

MYCIELSKI-REGULAR MEASURES

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Let  $\mu$  be a Radon probability measure on  $M$ , the  $d$ -dimensional Real Euclidean space (where  $d$  is a positive integer), and  $f$  a measurable function. Let  $P$  be the space of sequences whose coordinates are elements in  $M$ . Then, for any point  $x$  in  $M$ , define a function  $f_n$  on  $M$  and  $P$  that looks at the first  $n$  terms of an element of  $P$  and evaluates  $f$  at the first of those  $n$  terms that minimizes the distance to  $x$  in  $M$ . The measures for which such sequences converge in measure to  $f$  for almost every sequence are called Mycielski-regular. We show that the self-similar measure generated by a finite family of contracting similitudes and which up to a constant is the Hausdorff measure in its dimension on an invariant set  $C$  is Mycielski-regular.

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## CHAPTER 1

### INTRODUCTION

Consider the measure space  $(X, \Sigma, \nu)$ , where  $X = \mathbb{R}^d$ , for  $d \geq 1$ , and  $\Sigma$  is the domain of  $\nu$ . We say that  $\nu$  is a *topological measure* if  $\Sigma$  contains the open sets. Note that, in this case,  $\Sigma$  contains also all the closed sets and all Borel sets. We say that  $\nu$  is *locally finite* if every bounded set has finite outer measure. If  $\nu$  is a topological measure,  $\nu$  is *inner regular* with respect to the compact sets means

$$(1) \quad \nu(E) = \sup\{\nu(K) : K \subseteq E, K \text{ is compact}\}$$

for all  $E \in \Sigma$ . Finally,  $\nu$  is called a *Radon measure* if it is a complete, locally finite topological measure that is inner regular with respect to the compact sets (a *complete* measure includes all the subsets of sets of measure 0). Of course, when we say that  $\nu$  is a *probability* measure, we simply mean that  $\nu(X) = 1$  [5].

In his paper, *Learning Theorems* [7], Jan Mycielski poses the following scenario: Given a metric space  $M$  and a sequence of points  $(x_k)_{k=0}^{\infty}$  in  $M$  and an unknown real-valued function  $f : M \rightarrow \mathbb{R}$ , for which we have learned its values for  $x_0, x_1, \dots, x_{n-1}$  (but perhaps not for  $x_n$ ), we predict the value of  $f(x_n)$  by the following algorithm. Let  $f_n : M \rightarrow \mathbb{R}$  be the function  $x \mapsto f(x_k)$ , where  $x_k$  is the first term of the first  $n$  elements of the sequence that minimizes the distance from  $x$  to  $x_i$ , for  $0 \leq i \leq n-1$ . To make the dependence of  $f_n$  on the sequence  $\vec{x} = (x_k)_{k=0}^{\infty}$  clear, we denote  $f_n(x) = f_n(x; \vec{x}|_{n-1})$ . In his paper, Mycielski proves the following theorem:

**THEOREM 1.1.** *Let  $\nu$  be a Radon probability measure on the Euclidean space  $\mathbb{R}^d$ , and  $P = \nu^{\mathbb{N}}$  the product measure in  $(\mathbb{R}^d)^{\mathbb{N}}$ . If  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is  $\nu$ -measurable, then*

$$(2) \quad \lim_{n \rightarrow \infty} P(|f_n(x_n; \vec{x}|_{n-1}) - f(x_n)| < \epsilon) = 1$$

for every  $\epsilon > 0$ .

In other words, for every  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  and  $\delta > 0$  such that for all  $n \geq N$ ,

$$(3) \quad \nu^{\mathbb{N}}(E_{n,\epsilon}) = P(E_{n,\epsilon}) \geq 1 - \delta,$$

where  $\vec{x} \in E_{n,\epsilon}$  if and only if  $|f_n(x_n, \vec{x}_{\uparrow n-1}) - f(x_n)| < \epsilon$ . So given  $\epsilon > 0$ , the probability that you will choose a sequence in  $(\mathbb{R}^d)^{\mathbb{N}}$  such that  $f_n$  differs from  $f$  by less than  $\epsilon$  at the  $n$ th term of that sequence goes to 1 as  $n \rightarrow \infty$ .

Mycielski then noted that it would be interesting to estimate the rate of convergence in Theorem 1.1; and it seems that  $f_n \rightarrow f$  in  $\nu$  measure for  $P$ -almost every sequence  $(x_0, x_1, \dots)$ . That is, it seems in the context of Theorem 1.1 that for all  $\epsilon > 0$

$$(4) \quad \lim_{n \rightarrow \infty} \nu(\{x \in \mathbb{R}^d : |f_n(x; \vec{x}_{\uparrow n-1}) - f(x)| < \epsilon\}) = 1$$

for  $P$ -a.e. sequence in  $(\mathbb{R}^d)^{\mathbb{N}}$ . In other words, for  $P$ -a.e.  $\vec{x} \in (\mathbb{R}^d)^{\mathbb{N}}$ ,  $\nu(G(n, \vec{x})) \rightarrow 1$ , where  $x \in G(n, \vec{x})$  if and only if  $|f_n(x; \vec{x}_{\uparrow n-1}) - f(x)| < \epsilon$ . It seems likely that both (2) and (4) hold for Radon probability measures on  $\mathbb{R}^d$ . Theorem 1.1 certainly shows that for all Radon probability measures on  $\mathbb{R}^d$ , (2) holds. However, I do not know if it can be shown that (2) implies (4). If it could, then (4) would follow immediately by virtue of Theorem 1.1. Instead, I have used some results in a note by David H. Fremlin.

Fremlin has called measures that satisfy the condition that  $f_n \rightarrow f$  in  $\nu$ -measure for  $P$ -almost every sequence  $(x_0, x_1, \dots)$ , *Mycielski-regular*. He has proved that for the the unit cube with the Euclidean metric, Lebesgue measure is Mycielski-regular [4]. The purpose of this paper is to extend Fremlin's result to other measures on  $\mathbb{R}^d$  with the Euclidean metric. In particular, I show in Theorem 5.4 that the self-similar measure generated by a finite family of contracting similitudes and which up to a constant is the Hausdorff measure in its dimension on the invariant set  $C$  is Mycielski-regular.

I begin by tracing Fremlin's development of conditions for which a measure  $\mu$  is Mycielski-regular [4]. I concentrate on the functional  $\theta(E) = \limsup_{n \rightarrow \infty} \int F(\omega \upharpoonright n, \mathbb{1}_E) d\mu$  (where  $E$  is a measurable set) and note some of its properties and its relation to a Radon

measure  $\mu$  (I define  $\theta$  formally in chapter 2; for the moment, note that by the function  $F(\omega \upharpoonright n, \mathbb{1}_E)$  I mean what was denoted by the  $f_n$  above, where  $f = \mathbb{1}_A$ ). Some results are that  $\theta$  does not depend on  $\omega$  and that a Radon measure  $\mu$  is Mycielski-regular if and only if it is absolutely continuous with respect to  $\theta$ ; that is,  $\mu$  is Mycielski-regular if and only if  $\theta(E) = 0$  implies that  $\mu(E) = 0$  for all  $E \in \Sigma$  [6]. This leads to the main result, that a measure is Mycielski-regular if it has moderated Voronoi tessellations (I define this in chapter 3)- this is shown by proving that if a measure satisfies this condition, then it is absolutely continuous with respect to  $\theta$ . It is this implication which provides the basic foundation for our method. It is good to note at this point that Fremlin's analysis is somewhat more general than what is presented in this paper; in particular, he does not require the measure  $\mu$  to be atomless. However, for my purposes, it is useful to have  $\mu$  to be atomless. In any case, since the measures I study are atomless (since the Hausdorff measures  $\mathcal{H}^s$  for  $s > 0$ , are), this is all I need to have.

Since the self-similar measures I study have all their mass on an invariant set which is constructed via similitudes that obey the open set condition, I spend some time developing the theory in general of such self-similar measures and the open set condition. In doing so, I am following Falconer's treatment of the theory [3]. This is interesting not only as background material; it is also interesting because the techniques in the proofs to the theorems are in some instances mirrored in my own.

I present my results in two stages. First, in Theorem 5.1 I show that if the similitudes have the same contraction ratio, then the self-similar measure is Mycielski-regular. I believe it is a profitable exercise to do this first since the proof is slightly less complicated than the more general case, gives the basic idea for all similar cases, and thus prepares one for the proof of the general case. After giving an example, I then give in Theorem 5.4 my most general result - for a self-similar measure  $\mu$  that concentrates its mass on the invariant set  $C$  which is constructed via similitudes (not necessarily with the same contraction ratio) and which obey the open set condition, then  $\mu$  is Mycielski-regular.



Finally, I discuss the application of this method to some other measures; in particular, I indicate that the measures which are used in the paper by Mauldin and Cawley [1] to get some of the results in the multifractal analysis can similarly be shown to be Mycielski-regular. I end with a discussion of future research possibilities. In particular, it would be interesting to know if this extends to conformal measures. Of course, Mycielski posed the question for all Radon probability measures, and so there is still much work to be done here.

To begin, however, I provide the proper setting and definitions, and give a couple of examples of measures - one which is *not* Mycielski-regular, and one which is. The first example shows that this is a non-trivial question: there are measures which do not converge in measure for these sequences of functions.

## CHAPTER 2

### DEFINITIONS AND MOTIVATION

We begin with notation. When referring to sets, the absolute value notation will refer to the diameter of a set (otherwise, it carries its usual meaning). So, if  $A$  is a subset of some metric space  $(X, \rho)$ , then  $|A| = \sup\{\rho(x, y) : x, y \in A\}$ . When referring to the number of elements in a set - its cardinality - we will simply write  $\text{Card}(A)$ . The interior of a set  $A$  will be denoted by  $\text{int } A$ .

The following definitions are from David Fremlin [4]:

DEFINITION 2.1. Let  $(X, \rho)$  be a metric space. Let  $\omega = (x_k)_{k=0}^{\infty}$  be an infinite sequence in  $X^{\mathbb{N}}$  and let  $\omega[n] = \{x_0, \dots, x_{n-1}\}$ . Suppose that  $z \in \omega[n]$ . Define the Voronoi tile  $V(\omega \upharpoonright n, z)$  by

$$(5) \quad V(\omega \upharpoonright n, z) = \{x \in X : \rho(x, z) = \rho(x, \omega[n]) \text{ and if } i < j < n \text{ and } z = x_j \neq x_i, \\ \text{then } \rho(x, z) < \rho(x, x_i)\};$$

We call the collection of such  $V(\omega \upharpoonright n, z)$  the *Voronoi tessellation* defined by  $\omega[n]$ .

