, $\Sigma e_\alpha x$ is summable in the norm of A, and especially when $\{e_\alpha\}$ is a maximal family, $x=\Sigma e_\alpha x$ holds for every $x\in A$.

Proof: Since U is the completion of A so U is a Banach *-algebra satisfying weakened B*-condition. By Lemma 2.6.3, A is a dense ideal of U. Therefore, by Lemma 2.6.2 (1), we have that U is w.c.c. since $U=\overline{U}=\overline{U^2}$. By Theorem 2.3.7, U is dual and U is a completion of A. Hence we may assume that U is a B*-algebra with norm $|x|$, i.e., U becomes A*-algebra. Let (1) be satisfied. A is dual by Lemma 2.6.1.

By Corollary 2.3.3, $\Sigma e_\alpha x$ is summable in the norm of U. Since $\overline{A}=U$, so that $\{e_\alpha\}$ is also maximal in U and $x=\Sigma e_\alpha x$ by Theorem 2.3.2. As in the proof of Lemma 2.2.3, the mapping $z\to za$, $a\in A$ being fixed, from U into A is continuous. Therefore $\Sigma e_\alpha xy$ is summable in the norm of A for every $y\in A$. Using (1) we can take y_n such that $||x-xy_n||<\frac{1}{n}$ for n=1,2, 3,.... Except for a set of at most countable e_{α_k}, $e_\alpha xy_n=0$ (n=1,2,...). For $||(e_{\alpha_1}+e_{\alpha_2}+\dots+e_{\alpha_n})x||=||(e_{\alpha_1}+e_{\alpha_2}+\dots+e_{\alpha_n})(xy_n+x-xy_n)||\leq||(e_{\alpha_1}+e_{\alpha_2}+\dots+e_{\alpha_n})xy_n||+||(e_{\alpha_1}+e_{\alpha_2}+\dots+e_{\alpha_n})(x-xy_n)||\leq||(e_{\alpha_1}+e_{\alpha_2}+\dots+e_{\alpha_n})xy_n||+||(e_{\alpha_1}+e_{\alpha_2}+\dots+e_{\alpha_n})||\,||(x-xy_n)||$. But we know that $(e_{\alpha_1}+e_{\alpha_2}+\dots+e_{\alpha_n})xy_n$ is summable in the norm of A and $||x-xy_n||\to 0$ as $n\to\infty$. Therefore $\Sigma e_{\alpha_k}x$ is summable in the norm of A. This implies that $\Sigma e_\alpha x$ is summable in the norm of A. Next suppose that (2) is satisfied. Since A is locally weakly compact, then $x\in\overline{xA}$. By Lemma 2.6.1, A is dual. Except for a set of most countable e_{α_k}, $e_\alpha x=0$. $||(e_{\alpha_1}+\ldots+e_{\alpha_n})x||\leq c||x||$, that is, $\{(e_{\alpha_1}+\ldots+e_{\alpha_n})x\}$ is bounded. Since A is locally weakly compact, $\{(e_{\alpha_1}+\ldots+e_{\alpha_n})x\}$ has a weakly convergent subsequence in A. Therefore $\Sigma e_\alpha x$ is summable in the norm of A. Since $\bar{A}=U$, $\{e_\alpha\}$ is maximal in U and $x=\Sigma e_\alpha x$. Thus the proof is completed. Q.E.D.

Theorem 2.6.5: Let U be a w.c.c. B*-algebra and A be a Banach *-algebra which is a dense ideal of U. Let A satisfy one of the following:

(1) the closure of xA contains x for every $x\epsilon A$.

(2) A is reflexive.

Then A is dual and w.c.c.

Proof: Since U is B*-algebra then $U=U^2$ by Lemma 2.6.3. As A is a dense ideal of U and hence U is dual by Theorem 2.3.7. Since U is a B*-algebra hence A*-algebra. Therefore from Lemma 2.6.2, part (2), A is w.c.c.

Now let A satisfy (1), then by Lemma 2.6.1, A is dual. If A satisfies (2), then as in the proof of Lemma 2.7.2, we have that A is dual again. Since A is also locally weakly compact, therefore we complete that A is dual and w.c.c. Q.E.D.

Theorem 2.6.6: For a dual A*-algebra to be of the first kind it is necessary and sufficient that $||x||_1=\underset{||y||=1}{\text{l.u.b.}}||xy||$ satisfies $||x||_1{}^2\leq k||x^*x||_1$ for a constant k.

Proof: Assume that A is a dual A*-algebra of the first kind. By definition, A is a dense ideal of a B*-algebra

U with norm $|x|$. By Lemma 2.2.4, we have that $||xy|| \leq c|x|\ ||y||$ for every $x, y \in A$ and some constant c. This implies that $||x||_1 = \underset{||y||=1}{\text{l.u.b.}} ||xy|| \leq \underset{||y||=1}{\text{l.u.b.}}\ c \cdot |x|\ ||y|| \leq c|x| \underset{||y||=1}{\text{l.u.b.}} ||y|| \leq c|x|$. And $||xz||_1 = \underset{||y||=1}{\text{l.u.b.}} ||xzy|| \leq \underset{||y||=1}{\text{l.u.b.}} ||xzy^2|| = \underset{||y||=1}{\text{l.u.b.}} ||xyzy|| \leq \underset{||y||=1}{\text{l.u.b.}} ||xy||\ ||zy|| = \underset{||y||=1}{\text{l.u.b.}} ||xy|| \cdot \underset{||y||=1}{\text{l.u.b.}} ||zy|| = ||x||_1 \cdot ||z||_1$, that is, $||xz||_1 \leq ||x||_1 ||z||_1$. Therefore $||x||_1$ has the usual norm properties and the multiplication property (except for the completeness). Hence let U_1 be the completion of A by $||x||_1$. Therefore the mapping $x \to x$ from A with norm $||x||$ onto A with $||x||_1$ can be extended to a continuous homomorphism ϕ of U into U_1 [11, p. 180, Theorem 2]. Also we know that the kernel J of ϕ is a closed ideal of U [11, p. 166, Theorem 1] with the property $J \cap A = (0)$. Therefore from the proof of Lemma 2.2.1 it is clear that $J = (0)$, that is, ϕ is an isomorphism. As in the proof of Theorem 2.4.1, the minimal character of the usual norm in the algebra $C(\Omega)$ gives us that $|x|^2 = |x^*x| \leq ||x^*x||_1 \leq ||x^*||_1 ||x||_1 \leq c|x|\ ||x||_1$, i.e., $|x| \leq c||x||_1$ where c is a constant. Therefore two norms $|x|$ and $||x||_1$ are equivalent. Then $||x||_1$ satisfies $||x||_1^2 \leq k||x^*x||_1$.

