UNIFORMLY σ-FINITE DISINTEGRATIONS OF MEASURES

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A disintegration of measure is a common tool used in ergodic theory, probability, and descriptive set theory. The primary interest in this paper is in disintegrating σ-finite measures on standard Borel spaces into families of σ-finite measures. In 1984, Dorothy Maharam asked whether every such disintegration is uniformly σ-finite meaning that there exists a countable collection of Borel sets which simultaneously witnesses that every measure in the disintegration is σ-finite. Assuming Gödel’s axiom of constructability I provide answer Maharam's question by constructing a specific disintegration which is not uniformly σ-finite.

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CHAPTER 1

INTRODUCTION

1.1. Overview

Disintegrations of measures are used commonly in ergodic theory, probability, and descriptive set theory. One of the first known definitions of a disintegration is due to von Neumann [12] in the early 1930s. The concept was developed further in the early 1940s by Maharam [7], Rokhlin [10], and Halmos [4]. In [8] Maharam considers σ-finite disintegrations that are uniformly σ-finite but leaves as an open question whether every given σ-finite disintegration is necessarily uniformly σ-finite. In [3] Mauldin and Graf sharpen Maharam's results and also prove that any given σ-finite disintegration (μ_y) is uniformly σ-finite in a weak sense. This weaker sense being that Borel sets are replaced with sets that are μ_y-measurable for every y.

In this paper Maharam's question is answered in the negative. Assuming Gödel's axiom of constructibility, a specific σ-finite disintegration of a σ-finite measure is constructed which is *not* uniformly σ-finite. The construction of this disintegration relies on building counting measure over the fibers of a $\mathbf{\Pi}^1_1$ subset of $\omega^\omega \times \omega^\omega$ having countable sections but which is not the countable union of $\mathbf{\Pi}^1_1$ graphs. The existence of such a set was shown by Mauldin and Jackson in [5]. However, in order for counting measure on the fibers of this set to be a disintegration the proof of the existence of the $\mathbf{\Pi}^1_1$ set needs stronger additional properties.

1.2. Structure of the Paper

The paper is broken down into two parts. The first is on logic and set theory, and the second is on measure theory. The logic and set theory portion consists of the proof of the modified version of Mauldin and Jackson's theorem from [5] as well as preliminaries that are needed. The reader well-versed in logic and set theory may want to skip the majority of Chapter 2 which introduces these preliminaries. However in Section 2.2 a coding of models is developed that is essential to the proof of Theorem 2.15. Chapter 3 contains definitions of disintegrations of measures, theorems regarding their existence, and the construction of a σ-finite disintegration that is not uniformly σ-finite.

CHAPTER 2

TOOLS FROM LOGIC AND SET THEORY

2.1. The L Hierarchy

The objects of interest in set theory are, not surprisingly, sets. The class of all sets is denoted by $\mathbf{V}$ which can be defined by $\mathbf{V} = \{x : x = x\}$. Here $\mathbf{V}$ is defined as the collection of objects satisfying a particular formula, namely $x = x$. Defining objects by formulas is a common practice in set theory and we would like a manner of doing so that avoids any paradoxes. Gödel's $\mathbf{L}$ hierarchy of sets accomplishes this and its construction will be presented. Given a set A we wish to describe the subsets of A given by n-place relations on A defined by a formula relativized to A. This notion is made precise below. First, however, some preliminary notions are presented.

A **structure** in the language of set theory is a pair (M, E) where M is a set and E is a binary relation on M. The basic symbols of the language of set theory consist of $\vee$, $\wedge$, $\neg$, the relation E, $=$, (,), and variable symbols x_i for each natural number i. Since the allowable symbols in our language depend on the binary relation E, $\mathcal{L}^E$ will denote the language with the specific relation E. **Formulas** are finite strings of these basic symbols built recursively by

Definition 2.1. (1) $x_i E x_j$ and $x_i = x_j$ are formulas

(2) if ϕ and ψ are formulas then $(\phi \vee \psi)$, $(\phi \wedge \psi)$, and $(\neg\phi)$ are formulas

(3) if ϕ is a formula then $(\forall x_i \phi)$ and $(\exists x_i \phi)$ are formulas.

A variable in a formula is considered **free** if it occurs outside the scope of all existential and universal quantifiers. A **sentence** is a formula which contains no free variables. If S is a collection of sentences and ϕ is a sentence define $S \vdash \phi$ iff there is a finite sequence of sentences, $\phi_1, \ldots, \phi_n$ such that ϕ_n is ϕ and for each i, either ϕ_i is in S or ϕ_i follows from

$\phi_1, \ldots, \phi_{i-1}$ by logical rules of inference. If ϕ is a formula, define $S \vdash \phi$ iff $S \vdash \psi$ where ψ is the sentence obtained by quantifying all occurrences of free variables in ϕ with $\forall$.

Since formulas are finite strings of symbols from a countable alphabet the collection of formulas must be countable. The Gödel numbering is a standard method of ordering formulas of a language, $\mathcal{L}^E$, and its details can be found in [6] or [9]. Let ϕ_n denote that ϕ is the nth formula under the Gödel numbering of the formulas of $\mathcal{L}^\in$.

