

A GENESIS FOR COMPACT CONVEX SETS

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THESIS

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INTRODUCTION

This paper was written in response to the following question: what conditions are sufficient to guarantee that if a compact subset A of a topological linear space L^3 is not convex, then for every point x belonging to the complement of A relative to the convex hull of A there exists a line segment yz such that x belongs to yz and y belongs to A and z belongs to A ? Restated in the terminology of this paper the question may be given as follows: what conditions may be imposed upon a compact subset A of L^3 to insure that A is braced?

The pursuit of this problem gave rise to the concept of the genesis of a compact convex set. The class of braced sets having convex hull A could be considered as a propositional subset of the genesis of A .

In general, the terminology and language of this paper is that followed by Valentine (1) in his book on convex sets. Terms that are newly introduced herein are marked by an asterisk.

CHAPTER I

CONVEX HULLS

Convex sets were first studied by Brunn in 1887. Later, Minkowski added considerably to the body of knowledge that had begun to accumulate around convex sets. As a preliminary to the consideration of some theorems related to convex sets the following definitions are introduced:

Definition 1.1 Suppose S is a set of elements with a collection of subsets called open sets which satisfy the following axioms:

(a) The union of each collection of open sets is an open set.

(b) The intersection of each finite collection of open sets is an open set.

Then S together with its collection of open sets is called a topological space. The collection of open sets is called the topology of S . It can readily be deduced from (a) and (b) that ϕ and S are open sets. Henceforth, the better known properties of topological spaces will be assumed.

Definition 1.2 The sets $C \subseteq S$ and $D \subseteq S$ are complementary iff $C \cup D = S$ and $C \cap D = \emptyset$.

The complement of A relative to B is the set denoted by $B \setminus A = \{x \mid x \in B \text{ and } x \notin A\}$. Obviously if C and D are complementary, then $C = S \setminus D$. \bar{C} denotes the complement of C , i.e. $\bar{C} = S \setminus C = D$.

Definition 1.3 A closed set in a topological space S is the complement of an open set, i.e. A is closed iff $A = S \setminus B$, where B is open.

Definition 1.4 A linear space ℓ over the field of real numbers R is a space for which there exist two binary operations, vector addition (indicated by $+$) from $\ell \times \ell$ into ℓ and scalar multiplication (indicated by juxtaposition) from $R \times \ell$ into ℓ which satisfy the following axioms:

If $x, y, z \in \ell$ and $\alpha, \beta \in R$ then

- | | |
|-----------------------|--|
| (1) $x+y \in \ell$ | (4) $\alpha x \in \ell$ |
| (2) $x+y=y+x$ | (5) $\alpha(x+y)=\alpha x+\alpha y$ |
| (3) $(x+y)+z=x+(y+z)$ | (6) $\alpha(\beta x)=(\alpha\beta)x$ |
| | (7) $(\alpha+\beta)x=\alpha x+\beta x$ |

(8) for every pair $x, y \in \ell$, there exists a $z \in \ell$ such that $x+z=y$

(9) if 1 is the identity of the real field R , then $1x=x$ for every $x \in \ell$.

Number (8) implies the existence of an additive identity for vector addition since $x=y$ is a possible pair. θ will be used to denote the additive identity and is synonymous with the origin of a vector space.

Definition 1.5 Let ℓ be a linear space with a topology. Let $x \in \ell$. Then an open set N in the topology of ℓ such that $x \in N$ is called a neighborhood of x and is denoted by $N(x)$.

Definition 1.6 Let S be a set in ℓ . If $x \in S$ then x is an interior point of S iff there exists a neighborhood $N(x)$ of x such that $N(x) \subset S$. The set of all interior points of S is called the interior of S and is denoted by $\text{int}S$.

It is well known that $\text{int}S=S$ iff S is an open set.

Definition 1.7 Let S be a set in ℓ and $x \in \ell$. Then x is said to be a limit point of S iff for every neighborhood $N(x)$ of x , $N(x) \cap S \setminus \{x\} \neq \emptyset$.

Definition 1.8 Let S be a set in ℓ . Then S' denotes the set of all limit points of S and is called the derived set of S . $S \cup S'$ is called the closure of S and is denoted by $\text{cl}S$.

One may easily verify that $\text{cl}S$ is a closed set.

Definition 1.9 Let S be a set in ℓ . The set of points $x \in \ell$ such that for every neighborhood $N(x)$ of x , $N(x) \cap S \neq \emptyset$ and $N(x) \cap \bar{S} \neq \emptyset$ is called the boundary of S and is denoted by $\text{bd}S$.

A boundary point of S is said to be isolated if it is not a limit point of S .

Definition 1.10 Let f be a function which maps a subset S in a topological space ξ into a set S^* of a topological space ξ^* . The function f is said to be continuous at a point $x \in S$ iff for each neighborhood $V[f(x)]$ of $f(x)$ in ξ^* there exists a neighborhood $U(x)$ of x in ξ such that if $y \in S \cap U(x)$, then $f(y) \in V$; i.e., $f(S \cap U) \subset V$.

Definition 1.11 If a linear space ℓ has a topology then the topology is Hausdorff iff for each $x \in \ell$, $y \in \ell$, $x \neq y$, there exist two disjoint open sets containing x and y respectively.

Definition 1.12 A topological linear space L is a linear space with a Hausdorff topology such that the operations of vector addition, $f(x,y)=x+y$ denoted by $(x+y)/(x,y)$, and scalar multiplication, $f(\alpha,x)=\alpha x$ denoted by $(\alpha x)/(\alpha,x)$, where $x \in L$, $y \in L$, $\alpha \in R$ are continuous in all variables jointly.

Definition 1.13 If $x \in \ell$, $y \in \ell$, the line segment xy joining x and y is the set of all points of the form $\alpha x + (1-\alpha)y$ where $\alpha \in R$, $0 \leq \alpha \leq 1$.

Definition 1.14 A set $S \subset \ell$ is said to be star-shaped relative to a point $x \in \ell$ iff for each point $y \in S$ $xy \subset S$.

Definition 1.15 A topological linear space L is locally star-shaped iff for each neighborhood $U(\theta)$ of the origin θ of L , there exists a neighborhood $V(\theta)$ of θ such that $V(\theta) \subset U(\theta)$ and such that $V(\theta)$ is star-shaped relative to θ .

Definition 1.16 If A is a set in ℓ and if $x \in \ell$, then the set $A+x = \{y+x \mid y \in A\}$ is called the translate of A by x .

Definition 1.17 If A is a set in ℓ and $\alpha \in R$ $\alpha A = \{\alpha x \mid x \in A\}$ is called the scalar multiple of A by α .

Several well known theorems are tacitly assumed in this paper in order to devote time to properties and theorems whose consequences have a greater bearing on the subject under consideration. So it is without apology that the following theorem is offered without proof:

Theorem 1.1 The continuity of vector addition and scalar multiplication implies that each translate and non-zero scalar multiple of an open set is open.

Theorem 1.2 A topological linear space L is locally star-shaped and locally symmetric.