For the converse, assume that $||x||_1$ satisfies the condition $||x||_1^2 \leq k||x^*x||_1$. We know that A is a dual A*-algebra with an auxiliary norm $||x||_1$. By Lemma 2.2.3, A is a dense ideal of the completion U_1. Therefore A is a dual A*-algebra of the first kind. Q.E.D.

Theorem 2.6.7: Let A be a dual A*-algebra of the first kind. Then any closed *-subalgebra B of A is a dual A*-algebra of the first kind.

Proof: Since A is dual A*-algebra of the first kind, then by definition U is a dual B*-algebra with norm $|x|$ of which A is dense ideal. Let F be the closure of B in U, then F is a closed *-subalgebra of U. By Corollary 2.3.8, any closed *-subalgebra of a dual B*-algebra is dual and therefore F is a dual B*-algebra. Since U is B*-algebra, i.e., A*-algebra and $\overline{A}$=U, using Lemma 2.2.4, we have $||xy|| \leq c|x|\ ||y||$ for every $x \in U$, every $y \in A$. By Lemma 2.2.3, we have that B is a dense ideal of F.

Let $\{e_\alpha\}$ be a maximal family of orthogonal self-adjoint idempotents e_α in F. Since B is a dense ideal of F, so $e_\alpha \in F$ implies $e_\alpha \in B$. By Theorem 2.6.4, Σxe_α is summable in A for every $x \in A$, in particular, for every $x \in B$. And therefore Σxe_α is also summable in B. But $\{e_\alpha\}$ is a maximal family of orthogonal self-adjoint idempotents in F, so that $x = \Sigma xe_\alpha$. Therefore $x \in \overline{xB}$ in B. Hence B is dual. Q.E.D.

Corollary 2.6.8: Let A be an A*-algebra which is a dense ideal of a dual A*-algebra A, of the first kind. A closed *-subalgebra B of A is dual if the closure of xB contains x for every $x \in B$.

Proof: This proof follows from the proof of Theorem 2.6.7 and Lemma 2.6.1. Q.E.D.

7. Some Properties of Dual A*-algebras

Lemma 2.7.1: Let B be a dual Banach *-algebra which is a dense ideal of an A*-algebra A. If B has an approximate identity $\{U_\alpha\}$, then A is dual.

Proof: Let $a \in A$ be any element. Consider $b \in B$ an element such that $||a-b||_A < \epsilon$ where ϵ is a positive number. Then we have $||a-au_\alpha||_A \leq ||a-b||_A + ||b-bu_\alpha||_A + ||bu_\alpha - au_\alpha||_A$. Since the mapping $x \rightarrow x$ from B into A is continuous, therefore $||b-bu_\alpha||_A \leq c||b-bu_\alpha||_B$ [16, p. 626, Theorem 5.4]. Also $bu_\alpha - au_\alpha \in B$, hence $||bu_\alpha - au_\alpha||_A \leq c'||bu_\alpha - au_\alpha||_B$. But B is a dense ideal in A, therefore by Lemma 2.2.4, $||bu_\alpha - au_\alpha||_B \leq c''||b-a||_A ||u_\alpha||_B$. Therefore $||bu_\alpha - au_\alpha||_A \leq c'''||b-a||_A ||u_\alpha||_B$. That is, we have $||a-a_\alpha|| \leq ||a-b||_A + c||b-bu_\alpha||_B + c'''||b-a||_A ||u_\alpha||_B$, where c, c', c'' and c''' are constants. Therefore $\overline{\lim_\alpha} ||a-a_\alpha|| \leq \epsilon + \epsilon c''' = \epsilon(1+c''')$ since the right-hand side is independent of α. Hence $||a-au_\alpha|| \rightarrow 0$, i.e., $a \rightarrow au_\alpha$ for every $a \in A$. This implies $a \in aB \subset \overline{aB}$, but $\overline{aB} \subset \overline{aA}$, then $a \in \overline{aA}$. Hence A is dual [8, p. 694, Theorem 7]. Q.E.D.

Lemma 2.7.2: Let A be a dual A*-algebra with an approximate identity and B a Banach *-algebra which is a dense ideal of A. Then, if B is reflexive, B is dual.

Proof: B is locally weakly compact since B is reflexive. By Lemma 2.2.4, $||xu_\alpha||_B \leq c||x||_B ||u_\alpha||_A$, (c is a constant) for every $x \in B$ and $U_\alpha \in A$. Therefore $\{xu_\alpha\}$ is bounded in B for every $x \in B$. Hence $\{xu_\alpha\}$ contains a cofinal set converging weakly to an element $z\in B$. Since A has an approximate identity, then $||x-xu_\alpha||_A\to 0$. By Theorem 6.2, [16, p. 627], every continuous linear functional on A is continuous on B, i.e., $z=x$. Hence $||z-zu_\alpha||_B\to 0$. that is, $z\in \overline{zB}$. By Lemma 2.6.1, B is dual. Q.E.D.

Theorem 2.7.3: Let A be a dual A*-algebra of the second kind with norm $||x||$. Put $||x||_1=\underset{||y||=1}{\text{l.u.b.}}\ ||xy||$ for every $x\in A$. If $||x^*||_1=||x||_1$ and A has an approximate identity $\{u_\alpha\}$ with respect to the new norm $||x||_1$, then the completion A_1 of A by the norm $||x||_1$ is a dual A*-algebra of the second kind. Moreover,

(1) any dual A*-algebra with A as a dense ideal is considered as a subalgebra of A_1,

(2) there exists no A*-algebra with A_1 as a dense proper ideal,

(3) any reflexive Banach *-algebra which is a dense ideal of A_1 is dual.