That is,  $x \in V(\omega \upharpoonright n, z)$  either if the distance from  $x$  to  $z$  is smaller than its distance to any other element of  $\omega[n]$ , or if  $x$  is equidistant from two such points  $x_i$  and  $x_j$ , then  $x$  belongs to the  $V(\omega \upharpoonright n, z)$  such that  $z$  is equal to the first of the two elements  $x_i$  and  $x_j$ .

It is useful to note the fact that the Voronoi tessellation induces a partition on the space  $X$ . (If a point is repeated in the first  $n$  entries of the sequence  $\omega$ , say  $\omega(i) = \omega(j)$  for  $i < j$ , then  $V(\omega \upharpoonright n, \omega(j)) = \emptyset$ .) It is also easily seen that in a Banach space, the Voronoi tiles  $V(\omega \upharpoonright n, z)$  are convex sets and that  $\emptyset \neq \text{int } V(\omega \upharpoonright n, z) \subseteq V(\omega \upharpoonright n, z) \subseteq \overline{\text{int } V(\omega \upharpoonright n, z)}$ . The next definition is another way to define the function  $f_n$  above.

DEFINITION 2.2. Let  $f : X \rightarrow \mathbb{R}$ , and  $\omega[n]$  as above, and write  $x_i = x(i)$ . Let  $k(\omega[n], x)$  be the least  $i$  such that  $\rho(x, \omega[n]) = \rho(x, x(i))$ , so that  $x \in V(\omega \upharpoonright n, x(k(\omega[n], x)))$ . Define  $F(\omega \upharpoonright n, f)(x) = f(x(k(\omega[n], x)))$ .

DEFINITION 2.3. Let  $(X, \Sigma, \mu)$  be a measure space with  $\mu$  a topological probability measure. Let  $\lambda$  be the product measure  $\mu^{\mathbb{N}}$  on  $\Omega = X^{\mathbb{N}}$ . We say that  $\mu$  is *Mycielski-regular* provided for every measurable  $f : X \rightarrow \mathbb{R}$ , the sequence  $(F(\omega \upharpoonright n, f))_{n=1}^{\infty}$  converges in measure to  $f$  for  $\lambda$ -almost every  $\omega$ .

EXAMPLE 2.4. An easy example of a measure which is not Mycielski-regular is given by Fremlin [4]. Assume that there exists a countably additive extension  $\mu$  of Lebesgue measure to all subsets of  $X = [0, 1]$ . Let  $A = [0, 1/2)$  and  $B = [1/2, 1]$ , and let  $\mu_A$  and  $\mu_B$  be conditional measure induced by  $\mu$  on  $A$  and  $B$ , respectively.

Let  $\rho$  be the zero-one metric on  $X$ ; that is,

$$\rho(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{otherwise.} \end{cases}$$

Given a sequence  $\omega = (x_i)_{i=0}^{\infty}$ , then for any  $n$ ,  $V(\omega \upharpoonright n, x_0) = X \setminus \omega[n] \cup \{x_0\}$ , and has measure 1, while  $\mu\{z\} = 0$  for  $z \in \omega[n]$ . Hence, for almost every  $x \in X$ ,

$$F(\omega \upharpoonright n, f)(x) = f(x_0).$$

In particular, if  $f = \mathbb{1}_A$  (where  $\mathbb{1}_A$  is the characteristic function on the set  $A$ ), then

$$F(\omega \upharpoonright n, f)(x) = \begin{cases} 1, & \text{if } x_0 \in A, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that  $\mu\{x \in X : |F(\omega \upharpoonright n, f)(x) - f(x)| \geq \epsilon\} \geq 1 > 1/2$  for every sequence and for every  $n$ , and so  $F(\omega \upharpoonright n, f)$  never converges in measure to  $f$ , and hence is not Mycielski-regular.

EXAMPLE 2.5. An example of a measure that is Mycielski-regular - though it is rather uninteresting! - is one that concentrates all its mass on a single point. For example, let  $(X, \Sigma, \mu)$  be a measure space and let  $x_0 \in X$  such that  $\mu\{x_0\} = 1$ , and  $\mu(X \setminus \{x_0\}) = 0$ . If  $P = \mu^{\mathbb{N}}$ , then  $P$ -almost every sequence is the constant sequence  $(x_0)_{i=0}^{\infty}$ . Hence, using Mycielski's notation,  $f_n(x_0) = f(x_0)$  and so  $\mu(x \in X : |f_n(x) - f(x)| < \epsilon) = \mu\{x_0\} = 1$  for  $P$ -almost every sequence in  $X^{\mathbb{N}}$ . Indeed, Fremlin has shown that any probability measure such that  $\text{supp}(\mu)$  is countable is Mycielski-regular [4].

## CHAPTER 3

### CONDITIONS FOR A MEASURE TO BE MYCIELSKI-REGULAR

Here I trace David Fremlin's development of the conditions for a measure to be Mycielski-regular; the definitions and results in this chapter are taken from his note "Problem GO" [4]. In what follows,  $\Omega = X^{\mathbb{N}}$  and  $\lambda = \mu^{\mathbb{N}}$  is infinite product measure with domain  $\mathcal{B}(\Omega)$ . Following Fremlin, we define a functional  $\theta : \Sigma \rightarrow [0, 1]$  such that for any measurable  $E$ ,

$$(6) \quad \limsup_{n \rightarrow \infty} \int F(\omega \upharpoonright n, \mathbb{1}_E) d\mu = \theta(E)$$

for  $\lambda$ -almost every  $\omega \in \Omega$ . Fremlin has shown that  $\theta$  has the following properties:

- (i)  $\theta$  is a unital submeasure.
- (ii)  $\theta(H) \leq \mu(H)$  for every closed  $H \subseteq X$ , and  $\theta(G) \geq \mu(G)$  for every open  $G \subseteq X$ .
- (iii) If a measurable set  $E$  is such that  $\mu(\partial E) = 0$ , then  $\theta(E) = \mu(E)$ , where  $\partial E$  is the topological boundary of the set  $E$ .

To show these three properties, I note first that this function is measurable with respect to  $\mathcal{B}(\Omega)$ . To see this, write

$$(7) \quad F(\omega \upharpoonright n, f)(x) = \sum_{i=1}^n f(\omega(i)) \mathbb{1}_{V(\omega \upharpoonright n, \omega(i))}(x).$$

If  $f = \mathbb{1}_E$  for  $E \in \Sigma$ , then for all  $x \in X$ ,  $F(\omega \upharpoonright n, \mathbb{1}_E)(x) \leq 1$ , for every  $\omega \in \Omega$ . Hence,

$$(8) \quad \int_{\Omega} F(\omega \upharpoonright n, \mathbb{1}_E) d\lambda = \int_{\Omega \times X} F(\omega \upharpoonright n, \mathbb{1}_E) d(\lambda \times \mu) < \infty.$$

So  $F(\omega \upharpoonright n, \mathbb{1}_E) \in \mathcal{L}_1(\Omega \times X, \mathcal{B}(\Omega) \otimes \Sigma, \lambda \times \mu)$ . It follows by Fubini's theorem [6] that the function

$$(9) \quad \omega \mapsto \int_X F(\omega \upharpoonright n, \mathbb{1}_E) d\mu$$

is in  $\mathcal{L}_1(X^{\mathbb{N}}, \mathcal{B}(\Omega), \lambda)$  and, in particular, is  $\lambda$ -measurable.

At first sight, it appears that  $\theta$  depends both on  $E \subseteq X$  and  $\omega \in \Omega$ . I will show that if  $\omega$  and  $\omega' \in \Omega$  are eventually equal, then  $\lim_{n \rightarrow \infty} (F(\omega \upharpoonright n, f)(x) - F(\omega' \upharpoonright n, f)(x)) = 0$  for almost every  $x \in X$ , and so

$$(10) \quad \lim_{n \rightarrow \infty} \left( \int F(\omega \upharpoonright n, f) - \int F(\omega' \upharpoonright n, f) \right) d\mu = 0.$$

Hence the function  $h : \omega \in \Omega \mapsto \limsup_{n \rightarrow \infty} \int F(\omega \upharpoonright n, f) d\mu$  is measurable and is constant on all sequences that are eventually equal. By the Zero-One Law [6], the set  $\{\omega \in \Omega : h(\omega) > \alpha\}$  has measure 0 or 1 for every  $\alpha \in \mathbb{R}$ , and so there is an  $\alpha$  such that  $h(\omega) = \alpha$  for almost every  $\omega$ . To show (10), we enlist the aid of the following two propositions:

**PROPOSITION 3.1.** *Let  $(X, \rho)$  be a separable metric space and let  $\mu$  be a topological probability measure. If  $X_0$  is the support of  $\mu$ , then for every  $k \in \mathbb{N}$ ,  $X_0 = \overline{\omega[\mathbb{N} \setminus k]}$  for  $\lambda$ -a.e.  $\omega$ , where  $\overline{\omega[\mathbb{N} \setminus k]} = \overline{\{x_k, x_{k+1}, \dots\}}$ .*

**PROOF.** If  $X$  is separable metric, then any subspace is separable metric; in particular, it holds for  $X_0$ . Let  $\mathcal{U}$  be a countable base for  $X_0$ . Since  $X_0$  is the support of  $\mu$ , if  $U \in \mathcal{U}$ , then  $\mu(U) > 0$ , and so  $\lambda(\{\omega : \omega[\mathbb{N} \setminus k] \cap U \neq \emptyset\}) = 1$ , and since

$$(11) \quad \bigcap_{U \in \mathcal{U} \setminus \{\emptyset\}} \{\omega : U \cap \omega[\mathbb{N} \setminus k] \neq \emptyset\} \subseteq \{\omega : X_0 \subseteq \overline{\omega[\mathbb{N} \setminus k]}\},$$

it follows that  $\lambda(\{\omega : X_0 \subseteq \overline{\omega[\mathbb{N} \setminus k]}\}) = 1$  as well. □

**PROPOSITION 3.2.** *Let  $(X, \rho)$  be a separable metric space and let  $\mu$  be a topological probability measure such that  $\mu$  has no atoms. There exists  $\Omega_0 \subseteq \Omega$  with  $\lambda(\Omega_0) = 1$ , such that if  $\omega, \omega' \in \Omega_0$  are eventually equal, then for  $\mu$ -a.e.  $x \in X$ , there is an  $n \in \mathbb{N}$  such that  $F(\omega \upharpoonright m, f)(x) = F(\omega' \upharpoonright m, f)(x)$  for every  $m \geq n$  and for every  $f$  defined on  $X$ .*