DEFINITION 2.2. Two structures, (M_1, E_1) and (M_2, E_2), are **isomorphic** and denoted $(M_1, E_1) \cong (M_2, E_2)$ (or simply $M_1 \cong M_2$ if the relations are understood) provided there exists a bijection $F : M_1 \to M_2$ such that for all $x, y \in M_1$

$$xE_1y \iff F(x)E_2F(y).$$

DEFINITION 2.3. If (M_1, E_1) and (M_2, E_2) are structures such that $M_1 \subset M_2$ and $E_1 = E_2 \cap (M_1 \times M_1)$ then M_1 is a **substructure** of M_2 (or M_2 is an **extension** of M_1).

For each ordinal, α, define a structure, $(L_\alpha, \in)$, in the language, $\mathcal{L}^\in$, by considering the definable subsets of a given set.

DEFINITION 2.4. Let A be a set, let $n \in \omega$ and $i, j < n$. Define the **elementary relations** on A to be

$$Diag_\in(A, i, j, n) = \{x \in A^n : x(i) \in x(j)\}$$

$$Diag_=(A, i, j, n) = \{x \in A^n : x(i) = x(j)\}$$

$$Proj(A, R, n) = \{x \in A^n : \exists y \in R(y \restriction n = x)\}$$

where $y \restriction n$ denotes the restriction of y to n. Using recursion on k define $\forall n\ Def_k(A, n)$ by

(1) $Def_0(A, n) = \{Diag_\in(A, i, j, n) : i, j < n\} \cup \{Diag_=(A, i, j, n) : i, j < n\}$

(2) $Def_{k+1}(A, n) = Def_k(A, n) \cup \{A^n \setminus R : R \in Df_k(A, n)\} \cup \{R \cap S : R, S \in Def_k(A, n)\} \cup \{Proj(A, R, n) : R \in Def_k(A, n)$.

$Def(A, n) = \bigcup_{k\in\omega}\{Def_k(A, n)\}$.

The *definable power set* operation, $\mathcal{D}$ which produces the definable subsets of a given set may now be defined.

Definition 2.5.

$$\mathcal{D}(A) = \{X \subset A : \exists n \in \omega \; \exists s \in A^n \; \exists R \in Def(A, n+1)$$
$$(X = \{x \in A : s \frown < x > \in R\})\}.$$

The class $\mathbf{L}$ of all definable sets is built recursively on the ordinals as follows.

Definition 2.6. For each ordinal α define

(1) $L_0 = \emptyset$.
(2) $L_{\alpha+1} = \mathcal{D}(L_\alpha)$.
(3) $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$ when α is a limit ordinal.
(4) $\mathbf{L} = \bigcup_\beta L_\beta$ where the union ranges over all ordinals, β.

The axiom $\mathbf{V} = \mathbf{L}$ states that every set is definable. This axiom will be necessary in Section 2.15. A canonical well-ordering of $\mathbf{L}$ is constructed in [6] and [9]. This well-ordering will be referred to as $<_L$.

Each L_α and $\mathbf{L}$ has the property of being **transitive**, meaning that every element of L_α is a subset of L_α. Transitive sets or classes are important for finding isomorphisms between structures. If A is a set which is not transitive, the least transitive set containing A as a subset may be determined. This is made precise in the following definition.

Definition 2.7. If A is a set the **transitive closure** of A is the set $TC(A) = \bigcup_{n\in\omega} TC_n(A)$ where $TC_n(A)$ is defined recursively by

(1) $TC_0(A) = A$
(2) $TC_{n+1}(A) = \bigcup TC_n(A)$

Using the transitive closure a version of the L-hierarchy is defined starting from particular elements of ω^ω.

Definition 2.8. For each ordinal α and each $x \in \omega^\omega$ define

(1) $L_0(x) = TC(x)$

(2) $L_{\alpha+1}(x) = \mathcal{D}(L_\alpha(x))$

(3) $L_{\alpha+1}(x) = \bigcup_{\beta<\alpha} L_\beta(x)$ when α is a limit ordinal

(4) $L(x) = \bigcup_\beta L_\beta(x)$ where the union ranges over all ordinals, β.

Note that for every ordinal α, $L_\alpha = L_\alpha(\emptyset)$ and thus for every set A, $L_\alpha \subset L_\alpha(A)$.

2.2. A Coding of Models

In this section a particular coding of models is developed that will be essential in proving Theorem 2.15. A similar, but more powerful, coding is the concept of standard codes given in Devlin [1]. The coding given in this section is a tailored version of Devlin's. To accomplish this coding the concept of a Skolem Hull will be employed which is developed presently.

If (M, E) is a structure and ϕ is a formula in the language $\mathcal{L}^E$ then define the statement (M, E) **models** ϕ to mean that $(M, E) \models \phi$ (or simply $M \models \phi$ if the E relation is understood) provided that the formula is true when all variable assignment takes place within the set M. Given a structure, (M, E), the **theory** of (M, E) is the set of all sentences, ϕ, such that $(M, E) \models \phi$.

Definition 2.9. Suppose (M, E_M) and (N, E_N) are structures with $M \subset N$ and $E_M = E_N \cap (M \times M)$, *i.e.* (M, E_M) is a substructure of (N, E_N). If $\phi(x_1, \ldots, x_k)$ is a formula in $\mathcal{L}^{E_N}$ with free variables among $x_1, \ldots, x_k$ define the statement ϕ is **absolute for** M, N to mean

$$\forall x_1, \ldots, x_k \in M(M \models \phi(x_1, \ldots, x_k) \iff N \models \phi(x_1, \ldots, x_k)).$$

If E_N is $\in$, then ϕ is **absolute for** M if

$$\forall x_1, \ldots, x_k \in M(M \models \phi(x_1, \ldots, x_k) \iff \mathbf{V} \models \phi(x_1, \ldots, x_k)$$

Denote ZF_N to be a sufficiently large finite fragment of ZF such that Π^1_1 and Σ^1_1 formulas are absolute for transitive models of ZF_N.