Proof: Let U be a neighborhood of the origin θ . Now $\alpha x / (\alpha, x)$ is continuous at $(0, \theta)$, so that by the definition of continuity there exists a neighborhood $V(\theta)$ of θ and a number $\delta > 0$ such that if $|\alpha| < \delta$ and if $x \in V(\theta)$, then $\alpha x \in U(\theta)$. Immediately, $\alpha V \subset U$ whenever

$|\alpha| < \delta$. Consider the class $T = \{\alpha V \mid |\alpha| < \delta\}$. Then by definition of a topology and neighborhood $\cup T$ is a neighborhood of θ . Also, $N = \cup T \cap U(\theta)$, and N is star-shaped and symmetric relative to θ .

Theorem 1.3 If L^n is an n -dimensional topological linear space, then L^n is linearly isomorphic with Euclidean n -space E^n . This means there exists a one-to-one linear bicontinuous map of L^n onto E^n .

The immediate import of this theorem is that theorems holding in L^n are immediately applicable to E^n and vice-versa. For a proof of Theorem 1.3 see Valentine (1, pp. 7-8).

Definition 1.18 A set $S \subseteq \mathcal{L}$ is convex iff for each pair of points $x, y \in S$, $xy \subseteq S$.

Definition 1.19 A topological linear space L is locally convex iff for each neighborhood $U(\theta)$ of θ there exists a convex neighborhood $V(\theta)$ of θ such that $V(\theta) \subseteq U(\theta)$.

An obvious alternative to Definition 1.18 is that a set $S \subseteq \mathcal{L}$ is convex iff it is star-shaped relative to every point $x \in S$. Similarly, Definition 1.19 could be altered to conform with Definition 1.15.

Definition 1.20 A convex set in L which has an interior point is called a convex body.

Theorem 1.4 The intersection of any collection of convex sets in a linear space \mathcal{L} is a convex set.

Proof: Let C be a collection of convex sets in \mathcal{L} . Consider $\bigcap C$. Suppose $\bigcap C$ has two points x and y . Then x and y belong to every element of C . But this implies that $xy \subseteq A$ for every $A \in C$, so that $xy \subseteq \bigcap C$. Then $\bigcap C$ is convex. If $\bigcap C$ does not have at least two points, then the conclusion is obvious.

Definition 1.21 The convex hull of a set $S \subseteq L$ is the intersection of all convex sets in L containing S and is denoted by $\text{conv}S$.

S is convex iff $\text{conv}S = S$.

Definition 1.22 A set S of a topological linear space L is bounded iff for each neighborhood $N(\theta)$ of θ there exists a positive scalar multiple of $N(\theta)$, $\alpha N(\theta)$, such that $S \subseteq \alpha N(\theta)$.

Definition 1.23 If A is a subset of S where S is a topological space, then A is compact iff every open covering of A has a finite subcover. In other words, if T is the topology of S , A is compact iff for every class of open sets $C \subseteq T$ such that $A \subseteq \bigcup C$, there exists a finite subset B of C such that $A \subseteq \bigcup B$.

It is well known that $A \subseteq L^n$ is compact iff A is closed and bounded.

Definition 1.24 A flat or linear variety of a linear space \mathcal{L} is a translate of a linear subspace of

\mathcal{L} . Two flats are parallel iff one is a translate of the other; i.e., if F and F^* are flats in \mathcal{L} , then F and F^* are parallel iff there exists an $x \in \mathcal{L}$ such that $F = F^* + x$. Further, the dimension of a flat is the cardinality of the basis of the corresponding parallel linear subspace. A flat of dimension 1 is called a line.

Any good book on linear algebra can provide information concerning the basis and dimension of a vector space.

Theorem 1.5 Each finite dimensional flat of a topological linear space L is closed.

Proof: Suppose that V is a finite dimensional flat of L . Assume, by way of contradiction, that $x \in \text{cl}V \setminus V \neq \emptyset$. Then there exists a flat of minimal dimension containing V and x . Let W be such a flat. Describe a relative topology on W as follows: let $T = \{W \cap U \mid U \in T^*, \text{ where } T^* \text{ is the topology of } L\}$. Now if x is in the closure of V relative to W , Theorem 1.3 implies that $x \in V$, a contradiction. Thus, there exists a neighborhood $N(x)$ of x relative to W such that $N(x) \cap V = \emptyset$. But by definition of T there exists a neighborhood of x , $U(x) \in T^*$ such that $U(x) \cap W \subset N(x)$. Since $V \subset W$, $U(x) \cap V = \emptyset$. But then $x \notin \text{cl}V$. Thus $\text{cl}V \setminus V = \emptyset$ and V is closed.

The next theorem relates a convex set to its line segment subsets. For a proof see Valentine (1, p.16).

Theorem 1.6 Let S be a set in the n -dimensional linear space ℓ^n . Let S_i be defined recursively as follows:

$$S_1 = \bigcup_{\substack{x \in S \\ y \in S}} xy \qquad S_i = \bigcup_{\substack{x \in S_{i-1} \\ y \in S_{i-1}}} xy$$

Then $S_{i_n} = \text{conv} S$ if i_n satisfies the inequality $2^{i_n-1} \leq n+1 \leq 2^{i_n}$.

Theorem 1.7 If A and B are two convex sets in ℓ , then $\text{conv}(A \cup B) = \bigcup_{0 \leq \alpha \leq 1} [\alpha A + (1-\alpha)B] = D$.

Proof: If $x \in A$, $y \in B$, then $\alpha x + (1-\alpha)y \in \text{conv}(A \cup B)$ when $0 \leq \alpha \leq 1$. Thus $D \subseteq \text{conv}(A \cup B)$ immediately. Now suppose $z \in D$, $w \in D$. By definition of D there exist points $x, x^* \in A$ and $y, y^* \in B$ such that $z \in xy$, $w \in x^*y^*$. But $xx^* \in A$, $yy^* \in B$, so that $\text{conv}(xx^* \cup yy^*) \subseteq D$. Then zw is a subset of $\text{conv}(xx^* \cup yy^*)$ so that $zw \in D$. Thus D is convex and $(A \cup B) \subseteq D$. Immediately, $D = \text{conv}(A \cup B)$.

Definition 1.25 Let $S \subseteq \ell$. The kernel K of S is the set of all points $z \in \ell$ such that $zx \subseteq S$ for every $x \in S$. In other words, the kernel K of S is the set of all those points with respect to which S is star-shaped. Notice that S is convex iff $S=K$ where K is the kernel of S . Also, note that K is itself a convex set follows immediately from the definition of K , so that K is often referred to as the convex kernel of S .

Definition 1.26 The interior of a set $S \subseteq L$ relative to the minimal flat containing it is denoted by $\text{intv}S$. For a linear space ℓ , if $x \in \ell$, $y \in \ell$, $x \neq y$, then $\text{intv}xy$ denotes the set of all points of the form $\alpha x + (1-\alpha)y$, where $0 < \alpha < 1$. If $x \in \ell$, then $\text{intv}x = x$.

Definition 1.27 A point of $x \in S$ is a core point of S iff for each point $y \in \ell$, with $y \neq x$, there exists a point $z \in \text{intv}xy$ such that $xz \subseteq S$. The set of all such core points of S is called the $\text{core}S$.