Proof: Since A_1 is the completion of A with respect to $||x||_1$ and A has an approximate identity with respect to $||x||_1$ too, then $\{u_\alpha\}$ is also an approximate identity for A_1.

We show that A_1 is semi-simple, that $R\neq(0)$, so there would exist a self-adjoint primitive idempotent $e\in A$ such that $Re\neq(0)$, since $Re=(0)$ for every e implies that $RA=(0)$ and therefore $Ru_\alpha=(0)$ and $R=(0)$. Then $eRe=($the complex field)$\times$ e. This implies that $e\epsilon R$, which is a contradiction since zero is the only idempotent in the radical of any algebra [14, p. 56]. Therefore $R=(0)$, i.e., A_1 is semi-simple. Since $x^*x=0$ implies $x=0$ and $x\to x^*$ is continuous, then by Theorem 2.5.1, A_1 is a dual A*-algebra with a unique auxiliary norm $|x|_1$, where $|x|_1$ is also an auxiliary norm of A_1. Now we show that A_1 is of the second kind. Assume that A_1 is of the first kind, then, for every $x, y\epsilon A$, since $||x||_1=\text{l.u.b.}_{||y||=1}||xy||$ so that $||xu_\alpha y||\leq||xu_\alpha||_1||y||$. But A_1 is of the first kind, by Lemma 2.2.4, $||xu_\alpha||\leq c|x|_1$ $||u_\alpha||$ for some constant c. Therefore $||xu_\alpha y||\leq||xu_\alpha||_1$ $||y||\leq c|x|_1||u_\alpha||_1||y||\leq c|x|_1||y||$ since $||u_\alpha||_1\leq 1$. But $\lim_\alpha xu_\alpha y=xy$ and $c|x|_1||y||$ is independent of α, therefore $||xy||\leq c|x|_1||y||$. By Lemma 2.2.3, we see that A is of the first kind. This is a contradiction. Hence A_1 is of the second kind.

Proof of (1). Let B be any dual A*-algebra such that A is a dense ideal of B. By Lemma 2.2.4, we have that $||xy||_A\leq c||x||_B||y||_A$ for every $x, y\epsilon A$, c being a constant. As $||x||_1=\text{l.u.b.}_{||y||=1}||xy||$ for every $x\epsilon A$, so that $\text{l.u.b.}_{||y||_A=1}||xy||_A$ $\leq\text{l.u.b.}_{||y||_A=1}c||x||_B||y||_A$. Therefore $||x||_1\leq c||x||_B$. Hence $\phi:B\xrightarrow{\text{into}}A_1$ is a continuous *-homomorphism as a result of the extension of the mapping $x\to x$, $x\epsilon A$. By Lemma 2.2.1, the kernel of ϕ is a zero ideal.

Proof of (2). Let $x \in A_1$ and A_2 be any other A*-algebra with A_1 as a dense ideal. As $||xu_\alpha||_1 \to ||x||_1$ for every $x \in A_1$, we have $||x||_1 = \underset{||y||_1=1}{\text{l.u.b.}} ||xy||_1$. By Lemma 2.7.1, A_2 is dual. Hence (1) shows that A_2 is mapped *-isomorphically onto A, such that $x \leftrightarrow x$ for every $x \in A_1$, that is $A_1 = A_2$.

Proof of (3). It is a simple corollary of Lemma 2.7.2

Q.E.D.

<u>Theorem 2.7.4</u>: Let B be a dual Banach *-algebra which is a dense ideal of a dual A*-algebra A. Let B have an approximate identity $\{u_\alpha\}$ with bounded $\{||u_\alpha||_A\}$ (here we do not assume that $\{||u_\alpha||_B\}$ is bounded). If A' is a Banach *-algebra such that $B \subset A' \subset A$ where B is dense in A' and A' is an ideal of A, then A' is dual.

<u>Proof</u>: Take any element x of A' and let $b \in B$ such that $||x-b||_{A'} < \epsilon$ where ϵ is a positive number. As in the proof of Lemma 2.7.1, we have $||x-xu_\alpha||_{A'} \leq ||x-b||_{A'} + ||b-bu_\alpha||_{A'} + ||bu_\alpha - xu_\alpha||_{A'} \leq \epsilon + c'||b-bu_\alpha||_B + c''||b-x||_{A'}||u_\alpha||_A$, where c', c'' are constants. Therefore $||x-xu_\alpha|| \to 0$ implies $c \in \overline{xA'}$ for every $x \in A'$. By Lemma 2.6.1, A' is dual. Q.E.D.

<u>Theorem 2.7.5</u>: Let A be a dual A*-algebra with the property that any closed right ideal of A is the intersection of maximal regular ideals containing it. Any dual Banach *-algebra B which is a dense ideal of A has the property that any closed right ideal of B is the intersection of maximal regular ideals containing it.

<u>Proof</u>: Let I be any closed right ideal of B and $\bar{I}$ stand for the closure of I in A. Since B is dual, then we can see, as in the proof of Theorem 2.2.4, that $I=\bar{I}\cap A$. But A has the property that any closed right ideal of A is the intersection of maximal regular ideals containing it. Hence we can write $I=\cap M_\alpha$, where M_α is a regular maximal right ideal of A. Since A is a dual A*-algebra, then $L(M_\alpha)=Ae_\alpha$, where e_α is self-adjoint primitive idempotent of A, and $M_\alpha=\{z : e_\alpha z=0\}$. Put $N_\alpha=M_\alpha\cap A$. Since B is a dense ideal of A, then $e_\alpha \in B$. As $N_\alpha=M_\alpha\cap A$, $N_\alpha=\{x : e_\alpha x=0,\ x\in B\}$, which is a regular maximal right ideal of B. Since $\cap N_\alpha=\cap(M_\alpha\cap A)=\cap M_\alpha\cap A=I\cap A=I$, hence $I=\cap N_\alpha$. Therefore the proof is completed.