**PROOF.** Let  $\omega, \omega' \in \Omega_0$  such that  $\omega(m) = \omega'(m)$  for every  $m \geq l$ . Let  $X_0 = \overline{\omega[\mathbb{N} \setminus l]} \setminus I$  and  $I = \omega[l] \cup \omega'[l]$ . Then  $\mu(X_0) = 1$  since  $\mu(I) = 0$ . Now if  $x \in X_0$  then there exists  $n \geq l$

such that  $\rho(x, \omega[n \setminus l]) < \rho(x, I)$ , and the same is true for all  $m \geq n$ . So for any  $m \geq n$ ,  $k(\omega \upharpoonright m, x) = k(\omega' \upharpoonright m, x)$ , and hence that  $F(\omega \upharpoonright m, f)(x) = F(\omega' \upharpoonright m, f)(x)$ .  $\square$

It thus happens that the functional  $\theta$  is constant on measurable sets. We now establish the three properties mentioned above. First,  $\theta$  is a unital submeasure. By “unital” I simply mean that  $\theta : \Sigma \rightarrow [0, 1]$ , which is clear. That it is a submeasure is also easy to see. By “submeasure,” I mean that  $\theta$  has the following three properties:

- (i)  $\theta(A \cup B) \leq \theta(A) + \theta(B)$  for all  $A, B \in \Sigma$ ,
- (ii)  $\theta(A) \leq \theta(B)$  if  $A \subseteq B$ , and
- (iii)  $\theta(\emptyset) = 0$ .

These properties follow because we can write

$$(12) \quad \theta(E) = \limsup_{n \rightarrow \infty} \int F(\omega \upharpoonright n, \mathbb{1}_E) d\mu$$

$$(13) \quad = \limsup_{n \rightarrow \infty} \int \sum_{i=1}^n \mathbb{1}_E(\omega(i)) \mathbb{1}_{V(\omega \upharpoonright n, \omega(i))}(x) d\mu$$

$$(14) \quad = \limsup_{n \rightarrow \infty} \sum_{i=1}^n \int \mathbb{1}_E(\omega(i)) \mathbb{1}_{V(\omega \upharpoonright n, \omega(i))}(x) d\mu,$$

and because of the properties of the characteristic function. Thus,  $\mathbb{1}_{A \cup B}(x) \leq \mathbb{1}_A(x) + \mathbb{1}_B(x)$ , and if  $A \subseteq B$ , then  $\mathbb{1}_A \leq \mathbb{1}_B$ , and  $\mathbb{1}_\emptyset \equiv 0$ .

To show that  $\theta(H) \leq \mu(H)$  for every closed  $H \subseteq X$ , we first need the following lemma:

**LEMMA 3.3.** *Let  $f$  be a real-valued continuous function defined on  $X$ . Then for almost every  $\omega \in \Omega$  and for every  $x \in \text{supp}(\mu) = X_0$ ,  $F(\omega \upharpoonright n, f)(x)$  converges to  $f(x)$  as  $n \rightarrow \infty$ .*

**PROOF.** Let  $\epsilon > 0$ . By the continuity of  $f$ , there exists a  $\delta > 0$  such that if  $\rho(x, y) < \delta$  then  $|f(x) - f(y)| < \epsilon$ . Further, as  $n \rightarrow \infty$ , for every  $x \in X_0$ , we have that  $\rho(x, \omega[n]) \rightarrow 0$ . So there is an  $n_0 \in \mathbb{N}$ , such that if  $n \geq n_0$ , then  $\rho(x, \omega[n]) < \delta$ . So,

$$(15) \quad |F(\omega \upharpoonright n, f)(x) - f(x)| = \left| \sum_{i=1}^n f(\omega(i)) \mathbb{1}_{V(\omega \upharpoonright n, \omega(i))}(x) - f(x) \right|$$

$$(16) \quad = |f(\omega(j)) - f(x)|$$

for the  $1 \leq j \leq n$  such that  $x \in V(\omega \upharpoonright n, \omega(j))$ . As  $n \rightarrow \infty$ ,  $\rho(\omega(j), x) < \delta$ , and so we get that  $|f(\omega(j)) - f(x)| < \epsilon$ .  $\square$

Now let  $\epsilon > 0$ , and let  $H \subseteq X$  be closed. There is a continuous function  $f$ , such that  $\mathbb{1}_H \leq f$  and  $\int f d\mu < \mu(H) + \epsilon$ . Further, since  $\lim_{n \rightarrow \infty} F(\omega \upharpoonright n, f)(x) = f(x)$  for almost every  $x$ , then we have that

$$(17) \quad \theta(H) = \limsup_{n \rightarrow \infty} \int F(\omega \upharpoonright n, \mathbb{1}_H) d\mu$$

$$(18) \quad = \int f(x) d\mu \quad (\text{by Lemma 3.3})$$

$$(19) \quad < \mu(H) + \epsilon,$$

and hence we have that  $\theta(H) \leq \mu(H)$ .

On the other hand, if  $G \subseteq X$  is open, then  $\theta(G) = 1 - \theta(X \setminus G) \geq 1 - \mu(X \setminus G) = \mu(G)$ .

Finally, we show that if a measurable set  $E$  is such that  $\mu(\partial E) = 0$ , then  $\theta(E) = \mu(E)$ , where  $\partial E$  is the topological boundary of the set  $E$ . This follows from

$$(20) \quad \mu(E) = \mu(\overline{E}) = \mu(\text{int } E) \leq \theta(\text{int } E) \leq \theta(E) \leq \theta(\overline{E}) \leq \mu(\overline{E}) = \mu(E).$$

This lays the groundwork for the following two fundamental theorems, and which provide the key to proving which measures are in fact Mycielski-regular:

**THEOREM 3.4.** *Let  $(X, \rho)$  be a separable metric space,  $\mu$  a topological probability measure on  $X$  and  $\theta : \Sigma \rightarrow [0, 1]$  the functional defined above. Then the following are equivalent:*

- (i)  $\mu$  is Mycielski-regular;
- (ii)  $\theta$  is absolutely continuous with respect to  $\mu$ ;
- (iii)  $\theta = \mu$ .

**PROOF.** It is clear that (i)  $\implies$  (iii)  $\implies$  (ii). It is thus sufficient to show that (ii)  $\implies$  (i). Suppose then that  $\theta$  is absolutely continuous with respect to  $\mu$ . Let  $f : X \rightarrow \mathbb{R}$  be measurable, and for each  $k \in \mathbb{N}$ , let  $\delta_k > 0$  be such that  $\theta(E) \leq 2^{-k}$  whenever  $\mu(E) \leq \delta_k$ .



By Lusin's theorem [6], there exists a continuous function, call it  $g_k : X \rightarrow \mathbb{R}$ , and a set  $E_k = \{x \in X : g_k(x) \neq f(x)\}$ , such that  $\mu(E_k) \leq \min\{2^{-k}, \delta_k\}$ . Note that  $\{x \in X : F(\omega \upharpoonright n, f)(x) \neq F(\omega \upharpoonright n, g_k)(x)\} \subseteq \{x \in X : F(\omega \upharpoonright n, \mathbb{1}_{E_k}) = 1\}$  for every  $\omega \in \Omega$ . Define  $W_k \subseteq \Omega$  such that  $\omega \in W_k$  if and only if  $\lim_{n \rightarrow \infty} F(\omega \upharpoonright n, g_k)(x) = g_k(x)$  for almost every  $x$  and  $\limsup_{n \rightarrow \infty} \int F(\omega \upharpoonright n, \mathbb{1}_{E_k}) d\mu \leq 2^{-k}$ . Then  $\lambda(W_k) = 1$ . Let  $W = \bigcap_{k \in \mathbb{N}} W_k$ . For any  $\omega$ , we have that

$$(21) \quad \min\{|F(\omega \upharpoonright n, f) - F(\omega \upharpoonright n, g_k)|, \mathbb{1}_X\} \leq \min\{F(\omega \upharpoonright n, |f - g_k|), \mathbb{1}_X\}$$

$$(22) \quad \leq F(\omega \upharpoonright n, \mathbb{1}_{E_k}).$$

Hence,

$$(23) \quad \min\{|F(\omega \upharpoonright n, f) - f|, \mathbb{1}_X\} \leq \min\{|F(\omega \upharpoonright n, f) - F(\omega \upharpoonright n, g_k)|, \mathbb{1}_X\}$$

$$(24) \quad + \min\{|F(\omega \upharpoonright n, g_k) - g_k|, \mathbb{1}_X\} + \min\{|g_k - f|, \mathbb{1}_X\}$$

$$(25) \quad \leq F(\omega \upharpoonright n, \mathbb{1}_{E_k}) + \min\{|F(\omega \upharpoonright n, g_k) - g_k|, \mathbb{1}_X\} + \mathbb{1}_{E_k}.$$

Thus we have that if  $\omega \in W$ , then

$$(26) \quad \overline{\lim}_{n \rightarrow \infty} \int \min\{|F(\omega \upharpoonright n, f) - f|, \mathbb{1}_X\} d\mu \leq \overline{\lim} \int F(\omega \upharpoonright n, \mathbb{1}_{E_k}) d\mu$$

$$(27) \quad + \overline{\lim} \int \min\{|F(\omega \upharpoonright n, g_k) - g_k|, \mathbb{1}_X\} d\mu$$

$$(28) \quad + \overline{\lim} \int \mathbb{1}_{E_k} d\mu$$

$$(29) \quad \leq 2^{-k} + 0 + 2^{-k+1}.$$

Since this is true for every  $k \in \mathbb{N}$ , it follows that  $F(\omega \upharpoonright n, f)$  converges to  $f$  in measure. And since  $f$  is arbitrary, we have that  $\mu$  is Mycielski-regular.  $\square$

I now give a sufficient condition for a measure to be Mycielski-regular in terms of its tessellations. We need the following definition:

DEFINITION 3.5. Let  $X, \rho, \mu, \Omega$ , and  $\lambda$  be as defined above. We say that  $\mu$  has *moderated Voronoi tessellations* if for every  $\epsilon > 0$  there exists  $M \geq 0$  such that

$$(30) \quad \sum_{n=1}^{\infty} \lambda\{\omega : \mu\left(\bigcup\{V'(\omega \upharpoonright n, z) : z \in \omega[n], \mu(V'(\omega \upharpoonright n, z)) \geq M/n\}\right) \geq \epsilon\} < \infty,$$

where each  $V'(\omega \upharpoonright n, z)$  is the punctured Voronoi tile  $V(\omega \upharpoonright n, z) \setminus \{z\}$ .