Theorem 1.8 If $S \subseteq L$ is star-shaped relative to a point p , then the $\text{cl}S$ is star-shaped relative to p .

Proof: Let $y \in \text{cl}S$. Choose some point of yp , say $\lambda y + (1-\lambda)p$ where $0 \leq \lambda \leq 1$. Now, $\lambda x + (1-\lambda)p / (x)$ is continuous in x , so that for each neighborhood U of $\lambda y + (1-\lambda)p$ in L there exists a neighborhood $V(y)$ of y such that $\lambda x + (1-\lambda)p \in U$ whenever $x \in V(y)$. Since $y \in \text{cl}S$, let $x \in V \cap S$. Then $\lambda x + (1-\lambda)p \in U \cap S$. But this implies that $\lambda y + (1-\lambda)p \in \text{cl}S$. Then $yp \subseteq \text{cl}S$.

Corollary 1.8 Let C be a convex set in L . Then the closure of C is convex.

Proof: Let $z \in \text{cl}C$, $w \in \text{cl}C$. As in Theorem 1.8 $\lambda x + (1-\lambda)y / (x, y)$ is continuous, so that if some point of zw is chosen, say $\lambda z + (1-\lambda)w$ where $0 \leq \lambda \leq 1$, then, as in Theorem 1.8, $zw \subseteq \text{cl}C$, so that $\text{cl}C$ is convex.

Theorem 1.9 If S is a set in a topological linear space L , then the closed convex hull of S , denoted by $cconvS$ is the same as the closure of the convex hull of S .

Proof: Corollary 1.8 implies immediately that $cl(convS)$ is convex, and as a result $cconvS \zeta cl(convS)$. Note that $cconvS$ is a closed convex set which contains S . Hence, $convS \zeta cconvS$. Thus $cl(convS) \zeta cconvS$. Therefore, $cl(convS) = cconvS$.

Theorem 1.10 Suppose S is a set in L and S is star-shaped relative to a point p . If $p \in intK$, where K is the convex kernel of S , and if $y \in clS$, then $intvyp \zeta intS$.

Proof: Let $y \in clS$, and without loss of generality let $y = \phi$. Now all that is necessary is to show that $\alpha p \zeta intS$ for all $0 < \alpha < 1$ since $y = \phi$. Since $p \in intK$, there exists a neighborhood $V(\phi)$ of ϕ such that $p + V(\phi) \zeta intK$. Then, for $\beta = -\alpha/(1-\alpha)$, by Theorem 1.1, the set $\beta V(\phi)$ is open. Since $\phi \in clS$, there exists a point $u \in \beta[V(\phi)] \cap S$. Thus, there exists a point $v \in V(\phi)$ such that $\beta v \in S$. The set $U = (1-\alpha)\beta v + \alpha(p+V)$ is an open set contained in S , since $v \in S$, $p+V \zeta intK$. Also, $x = (1-\alpha)\beta v + \alpha(p+v) = \alpha p$ and $\alpha p \in U \zeta intS$. Thus $intvyp \zeta intS$.

Corollary 1.10 Let C be a convex set in a linear topological space L . If $x \in \text{int}C$, $y \in \text{cl}C$, then $\text{int}vxy \zeta \text{int}C$.

Proof: Since $\text{int}C = \text{int}K$, Theorem 1.10 implies that $\text{int}vxy \zeta \text{int}C$ immediately.

Theorem 1.11 Let S be a set in L and suppose that S is star-shaped relative to a point p . Then, if $p \in \text{int}S$; the $\text{int}S$ is star-shaped relative to p .

Proof: Let $y \in \text{int}S$ and without loss of generality suppose that $p = \theta$, and choose $x = \alpha y$, $0 < \alpha \leq 1$. Since $y \in \text{int}S$, there exists a neighborhood $U(y)$ of y such that $U(y) \zeta \text{int}S$. Since $p \in K$, $\alpha U(y) \zeta S$, and since $\alpha U(y)$ is open, $\alpha y \in \alpha U(y) \zeta \text{int}S$, for $0 < \alpha \leq 1$. Thus $yp \zeta \text{int}S$, and $\text{int}S$ is star-shaped relative to p .

Corollary 1.11 If S is a convex set in L , then $\text{int}C$ is convex.

Proof: The proof is immediate from Theorem 1.11 since C is star-shaped relative to every point in $\text{int}C$.

Theorem 1.12 If $S \zeta L$, then $\text{int}S \zeta \text{core}S$.

Proof: Suppose $z \in \text{int}S$. Let U be a neighborhood of z with $U \zeta \text{int}S$. Let $y \in L \setminus \{z\}$. Since $F(\alpha) = z + \alpha(y - z)$ is continuous at $\alpha = 0$, there exists a δ such that if $0 \leq \alpha < \delta$, then $x = F(\alpha) \in U$. Thus $x \in \text{int}vzy$ and $xy \zeta U \zeta \text{int}S$. Then $z \in \text{core}S$.

Theorem 1.13 If C is a convex set in L , then $\text{core}C$ is convex.

Proof: Let $x \in \text{core}C$, $y \in \text{core}C$. Hence $y \in C$ and $C=K$, where K is the convex kernel of C . Without loss of generality assume that $y=\theta$. Then, let $z=\alpha x$, where $0 \leq \alpha < 1$. Consider any vector u where $u \neq \theta$. Since $x \in \text{core}C$, there exists a constant $\delta > 0$ such that $x + \beta u \in C$ for $0 \leq \beta < \delta$. C is star-shaped relative to $y=\theta$ so that $\alpha(x + \beta u) \in C$ for $0 \leq \beta < \delta$. But this implies that $\alpha x \in \text{core}C$ and thus $xy \in \text{core}C$. Then $\text{core}C$ is convex.

Theorem 1.14 If C is a convex body in L , then $\text{core}C = \text{int}C = \text{int}(\text{cl}C)$.

Proof: Let $y \in \text{core}C$ and $x \in \text{int}C$. Since $y \in \text{core}C$, there exists a point z such that $y \in \text{int}vxz$ and such that $zy \in C$. Hence, $z \in \text{cl}C$. By Corollary 1.10 $\text{int}vxz \subset \text{int}C$, so that $y \in \text{int}C$. Consequently, $\text{core}C \subset \text{int}C$. Theorem 1.12 implies that $\text{int}C \subset \text{core}C$; hence, $\text{int}C = \text{core}C$. Now, let $y \in \text{int} \text{cl}C$, $x \in \text{int}C$. There exists a point $z \in \text{cl}C$ such that $y \in \text{int}vzx$. Again by Corollary 1.10, $\text{int}vzx \subset \text{int}C$. Hence, $y \in \text{int}C$ so that $\text{int} \text{cl}C \subset \text{int}C$. Since $C \subset \text{int} \text{cl}C$, then $\text{int} \text{cl}C = \text{int}C$.

Theorem 1.15 If S is an open set in L , then $\text{conv}S$ is open.