Q.E.D.

CHAPTER III

MINIMAL IDEALS IN TOPOLOGICAL RINGS

Throughout this chapter we let A be a ring with no nilpotent one-sided ideals $\neq (0)$. Such a ring is sometimes called SEMI-PRIME. We use S to denote the SOCLE of A. We use S^a to denote the ANTI-SOCLE of A. We write $S^a=L(S)=R(S)$. We use J to denote the radical of A.

In Section 1, we study some minimal ideals in topological rings. In Section 2, we work on some purely algebraic developments concerning minimal ideals.

1. On Rings with Minimal Ideals

Lemma 3.1.1: A left (right) ideal $I\neq(0)$ in A contains no minimal left (right) ideal of A, if and only if $I\subset S^a$.

Proof: Assume that $I\not\subset S^a$, then we must show that I contains a minimal left ideal of A. There exists a minimal idempotent e such that $eI\neq(0)$. For $eI=(0)$, so $e\in L(I)$. But $[L(I)]^2=(0)$ and hence $e=0$, which is impossible. Take $u\in I$ such that $eu\neq(0)$. Since no nilpotent one-sided ideals $\neq (0)$. and eA is minimal, then $eA=euA$. Therefore there exists a $z\in A$ such that $euz=e$. Since $(euz)^2=e^2=e$, we have that $euz\neq 0$. Note that $(euz)^2=euz$. As $u\in I$, we have $Aeuz\subset I$. To see that Aeuz is the desired minimal ideal, it is sufficient to see that euzA is a division ring [6, p. 65].

Assume that $I \subset S^a$, then I can not contain a minimal left ideal M of A. Otherwise $M \subset S \cap S^a$, which is impossible.

Q.E.D.

Lemma 3.1.2: Let A be a topological ring, M a maximal-closed modular right ideal. The following are equivalent:

(1) $M \not\supset S$ and $L(M) \neq (0)$;

(2) $L(M)$ is a minimal left ideal of A;

(3) $M=(1-e)A$ for a minimal idempotent e.

If $S^a=(0)$, then (2) and (3) are equivalent to (1′) $L(M) \neq (0)$.

Proof: (1)⇒(3). Assume that $L(M)$ contains no minimal left ideal of A. By Lemma 3.1.1, we have that $L(M) \subset S^a$. From the definition of S^a, we have that $S^a=R(S)=L(S)$. This implies that $L(M) \subset L(S)$ and hence $RL(M) \supset RL(S)$. But $S \subseteq RL(S)$ and therefore $S \subseteq RL(S) \subset RL(M)$. Now $RL(M)$ is a closed modular right ideal containing M. Since M is maximal-closed, we have $M=RL(M)$ or $RL(M)=A$. Since $L(M)$ is a left annihilator ideal, then $LRL(M)=L(M)$ [14, p. 96]. So if $RL(M)=A$, then $LRL(M)=L(A)=(0)$, i.e., $L(M)=(0)$ which is impossible. Therefore $RL(M)=M$. But $S \subset RL(M)=M$, which contradicts the assumption. Therefore there is a minimal idempotent e such that $Ae \subset L(M)$ and $M=RL(M)$. But $R(Ae) \supset RL(M)$. Since $Ae(1-e)A=AeA-AeA=(0)$, then $R(Ae)=(1-e)A$. Therefore $M \subset (1-e)A$. By definition of modular, $(1-e)A \subset M$ and hence $M=(1-e)A$.

(3)⇒(2). Assume that $M=(1-e)A$ for a minimal idempotent e then $L(M)=Ae$. Since Ae is a minimal left ideal of A so is $L(M)$.

(2)⇒(1). Assume that $L(M)$ is a minimal left ideal of A. If $S\subset M$ then $L(M)\subset M$ since $L(M)\subset S$. Since A is a semi-prime ring then $[L(M)]^2=(0)$. This implies that $L(M)=(0)$, which is impossible. Therefore $S\not\subset M$ and $L(M)\neq(0)$.

Finally consider the case where $S^a=(0)$ and suppose that $L(M)\neq(0)$. If $S\subset M$ then $L(M)\subset L(S)=S^a=(0)$. Therefore $L(M)=(0)$, which is a contradiction. Thus (1′)⇒(1). Q.E.D.

<u>Lemma 3.1.3</u>: Let M be a modular maximal right ideal of A. The following statements are equivalent:

(1) $S\not\subset M$;

(2) $L(M)$ is a minimal left ideal of A;

(3) $L(M)\neq(0)$;

(4) $M=(1-e)A$, where e is a minimal idempotent of A.

<u>Proof</u>: (1)⇒(4). Assume that $S\not\subset M$. As in the proof of Lemma 3.1.2, we know that there exists a minimal right ideal eA, where e is a minimal idempotent. Since $eA\subset S$, then $eA\not\subset M$, and hence $eA\cap M=(0)$. By the Peirce decomposition theorem, $A=M\oplus eA$. Consider a left identity j for A modular M. We can write $j=u+v$, where $u\in M$ and $v\in eA$. Since M is a modular ideal $(j-1)x\in M$ for all $x\in A$, i.e., $(u+v)x-x\in M$. This implies that $ux+vx-x\in M$, i.e., $vx-x\in M$. Therefore $(1-v)A\subset M$. As $eA\cap M=(0)$, we have that $eA\cap(1-v)A=(0)$. Hence $vx=x$ for all $x\in eA$ and hence $eA=vA$. Therefore v is a minimal idempotent. By the Peirce decomposition, $A=(1-v)A\oplus vA=M\oplus vA$. As $(1-v)A\subset M$, we see that $M=(1-v)A$.

(4)⇒(3). The proof follows by Lemma 3.1.2.