Note that if  $\mu$  has moderated Voronoi tessellations for  $M$  then  $\mu$  has moderated Voronoi tessellations for all  $M' \geq M$ . The reason for this is as follows: call  $A(n, M, \epsilon) = \{\omega : \mu\left(\bigcup\{V'(\omega \upharpoonright n, z) : z \in \omega[n], \mu(V'(\omega \upharpoonright n, z)) \geq M/n\}\right) \geq \epsilon\}$ , and  $B(n, M) = \bigcup\{V'(\omega \upharpoonright n, z) : z \in \omega[n], \mu(V'(\omega \upharpoonright n, z)) \geq M/n\}$ . If  $V'(\omega \upharpoonright n, z) \in B(n, M')$ , then  $V'(\omega \upharpoonright n, z) \in B(n, M)$  so that  $B(n, M') \subseteq B(n, M)$ . Thus, if  $\mu(B(n, M')) \geq \epsilon$  then  $\mu(B(n, M)) \geq \epsilon$ . So if  $\omega \in A(n, M', \epsilon)$  then  $\omega \in A(n, M, \epsilon)$ . Hence, if  $\sum A(n, M, \epsilon) < \infty$  then  $\sum A(n, M', \epsilon) < \infty$ . We now have the proper background to state the following theorem.

THEOREM 3.6. *Let  $(X, \rho)$  be a separable metric space,  $\mu$  a topological probability measure on  $X$  which has moderated Voronoi tessellations. Then  $\mu$  is Mycielski-regular.*

PROOF. Let  $\theta$  be the submeasure introduced above. I will show that  $\theta$  is absolutely continuous with respect to  $\mu$  and therefore by the previous theorem, it will follow that  $\mu$  is Mycielski-regular.

Let  $\epsilon > 0$ , and let  $M \geq 0$  such that

$$(31) \quad \sum_{n=1}^{\infty} \lambda\{\omega : \mu\left(\bigcup\{V(\omega \upharpoonright n, z) : z \in \omega[n], \mu(V(\omega \upharpoonright n, z)) \geq M/n\}\right) \geq \epsilon/3\} < \infty.$$

Let

$$(32) \quad \Omega_1 = \{\omega : \mu\left(\bigcup\{V(\omega \upharpoonright n, z) : z \in \omega[n], \mu(V(\omega \upharpoonright n, z)) \geq M/n\}\right) < \epsilon/3 \text{ for all but finitely } n\}.$$

It follows that  $\lambda(\Omega_1) = 1$ .

Now, let  $\delta > 0$  such that  $2M\delta \leq \epsilon/3$ ,  $\delta \leq \epsilon/3$ , and  $\delta \leq 1/2$ . Suppose that  $\mu(E) \leq \delta$ .

Let

$$(33) \quad \Omega_2 = \{\omega : \text{Card}\{n : \text{Card}\{i : i < n, \omega(i) \in E\} > 2\delta n\} < \infty\}.$$

It follows by the Strong Law of Large Numbers [6] that  $\lambda(\Omega_2) = 1$ . Let  $\omega \in \Omega_1 \cap \Omega_2$ . Let  $n$  be such that

$$(34) \quad \mu\left(\bigcup\{V(\omega \upharpoonright n, z) : z \in \omega[n], \mu(V(\omega \upharpoonright n, z)) \geq M/n\}\right) \leq \epsilon/3,$$

and

$$(35) \quad \text{Card}\{i : i < n, \omega(i) \in E\} \leq 2\delta n.$$

Set  $I = E \cap \omega[n]$ , and  $J = \{z : z \in \omega[n], \mu(V(\omega \upharpoonright n, z)) \geq M/n\}$ . Thus,

$$(36) \quad \int F(\omega \upharpoonright n, \mathbb{1}_E) d\mu = \sum_{z \in I} \mu(V(\omega \upharpoonright n, z))$$

$$(37) \quad = \sum_{z \in I \cap J} \mu(V(\omega \upharpoonright n, z)) + \sum_{z \in I \setminus J} \mu(V(\omega \upharpoonright n, z))$$

$$(38) \quad \leq \epsilon/3 + \text{Card}(I \setminus J) \cdot M/n$$

$$(39) \quad \leq \epsilon/3 + M \cdot \text{Card}(I)/n$$

$$(40) \quad \leq \epsilon/3 + 2M\delta \leq \epsilon.$$

As this is true for all but finitely  $n$ , it follows that  $\theta(E) \leq \epsilon$ , and thus that  $\theta$  is absolutely continuous with respect to  $\mu$ . □

I now present Fremlin's theorem and proof that  $\mu$  is Mycielski-regular when  $\mu$  is Lebesgue measure.

**THEOREM 3.7.** *Let  $r \geq 1$  be an integer. Let  $(X, \rho, \mu)$  be  $[0, 1]^r$  with its Euclidean metric and Lebesgue measure. Then  $\mu$  has moderated Voronoi tessellations, and so is Mycielski-regular.*

PROOF. The point of the first part of the proof is just to get some calculations out of the way that will be useful in the course of the proof. We begin by letting  $0 < \epsilon < 1$ ,  $q = (2 + 2\sqrt{r})^r$ ,  $M = 1 + \lceil \ln 10/\epsilon \rceil$ , and  $\gamma = 1/2^{\epsilon/4M}$ . Finally, let

$$(41) \quad n_0 = \left\lceil \max \left( \frac{2M}{(2^{1/r} - 1)^r}, \frac{4M}{\epsilon}, \frac{20}{\epsilon} \right) \right\rceil.$$

Now let  $n \geq n_0$  and let

$$(42) \quad l = \left\lfloor \left( \frac{n}{M} \right)^{1/r} \right\rfloor.$$

Observe that  $n \leq M(l+1)^r$  and so  $l+1 \geq \frac{2^{1/r}}{2^{1/r} - 1}$ ,  $l \geq \frac{1}{2^{1/r} - 1}$ ,  $l+1 \leq 2^{1/r}l$ , and  $n \leq 2Ml^r$ .

Let  $m = \lceil \frac{\epsilon n}{4M} \rceil$ . It follows that  $\frac{\epsilon n}{5M} \leq \frac{\epsilon n}{4M} - 1$  and  $\frac{l^r}{m} \leq \frac{n}{Mm} \leq \frac{5}{\epsilon}$ ; also  $m \leq \frac{n}{4M} \leq l^r$ .

Now that we are finished with the preliminaries, we begin the proof proper. Let  $\mathcal{J}$  be the set of hypercubes of the form

$$(43) \quad \prod_{j < r} \left[ \frac{i_j}{l}, \frac{i_j + 1}{l} \right],$$

where  $i_j < l$  for  $j < r$ . It follows that  $\text{Card } \mathcal{J} = l^r$ ,  $\cup \mathcal{J} = X$ ,  $\mu J = 1/l^r$ , and  $|J| = \frac{\sqrt{r}}{l}$  for  $J \in \mathcal{J}$ .

The proof is based on the following claim:

CLAIM 3.8. *Now suppose that  $V \subseteq [0, 1]^r$  is convex and  $x \in V$ . Let*

$$(44) \quad V_x = V \cap \bigcup_{y \in V} \{J \in \mathcal{J} : J \subseteq \text{int } B(y, \rho(y, x))\}.$$

*Then,  $\mu(V \setminus V_x) \leq 2Mq/n$ .*

To show the claim, let  $\mathcal{K}$  be the set of members of  $\mathcal{J}$  meeting  $B(x, \frac{\sqrt{r}}{l})$ . Then  $\text{Card } \mathcal{K} \leq q$  since each projection of  $B(x, \frac{\sqrt{r}}{l})$  onto a coordinate has length at most  $\frac{2\sqrt{r}}{l}$  and meets at most  $2 + 2\sqrt{r}$  intervals of the form  $[\frac{i}{r}, \frac{i+1}{r}]$ . If  $y \in V \setminus \cup \mathcal{K}$ , let  $J \in \mathcal{J}$  such that  $y \in J$ . Then  $J \subseteq B(y, \frac{\sqrt{r}}{l})$ , while  $\rho(y, x) > \sqrt{r}/l$ , so that  $J \subseteq \text{int } B(y, \rho(x, y))$  and  $y \in V_x$ . Accordingly,  $V \setminus V_x \subseteq \cup \mathcal{K}$  is covered by  $q$  members of  $\mathcal{J}$ , and has measure at most  $\frac{q}{l^r} \leq \frac{2Mq}{n}$ . This finishes the proof for Claim 3.8.

CLAIM 3.9. For  $\omega \in \Omega$ , set

$$(45) \quad H_n(\omega) = \bigcup \{V(\omega \upharpoonright n, z) : z \in \omega[n], \mu V(\omega \upharpoonright n, z) > \frac{4Mq}{n}\},$$

$$(46) \quad \mathcal{K}_\omega = \{J \in \mathcal{J} : J \cap \omega[n] = \emptyset\}.$$

Then  $\mu H_n(\omega) \leq \frac{4M}{n} \cdot \text{Card}(\mathcal{K}_\omega)$ .

To show this claim, note that if  $z \in \omega[n]$  and  $V(\omega \upharpoonright n, z)$  has measure greater than  $\frac{4Mq}{n}$ , then for every  $y \in V(\omega \upharpoonright n, z)$ ,  $\text{int } B(y, \rho(y, z))$  does not meet  $\omega[n]$ , and every member of  $\mathcal{J}$  included in  $\text{int } B(y, \rho(y, z))$  belongs to  $\mathcal{K}_\omega$ . By Claim 3.8,  $V(\omega \upharpoonright n, z) \setminus \bigcup \mathcal{K}_\omega$  has measure at most  $2Mq/n$ . Consequently,

$$(47) \quad \mu(V(\omega \upharpoonright n, z)) \leq 2\mu(V(\omega \upharpoonright n, z) \cap \bigcup \mathcal{K}_\omega).$$

Summing over relevant  $z$ ,

$$(48) \quad \mu H_n(\omega) \leq 2\mu(H_n(\omega) \cap \bigcup \mathcal{K}_\omega) \leq \frac{2}{l^r} \cdot \text{Card}(\mathcal{K}_\omega) \leq \frac{4M}{n} \cdot \text{Card}(\mathcal{K}_\omega).$$

This shows the claim.