Proof: Corollary 1.11 implies that $\text{int} \text{conv}S$ is convex. Since $S \subset \text{conv}S$, and $S \cap (\text{bd} \text{conv}S) = \emptyset$, then

$S \subseteq \text{int conv} S$. But $\text{int conv} S$ is convex, so that the previous sentence implies $\text{conv} S \subseteq \text{int conv} S$. Also, $\text{int conv} S \subseteq \text{conv} S$, so that $\text{conv} S = \text{int conv} S$. Thus, $\text{conv} S$ is open.

It should be pointed out that the convex hull of a closed set need not be closed. As an example consider the set S in Euclidean two-space E^2 with rectangular coordinates (x,y) defined by

$$S = \{(x,y) \mid x^2 y^2 = 1, 0 < y < \infty\}$$

S is closed but $\text{conv} S = \{(x,y) \mid y < 0\}$ is not closed.

CHAPTER BIBLIOGRAPHY

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CHAPTER II

SUPPORT THEOREMS

The concepts of hyperplanes and half-spaces are very useful in the formulation and proof of various theorems concerning convex sets. As soon shall become apparent, hyperplanes are related to flats in a very intimate manner. In order to define a hyperplane the notion of a linear functional is introduced.

Definition 2.1 Recall that a real valued function is often called a functional. A linear functional is a function from ℓ to \mathbb{R} which is additive and homogeneous, that is

$$f(x+y)=f(x)+f(y) \quad x \in \ell, y \in \ell$$

$$f(\alpha x)=\alpha f(x) \quad x \in \ell, \alpha \in \mathbb{R}.$$

Definition 2.2 Suppose $H \subseteq \ell$, then H is said to be a hyperplane iff there exists a non-identically zero linear functional f and a real constant $\alpha \in \mathbb{R}$ such that $H = \{ x \in \ell \mid f(x) = \alpha \}$ denoted by $H = [f:\alpha]$.

Theorem 2.1 A hyperplane is a flat.

Proof: Suppose that f is a linear functional and that $f(x) \neq 0, x \in \ell$. Let $H = [f:\alpha]$. Since $f \neq 0$, let

$z \in \ell$, with $f(z) \neq 0$. Choose an arbitrary point $q \in \ell$, and since $f(z) \neq 0$, define $p = [-f(q)/f(z)](z) + y$ where $y \in H$. Since f is linear,

$$f(p) = f\{[-f(q)/f(z)](z) + y\}$$

$$f(p) = -f(q) + f(y) \quad \text{so that}$$

$$f(p) + f(q) = f(p+q) = f(y) = \alpha.$$

Thus $p+q \in H$. But this statement together with the definition of p implies that $\ell = H + Rz$. Consequently, H is a flat.

Definition 2.3 If f is a linear functional and if $A \zeta \ell$, then $f(A) \geq \alpha$ means $f(x) \geq \alpha$ for every $x \in A$. Similar definitions may be made for inequalities that are strict or reversed.

Definition 2.4 The hyperplane $H = [f:\alpha]$ bounds the set $A \zeta \ell$ if either $f(A) \geq \alpha$ or $f(A) \leq \alpha$ holds.

Definition 2.5 The sets $\{x \in \ell \mid f(x) > \alpha\}$, $\{x \in \ell \mid f(x) < \alpha\}$, $\{x \in \ell \mid f(x) \geq \alpha\}$, and $\{x \in \ell \mid f(x) \leq \alpha\}$ are called half-spaces of ℓ .

Axiom 2.1 A partial order on a set S is a subset O of $S \times S$ such that if $(x, y) \in O$, then $(y, x) \notin O$, and if $(x, y) \in O$ and $(y, z) \in O$ then $(x, z) \in O$. Suppose that S is a partially ordered set, then S contains at least one maximal linearly ordered subset, where a linear order is a partial order with the additional condition that $(x, y) \notin O$ implies that $(y, x) \in O$.

Examples of partially ordered relations are less than or equal " \leq " and set inclusion " ζ ." Zorn's maximal principle is useful in the proof of a number of theorems concerning convex sets. It is equivalent to the axiom of choice and also to the Hausdorff maximal principle. For a proof of this equivalence and a discussion of related topics see Kelley (1, pp. 31-36).

Zorn's Maximal Principle If T is a partially ordered set and each linearly ordered subset of T has an upper bound in T , then T contains at least one maximal element.

Theorem 2.2 If A and B are disjoint convex sets in a linear space ℓ , then there exist complementary convex sets C and D of ℓ such that $A\zeta C$ and $B\zeta D$.

Proof: Consider the class $P = \{(A_i, B_i)\}$, where A_i and B_i are convex sets in ℓ such that $A_i \cap B_i = \phi$, $A_i \zeta A_j$, $B_i \zeta B_j$. P is not empty since $(A, B) \in P$. Now define a partial order on P as follows:

$$(A_i, B_i) < (A_j, B_j) \text{ if } A_i \zeta A_j \text{ and } B_i \zeta B_j.$$

The union of every linearly ordered subset of elements in P belongs to P . By Zorn's Maximal Principle there exists a maximal element (C, D) in P .

Now all that remains is to show that $\bar{C} = D$.

Suppose that $p \in \ell \setminus (C \cup D) \neq \phi$. By Theorem 1.7

$$\text{conv}(C \cup \{p\}) = \bigcup_{0 \leq \alpha \leq 1} [\alpha C + (1-\alpha)\{p\}] \text{ and also}$$

$\text{conv}(D \cup \{p\}) = \bigcup_{0 \leq \alpha \leq 1} [\alpha D + (1-\alpha)\{p\}]$. Since (C, D) is maximal, there exist points $x \in D \cap \text{conv}(C \cup \{p\})$, $y \in C \cap \text{conv}(D \cup \{p\})$, such that $x \notin C$ and $y \notin D$. But the above implies that there exist points $c \in C$, $d \in D$ such that $x \in \text{int}vc_p$, $y \in \text{int}vd_p$. The points c, d, p determine a triangle and, like medians, $dx \cap cy \neq \emptyset$. Since $dx \subset D$, $cy \subset C$, then $C \cap D \neq \emptyset$, a contradiction. Thus $\bar{C} = D$.

Theorem 2.3 If f and g are linear functionals in ℓ such that $[f:\alpha] = [g:\beta]$, then there exists a constant $\lambda \neq 0$ such that $f = \lambda g$, $\alpha = \lambda \beta$.

Proof: If f or g is identically zero, the theorem is trivial. Suppose $f \neq 0$ and $g \neq 0$. If $\beta = 0$, choose $z \in \ell \setminus [g:\beta]$. If $\beta \neq 0$, choose $z \in [g:\beta]$. Thus $g(z) \neq 0$. Let $p \in \ell$. Then there exists a $t \in [g:\beta]$ such that $p = t + \gamma z$, where $\gamma \in \mathbb{R}$. Now, $f(p-t) = \gamma f(z)$, and $g(p-t) = \gamma g(z)$. Then $g(p-t)/g(z) = \gamma$ so that $f(p-t) = [f(z)/g(z)][g(p-t)]$. Let $\lambda = f(z)/g(z)$. Thus $f(p) - f(t) = \lambda g(p) - \lambda g(t)$, and hence $f(p) - \alpha = \lambda g(p) - \lambda \beta$. Since the previous statement is true for all $p \in \ell$, then it is true for $p = z$. Consequently, $\alpha - \lambda \beta = f(z) - \lambda g(z) = 0$. Hence $\alpha = \lambda \beta$. Immediately, $f(p) = \lambda g(p)$.