For each n let ϕ_n be the n-th formula in the Gödel numbering of the formulas in the language $\mathcal{L}^{\in}$. Given an $x \in \omega^\omega$ such that $x(n) \in \{0,1\}$ for each $n \in \omega$ define the theory Th_x by $\phi_n \in Th_x \iff x(n) = 1$. Let $\phi_{<L}$ be a formula defining the canonical well-ordering of $\mathbf{L}$ and let $M \in \omega$ be the integer such that $\phi_{<L} = \phi_M$.

Define a set of codes $C \subset \omega^\omega$ by $x \in C$ iff:

(1) $\forall n \in \omega \quad x(n) \in \{0,1\}$

(2) Th_x is a consistent and complete theory of $ZF_N + (\mathbf{V} = \mathbf{L})$

(3) $x(M) = 1$.

Given a formula $\phi_n(x_1, \ldots, x_k)$ with free variables $x_1, \ldots, x_k$ the **Skolem function** for ϕ_n is the function $\tau_n : \omega^k \to \omega$ such that:

(1) if ϕ_n is $\exists z\ \phi_j(z, x_1, \ldots, x_k)$ and $\exists y \in \omega\ \phi_j(y, x_1, \ldots, x_k)$ then $\tau_n(x_1, \ldots, x_k)$ is the $<_L$ least such y, or

(2) if ϕ_n is $\exists z\ \phi_j(z, x_1, \ldots, x_k)$ and $\neg\exists y \in \omega\ \phi_j(y, x_1, \ldots, x_k)$ then $\tau_n(x_1, \ldots, x_k) = 0$, or

(3) if ϕ_n is not of the form $\exists z\ \phi_j(z, x_1, \ldots, x_k)$ or $k = 0$ then $\tau_n(x_1, \ldots, x_k) = 0$.

Fix $x \in C$ and let S_x be the collection of Skolem functions for the theory Th_x. Define an equivalence relation, $\equiv$, on S_x by

$$\tau_n \equiv \tau_m \iff Th_x \vdash \tau_n(x_1, \ldots x_k) = \tau_m(y_1, \ldots, y_l).$$

Define M_x to be the set of equivalence classes of all Skolem functions arising from formulas, ϕ, such that $Th_x \vdash \phi$. Here, M_x is the **Skolem Hull** of the theory, Th_x. Define the relation E_x on $M_x \times M_x$ by

$$[\tau_i] E_x [\tau_j] \iff Th_x \vdash \tau_i(x_1, \ldots, x_k) \in \tau_j(y_1, \ldots, y_k).$$

It will now be shown that if an element of ω^ω is constructed at an ordinal α then there exists a code $x \in C$ for a structure (M_x, E_x) that is isomorphic to L_α.

Proposition 2.10. *If $w \in \omega^\omega \cap L_{\alpha+1} \setminus L_\alpha$ then $\exists x \in \omega^\omega$ such that $M_x \cong L_\alpha$.*

Proof. Let T be the theory of L_α and let $x \in \omega^\omega$ such that $Th_x = T$. Then (M_x, E_x) is an elementary submodel of $(L_\alpha, \in)$. Since L_α is well-founded M_x is well-founded. Then $\in$ is well-founded on the transitive collapse $TC(M_x)$ and thus $(M_x, E_x) \cong (TC(M_x), \in) \cong (L_\beta, \in)$ for some β and $w \in L_{\beta+1}$. Thus $\beta = \alpha$. □

2.3. Uniformizations and Nonuniformizations

In this section the concept of uniformizations is discussed, and theorems guaranteeing their existence are presented. Also a modified version of a result by S. Jackson and R. D. Mauldin is proved which constructs a coanalytic set not having a particular uniformization property. This construction will be used in Section 3.2.

Definition 2.11. Let X and Y be Polish spaces and $B \subset X \times Y$. A set $C \subset B$ is called a **uniformization** of B provided that $\forall x \in X$, the section $C_x = \{y \in Y : (x, y) \in C\}$ contains no more than one point and $\pi_X(C) = \pi_X(B)$ where $\pi_X : X \times Y \to X$ is the projection map. In other words, C is a uniformization of B if C is the graph of a function $f : \pi_X(B) \to Y$.

Using the axiom of choice we can always produce $C \subset B$ which is a uniformization of B however we may not know much about the set C (is it Borel, analytic, PCA, etc.). There are a number of theorems guaranteeing the existence of "nice" uniformizations of sets satisfying particular conditions on their sections. This section contains a few existence theorems which will be useful. For further detail see [11]. The following theorem guarantees the existence of a uniformization that is a Borel set and is due to Lusin. For a proof of the following two theorems see [11].

Theorem 2.12. *Let X and Y be Polish spaces and $B \in \mathcal{B}(X \times Y)$ with B_x countable for each $x \in X$. Then there exists a Borel uniformization of B.*

Kondo proved that coanalytic uniformizations exist for coanalytic sets regardless of what their sections look like.