It follows that if  $\mu H_n(\omega) \geq \epsilon$ , then  $\text{Card}(\mathcal{K}_\omega) \geq \frac{\epsilon n}{4M} \geq m$ . Hence, we have that

$$(49) \quad \{\omega : \mu H_n(\omega) \geq \epsilon\} \subseteq \{\omega : \text{Card}(\mathcal{K}_\omega) \geq m\}.$$

Define  $[\mathcal{J}]^m = \{\mathcal{K} \subseteq \mathcal{J} : \text{Card}(\mathcal{K}) = m\}$ . Then,

$$(50) \quad \lambda\{\omega : \text{Card}(\mathcal{K}_\omega) \geq m\} \leq \sum_{\mathcal{K} \in [\mathcal{J}]^m} \lambda\{\omega : \omega[n] \text{ does not meet } \bigcup \mathcal{K}\}$$

$$(51) \quad \leq \text{Card}([\mathcal{J}]^m) \left(1 - \frac{m}{l^r}\right)^n$$

$$(52) \quad \leq \frac{l^{rm}}{m!} \left(1 - \frac{m}{l^r}\right)^{Ml^r} \quad (\text{since } Ml^r \leq n)$$

$$(53) \quad \leq \frac{e^m l^{rm}}{m^m} \left(1 - \frac{1}{l^r}\right)^{Mml^r} \quad (\text{since } 1 - ml^r \leq (1 - l^r)^m)$$

$$(54) \quad \leq \frac{e^m l^{rm}}{m^m} \left(\frac{1}{e}\right)^{Mm} \quad (\text{since } (1 - x)^{1/x} \leq 1/e \text{ for every } x > 0)$$

$$(55) \quad = \left(\frac{el^r}{me^M}\right)^m$$

$$(56) \quad \leq \left( \frac{5e}{\epsilon e^M} \right)^m$$

$$(57) \quad \leq \frac{1}{2^m}$$

$$(58) \quad \leq \frac{1}{2^{\epsilon n/4M}} = \gamma^n.$$

The above applies for all  $n \geq n_0$ , and so, summing over  $n$ , we get that

$$(59) \quad \sum_{n=1}^{\infty} \lambda\{\omega : \mu \left( \bigcup \{V(\omega \upharpoonright n, z) : z \in \omega[n], \mu(V(\omega \upharpoonright n, z)) \geq 4Mq/n\} \right) \geq \epsilon\}$$

$$(60) \quad \leq n_0 + \sum_{n=n_0}^{\infty} \lambda\{\omega : \mu H_n(\omega) \geq \epsilon\}$$

$$(61) \quad \leq n_0 + \sum_{n=n_0}^{\infty} \gamma^n < \infty.$$

Since  $\epsilon$  is arbitrary, we have that  $\mu$  has moderated Voronoi tessellations. It follows that  $\mu$  is Mycielski-regular. □

## CHAPTER 4

### SELF-SIMILAR MEASURES AND THE OPEN SET CONDITION

In this section, I am basically following Kenneth Falconer's treatment of self-similar measures; the theorems and their proofs are from material in his book *The Geometry of Fractal Sets* [3]. As stated, the goal is to extend Fremlin's results to other measures besides Lebesgue measure. Here I am concerned with self-similar measures on bounded subsets of  $\mathbb{R}^d$ , for  $d \geq 1$ . Our setting is as follows. We begin with two definitions.

**DEFINITION 4.1.** A *contraction* is a mapping  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $|\phi(x) - \phi(y)| < c|x - y|$  for  $c < 1$  and all  $x, y \in \mathbb{R}^n$ .

**DEFINITION 4.2.** A *similitude* is a mapping  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that there exists a constant  $c > 0$  for which  $|\phi(x) - \phi(y)| = c|x - y|$  for all  $x, y \in \mathbb{R}^n$ .

Let  $X$  be a subset of  $\mathbb{R}^d$ ,  $\rho$  the Euclidean metric, and let  $\phi_i : X \rightarrow X$  for  $1 \leq i \leq l$ , be similitudes with contraction ratios  $r_1, r_2, \dots, r_l$ , and for which the open set condition holds (to be explained shortly). Let

$$(62) \quad \phi(F) = \bigcup_{i=1}^l \phi_i(F).$$

A set  $C$  is called an *invariant set* if  $C = \bigcup_{i=1}^l \phi_i(C)$ . Moreover, if  $\mathcal{H}^s(\phi_i(C) \cap \phi_j(C)) = 0$  (where  $\mathcal{H}^s$  is the  $s$ -dimensional Hausdorff measure, see definition below), then we call  $C$  a *self similar set*. We will be looking at measures that concentrate their mass on a compact subset of  $X$ , which is constructed via these similitudes. Theorem 4.5 guarantees the existence of the type of compact set that we are looking for. Though I am primarily interested in self-similar measures on bounded subsets of  $\mathbb{R}^d$  with the Euclidean metric, I now define both the Hausdorff measure and Hausdorff metric which will be used in establishing several of the

following theorems. It will also be seen that Hausdorff measure and the self-similar measures I am interested in are closely related.

DEFINITION 4.3. Let  $F$  be a subset of  $\mathbb{R}^d$  and let  $s$  be a non-negative real number. For  $\delta > 0$ , define

$$(63) \quad \mathcal{H}_\delta^s(F) = \inf \sum_{i=1}^{\infty} |U_i|^s,$$

where the infimum is taken over all countable covers  $\{U_i\}$  of  $F$  with diameter less than or equal to  $\delta$ . The  $s$ -dimensional Hausdorff measure is defined to be  $\lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(F)$  and is written  $\mathcal{H}^s(F)$ .

Note that, given  $\delta < 1$ , and a measurable set  $E$ ,  $\mathcal{H}_\delta^s(E)$  is a non-increasing function of  $s$ . Now suppose that  $\mathcal{H}^s(E) < \infty$ . Then for any  $t > s$ , and for any  $\delta$ -cover  $\{U_i\}$  of  $E$ ,

$$(64) \quad \mathcal{H}_\delta^t(E) \leq \sum_{i=1}^{\infty} |U_i|^t \leq \delta^{t-s} \sum_{i=1}^{\infty} |U_i|^s.$$

Letting  $\delta \rightarrow 0$ , we see that  $\mathcal{H}^t(E) = 0$  for all  $t > s$ . This also implies that for all  $t < s$ ,  $\mathcal{H}^t(E) = \infty$ . Hence, because of the properties of Hausdorff measure, graphing the Hausdorff measure of a set versus the number  $s$  shows that the graph is always infinity or 0, except at one point, where the graph jumps from infinity to 0. This point is called the *Hausdorff dimension* of a set, and is of considerable interest. In particular, it is often of interest to know if  $0 < \mathcal{H}^s(F) < \infty$ . If so, then the set  $F$  is called an  $s$ -set.

DEFINITION 4.4. Let  $(X, \rho)$  be a metric space and let  $\mathcal{K}(X)$  be the collection of all non-empty compact subsets of  $X$ . The Hausdorff metric on  $\mathcal{K}(X)$  is given by

$$(65) \quad \rho_H(K_1, K_2) = \max \left\{ \sup_{x \in K_2} \{\rho(x, K_1)\}, \sup_{x \in K_1} \{\rho(x, K_2)\} \right\}$$

$$(66) \quad = \inf \{ \epsilon > 0 : K_1 \subseteq B(K_2, \epsilon), K_2 \subseteq B(K_1, \epsilon) \},$$

where  $B(K, \epsilon) = \{x \in X : \rho(x, K) < \epsilon\}$  and  $\rho(x, K) = \inf \{\rho(x, y) : y \in K\}$ .



THEOREM 4.5. Let  $\{\phi_i\}_{i=1}^l$  be contractions on  $\mathbb{R}^d$  with contraction ratios  $r_i < 1$ . Then there exists a unique non-empty compact set  $C$  such that

$$(67) \quad C = \phi(C) = \bigcup_{i=1}^l \phi_i(C).$$

Further, if  $F$  is any non-empty compact subset of  $\mathbb{R}^d$  the iterates  $\phi^k(F)$  converge to  $C$  in the Hausdorff metric as  $k \rightarrow \infty$ .

PROOF. Let  $\mathcal{K}$  be the collection of non-empty compact subsets of  $\mathbb{R}^d$ . First we show that  $\mathcal{K}$  is a complete metric space when equipped with the Hausdorff metric. The result will then follow by an application of the Banach Fixed Point Theorem [6].

To show that  $\mathcal{K}$  is a complete metric space when equipped with the Hausdorff metric, let  $\{E_i\}$  be a Cauchy sequence of compact sets of  $\mathbb{R}^d$  so that  $\rho_H(E_i, E_j) \leq 1/\min(i, j)$ , and let

$$(68) \quad E = \bigcap_{j \geq 1} \overline{\bigcup_{i \geq j} E_i}.$$

$E$  is non-empty compact since it is the intersection of a decreasing sequence of non-empty compact sets. It will now be shown that the sequence  $\{E_i\}$  converges to  $E$  in the Hausdorff metric. To see this, note first that

$$(69) \quad E \subseteq \overline{\bigcup_{i \geq j} E_i} \subseteq B(E_j, 1/j)$$

for all  $j$ . On the other hand, let  $x \in E_j$ ; it follows that  $x \in B(E_i, 1/j)$  for all  $i \geq j$  and so  $x \in B(\overline{\bigcup_{i=k}^{\infty} E_k}, 1/j)$  for  $k \geq j$ . Now choose a sequence  $\{y_k\}$  such that  $y_k \in \overline{\bigcup_{i=k}^{\infty} E_k}$  and  $|x - y_k| \leq 1/j$ . By compactness, this sequence has a subsequence that converges to some  $y \in \bigcap_{k=1}^{\infty} \overline{\bigcup_{i=k}^{\infty} E_k} = E$  and  $|x - y| \leq 1/j$ . Hence,  $x \in B(E, 1/j)$ , and so  $E_j \subseteq B(E, 1/j)$ . Therefore,  $\rho_H(E_j, E) \leq 1/j$  and it follows that the  $E_j$  converge to  $E$  in the Hausdorff metric.