Definition 2.6 A hyperplane H is said to support a set S at a point $x \in S$ iff $x \in H$ and H bounds S .

Theorem 2.4 Suppose A and B are convex subsets of a linear space ℓ , and that $\text{core} B \neq \emptyset$, $A \neq \emptyset$, and $A \cap \text{core} B = \emptyset$. Then there exists a hyperplane $H = [f:\alpha]$ such that $f(A) \leq \alpha$ and $f(B) \geq \alpha$.

Note that in a finite-dimensional linear topological space "core" corresponds to "interior." For a proof of Theorem 2.4, which is a fundamental separation theorem, see Valentine (3, p. 24).

Theorem 2.5 Suppose that B is a convex body in a topological linear space L , and suppose that F is a flat in L such that $F \cap \text{int} B = \emptyset$. Then there exists a hyperplane H containing F which bounds B .

Proof: By Theorem 2.4 there exists a hyperplane $H^* = [f:\alpha]$ such that $f(F) \leq \alpha$ and $f(B) \geq \alpha$. Then there exists a translate $H = H^* + x$ of H^* such that $F \subset H$ and H bounds B .

Theorem 2.6 A hyperplane $H = [f:\alpha]$ in L bounds a nonempty open set iff f is continuous and $f \neq 0$.

For a proof of Theorem 2.6 consult Valentine (3, pp.25-26).

Theorem 2.7 A hyperplane $H = [f:\alpha]$ is closed iff f is continuous with $f \neq 0$.

Proof: Suppose H is closed. Since $H \neq L$, there exists a point $x \notin H$ and since H is closed, $x \notin \text{cl} H$ so that there exists a neighborhood $N(x)$ of x such that $N(x) \cap H = \emptyset$. Now by Theorem 2.6, f is continuous since H bounds $N(x)$.

Suppose that f is continuous with $f \neq 0$. Then $\{x \mid f(x) < \alpha\}$ and $\{x \mid f(x) > \alpha\}$ are open convex sets and $H = L \setminus [\{x \mid f(x) < \alpha\} \cup \{x \mid f(x) > \alpha\}]$ so that $H = [f:\alpha]$ is closed.

Theorem 2.8 Suppose that C is a closed convex set in a locally convex space L and that $x \in L \setminus C$. Then there exists a closed hyperplane H through x such that $H \cap C = \emptyset$.

Proof: $L \setminus C$ is a neighborhood of x . Then there exists a convex neighborhood $N(x)$ of x such that $N(x) \cap L \setminus C$ and $N(x) \cap C = \emptyset$ since L is Hausdorff, locally convex, and C is closed. The interior of $N(x) \neq \emptyset$, so that Theorem 2.4 implies the existence of a closed hyperplane H^* separating $N(x)$ and C . Now consider the translate H of H^* such that $x \in H$. $H \cap C = \emptyset$.

Theorem 2.9 If C is a convex body in a topological linear space L , then through each boundary point of C there passes a closed hyperplane of support.

Proof: If $C=L$ the conclusion is trivial. Suppose that $C \neq L$. Let $x \in \text{bd}C$, and note that $\{x\}$ is a convex set. By Corollary 1.11 $\text{int}C$ is convex. But Theorem 2.4 implies that there exists a hyperplane H^* which separates $\text{int}C$ and $\{x\}$. Now consider a translate H of H^* such that $x \in H$. Then H is a hyperplane of support at x .

Theorem 2.10 Suppose that S is a set in a topological linear space L and that $\text{int conv}S \neq \emptyset$. A point $p \in \text{int conv}S$ iff each hyperplane H through p strictly separates at least two points of S .

Proof: Suppose that $p \in \text{int conv}S$ and that there exists a hyperplane $H = \{f: \alpha\}$ such that $p \in H$ and H does

not separate at least two points of S . But this statement means that all points of S belong to a closed half-space determined by H , say $\{x | f(x) \leq \alpha\}$, since f is continuous by Theorem 2.6. But now it is apparent that $p \in \text{bd conv}S$, a contradiction. Thus each hyperplane through p separates points of S .

Now suppose that each hyperplane through p strictly separates at least two points of S . And further suppose, by way of contradiction, that $p \notin \text{int conv}S$. By Theorem 2.5 and Theorem 2.7, since $\text{int conv}S \neq \emptyset$, there exists a closed hyperplane through p bounding S , a contradiction. Thus $p \in \text{int conv}S$.

Theorem 2.11 A line through the interior of a compact convex body A intersects $\text{bd}A$ twice.

Proof: Let $p \in \text{int}A \neq \emptyset$. Since A is compact, $A \neq L$. Consider a ray Q from p . A is bounded, but Q is unbounded so that there exists some point $y \in (L \cup A) \cap Q$. Now let $Q = \{x \in L \mid x = \alpha y + (1-\alpha)p, \alpha \geq 0\}$. Consider line segment $py = \{x \in Q \mid 0 \leq \alpha \leq 1\}$. Order the elements of py as follows: if $w \in py, z \in py$, where $w = \beta y + (1-\beta)p$ and $z = \lambda y + (1-\lambda)p$ for some $0 \leq \beta, \lambda \leq 1$, then $z > w$ iff $\lambda > \beta$. The $py \cap A$ is bounded above by y . Then there exists a least upper bound k for $py \cap A$. Obviously, $k \in \text{bd}A$ and also $k \in Q \cap \text{bd}A$. Similarly, for a ray $Q^* = \{x \in L \mid x = \alpha y + (1-\alpha)p, \alpha \leq 0\}$ the same conclusion as before is obtained. So that $Q \cup Q^*$ is a line through p and $Q \cup Q^*$ intersects $\text{bd}A$ in at

least two points. In addition, to intersect in more than two points contradicts the convexity of A .

Theorem 2.12 Let S be a closed set in a topological linear space L , and suppose that $\text{int}S \neq \emptyset$. Then S is convex iff through each boundary point of S there passes a hyperplane of support to S .

Proof: The necessity is provided immediately by Theorem 2.9.

Now suppose that through each boundary point of S there passes a hyperplane of support to S . If $S=L$, then S is convex. Suppose $S \neq L$. Then let $x \in \text{int}S$ and $y \in L \setminus S \neq \emptyset$. As in Theorem 2.11, there exists a $z \in \text{bd}S$ such that $z \in xy$. By hypothesis, there exists a hyperplane of support $H=[f:\alpha]$ through z . $y \notin H$, for if $y \in H$, then $x \in H$ since x, y , and z are colinear. But $x \in H$ would contradict that H is a hyperplane of support. Thus the closed half-space $H^+ = \{x \mid f(x) \geq \alpha\}$ determined by H and containing x contains S but not y where y is any point in the complement of S . Let T be the class of all such half-spaces that contain S but no point y external to A . Thus, $\Omega T = S$. By Theorem 1.4, S is convex since each element of T is convex.

A finite dimensional normed linear space is sometimes called a Minkowski space. Consult the Appendix for a more formal definition of a Minkowski space. For a proof of the following theorem see Valentine (3, p. 40).