THEOREM 2.13. *Let X and Y be Polish spaces and let C be a coanalytic subset of $X \times Y$. Then C admits a coanalytic uniformization.*

In [5], Jackson and Mauldin prove that coanalytic sets with finite sections may be filled up by countably many coanalytic graphs.

THEOREM 2.14. *If X and Y are Polish spaces and C is a coanalytic subset of $X \times Y$ such that each section C_x is finite then there exist for each n a coanalytic function $f_n : \pi_X(C) \to Y$ such that C is the union of the graphs of the functions f_n.*

However, Jackson and Mauldin also prove in [5] that within ZF coanalytic sets with countable sections may fail to be the union of countably many coanalytic graphs. The following theorem is a modification of their result and requires the additional assumption $\mathbf{V} = \mathbf{L}$.

THEOREM 2.15. *Assume $\mathbf{V} = \mathbf{L}$. Let $X = Y = \omega^\omega$. Let P be a closed subset of $X \times Y$ such that $\forall x \in X,\ \ P_x$ is nonempty and perfect and if $x \neq x', P_x \cap P_{x'} = \emptyset$. Then there exists a $\mathbf{\Pi}^1_1$ set $G \subset P$ with the following properties:*

(1) $\forall x \in X, |G_x| = \omega_0$
(2) *For every $n \in \omega$ and for every $\mathbf{\Delta}^1_1$ set $B \subset Y, \{x \in X : |B \cap G_x| \geq n\}$ is $\mathbf{\Delta}^1_1$*
(3) *G is not the union of countably many $\mathbf{\Pi}^1_1$ uniformizations over X.*

PROOF. Fix a pair of recursive bijections, $x \mapsto (x_n)_{n=0}^\infty$ from ω^ω onto $(\omega^\omega)^\omega$ and $x \mapsto (x^0, x_1)$ from ω^ω onto $\omega^\omega \times \omega^\omega$. Denote the inverse of the second bijection by $(y, z) \mapsto \langle y, z \rangle$. Call an ordinal β **good** (with respect to x) if $L_\beta(x) \models ZF_N + (\mathbf{V} = \mathbf{L})$. Let $p \in \omega^\omega$ be a code for P.

Recall from Section 2.2 that $C \subset \omega^\omega$ such that each $x \in C$ is a code for a model (M_x, E_x) of $ZF_N + (\mathbf{V} = \mathbf{L})$. Define $U \subset C$ by $x \in U$ if and only if there exists an ordinal $\alpha(x) \geq \omega_0$ such that $M_x \cong L_{\alpha(x)}$ and $p \in L_{\alpha(x)}$. Define $V \subset C$ by $x \in V$ iff M_x is an ω-model, and $M_x \models$ "$p \in \omega^\omega$". Note that $U \subset V$ and that V is $\mathbf{\Delta}^1_1$.

For each $n \in \omega$ let $f_n : X \to Y$ be $\mathbf{\Delta}^1_1$ functions such that $\forall x \in X$ and for $n \neq m$ $f_n(x) \neq f_m(x)$ and such that $\forall x \in X\ \forall n \in \omega\ f_n(x) \in P_x$.

Define the set $G' \subset X \times Y$ by $(x, y) \in G' \iff$

$$[x \notin V \wedge \exists n(y = f_n(x))] \vee [x \in V \wedge (x,y) \in P \wedge [M_x \models \text{“}y \in \omega^\omega\text{”} \vee$$

$$\text{there exists a well-founded extension } M \text{ of } M_x\ \exists \alpha < \omega_1$$

$$(M_x \subset M \cong L_\alpha \wedge y \in L_\alpha(x) \wedge$$

$$[\forall \gamma < \alpha(\neg(\gamma \text{ is good and a limit of good ordinals}) \vee$$

$$\exists \phi \in \Sigma^1_2\ \exists \tau > \gamma\, (L_\gamma(x) \models \phi \wedge L_\tau(x) \models \neg\phi))])]].$$

Note that G' is Σ^1_2 and let $\Omega'(x, y)$ be the above Σ^1_2 formula defining G'.

It will first shown that the sections of G' are countable. Clearly G'_x is countable for every $x \notin V$. Since each model M_x is countable G'_x is countable for every $x \in V \setminus U$. Finally suppose $x \in U$. Then there is a well-founded extension M of M_x and an ordinal α such that $M \cong L_{\alpha(x)}$. Let β be the least ordinal less than ω_1 such that $L_\beta(x)$ is a Σ_2 elementary substructure of $L(x)$. Then for every $\beta' > \beta$ and every Σ^1_2 formula ϕ,

$$L_\beta(x) \models \phi \iff L_{\beta'}(x) \models \phi \iff L(x) \models \phi.$$

By the definition of G', $\beta \geq \alpha$. Thus $G'_x \subset L_\beta(x)$ and is therefore countable.

Let $G \subset \omega^\omega \times \omega^\omega$ be such that for every $x \in \omega^\omega$

$$G'(x, y) \iff \exists z\, G(x, \langle y, z\rangle) \iff \exists! z\, G(x, \langle y, z\rangle).$$

Let Ω be a Π^1_1 formula defining G. It is assumed that ZF_N was chosen large enough so that the following is a theorem of ZF_N.

$$\forall x\, \forall w\, [\Omega'(x, y) \iff \exists z\, \Omega(x, \langle y, z\rangle) \iff \exists! z\, \Omega(x, \langle y, z\rangle)].$$

Note that since the sections of G' are countable, so too are the sections of G. Next it is shown that the Borel condition in property (2) holds for G.