(3)⇒(2). Assume that $L(M)\neq(0)$. Let π be the natural homomorphism of A onto $^A/J$, where J is the radical of A. Then $J\cap L(J)=(0)$ by semi-simplicity, and $\pi(M)$ is a modular maximal right ideal of the semi-simple ring $^A/J$. Since $J\subset M$, then $L(M)\subset L(J)$, i.e., $L(M)\cap J\subset L(J)\cap J$. Since $[L(J)\cap J]^2=(0)$, then we have $L(J)\cap J=(0)$, and therefore $L(M)\cap J=(0)$. This implies that $\pi[L(M)\cap J]=\pi[(0)]=(0)$. Therefore π is an isomorphism, i.e., π is one-to-one on $L(M)$. Since $L(M)\neq(0)$ so that $\pi[L(M)]\neq(0)$, which is a non-zero left ideal lying in the left ideal $L[\pi(M)]$. Therefore $L[\pi(M)]\neq(0)$. Since $^A/J$ is semi-simple, then $L[\pi(M)]$ is a minimal left ideal of $^A/J$ [18, p. 96]. This implies that $L[\pi(M)]\subset\pi[L(M)]$, and hence $L[\pi(M)]=\pi[L(M)]$. Suppose that $L(M)$ contains a left ideal $I\neq(0)$ of A and $I\neq L(M)$. Then $\pi(I)$ is a left ideal of $^A/J$. Since $I\neq L(M)$ then $\pi(I)\neq\pi[L(M)]$, i.e., $\pi(I)\neq L[\pi(M)]$. But $L[\pi(M)]$ is a minimal left ideal of $^A/J$. Therefore $\pi(I)=(0)$, i.e., $I=(0)$ which is a contradiction. Hence $L(M)$ is a minimal left ideal of A.

(2)⇒(1). The proof follows by Lemma 3.1.2. Q.E.D.

<u>Lemma 3.1.4</u>: Let M be a modular maximal left ideal of A. The following statements are equivalent:

(1) $S\not\subset M$;

(2) $R(M)$ is a minimal right ideal of A;

(3) $R(M)\neq(0)$;

(4) $M=A(1-e)$, where e is a minimal idempotent of A.

Proof: The proof is similar to that of Lemma 3.1.3.

Q.E.D.

Theorem 3.1.5: Let A be an annihilator ring. Then $D_\ell=D_r$. If also S is dense in A, then every maximal-closed modular right (left) ideal in A is a maximal right (left) ideal, where D_ℓ (D_r) is the intersection of the closed modular maximal right (left) ideals of R, which is a topological ring.

Proof: Let M be a closed modular right ideal, then $L(M)\neq(0)$. By Lemma 3.1.3, $M=(1-e)A$ for a minimal idempotent e. Conversely any such M is a closed modular maximal right ideal. Therefore $M=(1-e)A=R(Ae)$. But $Ae\subset S$, i.e., $R(S)=S^a\subset R(Ae)$. Thus $S^a\subset M$. It follows that $S^a\subset D_r$. On the other hand, if $x\in D_r$, then $x\in R(Ae)$ for every minimal idempotent e, i.e., $x\in J$. Thss implies that $x\in S^a$ since $J\subset S^a$. Therefore, $S^a=D_r$. In a same manner, we can prove that $S^a=D_\ell$. Thus $S^a=D_r=D_\ell$.

Suppose that S is dense in A, i.e., $\overline{S}=A$. Let N be a maximal-closed modular ideal right ideal. Let $x\in L(S)=S^a$, then $xS=(0)$. By continuity of multiplication, $x\overline{S}=(0)$, i.e., $xA=(0)$. Since A is an annihilator ring, then $x=0$, i.e., $S^a=(0)$. By Lemma 3.1.2, N is a maximal right ideal.

Q.E.D.

2. Algebraic Developments Concerning Minimal Ideals

Lemma 3.2.1: Any two-sided ideal I of A has no nilpotent one-sided ideals $\neq$ (0).

Proof: Let R be a right ideal of I, $R\neq(0)$. We show that $RI\neq(0)$. Suppose that $RI=(0)$. Since I is two-sided ideal of A, then $IA\subset I$ and $AI\subset I$. As R is a right ideal of I, $R\subset I$. Then $RA\subset IA\subset I$. Hence $ARA\subset AI\subset I$, i.e., $ARA\subset I$. Therefore $RARA\subset RI$. This implies that $(RA)^2\subset RI=(0)$ and therefore $(RA)^2=(0)$. This implies that RA is a nilpotent right ideal of A. Since A is a ring with no nilpotent one-sided ideals $\neq$ (0), then $RA=(0)$. Since $R\subset I\subset A$, then $R\subset A$ and $L(A)=\{x\in A: xA=(0)\}$. Therefore $R\subset L(A)$. But $[L(A)]^2=(0)$ so that $R\subset L(A)=(0)$. Hence $R=(0)$ which is a contradiction. Therefore $RI\neq(0)$.

If $R^n=(0)$ for some positive integer $n>1$ then $(RI)^n=(0)$, since $RI\subset R$. Since $IA\subset I$, i.e., $RIA\subset RI$, it implies that RI is also a right ideal of A. Therefore $RI=(0)$, which is impossible. Q.E.D.

Lemma 3.2.2: Let I be a two-sided ideal of A and M a modular maximal right ideal of I. If $L(M)\cap I=(0)$, then M is contained in a maximal modular right ideal N of A with $L(N)=(0)$.

Proof: We show first that M is also a right ideal of A. Suppose otherwise and let j be a left identity for I modulo M. Since M is not a right ideal of A, then there is a $x\in A$, $v\in M$, such that $vx\notin M$. But M is a modular maximal right ideal of I, hence $vx \in I$. As M is maximal, there exists $w \in I$, $z \in M$, and an integer k such that $j=z+(vx)w+kvx$. This implies $j^2=[z+(vx)w+kvx]j$, i.e., $j^2=zj+(vx)wj+kvxj=xj+v(xwj+kxj)$. Since $MI \subset M$, so $zj \in M$. Also $AI \subset I$, i.e., $xw \in I$, $xwj \in I$ and $xj \in I$. Hence $xwj+kxj \in I$ and $v(xwj+kxj) \in M$. We have $(j-1)I \subset M$, then $(j-1)j \in M$, i.e., $j^2-j \in M$. Therefore we see that $j^2 \in M$, then $j \in M$, which is impossible. Hence M is a right ideal of A.