Now, let  $F_1$  and  $F_2 \in \mathcal{K}$ . Then,

$$(70) \quad \rho_H(\phi(F_1), \phi(F_2)) = \rho_H \left( \bigcup_{i=1}^l \phi_i(F_1), \bigcup_{i=1}^l \phi_i(F_2) \right)$$

$$(71) \quad \leq \max_j \rho_H(\phi_j(F_1), \phi_j(F_2))$$

$$(72) \quad \leq (\max_j r_j) \rho_H(F_1, F_2).$$

By the Banach Fixed Point Theorem, it follows immediately that there is a unique  $F \in \mathcal{K}$  such that  $F = \phi(F)$ .  $\square$

The number  $s$  for which  $\sum_{j=1}^l r_j^s = 1$  is called the *similarity dimension*. If we raise both sides to the  $k$ th power, we obtain,

$$(73) \quad \sum_{j_1 j_2 \dots j_k} (r_{j_1} r_{j_2} \dots r_{j_k})^s = 1,$$

where we are summing over all the  $k$ -tuples  $\{j_1 j_2 \dots j_k\}$  for  $1 \leq j \leq l$ . Let  $F_{j_1 \dots j_k} = \phi_{j_1} \circ \dots \circ \phi_{j_k}(F)$ . We now come to the self-similar measure which is defined on the compact set  $C$ .

PROPOSITION 4.6. *There exists a Borel measure  $\mu$  with support contained in  $C$ , such that  $\mu(\mathbb{R}^d) = 1$  and such that for any measurable set  $F$ ,*

$$(74) \quad \mu(F) = \sum_{j=1}^l r_j^s \mu(\phi_j^{-1}(F)).$$

PROOF. Let  $x \in C$  and let  $x_{j_1 \dots j_k} = \phi_{j_1} \circ \dots \circ \phi_{j_k}(x)$ . For  $k = 1, 2, \dots$ , define a positive linear function defined on the space of continuous functions by

$$(75) \quad L_k(f) = \sum_{j_1 \dots j_k} (r_{j_1} r_{j_2} \dots r_{j_k})^s f(x_{j_1 \dots j_k}).$$

If  $f$  is continuous, then  $f$  is uniformly continuous on  $C$ , so given  $\epsilon > 0$ , we can find a  $k_0$  such that for all  $k \geq k_0$ , we have that  $f$  varies by no more than  $\epsilon$  on each set  $C_{j_1 \dots j_k}$  since  $|C_{j_1 \dots j_k}| \leq (\max r_j)^k |E|$ . For  $k \geq k_0$ , we have that  $x_{j_1 \dots j_k} \in C_{j_1 \dots j_k}$  and therefore  $|L_k(f) - L_{k'}(f)| \leq \epsilon$  as long as both  $k, k' \geq k_0$ . It follows that  $L_k(f)$  converges for each  $f$  and the limit function defines a positive linear function on the space of continuous functions on  $C$ . By the Riesz Representation Theorem, there exists a Borel measure  $\mu$  such that

$$(76) \quad \int f d\mu = \lim_{k \rightarrow \infty} L_k(f)$$

for each continuous function  $f$ . Moreover,

$$(77) \quad L_k(f) = \sum_{j_1} r_{j_1}^s \sum_{j_2 \dots j_k} (r_{j_2} \dots r_{j_k})^s f(x_{j_1 \dots j_k})$$

$$(78) \quad = \sum_{j_1} r_{j_1}^s \sum_{j_2 \dots j_k} (r_{j_2} \dots r_{j_k})^s f(\phi(x_{j_2 \dots j_k}))$$

$$(79) \quad = \sum_j r_j^s L_{k-1}(f \circ \phi_j).$$

Letting  $k \rightarrow \infty$ , we obtain

$$(80) \quad \int f d\mu = \sum_j r_j^s \int f \circ \phi_j d\mu.$$

By the Monotone Convergence Theorem, it follows that this also holds for all non-negative functions, and so, letting  $f = \mathbb{1}_F$ , we obtain

$$(81) \quad \mu(F) = \int \mathbb{1}_F d\mu = \sum_j r_j^s \int \mathbb{1}_F \circ \phi_j d\mu = \sum_j r_j^s \mu(\phi_j^{-1}(F)).$$

Moreover, if  $f \equiv 1$ , then by equation (73), we have that  $\mu(\mathbb{R}^d) = 1$ . Finally,  $\mu$  has its support in  $C$ , because if  $f$  is any continuous function vanishing on  $C$ ,  $L_k(f) = 0$  for every  $k$ , and so  $\int f d\mu = 0$ .  $\square$

The *open set condition* means that there exists a bounded open set  $V$  such that

$$(82) \quad \phi(V) = \bigcup_{j=1}^l \phi_j(V) \subseteq V,$$

and this union is disjoint. Let  $\phi_{j_1} \circ \dots \circ \phi_{j_k} = \phi_{j_1 \dots j_k}$ , and  $V_{j_1 \dots j_k} = \phi_{j_1} \circ \dots \circ \phi_{j_k}(V)$ . Then by applying  $\phi_{j_1 \dots j_k}$ , we have that

$$(83) \quad \bigcup_{j=1}^l V_{j_1 \dots j_k, j} \subseteq V_{j_1 \dots j_k},$$

and this union is also disjoint. So the sets  $V_{j_1 \dots j_k}$  form a net in the sense that each one is disjoint from the other, contains or is contained in the other.

We also have that

$$(84) \quad C = \bigcap_{k=0}^{\infty} \phi^k(\bar{V}).$$

This follows because the  $\phi^k(\bar{V})$  are a decreasing sequence of non-empty compact sets and which converge to  $C$  in the Hausdorff metric by Theorem 4.5, which couldn't happen if  $C$  has points outside of  $\bar{V}$ . Moreover, if we apply  $\phi_{j_1} \circ \cdots \circ \phi_{j_k}$ , we get that  $C_{j_1 \dots j_k} \subseteq \bar{V}_{j_1 \dots j_k}$ .

It turns out that if you have the open set condition for the similitudes  $\{\phi_i\}$  with corresponding reduction ratios  $r_i$  such that  $\sum_{i=1}^l r_i^s = 1$ , then the set  $C$  is an  $s$ -set; that is,  $0 < \mathcal{H}^s(C) < \infty$ . In this case, we say that the self-similar measure  $\mu$  is associated with the  $s$ -dimensional Hausdorff measure. Moreover, if the open set condition holds for the similitudes, then it follows that  $C$  is self-similar, and as  $\mu$  concentrates its mass on a self-similar set, we call  $\mu$  a self-similar measure. So we end this section with establishing these final results. The proof of Theorem 4.8 is not only interesting in itself, but also because its method gave us an idea of how to deal with differing contraction ratios in Theorem 5.4. I first give an important geometric lemma.

LEMMA 4.7. *Let  $\{V_i\}$  be a collection of disjoint open subsets of  $\mathbb{R}^d$  such that  $V_i$  contains a ball of radius  $c_1\zeta$  and is contained in a ball of radius  $c_2\zeta$ . Then any ball  $B$  of radius  $\zeta$  intersects no more than  $\Gamma = (1 + 2c_2)^d c_1^{-d}$  of the sets  $\bar{V}_i$ .*

PROOF. Suppose that  $\bar{V}_i \cap B \neq \emptyset$ .  $B$  is of radius  $\zeta$  and since  $V_i$  is contained in a ball of radius  $c_2\zeta$ , it follows that  $\bar{V}_i$  is contained in a ball concentric with  $B$  and of radius  $(1 + 2c_2)\zeta$ . Suppose now that  $\Gamma$  of the  $\bar{V}_i$  meet  $B$ , then summing up the volumes of the corresponding balls of radius  $c_1\zeta$ , we get that  $\Gamma(c_1\zeta)^d \leq (1 + 2c_2)^d \zeta^d$ .  $\square$

THEOREM 4.8. *Suppose that the open set condition holds for the similitudes  $\phi_i$  with ratios  $r_i$  for  $1 \leq i \leq l$ . Then the associated compact invariant set  $C$  is an  $s$ -set, where  $s$  is determined by  $\sum_{i=1}^l r_i^s = 1$ .*

PROOF. Iterate (67) to obtain  $C = \cup_{j_1 \dots j_k} C_{j_1 \dots j_k}$ . By (73), we have that

$$(85) \quad \sum_{j_1 \dots j_k} |C_{j_1 \dots j_k}|^s = \sum_{j_1 \dots j_k} (r_{j_1} \cdots r_{j_k})^s |C|^s = |C|^s,$$

and since  $|C_{j_1 \dots j_k}| \leq \max r_i^s |C| \rightarrow 0$  as  $k \rightarrow \infty$ , it follows that  $\mathcal{H}^s(C) \leq |C|^s < \infty$ .

To show that  $0 < \mathcal{H}^s(C)$ , let  $V$  be the bounded open set given by the open set condition, and suppose that it contains a ball of radius  $c_1$  and is contained in a ball of radius  $c_2$ . Let  $\zeta > 0$ . For each infinite sequence  $\{j_1, j_2, \dots\}$ ,  $1 \leq j_i \leq l$ , truncate the sequence at the first  $k \geq 1$  for which  $(\min_j r_j)\zeta \leq r_{j_1}r_{j_2} \cdots r_{j_k} \leq \zeta$ , and let  $\mathcal{S}$  denote the set of finite sequences obtained in this manner. From an earlier remark, the collection  $\{V_{j_1 \dots j_k} : j_1 \dots j_k \in \mathcal{S}\}$  is a disjoint collection of open sets; each member containing a ball of radius  $c_1 r_{j_1} \cdots r_{j_k} \geq c_1 (\min_j r_j)\zeta$  and contained in a ball of radius  $c_2 \zeta$ . By Lemma 4.7, any ball  $B$  of radius  $\zeta$  intersects, at most  $\Gamma$  sets of the collection  $\{\bar{V}_{j_1 \dots j_k} : j_1 \dots j_k \in \mathcal{S}\}$ . Also note that  $\mu_{j_1 \dots j_k}(\mathbb{R}^d) = 1$  and that  $\mu_{j_1 \dots j_k}$  has its support in the set  $C_{j_1 \dots j_k} \subseteq \bar{V}_{j_1 \dots j_k}$ .