Fix a $\mathbf{\Delta}^1_1$ set $B \subset Y$, fix an $n \in \omega$, let $K_n = \{x \in X : |B \cap G_x| \geq n\}$, let $b \in \omega^\omega$ be a code for B, and let τ be the level of L at which b is constructed. Partition V into the following $\mathbf{\Delta}^1_1$ sets: $C = \{x \in V : \text{“}b \notin M_x\text{”}\}$ and $D = \{x \in V : \text{“}b \in M_x\text{”}\}$.

First consider $x \in V \setminus U$. Then by the definition of G, $x \in K_n \cap V \setminus U \iff (x \in V \setminus U) \wedge (M_x \models \text{“}\exists a_1, \ldots, a_n \in B\text{”})$. Now consider $x \in D \cap U$. Also by the definition of G, there is an ordinal α and an extension M of M_x such that $M \cong L_\alpha$ with $b \in L_\alpha$. Since $M_x \models ZF_N$, $M_x \models \text{“}\exists a_1, \ldots, a_n \in B\text{”} \iff M \models \text{“}\exists a_1, \ldots a_n \in B\text{”} \iff L_\alpha(x) \models \exists a_1, \ldots, a_n \in B$. Hence $x \in K_n \cap D \cap V \iff (x \in D \cap V) \wedge (M_x \models \exists a_1, \ldots, a_n \in B)$.

Since $\tau < \omega_1$ and each $x \in U$ uniquely determines a well-founded L_α there can be only countably many $x \in U$ which are constructed before τ. Here $\mathbf{V} = \mathbf{L}$ is used and so every $x \in U$ is constructed at some L_α. Let C_0 be this countable collection and let $C^* = C \setminus C_0$. Then $C^* \subset V \setminus U$. Now $x \in K_n \cap C^* \iff (x \in C^*) \wedge (M_x \models \exists \text{“}a_1, \ldots, a_n \in B\text{”})$.

For every $x \in V$, M_x is a countable model and thus all objects that M_x models as reals are recursive $\Delta^1_1(x)$. Thus for a given $a \in \omega^\omega$ $\{x : M_x \models \text{“}a \in B\text{”}\}$ is $\mathbf{\Delta^1_1}$. To see that $K_n \cap (C^* \cup D)$ is $\mathbf{\Delta}^1_1$ write $\{x : M_x \models \text{“}\exists a_1, \ldots, a_n \in B\text{”}\}$ as the following countable projection of a $\mathbf{\Delta}^1_1$ set:

$$\operatorname{proj}_1 \left(\bigcup_{(a_1, \ldots, a_n) \in \omega^n} \{(x, a_1, \ldots, a_n) : (M_x \models \text{“}a_1, \ldots, a_n \in \omega^\omega\text{”}) \wedge (TC(a_1), \ldots, TC(a_n) \in B)\} \right).$$

Thus $K_n \cap (C^* \cup D)$ is $\mathbf{\Delta}^1_1$ and hence so is $K_n \cap V$.

For $x \in X \setminus V$ each section $G_x = \bigcup_{n=1}^\infty f_n(x)$. Thus for each $x \in X \setminus V$,

$$\begin{aligned} |B \cap G_x| \geq n &\iff \exists k_1, \ldots, k_n [f_{k_1}(x) \in B, \ldots, f_{k_n}(x) \in B] \\ &\iff x \in \bigcup_{(k_1, \ldots, k_n)} f^{-1}_{k_1}(B) \cap \ldots \cap f^{-1}_{k_n}(B). \end{aligned}$$

Therefore $K_n \cap X \setminus V$ is $\mathbf{\Delta}^1_1$.

Finally it will be shown that property (3) holds for G. Proceeding by contradiction suppose that G could be written as a countable union of $\mathbf{\Pi}^1_1$ graphs G_m. Choose a sequence (x_m) from ω^ω and formulas $\psi_m(x, y)$ so that ψ_m are $\mathbf{\Pi}^1_1(x_m)$ formulas defining the G_m. Let

$x' \in \omega^\omega$ be such that $x' \mapsto (x_m)_{m=0}^{\infty}$ and choose $x \in U$ and α such that $M_x \subset M \cong L_\alpha(x)$ and $x' \in L_\alpha(x)$. Next let β be the least ordinal such that (β is good and a limit of goods) $\wedge \forall \phi \in \Sigma^1_2 \, (L_\beta(x) \models \neg\phi \Rightarrow \forall \tau > \beta \, L_{\tau(x)} \models \neg\phi)$.

From the definition of G', $\omega^\omega \cap L_\beta(x) \subset G'_x$. Furthermore if $y \in L_\beta(x)$ then for some good ordinal $\alpha < \beta$, $y \in L_\alpha(x)$. Since β was chosen to be minimal then $\forall \gamma < \alpha[\neg(\gamma$ is good and a limit of good ordinals$) \vee \exists \phi \in \Sigma^1_2 \, (L_\alpha(x) \models \neg\phi \wedge \exists \tau > \gamma(L_\tau(x) \models \phi))]$. In fact we may replace "$\exists \tau > \gamma$" in the previous statement with "$\exists \tau > \gamma, \tau < \beta$". Thus α witnesses that $L_\beta(x) \models \Omega'(x, y)$.