We are given that $L(M) \cap I=(0)$. Since $IL(M) \subset I \cap L(M)$, then $IL(M)=(0)$. We wish to show that $L(M)=R(M)=L(I)=R(I)$. Since $IL(M)=(0)$, then $L(M) \subset R(I)$. But I is two-sided ideal of A, hence $R(I)=L(I)$. Therefore $L(M) \subset R(I)=L(I)$. As $M \subset I$, $L(I) \subset L(M)$. Therefore $L(M)=L(I)=R(I)$. Since M is a right ideal of A, i.e., $MA \subset M$, this implies $R(M)MA \subset R(M)M$. Hence $R(M)M$ is also a right ideal of A. Since $R(M)M \subset R(M) \cap M=(0)$, then $[R(M)M]^2=(0)$, i.e., $R(M)M$ is nilpotent. It follows that $R(M) \subset L(M)=R(I)$. As $M \subset I$, $R(I) \subset R(M)$. Therefore $R(I)=R(M)$. Hence $R(M)=R(I)=L(M)=L(I)$.

From this we will prove that $L(M+R(M))=(0)$. Let $x \in L(M+R(M))$, then $x \in L(M)$ and $x \in LR(M)$. Therefore $x^2 \in L(M)M$, i.e., $x^2 \in R(M)M=(0)$. Hence $x^2=0$, $x=0$. Thus $L(M+R(M))=(0)$.

Next set $\beta(M)=\{w \in A: wy \in M \text{ for all } y \in I\}$. Since M is right ideal of I, then $xy \in M$ for $x \in M$, $y \in I$. This implies $xy \in \beta(M)$, i.e., $M \subset \beta(M)$. Let $x \in A$, $y \in I$ and j be a left identity for I modulo M. Then $(jx-x)y=j(xy)-(xy)=(j-1)(xy)$. By definition of modular, $(j-1)(xy) \in M$ as $xy \in I$. But $M \subset \beta(M)$, hence $(j-1)(xy)\in\beta(M)$. Therefore j is also a left identity for A modulo $\beta(M)$. We wish to show that $\beta(M)$ is contained in a modular maximal right ideal N of A, with $L(N)=(0)$. Let j be a left identity of A modulo N. We claim $j\notin\beta(M)$. For otherwise $j\in\beta(M)$. Since $j\in I$ and $\beta(M)=\{w\in A: wy\in M, \text{ for all } y\in I\}$, then $j^2\in M$ by taking $w=j$, $y=j$. This implies $j\in M$ which is impossible. Therefore $j\notin\beta(M)$ and hence $\beta(M)\subseteq N$. But $L(M)I=(0)=L(M)\cap I$ so that $L(M)\subset\beta(M)$. Since $L(M)=R(M)$, hence $R(M)\subset\beta(M)\subset N$. As $M\subset N$, then $M+R(M)\subset N$. This gives us $L(N)\subset L(M+R(M))$. But $L(M+R(M))=(0)$, i.e., $L(N)=(0)$. Q.E.D.

<u>Theorem 3.2.3</u>: Let A be a modular annihilator ring and I a two-sided ideal of A. Then I is also a modular annihilator ring.

<u>Proof</u>: As in the proof of Lemma 3.2.2, we have that for every modular maximal ideal M of I is contained in a modular maximal ideal N of A, i.e., $M\subset N$. Since A is a modular annihilator ring, hence $L(N)\neq(0)$ and therefore $R(M)\neq(0)$. As $M\subset N$, we have that $L(N)\subset L(M)$. This implies that $L(M)\neq(0)$. Therefore I is also a modular annihilator ring. Q.E.D.

<u>Theorem 3.2.4</u>: A semi-simple modular annihilator ring A is the subdirect sum of primitive modular annihilator rings.

<u>Proof</u>: Since any semi-simple ring is a subdirect sum of primitive rings [6, p. 14], then we need only to show that $^A/P$ is a modular annihilator ring, where P is a primitive ideal of A. Let N be a modular maximal

right ideal of A/P and π the natural homomorphism of A onto A/P. We must show that $L(N)\neq(0)$ in A/P.

Suppose that $L(N)=(0)$. Since $\pi^{-1}(N)$ is a modular maximal right ideal of A and let $M=\pi^{-1}(N)$, then M is also a modular maximal right ideal of A. We have $L(N)\subset P$, i.e., $R(P)\subset RL(M)=M$, because M is maximal and $M\subset R(L(M))$. By definition, $P=M:A\subset M$. Thus $P\subset M$ and therefore $R(P)+P\subset M$, i.e., $L(M)\subset L(R(P)+P)$. As in the proof of Theorem 3.2.2, $L(R(P)+P)=(0)$, and this implies that $L(M)=(0)$, which is a contradiction. Therefore $L(N)\neq(0)$. Q.E.D.

Lemma 3.2.5: Let I be a two-sided ideal of A and let $S_o(T_o)$ be the socle (anti-socle) of I. Then $S_o=S\cap I=SI=IS$ and $T_o=S^a\cap I$.

Proof: Let eA be a minimal right ideal of A, where $e^2=e$. We show first that either eA is a minimal right ideal of I or $eA\subset L(I)=R(I)$. We have either $eA\cap I=(0)$ or $eA\cap I=eA$. If $eA\cap I=eA$, then $eA\subset I$, i.e., $e\in I$. Since $eA\subset I$, then $e^2A\subset eI$, i.e., $eA\subset eI$. But $I\subset A$ implies that $eI\subset eA$. Therefore $eA=eI$. It follows that $eAe=eIe$. Since eA is a minimal right ideal of A, $e^2=e$, then eAe is a division ring [6, p. 65]. This implies that eIe is a division ring too. By Lemma 3.2.1, I has no nilpotent one-sided ideals $\neq(0)$. By [6, p. 65], we have that eI is a minimal right ideal of I. Now suppose that $eA\cap I=(0)$. Since $I\subset A$, then $eI\subset eA$, for I is a two-sided ideal of A, thus $AI\subset I$. But $e\in A$ implies that $eI\subset I$. Therefore

$eI \subset eA \cap I = (0)$, i.e., $eI = (0)$. Hence $e \in L(I) = R(I)$, since I is a two-sided ideal of A. Therefore $eA \subset L(I) = R(I)$.