Define  $\mu_{j_1 \dots j_k}(F) = \mu((\phi_{j_1} \circ \cdots \circ \phi_{j_k})^{-1}(F)) = \mu(\phi_{j_k}^{-1} \circ \cdots \circ \phi_{j_1}^{-1}(F))$ . Then the measure  $\mu_{j_1 \dots j_k}$  is supported on  $C_{j_1 \dots j_k}$  and

$$(86) \quad \mu_{j_1 \dots j_k} = \sum_j r_j^s \mu_{j_1 \dots j_k j}.$$

By iterating (86) where appropriate, we get that

$$(87) \quad \mu = \sum_{j_1 \dots j_k \in \mathcal{S}} (r_{j_1 \dots j_k})^s \mu_{j_1 \dots j_k}.$$

Hence,  $\mu(B) \leq \sum (r_{j_1 \dots j_k})^s \mu_{j_1 \dots j_k}(\mathbb{R}^d)$ , where the sum is taken over those sequences in  $\mathcal{S}$  for which  $\bar{V}_{j_1 \dots j_k} \cap B \neq \emptyset$ . It follows that  $\mu(B) \leq \Gamma \zeta^s = \Gamma 2^{-s} |B|^s$  for any ball with  $|B| < |V|$ . However, given any cover  $\{U_i\}$  of  $C$ , we can cover  $C$  by balls of diameter less than or equal to  $2|U_i|$ , and so

$$(88) \quad 1 = \mu(C) \leq \sum \mu(B_i) \leq \Gamma 2^{-s} \sum |B_i|^s \leq \Gamma \sum |U_i|^s.$$

Since the cover  $\{U_i\}$  is arbitrary, it follows that  $\mathcal{H}^s(C) \geq 1/\Gamma > 0$ . □

**COROLLARY 4.9.** *If the open set condition holds, then  $\mathcal{H}^s(\phi_i(C) \cap \phi_j(C)) = 0$  for  $i \neq j$ , so the set  $C$  is self-similar.*

PROOF. Since the  $\phi_i$  are similitudes, we have that

$$(89) \quad \sum_{j=1}^l \mathcal{H}^s(\phi_j(C)) = \sum_{j=1}^l r_j^s \mathcal{H}^s(C) = \mathcal{H}^s(C),$$

Since  $0 < \mathcal{H}^s(C) < \infty$ , this can only happen if  $\mathcal{H}^s(\phi_i(C) \cap \phi_j(C)) = 0$  by (67) and the additive properties of Hausdorff measure. □

## CHAPTER 5

### SELF-SIMILAR MEASURES AND MYCIELSKI-REGULARITY

In this chapter,  $X$  is a convex and bounded subset of  $\mathbb{R}^d$ ,  $\rho$  is the Euclidean metric, and  $\mu$  is the self-similar measure which up to a constant is Hausdorff measure on the invariant set  $C$ . The maps  $\phi_i$  are similitudes and  $r_i < 1$  are the corresponding contraction ratios, where  $1 \leq i \leq l$ . I assume that the open set condition is satisfied (taking  $V$  as the interior of  $X$ ), so that the theorems in the previous chapter apply. First, I show that if the contraction ratios are the same for each map  $\phi_i$ , then  $\mu$  has moderated Voronoi tessellations.

**THEOREM 5.1.** *Let  $(X, \rho, \mu), \phi_i$  be as defined above, and such that  $r_i = r_j$  for each  $i, j$ . Then  $\mu$  has moderated Voronoi tessellations.*

**PROOF.** I use the basic argument of Fremlin, which he uses for Lebesgue measure; however, it is sufficiently remodeled as to be a new result; this is especially seen in that his proof really depends on the fact that Lebesgue measure is uniformly distributed over the unit cube. I do, however, use much of the same terminology and notation.

Let  $\epsilon > 0$ . As noted above, I assume that OSC is satisfied for the similitudes. Let  $J_1 = \phi_1(X)$ ,  $J_2 = \phi_2(X)$ ,  $\dots$ ,  $J_l = \phi_l(X)$ . In general, for  $\sigma \in \{1, 2, \dots, l\}^t$  (so  $\sigma = (j_1, \dots, j_t)$ , where  $1 \leq j_k \leq l$  and  $1 \leq k \leq t$ ), let  $J_\sigma = \phi_{j_1} \circ \dots \circ \phi_{j_t}(X)$ . Further, let  $\mathcal{J}_t = \{J_\sigma : \sigma \in \{1, 2, \dots, l\}^t\}$ . Then  $\text{Card}(\mathcal{J}_t) = l^t$ , and  $\mu(J_\sigma) = 1/l^t$ . Let  $M = 1 + \lceil \ln(10/\epsilon) \rceil$ . Also, for large  $n \in \mathbb{N}$ , choose  $t(n) \in \mathbb{N}$  such that

$$(90) \quad \log_l n - \log_l M \geq t(n) \geq \log_l n - \log_l M - 1 \geq \log_l(\log_l n) + \log_l 2 - \log_l(\log_l 2) - \log_l \epsilon/2.$$

Note that as  $n \rightarrow \infty$ , so does  $t(n)$ . Also note that usually I denote  $t(n)$  by  $t$ , unless there is a reason to specifically highlight its dependence on  $n$ .

Let  $z \in C$ , and let  $J_\sigma \in \mathcal{J}_t$  for  $\sigma \in \{1, \dots, l\}^t$ . Then the  $|J_\sigma| = r^t |X| = |X|/l^{t/s}$  (where  $r$  is the common contraction ratio, and  $s$  is the similarity dimension of the set  $C$ ), and so each  $J_\sigma$  is contained in a ball of radius  $|X|/2l^{t/s}$  and contains a ball of radius  $c_1/l^{t/s}$ , where  $c_1 < |X|/2$ . Then if  $E = B(z, |X|/l^{t/s})$ , by Lemma 4.7 there is a uniform bound (in  $z$  and  $t$ )  $\Gamma$  on the number of the  $J_\sigma$  that meet  $E$ . Let  $\mathcal{K}$  be the set of members of  $\mathcal{J}_t$  meeting  $E$ ; it follows that  $\text{Card}(\mathcal{K}) \leq \Gamma$ .

Let  $V \subseteq X$  be convex. Suppose that  $y \in (V \cap C) \setminus \cup \mathcal{K}$  such that  $y \in J_\sigma$  for  $J_\sigma \in \mathcal{J}_t$ . Since  $y \notin E$ , it follows that  $\rho(y, z) > |X|/l^{t/s}$ . Further, since  $|J_\sigma| = |X|/l^{t/s}$ , it follows that  $J_\sigma \subseteq \text{int } B(y, \rho(y, z))$ . Hence,  $y \in V_z = V \cap \bigcup_{y \in V \cap C} \{J \in \mathcal{J}_t : J \subseteq \text{int } B(y, \rho(y, z))\}$ . Accordingly,  $(V \cap C) \setminus V_z \subseteq \cup \mathcal{K}$  is covered by at most  $\Gamma$  members of  $\mathcal{J}_t$  and so  $V \setminus V_z$  has measure at most  $\Gamma/l^t$ . (Note that  $\mu((V \cap C) \setminus V_z) = \mu(V \setminus V_z)$  since the measure of the complement of  $C$  has measure 0.)

Recall that the Voronoi tiles are convex. Let  $\omega \in \Omega$  and let

$$(91) \quad H_n(\omega) = \bigcup \{V(\omega \upharpoonright n, z) : z \in \omega[n], \mu V(\omega \upharpoonright n, z) \geq 2\Gamma l^{-t}\}.$$

Let

$$(92) \quad \mathcal{K}_\omega = \{J_\sigma \in \mathcal{J}_t : J_\sigma \cap \omega[n] = \emptyset\},$$

Suppose that  $\mu(V(\omega \upharpoonright n, z)) \geq 2\Gamma l^{-t}$ . It follows that if  $y \in V(\omega \upharpoonright n, z)$ , ( $y \neq z$ ), then  $\text{int } B(y, \rho(y, z)) \cap \omega[n] = \emptyset$  and if  $J_\sigma \subseteq \text{int } B(y, \rho(y, z))$  then  $J_\sigma \in \mathcal{K}_\omega$ . By above, we have that  $\mu(V(\omega \upharpoonright n, z) \setminus \bigcup \mathcal{K}_\omega) \leq \Gamma l^{-t}$  and so  $\mu(V(\omega \upharpoonright n, z)) \leq 2\mu(V(\omega \upharpoonright n, z) \cap \bigcup \mathcal{K}_\omega)$ .

Hence,

$$(93) \quad \mu H_n(\omega) \leq 2\mu(H_n(\omega) \cap \bigcup \mathcal{K}_\omega)$$

$$(94) \quad \leq 2l^{-t} \cdot \text{Card}(\mathcal{K}_\omega).$$



It follows that if  $\mu H_n(\omega) \geq \epsilon$  then  $\text{Card}(\mathcal{K}_\omega) \geq \frac{\epsilon l^t}{2} \geq m$ , where  $m = \lfloor \frac{\epsilon l^t}{2} \rfloor$ . Therefore,  $\{\omega \in \Omega : \mu H_n(\omega) \geq \epsilon\} \subseteq \{\omega \in \Omega : \text{Card}(\mathcal{K}_\omega) \geq m\}$ . Also,

$$(95) \quad \lambda\{\omega \in \Omega : \text{Card}(\mathcal{K}_\omega) \geq m\} \leq \sum_{\mathcal{K} \in [\mathcal{J}]^m} \lambda\{\omega : \omega[n] \text{ does not meet } \cup \mathcal{K}\},$$

where  $[\mathcal{J}]^m = \{\mathcal{K} \subseteq \mathcal{J}_t : \text{Card}(\mathcal{K}) = m\}$ , and so

$$(96) \quad \lambda\{\omega \in \Omega : \text{Card}(\mathcal{K}) \geq m\} \leq \frac{l^{tm}}{m!} (1 - ml^{-t})^n$$

$$(97) \quad \leq \frac{l^{tm}}{m!} (1 - ml^{-t})^{Ml^t} \quad (\text{since } Ml^t \leq n)$$

$$(98) \quad \leq \frac{e^m l^{tm}}{m^m} (1 - l^{-t})^{Mml^t} \quad (\text{since } 1 - ml^{-t} \leq (1 - l^{-t})^m)$$

$$(99) \quad \leq \frac{e^m l^{tm}}{m^m} \left(\frac{1}{e}\right)^{Mm} \quad (\text{since } (1 - x)^{1/x} \leq 1/e \text{ if } x > 0)$$

$$(100) \quad = \left(\frac{el^t}{me^M}\right)^m$$

$$(101) \quad = \left(\frac{el^t}{\lfloor \frac{\epsilon l^t}{2} \rfloor e^M}\right)^m$$

$$(102) \quad \leq \left(\frac{5e}{\epsilon e^M}\right)^m$$

$$(103) \quad \leq \frac{1}{2^m} \quad (\text{by choice of } M)$$

$$(104) \quad \leq \frac{1}{2^{\epsilon l^t / 2}}.$$

By (90), it follows that for  $n$  sufficiently large,

$$(105) \quad \left(\frac{1}{2}\right)^{\epsilon l^t / 2} \leq \frac{1}{n^2}.$$

Let  $n_0$  be such that for all  $n \geq n_0$ , (90) holds. Then,

$$(106) \quad \sum_{n=1}^{\infty} \lambda\{\omega : \mu(H_n(\omega)) \geq \epsilon\} \leq n_0 + \sum_{n=n_0}^{\infty} \lambda\{\omega : \mu H_n(\omega) \geq \epsilon\}$$

$$(107) \quad \leq n_0 + \sum_{n=n_0}^{\infty} \frac{1}{n^2} < \infty.$$

Of course, the definition of moderated Voronoi tessellations is that for every  $\epsilon > 0$  there exists  $M \geq 0$  such that

$$(108) \quad \sum_{n=1}^{\infty} \lambda\{\omega : \mu\left(\bigcup\{V(\omega \upharpoonright n, z) : z \in \omega[n], \mu(V(\omega \upharpoonright n, z)) \geq M/n\}\right) \geq \epsilon\} < \infty,$$

but this is true if and only if

$$(109) \quad \sum_{n=1}^{\infty} \lambda\{\omega : \mu\left(\bigcup\{V(\omega \upharpoonright n, z) : z \in \omega[n], \mu(V(\omega \upharpoonright n, z)) \geq 2\Gamma l^{-t}\}\right) \geq \epsilon\} < \infty.$$