Since β was defined to be a Σ_2 elementary substructure of $L(x)$, we have that $L_\beta(x) \models$ "$\{y : \exists m \, \psi_m(x, y)\}$ is countable". However, $L_\beta(x) \models$ "ω^ω is uncountable". Thus we may let $y, z \in L_{\beta(x)}$ such that

$$L_{\beta(x)} \models \Omega(x, \langle y, z\rangle) \text{ and}$$

$$L_{\beta(x)} \models \forall m \, \neg\psi_m(x, \langle y, z\rangle).$$

Then from the definition of β, $L(x) \models \forall m \, \neg\psi(x_m, \langle y, z\rangle)$ and therefore $\mathbf{V} \models \forall m \, \neg\psi(x_m, \langle y, z\rangle)$. Thus $\forall m(x, \langle y, z\rangle) \notin G_m$. However this contradicts the fact that $\mathbf{V} \models \Omega(x, \langle y, z\rangle)$ by absoluteness and therefore $(x, \langle y, z\rangle) \in G$.

$\square$

CHAPTER 3

MEASURE THEORY

3.1. Disintegrations of Measures

Throughout this section let $(X, \mathcal{B}(X))$ and $(Y, \mathcal{B}(Y))$ be uncountable standard Borel spaces, i.e. measure spaces isomorphic to uncountable Polish spaces equipped with the σ-algebra of Borel sets, let $\phi : X \to Y$ be measurable, and let μ and ν be measures on $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ respectively.

Definition 3.1. A **disintegration** of μ with respect to (ν, ϕ) is a family, $\{\mu_y : y \in Y\}$, of measures on $(X, \mathcal{B}(X))$ satisfying:

(1) $\forall B \in \mathcal{B}(X),\ y \mapsto \mu_y(B)$ is $\mathcal{B}(Y)$-measurable
(2) $\forall y \in Y,\ \mu_y(X \setminus \phi^{-1}(y)) = 0$ and
(3) $\forall B \in \mathcal{B}(X),\ \mu(B) = \int \mu_y(B) d\nu(y)$.

Remark 3.2. Suppose that $\{\mu_y : y \in Y\}$ is a disintegration of μ with respect to (ν, ϕ). If $N \in \mathcal{B}(Y)$ with $\nu(N) = 0$ then combining properties (2) and (3) we have

$$\begin{aligned}\mu \circ \phi^{-1}(N) &= \int \mu_y(\phi^{-1}N) d\nu(y) \\ &= \int_N \mu_y(X) d\nu(y) \\ &= 0.\end{aligned}$$

Therefore the image measure, $\mu \circ \phi^{-1}$, is absolutely continuous with respect to ν.

Definition 3.3. If $\{\mu_y : y \in Y\}$ is a disintegration of μ with respect to (ν, ϕ) such that $\forall y \in Y$, μ_y is σ-finite then define the disintegration to be **σ-finite**. If $\{\mu_y : y \in Y\}$ is a σ-finite disintegration of μ with respect to (ν, ϕ) define the disintegration to be **uniformly σ-finite** provided there exists a sequence, (B_n), from $\mathcal{B}(X)$ such that

(1) $\forall n \in \mathbb{N}\ \forall y \in Y,\ \mu_y(B_n) < \infty$ and

(2) $\forall y \in Y,\ \mu_y(X \setminus \bigcup_n B_n) = 0.$

The following existence theorem for disintegrations can be found in Fabec [2].

THEOREM 3.4. *Suppose $(X, \mathcal{B}(X))$ and $(Y, \mathcal{B}(Y)$ are standard Borel spaces, μ is a σ-finite measure on $\mathcal{B}(X)$, ν is a σ-finite measure on $\mathcal{B}(X)$, and $\phi : X \to Y$ is a Borel measurable function. If $\mu \circ \phi^{-1} << \nu$ then there exists a σ-finite disintegration $\{\mu_y : y \in Y\}$ of μ with respect to (ν, ϕ). Moreover this disintegration is unique in the sense that if $\{\hat{\mu}_y : y \in Y\}$ is any σ-finite disintegration of μ with respect to (ν, ϕ) then there exists $N \subset Y$ such that $\nu(N) = 0$ and $\forall y \notin N\ \mu_y = \hat{\mu}_y$.*

Dorothy Maharam asked in [8] whether every σ-finite disintegration was in fact uniformly σ-finite. This chapter provides an answer to Maharam's question in the form of a counterexample.

The following theorem demonstrates in what manner a given disintegration is "almost" uniformly σ-finite.

THEOREM 3.5. *Suppose $\{\mu_y : y \in Y\}$ is a σ-finite disintegration of the σ-finite measure μ with respect to (ν, ϕ). Then there exists a sequence, (D_n), from $\mathcal{B}(X)$ such that*

(1) $\forall y \in Y,\ \mu_y(D_n) < \infty$

(2) *for ν-a.e. $y \in Y$, $\mu_y\left(X \setminus \bigcup_n D_n\right) = 0$.*

PROOF. Define $F : \mathcal{B}(X) \to \mathcal{B}(Y)$ by

$$F(B) = \{y \in Y : \mu_y(B) < \infty\}.$$

Note that $\forall B \in \mathcal{B}(X)$, $F(B) = \bigcup_n \{y \in Y : \mu_y(B) < n\}$. Thus F properly maps $\mathcal{B}(X)$ into $\mathcal{B}(Y)$.