We know that eI is a minimal right ideal of I, that is, $eI \subset I$. This implies that $eIA \subset IA$, but $IA \subset I$, hence $eIA \subset I$. Since $e^2 = e$, then $e^2 IA \subset eI$, i.e., $eIA \subset eI$. Therefore eI is also a right ideal of A and hence a minimal right ideal of A. Thus $S_o \subset SI \subset S \cap I$. Now let $y = e_1 x_1 + e_2 x_2 + \ldots + e_n x_n$ be any element of $S \cap I$ where each e_k is a minimal idempotent of A, $x_k \in A$ and $e_k x_k \neq 0$. We had proved that $e_k A \subset I$ or $e_k A \subset R(I)$. Since $[I \cap R(I)]^2 = (0)$, then $I \cap R(I) = (0)$, and then both $e_k A \subset I$ and $e_k A \subset R(I)$ can not happen. Let u be the sum of $e_k x_k$ contained in I and v be the sum of $e_k x_k$ contained in R(I). Then we write $y = u + v$. Since $y \in I$ and $u \in I$, thus $y - u \in I$, i.e., $v \in I$. But $v \in R(I)$ and $I \cap R(I) = (0)$. Hence $v = 0$. Therefore we may suppose that each $e_k A \subset I$ so that $y \in S_o$. This implies that $S \cap I \subset S_o$, i.e., $SI \subset S \cap I \subset S_o$. Therefore we have that $S_o = SI = S \cap I$, but $SI = IS$, since I is two-sided ideal, hence $S_o = S \cap I = SI = IS$.

Since $S_o = S \cap I$, then $S_o \subset S$, i.e., $L(S) \subset L(S_o)$. This implies that $S^a \subset T_o$, i.e., $S^a \cap I \subset T_o \cap I$. We see that $T_o \subset I$ and therefore $S^a \cap I \subset T_o$. Let $x \in T_o$ and let N be a minimal right ideal of A. If N is a minimal right ideal of I, then $xN = (0)$ since $x \in T_o = L(S_o) = \{y \in I : yS_o = (0)\}$, and $N \subset S_o$. On the other hand, if N is not a minimal right ideal of I, then we have that $N \subset R(I) = \{y \in A : Iy = (0)\}$. This implies that $xN = (0)$. Therefore $x \in S^a = L(S) = \{y \in A : yS = (0)\}$. Also we see that $x \in T_o = L(S_o)$, i.e., $x\epsilon I$. Thus $x\epsilon S^a\cap I$. Therefore $T_o\subset S^a\cup\epsilon$. It follows that $T_o=S^a\cap I$. Q.E.D.

Lemma 3.2.6: Let A be a topological ring. The following are equivalent:

(1) S is dense in A;

(2) A is the topological direct sum of its minimal closed two-sided ideals, and $S^a=(0)$.

Proof: (1)⇒(2). By Theorem 5 [3, p. 158], each minimal right ideal of A is contained in a minimal closed two-sided ideal of A, then S is contained in the direct sum of minimal closed two-sided ideals, i.e., $S\subseteq D$, where D denotes the algebraic direct sum of minimal closed two-sided ideals in A. Therefore $\overline{S}\subseteq\overline{D}$. As $\overline{D}\subseteq A$ and $A=\overline{S}$, then $\overline{D}\subseteq\overline{S}$. This implies that $A=\overline{S}=\overline{D}$. By definition, $\overline{D}$ is the topological direct sum.

Let $x\epsilon L(S)$, i.e., $xs=(0)$. Hence $x\overline{S}=(0)$, i.e., $xA=(0)$. Therefore $x=0$, so that $L(S)=S^a=(0)$.

(2)⇒(1). Let P be a minimal closed two-sided ideal of A. Since $S\neq(0)$, then $PS\subset P\cap S=(0)$. If $PS=(0)$, then $S\subset R(P)$, i.e., $S=(0)$, which is a contradiction. As $SP\subset S\cap P$ and $S\subset P$, we have that $SP\subset S$, i.e., $\overline{SP}\subset\overline{S}$. Also $SP\subset S\subset P$, i.e., $\overline{SP}\subset\overline{P}=P$. Since S is a two sided ideal of A, $AS\subset S$, hence $ASP\subset SP$, i.e., SP is a left ideal of A. But P is a minimal two-sided ideal of A. Hence $P=\overline{P}\subset\overline{SP}$. Therefore $P=\overline{SP}\subset\overline{S}$. This by hypothesis implies $A\subset\overline{S}$. As $S\subset A$, $\overline{S}\subset\overline{A}=A$. Thus $A=\overline{S}$, and S is dense in A.

Q.E.D.

Theorem 3.2.7: Let A be a modular annihilator ring which is a topological ring with dense socle S. Then A is the topological direct sum of topologically simple modular annihilator rings with dense socle.

Proof: By Lemma 3.2.6, A is the topological sum of its minimal closed two-sided ideals. Consider such an ideal I. By Theorem 3.2.3, I is a modular annihilator ring. From Lemma 3.2.5, we see that SI is the socle of I. By Lemma 2.1.11, [14, p. 46], $SI \neq (0)$. Since A has a dense socle, then I also has a dense socle. Therefore $I = \overline{SI}$. From Lemma 3.2.1 and Theorem 2.1 [6, p. 65], then I is topologically simple. Q.E.D.

APPENDIX

[3, p. 157, Remark]. The annihilator of a maximal closed right ideal in a semi-simple annihilator algebra is a minimal closed left ideal and conversely.

[3, p. 158, Theorem 5]. If I is a minimal right ideal algebra annihilator in a semi-simple A then the closed two-sided ideal (I) generated by I is a minimal closed two-sided ideal of A.