Theorem 2.13 If S is a compact set in a Minkowski space L^n , then $\text{conv}S$ is compact.

The closed convex hull of a compact set is also compact in a Banach space. A Banach space is a normed linear space in which the Cauchy criterion is sufficient for the convergence of sequences. For a proof of Theorem 2.13 in Banach spaces see Taylor (2).

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CHAPTER III

EXTREMAL CHARACTERIZATION OF CONVEX SETS

Sufficient notions have now been introduced to enable an examination of some sets which share a convex hull, in particular, those sets which are minimal and generate a given convex hull. The theorems in this section culminate in extremal characterization of convex sets including the Krein-Milman Theorem and other related results.

Definition 3.1 A point $x \in \text{bd}C$, where C is convex and $C \subseteq L$, is called an exposed point of C iff there exists a hyperplane of support H to C through x such that $H \cap C = \{x\}$.

Rather than introduce a norm in order to define a Minkowski space L^n , the following theorem is offered without proof and only the comment that E^n is a Minkowski space so that the theorem is intuitively correct. For a proof consult Valentine (2, p. 52).

Theorem 3.1 Let C be a closed convex set in a Minkowski space L^2 (E^2). Each compact connected portion of the boundary of C which is not contained in a line

segment contains an exposed point of C . If C is a line segment, it has two exposed points.

Definition 3.2 If S is a convex set in ℓ , then a point $x \in S$ is an extreme point of S iff no non-degenerate segment in S exists which contains x in its relative interior.

Note that x is an extreme point of a convex set S iff $S \setminus \{x\}$ is convex. Also, the above definitions imply that the exposed points of S are extreme points of S .

Definition 3.3 A non-empty set M of a set $K \subseteq L$ is called an extremal subset of K iff $x \in K$, $y \in K$, and $M \cap \text{int} \{xy\} \neq \emptyset$ implies that $\{x\} \cup \{y\} \subseteq M$.

An extreme point of a set K is an extremal subset of K consisting of just a single point.

Definition 3.4 If C is a compact convex set of a topological linear space L , then $\text{ext}C$ denotes the set of extreme points of C and $\text{exp}C$ denotes the set of exposed points of C .

Definition 3.5 A real valued function is often called a functional. Let p be a function defined on a subset of ℓ . Then p is a convex functional defined on a convex subset S of ℓ , iff

$$p[\alpha x + (1-\alpha)y] = \alpha p(x) + (1-\alpha)p(y)$$

holds for all $x, y \in S$ and $0 \leq \alpha \leq 1$.

Definition 3.6* Let A be a compact convex subset of L^n . Then the set denoted by $gA = \{x \mid x \zeta A \text{ and } \text{conv}x = A\}$ is called the genesis of A .

The elements of gA are called the generators of A . The set $y \in gA$ such that $y \zeta x$ for every $x \in gA$ (if such a set exists) is called the antecedent of A . If gA is considered to be partially ordered in the usual sense of partially ordering sets, then the antecedent of A is a minimal set analogous to Zorn's maximal element.

The following examples provide some clue as to the nature of sets having antecedents:

Example 3.1 The antecedent of a closed line segment is the set of its endpoints.

Example 3.2 The antecedent of a closed triangular region is the set of its vertices. Further, the antecedent of a solid polyhedron is the set of vertices of the polyhedron.

Example 3.3 The antecedent of $A = \{x \in E^n : |x-p| \leq r\}$ is the set $\{x \in E^n : |x-p| = r\}$.

Example 3.4 Consider the open ball A where $A = \{x \in E^n : |x-p| < r\}$. Although A is not closed and hence not compact, it is informative to note that there are infinitely many subsets of A which share A as a convex hull, but no smallest such set. On the other hand, the set $B = \{(x,y) \in E^2 \mid y \geq x^2\}$ is unbounded and thus not

compact but has a smallest set $\{(x,y) \in E^2 \mid y=x^2\} \zeta B$ whose convex hull is B .

Notice that in examples 3.1 - 3.3 the antecedent of each convex set was the set of exposed points of the convex set. One might be led to conjecture that the exposed points of a compact convex set in L^n is the antecedent of the compact convex set. Example 3.5 quickly dispels this notion.

Example 3.5 Let $A = \{x \in E^2 : |x-s|=r\}$ and $p \in E^2 \setminus A$. Let $B = \text{conv}(A \cup \{p\})$. The antecedent of B is not the set of exposed points of B . Let pw and pz be the tangents from p to circle A where $w, z \in A$. The antecedent of B is the set $\{\text{closed major arc } zw\} \cup \{p\}$. But z and w are not exposed points of B . They are, however, extreme points of B .

From the definitions of $\text{exp}C$ and $\text{ext}C$, it is obvious that $\text{exp}C \zeta \text{ext}C$. But Example 3.5 clearly shows that $\text{ext}C$ is not a subset of $\text{exp}C$. However, both $\text{ext}C$ and $\text{exp}C$ are subsets of $\text{bd}C$.

Theorem 3.2 If the antecedent of a compact convex set A in L^n exists, it is unique.

Proof: Suppose x is the antecedent of a compact convex set A . Now suppose, by way of contradiction, that there exists a $y \in \text{g}A$ such that y is an antecedent for A and $y \neq x$. But by definition of antecedent $x \zeta y$ and

$y \zeta x$. Then $y=x$, a contradiction. So that if the antecedent of A exists, it is unique.

Theorem 3.3 If F is a family of extremal subsets of a set K , then a non-empty intersection of any subfamily of F is an extremal subset of K .

Proof: Let $f_i \in F, i \in A$. Choose $x \in K, y \in K$. If $(\bigcap_{i \in A} f_i) \cap \text{int}vxy \neq \emptyset$, then $f_i \cap \text{int}vxy \neq \emptyset$, so that for each $f_i, \{x\} \cup \{y\} \zeta f_i, i \in A$. Hence, $\{x\} \cup \{y\} \zeta \bigcap_{i \in A} f_i$.

Definition 3.7 Let S be a set. Let T be a collection of subsets of S , then (S, T) is said to be a restricted topological space iff for every non-empty subcollection $F \zeta T$

- (a) $\cup F \in T$ and
- (b) if F is finite, $\bigcap F \in T$.

Theorem 3.4 The genesis of a compact convex set having an antecedent is a restricted topological space.

Proof: Let F be a finite subset of gA where A is a compact convex subset of L^n such that A has an antecedent. Let x be the antecedent of A . Obviously, $x \zeta y$ for every $y \in F$. Then $x \zeta \bigcap F \zeta A$. But, $\text{conv}x \zeta \text{conv}(\bigcap F)$ and $\text{conv}(\bigcap F) \zeta A$. Now $\text{conv}x = A$, so that $\text{conv}(\bigcap F) = A$. But this implies that $\bigcap F \in gA$. A similar argument produces that if $F^* \zeta gA$, then $\cup F^* \in gA$.

Theorem 3.5 If K is a non-empty compact set in a locally convex topological linear space L , then K has at least one extreme point.