Let (B_n) be a sequence from $\mathcal{B}(X)$ such that $\forall n \in \mathbb{N},\ \mu(B_n) < \infty$ and $X = \bigcup_n B_n$. Note that for every n we have that $\mu(B_n) = \int \mu_y(B_n) d\nu(y) < \infty$. Thus $\mu_y(B_n) < \infty$ for

ν-a.e. y and thus $\nu(Y \setminus F(B_n)) = 0$. Let $E = \bigcap_n F(B_n)$. Note that

$$\begin{aligned}
\nu(Y \setminus E) &= \nu\left(Y \setminus \bigcap_n F(B_n)\right) \\
&= \nu\left(\bigcup_n Y \setminus F(B_n)\right) \\
&\leq \sum_n \nu(Y \setminus F(B_n)) \\
&= 0,
\end{aligned}$$

and consequently

$$\begin{aligned}
\mu(X \setminus \phi^{-1}(E)) &= \mu(\phi^{-1}(Y \setminus E)) \\
&= \int_{Y \setminus E} \mu_y(X) d\nu(y) \\
&= 0.
\end{aligned}$$

For each $n \in \mathbb{N}$ define $D_n = \phi^{-1}(E) \cap B_n$. For every $y \in E$ we have that $\mu_y(D_n) = \mu_y(\phi^{-1}(E) \cap B_n) \leq \mu_y(B_n) < \infty$ and for every $y \in Y \setminus E$ we have that $\mu_y(D_n) = \mu_y(\phi^{-1}(E \cap B_n)) \leq \mu_y(\phi^{-1}(E)) = 0$. Furthermore

$$\begin{aligned}
\mu_y\left(X \setminus \bigcup_n D_n\right) &= \mu_y\left(X \setminus \left(\phi^{-1}(E) \cap \bigcup_n B_n\right)\right) \\
&= \mu_y\left(X \setminus \phi^{-1}(E) \cup \left(X \setminus \bigcup_n B_n\right)\right) \\
&\leq \mu_y\left(X \setminus \phi^{-1}(E)\right) + \mu_y\left(X \setminus \bigcup_n B_n\right) \\
&= 0 \text{ for } \nu\text{-a.e. } y.
\end{aligned}$$

□

Corollary 3.6. *Suppose $\{\mu_y : y \in Y\}$ is a σ-finite disintegration of the σ-finite measure μ with respect to (ν, ϕ). There exists a uniformly σ-finite disintegration $\{\hat{\mu}_y : y \in Y\}$ of μ with respect to (ν, ϕ) such that $\mu_y = \hat{\mu}_y$ for ν-almost every $y \in Y$.*

PROOF. Let (D_n) be the sequence from $\mathcal{B}(X)$ that is constructed in Theorem 3.5. Let $N \in \mathcal{B}(Y)$ be such that $\nu(N) = 0$ and such that $\mu_y(X \setminus \bigcup_n D_n) = 0$ for every $y \notin N$. Define $\hat{\mu}_y$ by

$$\hat{\mu}_y(B) = \begin{cases} \mu_y, & \text{if } y \notin N \\ 0, & \text{if } y \in N \end{cases}$$

□

3.2. A Nonuniformly σ-finite Disintegration

In this section a specific example is constructed of a disintegration of a σ-finite measure which is not uniformly σ-finite. A σ-finite measure on a subset of ω^ω will be constructed by integrating over a family, $\{\mu_x : x \in X\}$ of σ-finite measures on the Borel subsets of ω^ω.

Let $X = Y = \omega^\omega$. Let $P = \{((x_i), (y_i)) \in \omega^\omega \times \omega^\omega : \forall i \in \omega[y_{2i} = x_i]\}$. Let $\pi_i : \omega^\omega \times \omega^\omega \to \omega^\omega$ be the projection map onto the ith coordinate. P is closed, $\pi_1(P) = \omega^\omega = \pi_2(P)$, and if $x, x' \in \omega^\omega$ with $x \neq x'$ then $P_x \cap P_{x'} = \emptyset$. Since the sections, P_x, are disjoint define the function, $\phi : Y \to X$, by $\phi(y) = x \iff y \in P_x$. If $E \in \mathcal{B}(X)$ then $\phi^{-1}(E) = \bigcup_{x \in E} P_x = \pi_2((E \times \omega^\omega) \cap P) \in \Sigma^1_1(Y)$ and $Y \setminus \phi^{-1}(E) = \bigcup_{x \notin E} P_x = \pi_2((X \setminus E \times \omega^\omega) \cap P) \in \Sigma^1_1(Y)$. Thus ϕ is a Borel measurable function. Let G be the set constructed in Theorem 2.15. To recall, G is a $\mathbf{\Pi^1_1}$ subset of $X \times Y$ such that

(1) $\forall x \in X \ |G_x| = \omega_0$
(2) For every $n \in \omega$ and for every $B \in \mathcal{B}(Y), \{x \in X : |B \cap G_x| = n\} \in \mathcal{B}(X)$
(3) G is not the union of countably many $\mathbf{\Pi}^1_1$ uniformizations over X.

Also recall that in the proof of Theorem 2.15 a Borel set $V \subset X$ and Borel functions $f_n : X \to Y$ were constructed such that for $x \notin V$ $G_x = \bigcup_n \{f_n(x)\}$. Let $H = X \setminus V$ and let ν be a nonatomic probability measure on $\mathcal{B}(X)$ such that $\nu(H) = 1$.