[5, p. 13, Theorem 5]. If U is a ring that contains a minimal right ideal but no nilpotent ideals, then U contains a minimal left ideal.

[7, p. 539, Theorem 2]. Let U be a semi-simple algebra over R (real) or C (complex). Let $|| \ ||$, $|| \ ||'$ be norms on U such that (U, $|| \ ||$) and (U, $|| \ ||'$) are Banach algebras. Then the norms $|| \ ||$, $|| \ ||'$ define the same topology on U.

[8, p. 690, Theorem 2]. Let A be a dual ring without nil-potent ideals. Then any closed two-sided ideal B in A is a dual ring. Moreover any closed right ideal in B is a closed right ideal in A.

[8, p. 692, Theorem 5]. Let A be a dual ring in which the intersection of the closed regular maximal right ideals is (0); in particular, A may be a semi-simple dual Q-ring. Then A is the direct sum of simple dual rings R_i.

[8, p. 694, Theorem 7]. Let A be a topological ring such that for every x, x is the closure of xA and in the closure of Ax. Suppose further that A has a dense two-sided ideal B which is a dual ring in the relative topology. Then A is a dual ring.

[9, p. 403, Theorem 4.4]. Let A be a *-algebra where x-x* has a quasi-inverse; in particular, A may be any C-symmetric algebra. Then the primitive ideals of A are self-adjoint.

[9, p. 406, Theorem 5.1]. The structure space of a complex completely continuous Banach algebra is discrete.

[9, p. 407, Theorem 6.2]. Let C be the Banach algebra of all real or complex continuous functions vanishing at ∞ on a locally compact Hausdorff space X. Then any other norm on C (whether complete or not) is at least as large as the functions norm.

[9, p. 408, Theorem 6.3]. If the algebra C of Theorem 6.2 (above) is given any other norm in which completion is semi-simple, this second norm gives the same topology as the function norm.

[9, p. 411, Lemma 8.1]. Let I, J be closed ideals in a B*-algebra A with the property that $I \cap J=(0)$ and I+J is dense in A. Then I+J is all of A, and A is the B*-sum of I and J.

[9, p. 412, Theorem 8.3]. A dual B*-algebra is the B*(∞)-sum of algebras, each of which is the algebra of all completely continuous operators on a complex Hilbert space.

[13, p. 360, Theorem 1]. Let U be a *-algebra such that every maximal commutative *-subalgebra of U is semi-simple, finite-dimensional, and U has the property that $x^*x=0$ implies $x=0$. Then U is *-isomorphic with a direct sum of full matrix algebras over the complex field of finite orders. Therefore U is semi-simple and finite-dimensional.

[13, p. 361, Theorem 2]. Let U be a Banach *-algebra in which $k||x||^2 \leq ||x^*x||$, k being a positive constant, for every normal x, and $x^*x=0$ implies $x=0$. Then U is finite-dimensional if and only if any one of the following conditions is satisfied:

(a) U is reflexive,

(b) U is weakly complete,

(c) the bi-conjugate space of U is separable.

[13, p. 362, Remark]. Let U be a Banach *-algebra. U is called w.c.c. if every element of U is w.c.c. The set of w.c.c. elements of U is a closed two-sided ideal.

[13, p. 362, Theorem 4]. A bounded linear operator T in H is completely continuous if and only if T is a w.c.c element of B, where B is the Banach algebra of bounded linear operatcr in a Hilbert space H.

[14, p. 97, Theorem 2.8.5]. If I is a maximal-closed right ideal in A, then I is maximal modular right ideal, L(I) is a minimal left ideal, and I=R(L(I)). Similarly, if I′ is a minimal left ideal in A, then R(I′) is a maximal modular right ideal and I′=L(R(I′)).

[14, p. 99, Lemma 2.8.10]. Let I be any two-sided ideal in a semi-simple annihilator Banach algebra A. Then I∩L(I)=(0), L(I)=R(I) and I+L(I) is dense in A.

[14, p. 100, Theorem 2.8.15]. A semi-simple annihilator Banach algebra A is equal to the topological sum of its minimal left (right) ideals, and is equal to the direct topological sum of its minimal closed two-sided ideals.

[14, p. 100, Corollary 2.8.16]. The socle S of a semi-simple annihilator Banach algebra A is dense in A.

[14, p. 182, Theorem 4.1.3]. Every maximal normal subset C of a normed *-algebra A is a closed maximal commutative *-algebra of A.

[14, p. 187, Theorem 4.1.15]. The involution in an A*-algebra is necessarily continuous with respect to both norms.

[14, p. 188, Theorem 4.1.19]. Any *-subalgebra of an A*-algebra A is semi-simple.

[14, p. 244, Theorem 4.8.11]. Every complex B*-algebra is isometrically *-isomorphic to a C*-algebra.

[14, p. 249, Theorem 4.9.2]. Let I be a closed two-sided ideal in a B*-algebra. Then I*=I and $^{U}/I$ is also a B*-algebra.

[14, p. 261, Theorem 4.10.3]. Let U be a complex Banach $*$-algebra in which $x^*x=0$ implies $x=0$ and let I be a minimal left ideal in U. Then an inner product (x, y) can be introduced into I such that the left regular representation, $a \to T_a$, of U on I is a $*$-representation relative to (x, y) and every T_a is bounded relative to the inner product norm $|x|_0=(x,x)^{\frac{1}{2}}$.

[15, p. 29, Lemma 2.1]. If I is a minimal right ideal in a ring R, then there exists a unique element $e \in I$ such that $e^2=e \neq 0$, $e^*=e$ and $I=eR$. A similar result holds for left ideals.

[16, p. 626, Theorem 5.4]. Let A denote an A*-algebra and let A_1 denote a Banach algebra which is a $*$-subalgebra of A. Then there exists a constant k such that $||x|| \leq k||x||_1$ for $x \in A_1$, where $||x||$, $||x||_1$ are the norms in A, A_1, respectively.

[17, p. 193, Theorem 1]. Let U be a semi-simple regular Banach algebra which is algebraically embedded in a second Banach algebra B. Then every ϕ in Φ_U can be extended to an element of Φ_B.

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