Proof: Let F be the collection of all compact extremal subsets of K . Then F is not empty since $K \in F$. Partially order the collection F by set inclusion. Since K is compact, any non-empty linearly ordered subset of F has a compact non-empty intersection, which by Theorem 3.3 is an extremal subset of K . By Zorn's Maximal Principle, the set F has a minimal element, denoted by F^* .

Now all that remains is to show that F^* is an extremal subset of K consisting of exactly one point and thus an extreme point of K . Suppose, by way of contradiction, that F^* contains two distinct points x and y . Now $\{x\}$ is a closed convex set and $y \in L \setminus \{x\}$. By Theorem 2.8, there exists a closed hyperplane H such that $y \in H$ and $H \cap \{x\} = \emptyset$. Then $x \notin H$ and $F^* \cap H \neq \emptyset$. Since $F^* \cap H$ is clearly an extremal subset of K , and since $x \notin F^* \cap H$, the set F^* is not minimal. Hence F^* can contain at most one point. This statement implies that $F^* \in K$ is an extreme point of K , since an extreme point is an extremal set consisting of exactly one point.

Theorem 3.6 Let S be a compact convex set in a locally convex topological linear space L . Then for each closed hyperplane of support to S , $H \cap \text{ext} S \neq \emptyset$.

Proof: Let H be a closed hyperplane of support to S . $H \cap S \neq \emptyset$, where $H = [f:\alpha]$ and S is a subset of one of the half-spaces determined by H , say H^+ . Since H is closed, $H \cap S$ is a non-empty, compact convex set and by Theorem 3.5 $H \cap S$ has an extreme point x . If $y \in S \setminus H$, $z \in S$, then $f(y) < \alpha$, $f(z) \leq \alpha$, where without loss of generality $H^+ = \{x \in L \mid f(x) \leq \alpha\}$. Since $f(x) = \alpha$, then $x \notin \text{int} \text{conv} S$. Also, if $y \in H \cap S$, $z \in H \cap S$, then since x is an extreme point of $S \cap H$, $x \notin \text{int} \text{conv} S$. Thus x is an extreme point of S .

The Krein-Milman Theorem, which follows, is a well known theorem that describes a convex set in terms of its extreme points. For a different proof of Theorem 3.7 see Krein and Milman (1).

Theorem 3.7 If S is a compact set in a locally convex topological linear space L , then the closed convex hull of S is the same as the closed convex hull of the set of extreme points of S .

Proof: Since $\text{ext} S \subseteq S$, then $\text{cl conv ext} S$ is a subset of the closed convex hull of S . Now it remains to show that $S \subseteq \text{cl conv ext} S$. Suppose, by way of contradiction, that S is not a subset of $\text{cl conv ext} S$, and then choose x from $S \setminus \text{cl conv ext} S \neq \emptyset$. Since L is a locally convex space, Theorem 2.8 implies that there exists a closed hyperplane $H^* = [f:\beta]$ through x , where

without loss of generality, $f(y) < \beta$ for $y \in \text{cl conv ext} S$.
 But S is compact, so that a hyperplane $H = \{f: \alpha\}$, $\alpha \geq \beta$,
 exists such that $H \cap S \neq \emptyset$, $f(S) \leq \alpha$, and thus $H \cap \text{cl conv ext} S = \emptyset$.
 But Theorem 3.6 implies that since H is closed, H
 contains an extreme point of S . This is a contradiction.
 Thus, it is false that S is not a subset of $\text{cl conv ext} S$.
 Then $\text{cl conv } S = \text{cl conv ext} S$.

Theorem 3.7 may be restated as follows:

Theorem 3.8 Suppose that C is a compact convex subset of a locally convex Hausdorff linear space L and x is a subset of C . Then C is the closed convex hull of x iff each extreme point of C lies in the closure of x .

A finite dimensional normed linear space L^n is locally convex, so that Theorems 2.13, 3.7, and 3.8 now imply that every compact convex subset of L^n has an antecedent, namely the set of extreme points of the compact convex set. Recall also that L^n is linearly isomorphic to E^n . These statements then form the proof of the following:

Theorem 3.9 If S is a compact convex set in a finite dimensional normed linear (Minkowski) space L^n or E^n , then S has an antecedent.

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CHAPTER IV

A PROPOSITIONAL SUBSET OF THE GENESIS OF A SET

An advantage of defining the genesis of a compact convex subset of L^n is to discuss some properties of some of the propositional subsets of the genesis. As a case in point, the following definitions lead to one such subset of gA , where $A \subseteq E^3$:

Definition 4.1* If X is a set property and A is a compact convex subset of L^n , then $P_A(X)$ will be used to denote the subset of gA consisting of those elements having property X , i.e., $P_A(X) = \{y \in gA \mid y \text{ has } X\}$.

Definition 4.2* Let A be a subset of L , then the set denoted by $\text{aug}A = \bigcup_{\substack{x \in A \\ y \in A}} xy$ is the augmentation of A .

Definition 4.3 A maximal connected subset of a set A in a topological space S is called a component of A .

A set A in L is simply connected iff each component of the complement of A is unbounded.

Alternately, a set is simply connected iff any closed curve within it can be deformed continuously to a point of the set without leaving the set.

Definition 4.4* Let $A \subseteq L^n$, A not convex. Suppose $x \in \text{conv} A \setminus A$ and that there exists points z and y belonging to A such that $x = \alpha z + (1 - \alpha)y$ for some $0 < \alpha < 1$, then x is said to be braced relative to A . zy is an A -relative brace of x . If every point of $\text{conv} A \setminus A$ has an A -relative brace, then A is said to be a braced set.

Definition 4.5 A crosscut of a set $S \subseteq L$ is a closed segment xy such that $\text{int} vxy \subseteq \text{int} S$ and such that $x \in \text{bd} S$ and $y \in \text{bd} S$.

Theorem 4.1 If an open set K has no crosscuts, then it is the complement of a convex set.

Proof: Let $x, y \in \bar{K}$. If $xy \cap K \neq \emptyset$, then K has a crosscut which is a subinterval of xy since K is open. Hence $xy \cap K = \emptyset$ and \bar{K} is convex.

Definition 4.6 If S is a set in a linear space ℓ and if V is a K -dimensional flat, then $S \cap V$ is called a K -dimensional section of S . If H is a hyperplane of ℓ , then $S \cap H$ is called a plane section of S .

Definition 4.7 A continuum in a topological linear space L is a compact connected set. A closed connected set in a Minkowski space L^n is called a generalized continuum. A generalized continuum is automatically boundedly compact.

Definition 4.8* If $A \subseteq L^3$ and A is not convex and $x \in \text{conv} A \setminus A$ implies that there exists a plane H such

that $x \in H$ and $A \cap H$ is connected and $x \in \text{conv}(A \cap H)$, then A is said to be rigid.

Note that the set of braced sets whose convex hull is $\text{conv}A$ is a propositional subset of $g(\text{conv}A)$. If B is the property of being braced and C is a convex set then $P_C(B) \subseteq gC$. The following lemma is presented preparatory to determining a sufficient condition for a set in L^2 to be braced. Because of Theorem 1.3, a proof in E^2 is adequate.