For each $x \in X$ and $B \in \mathcal{B}(Y)$ define $\mu_x(B) = |B \cap G_x|$, i.e. counting measure on the fibers of G. Since each fiber, G_x, is countably infinite, μ_x is σ-finite $\forall x \in X$. Also since the fibers are pairwise disjoint, $\mu_x(Y \setminus \phi^{-1}(x)) = 0$. If $B \in \mathcal{B}(Y)$ then $\{x : \mu_x(B) \geq n\} = \{x : |B \cap G_x| \geq n\}$ which is a Borel subset of X by Theorem 2.15. Thus for every $B \in \mathcal{B}(Y)$ the function $x \to \mu_x(B)$ is $\mathcal{B}(X)$-measurable.

Define a measure μ on the Borel subsets of Y by

$$\mu(B) = \int \mu_x(B) d\nu(x).$$

It will first be shown that μ is σ-finite. Let $B_n = f_n(H)$ and note that $\forall x \in H$, $G_x \subset \bigcup_n B_n$. Each B_n is Borel since each f_n is injective, and $\forall x \in H$, $\mu_x(B_n) = |B_n \cap G_x| = 1$. Furthermore

$$\begin{aligned}
\mu\left(Y \setminus \bigcup_n B_n\right) &= \int \mu_x\left(Y \setminus \bigcup_n B_n\right) d\nu(x) \\
&= \int_{X\setminus H} \left|\left(Y \setminus \bigcup_n B_n\right) \cap G_x\right| d\nu(x) + \int_H \left|\left(Y \setminus \bigcup_n B_n\right) \cap G_x\right| d\nu(x) \\
&= \int_H \left|\left(Y \setminus \bigcup_n B_n\right) \cap G_x\right| d\nu(x) \\
&= 0.
\end{aligned}$$

The measure μ is thus a σ-finite measure on Y and the family $\{\mu_x : x \in X\}$ is a disintegration of μ with respect to (ν, ϕ) into σ-finite measures. However, this disintegration cannot be uniformly σ-finite. If it were, there would exist countably many Borel sets $E_n \subset Y$ such that $\forall x \in X$, $\mu_x(E_n) < \infty$ and $\mu_x(Y \setminus \cup_n E_n) = 0$. Thus for each $x \in X$, $|G_x \cap E_n| < \infty$ and $G \subset \bigcup_n X \times E_n$. For each n, $G \cap (X \times E_n)$ is $\mathbf{\Pi}^1_1$ with finite sections and is thus a countable union of $\mathbf{\Pi}^1_1$ graphs implying that $G = \bigcup_n G \cap E_n$ is a countable union of $\mathbf{\Pi}^1_1$ graphs, a contradiction.

REMARK 3.7. In order for the measure μ in the previous construction to be σ-finite a particular Borel set $H \subset X$ was used such that $\nu(H) = 1$ and so that $G \cap (H \times Y)$ is a Borel subset of $X \times Y$. The set H was the complement of a set V which was defined in the proof of Theorem 2.15. However it should be noted that the existence of such an H does not generally hold for $\mathbf{\Pi}_1^1$ subsets G of the product of Polish spaces $X \times Y$. The following demonstrates this fact indicating the delicate nature of the construction.

Let $W \subset [0,1] \times [0,1]$ be a $\mathbf{\Sigma}_2^1$ (or PCA) well-ordering of $[0,1]$ into type ω_1. As Gödel showed, if $\mathbf{V} = \mathbf{L}$ then there exists such a well-ordering with stronger properties. For further details see [9]. Let C be a $\mathbf{\Pi}_1^1$ subset of $[0,1]^3$ such that $\pi_{12}(C) = W$. By Kondo's uniformization theorem we may assume this projection is one-to-one on C. For every $y \in [0,1]$ $C_y = \{(x,z) \in [0,1]^2 : (x,y,z) \in C\}$ is countable since only countably many x can precede a fixed y. Also for every $x \in [0,1]$ $C_x = \{(y,z) \in [0,1]^2 : (x,y,z) \in C\}$ is co-countable. Let ν be a nonatomic probability measure on $[0,1]$. If there were an $H \in \mathcal{B}([0,1]^2)$ with $(\nu \times \nu)(H) > 0$ and $W \cap (H \times [0,1]) \in \mathcal{B}([0,1]^3)$ then we may use Fubini's theorem to calculate

$$\begin{aligned}(\nu \times \nu \times \nu)(C \cap (H \times [0,1])) &= \int (\nu \times \nu)([C \cap (H \times [0,1])]_y) d\nu(y) \\ &= 0.\end{aligned}$$

However applying Fubini's theorem again we see that

$$\begin{aligned}(\nu \times \nu \times \nu)(C \cap (H \times [0,1])) &= \int (\nu \times \nu)([C \cap (H \times [0,1])]_x) d\nu(x) \\ &= \nu(H) > 0.\end{aligned}$$

As an alternative a specific Borel set $H \subset X$ may be chosen with $\nu(H) = 1$ and $|X \setminus H| = \omega_1$. Then build the non-uniformizable set G over $X \setminus H$ and build a set over H that *is* uniformizable (similar to the construction of G over $X \setminus V$ in the proof of Theorem 2.15). This may be a more satisfactory method for constructing μ as it relies only on the conclusion of Theorem 2.15 and not on the specific details of its proof.

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