Lemma 4.1: If A is a connected subset of E^2 and $p \in A$ and $q \in \text{aug}A \cap A$ and $x = \alpha p + (1-\alpha)q$ for $0 < \alpha < 1$, then $x \in \text{aug}A$.

Proof: If $x \in A$, the result follows immediately. Suppose $x \notin A$. $q \in \text{aug}A \cap A$ iff there exist $0 < \beta < 1$ and $w, z \in A$ such that $q = \beta w + (1-\beta)z$. Let

$$[f:\alpha] = \{y \in E^2 \mid y = \lambda p + (1-\lambda)q, \lambda \in R\},$$

$$[g:\delta] = \{y \in E^2 \mid y = \lambda z + (1-\lambda)x, \lambda \in R\},$$

$$[h:\gamma] = \{y \in E^2 \mid y = \lambda w + (1-\lambda)x, \lambda \in R\}.$$

Note that if $f(w) > \alpha$, then $f(z) < \alpha$.

If $g(p) > \delta$, then $g(q) < \delta$, $g(w) < \delta$.

If $h(p) > \gamma$, then $h(z) < \gamma$, $h(q) < \gamma$.

The above statements may be adopted without loss of generality.

Clearly, p and w are elements of a connected set A separated by $[g:\delta]$. Then $D = A \cap [g:\delta] \neq \emptyset$, otherwise A is

a subset of the union of the two open half-planes determined by $[g:\delta]$, contradicting that A is connected.

Case I: Suppose $T_1 = D\Omega\{x | f(x) > \alpha\} \neq \emptyset$. Let $t_1 \in T_1$. Then there exists $0 < \Delta < 1$ such that $x = \Delta z + (1-\Delta)t_1$. But $t_1 \in A$, thus $x \in \text{aug}A$.

Case II: Suppose that $T_1 = \emptyset$. Then $D\Omega[f:\alpha] \neq \emptyset$ or $T_2 = D\Omega\{x | f(x) < \alpha\} \neq \emptyset$. But $D\Omega[f:\alpha] \neq \emptyset$ implies that $x \in A$, a contradiction. Thus $T_2 \neq \emptyset$. Let $t_2 \in T_2$. Then there exists $0 < \Delta < 1$ such that $t_2 = \Delta z + (1-\Delta)x$. Thus $h(t_2) < \gamma$, and t_2 and p are separated by $[h:\gamma]$.

Let $t_3 \in T_3 = A \cap [h:\gamma] \cap \Omega\{x | f(x) < \alpha\} \cap \Omega\{x | g(x) > \delta\} \neq \emptyset$. Then $t_3 = \Delta x + (1-\Delta)w$ where $\Delta > 1$. Now $x = (1/\Delta)t_3 - [(1-\Delta)/\Delta]w$, and $x = -[(1/\Delta) - 1]w + (1/\Delta)t_3$. Let $\lambda = 1/\Delta$. Then $0 < \lambda < 1$ so that $x = \lambda t_3 + (1-\lambda)w$. The last statement implies that $x \in \text{aug}A$.

Theorem 4.2 If A is a compact connected subset of E^2 , then $\text{aug}A = \text{conv}A$.

Proof: Obviously $\text{aug}A \subseteq \text{conv}A$.

Now suppose that $x \in \text{conv}A$. Since A is compact, if $x \in \text{bd conv}A$, then $x \in \text{ext}A \subseteq A$ or x belongs to the relative interior of some maximal line segment zy where $zy \subseteq \text{bd conv}A$. Now line zy bounds A so that by Theorem 3.6 and Definition 3.2, $z, y \in \text{ext}A \subseteq A$. Thus if $x \in \text{bd conv}A$, then $x \in \text{aug}A$.

Suppose that $x \in \text{int conv}A$. Let p be an extreme point of A , thus $p \in \text{bd conv}A$. Consider the line $q = \lambda x + (1-\lambda)p$. $q \in \text{int conv}A \neq \emptyset$ so that by Theorem 2.11, $q \in \text{bd conv}A \cup \{p\}$ consists of exactly one point, say k . Note that $q = \Delta k + (1-\Delta)p$ and for some $0 < \Delta < 1$, $x = \Delta k + (1-\Delta)p$.

If $k \in A$, then $x \in \text{aug}A$.

If $k \notin A$, $k \in \text{aug}A$ and by Lemma 4.1, $x \in \text{aug}A$.

Thus, in either case, $\text{conv}A \subseteq \text{aug}A$, so that $\text{aug}A$ and $\text{conv}A$ are identical, i.e., $\text{aug}A = \text{conv}A$.

It is now obvious that if A is a compact connected subset of L^2 , then $\text{aug}A$ is the convex hull of A .

Theorem 4.3 now follows immediately.

Theorem 4.3 If A is a compact connected subset of L^2 or E^2 , then A is braced.

Observe also that Theorem 4.2 together with Theorem 1.7 implies the following:

Theorem 4.4 If A and B are disjoint subsets of E^2 , and A is a compact connected set and B is a compact connected set, then $A \cup B$ is braced.

One should not be too hasty and conclude that connectedness is sufficient for a compact set to be braced in L^3 . The following example illustrates:

Example 4.1 Let A be a subset of E^3 formed by taking the union of three non-coplanar line segments having a common point of origin. A is compact and

connected, but no point in the interior of the closed solid tetrahedron which is A 's convex hull has a brace relative to A .

Other examples may be easily constructed to show that connectedness is not necessary for a set to be braced in E^2 or E^3 and consequently L^2 and L^3 .

For a proof of the following theorem consult Valentine (1, pp. 91-93).

Theorem 4.5 If S is a compact set in L^3 and if each plane section of S is a simply connected continuum, then S is convex. (The converse is obviously true.)

Theorem 4.6 If A is a subset of E^3 or L^3 and A is rigid, then A is braced.

Proof: The theorem follows immediately from the definition of rigid and Theorem 4.3.

The results of Chapter IV are now summarized in terms of Definition 4.1.

Theorem 4.7 Let B be the property of being braced. Let C be the property of being connected. Let R be the property of being rigid. Now suppose that X is a convex subset of L^2 and Y is a convex subset of L^3 , then

$$P_X(C) \zeta P_X(B) \text{ and } P_Y(R) \zeta P_Y(B).$$

Proof: The results are immediate from Theorem 4.2 and Theorem 4.5.

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APPENDIX

Definition A.1 A set $S \subseteq L$, star-shaped with respect to the origin θ , is called linearly bounded iff each line through θ intersects S in a line segment.

Definition A.2 Let S be an open set in L which is star-shaped with respect to θ and which is linearly bounded. Then the Minkowski distance functional ρ is a real valued function defined on L as follows:

$$\rho(x) = \lambda \geq 0 \text{ where } \lambda x^* = x, x^* \in \text{bd}S, \text{ and } \alpha x^* \in \text{int}S, 0 \leq \alpha \leq 1.$$

If $x \in S$, then $\rho(x) \leq 1$.

Definition A.3 A topological linear space L is said to be normable if it is locally convex and if it contains a non-empty bounded open set.

Definition A.4 A normable topological linear space L with a norm (as for example, the Minkowski distance functional) is called a normed linear space.

Definition A.5 A finite-dimensional normed linear space is called a Minkowski space.

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