Note that if we call

$$(110) \quad A_n = \{\omega : \mu\left(\bigcup\{V(\omega \upharpoonright n, z) : z \in \omega[n], \mu(V(\omega \upharpoonright n, z)) \geq M/n\}\right) \geq \epsilon\}$$

and

$$(111) \quad B_{t(n)} = \{\omega : \mu\left(\bigcup\{V(\omega \upharpoonright n, z) : z \in \omega[n], \mu(V(\omega \upharpoonright n, z)) \geq 2\Gamma l^{-t(n)}\}\right) \geq \epsilon\},$$

then for every  $n \in \mathbb{N}$ , we have that  $B_{t(n)} \subseteq A_n$  since  $M/n \leq 2\Gamma l^{-t(n)}$  by (90), and it follows that if  $\sum \lambda(A_n) < \infty$ , then  $\sum \lambda(B_{t(n)}) < \infty$ .

Now suppose that  $\sum \lambda(B_{t(n)}) < \infty$ . By (90), we also have that  $A_n \subseteq B_{t(n)}$  since  $2\Gamma M l/n \geq 2\Gamma l^{-t(n)}$  (let  $M$  in (110) be replaced by  $M' = 2\Gamma M l$ ). It follows that  $\sum \lambda(A_n) < \infty$ . Hence,  $\mu$  has moderated Voronoi tessellations.  $\square$

**COROLLARY 5.2.** *The measure  $\mu$  as defined above is Mycielski-regular.*

**EXAMPLE 5.3.** Let  $(X, \rho, \mu)$  be defined as follows: Let  $X$  be the two-dimensional unit square, let  $\rho$  be the Euclidean metric, and let  $\mu$  be the self-similar measure described above and which concentrates its mass on the two-dimensional Cantor set (with similarity dimension = Hausdorff dimension =  $\ln 4 / \ln 3$ ). In this case, the similitudes are  $\phi_1(x, y) = (x/3, y/3)$ ,  $\phi_2(x, y) = ((x+2)/3, y/3)$ ,  $\phi_3(x, y) = (x/3, (y+2)/3)$ , and  $\phi_4(x, y) = ((x+2)/3, (y+2)/3)$ . Note that the contraction ratios are obviously all the same and are equal to  $1/3$ . As the above theorem implies, this measure is Mycielski-regular.

I will now do the case for which the contraction ratios are different. This theorem, our most general result, will show that any self-similar measure which is Hausdorff measure (up to a constant) on an invariant set is Mycielski-regular.

**THEOREM 5.4.** *Let  $(X, \rho, \mu)$ ,  $\phi_i$  and  $r_i$  be as defined above. Then  $\mu$  has moderated Voronoi tessellations.*

**PROOF.** Suppose that the open set  $V$  given by the OSC contains a ball of radius  $c_1$  and is contained in a ball of radius  $c_2$ . Let  $\zeta = 1/l^{t/s}$ , let  $\beta = \min r_j$ , and  $\mathcal{S}_t$  be the set of finite sequences obtained in the following way: for each infinite sequence  $\{j_1, j_2, \dots\}$ ,  $1 \leq j_i \leq l$ , truncate the sequence at the first  $k \geq 1$  for which  $\beta\zeta \leq r_{j_1}r_{j_2} \cdots r_{j_k} \leq \zeta$ . It follows from the net property of the open sets that  $\{V_{j_1 \dots j_k} : j_1 \dots j_k \in \mathcal{S}_t\}$  is a disjoint collection. Each such  $V_{j_1 \dots j_k}$  contains a ball of radius  $c_1 r_{j_1} \cdots r_{j_k}$  and hence a ball of radius  $c_1 \beta \zeta$  and similarly is contained in a ball of radius  $c_2 \zeta$ . By Lemma 4.7, any ball of radius  $\zeta$  intersects, at most,  $(1 + 2c_2)^d c_1^{-d} (\min r_j)^{-d}$  sets of the collection  $\mathcal{V}_t = \{\bar{V}_{j_1 \dots j_k} : j_1 \dots j_k \in \mathcal{S}_t\}$ . Also, by (87) we have that

$$(112) \quad \mu = \sum_{j_1 \dots j_k \in \mathcal{S}_t} (r_{j_1} \cdots r_{j_k})^s \mu_{j_1 \dots j_k} \leq \sum_{j_1 \dots j_k \in \mathcal{S}_t} (r_{j_1} \cdots r_{j_k})^s.$$

Now let  $z \in C$  and let  $E = B(z, 2c_2\zeta)$ . Then  $E$  intersects at most  $\Gamma = (6c_2)^d c_1^{-d} (\min r_j)^{-d}$  members of  $\mathcal{V}_t$ . Let  $\mathcal{K} = \{\bar{V}_\sigma \in \mathcal{V}_t : \bar{V}_\sigma \cap E \neq \emptyset\}$  (where  $\sigma \in \{1, 2, \dots, l\}^k$  for some  $k$ ). So  $\text{Card}(\mathcal{K}) \leq \Gamma$ .

Let  $W \subseteq X$  be convex and let  $y \in W \setminus \cup \mathcal{K}$ ; suppose that  $y \in \bar{V}_\sigma$ ,  $\sigma \in \mathcal{S}_t$  for some  $\bar{V}_\sigma \notin \mathcal{K}$ . Then  $\bar{V}_\sigma \subseteq \text{int} B(y, \rho(y, z))$ , since  $\rho(y, z) > 2c_2\zeta$ . So  $y \in V_z = W \cap \cup_{y \in W} \{\bar{V}_\sigma : \bar{V}_\sigma \subseteq \text{int} B(y, \rho(y, z))\}$ . So  $W \setminus V_z \subseteq \cup \mathcal{K}$ .

**CLAIM 5.5.**  $\cup \mathcal{K}$  has measure at most  $\Gamma \zeta^s = \Gamma/l^t$  (and hence so does  $W \setminus V_z$ ).

**PROOF.** To see this, let  $\bar{V}_\sigma \in \mathcal{K}$  and let  $\sigma \in \mathcal{S}_t$  be such that  $\sigma = (a_1, a_2, \dots, a_m)$ , and so

$$(113) \quad \mu(V_\sigma) = \sum_{j_1 \dots j_k \in \mathcal{S}_t} (r_{j_1} \cdots r_{j_k})^s \mu_{j_1 \dots j_k}(V_\sigma)$$

$$(114) \quad = \sum_{j_1 \dots j_k \in \mathcal{S}_t} (r_{j_1} \dots r_{j_k})^s \mu(\phi_{j_k}^{-1} \circ \dots \circ \phi_{j_1}^{-1}(\phi_{a_1} \circ \dots \circ \phi_{a_m}(V)))$$

$$(115) \quad = (r_{a_1} \dots r_{a_m})^s$$

$$(116) \quad \leq \zeta^s,$$

since  $(a_1, a_2, \dots, a_m) \in \mathcal{S}_t$  and since  $\mu(\phi_{a_m}^{-1} \circ \dots \circ \phi_{a_1}^{-1}(\phi_{a_1} \circ \dots \circ \phi_{a_m}(V))) = 1$ , whereas every other term in the sum has measure zero, since  $\phi_i^{-1} \circ \phi_j(F) = \emptyset$  when  $i \neq j$  and for any  $F \subseteq X$ . This shows the claim.  $\square$

Let  $\omega \in \Omega$ . Define  $H_n(\omega)$  as before and let

$$(117) \quad \mathcal{K}_\omega = \{V_\sigma \in \mathcal{S}_t : V_\sigma \cap \omega[n] = \emptyset\}.$$

Let  $\epsilon > 0$ . By an argument identical to the one I gave in Theorem 5.1, we get

$$(118) \quad \mu H_n(\omega) \leq \frac{2}{l^t} \cdot \text{Card}(\mathcal{K}_\omega),$$

and so if  $H_n(\omega) \geq \epsilon$ , then

$$(119) \quad \text{Card}(\mathcal{K}_\omega) \geq \frac{\epsilon l^t}{2} \geq m,$$

where  $m = \lfloor \frac{\epsilon l^t}{2} \rfloor$ . Again, as before, we get that  $\{\omega \in \Omega : \mu H_n(\omega) \geq \epsilon\} \subseteq \{\omega \in \Omega : \text{Card}(\mathcal{K}) \geq m\}$  and so

$$(120) \quad \lambda\{\omega \in \Omega : \text{Card}(\mathcal{K}) \geq m\} \leq \sum_{\mathcal{K} \in [\mathcal{V}_t]^m} \lambda\{\omega : \omega[n] \text{ does not meet } \cup \mathcal{K}\},$$

CLAIM 5.6.  $\text{Card } \mathcal{K} \leq l^t / \beta^s$ .

PROOF. This follows because

$$(121) \quad 1 \geq \mu(\cup \mathcal{K}) = \sum_{V_\sigma \in \mathcal{K}} \mu(V_\sigma) \geq \sum_{V_\sigma \in \mathcal{K}} (\beta \zeta)^s \geq \sum_{V_\sigma \in \mathcal{K}} \frac{\beta^s}{l^t} = \text{Card } \mathcal{K} \frac{\beta^s}{l^t},$$

and so

$$(122) \quad \text{Card } \mathcal{K} \leq \frac{l^t}{\beta^s}.$$

$\square$

Let

$$(123) \quad M = \frac{1 + \ln(10 \lceil \beta^{-s} \rceil / \epsilon)}{\beta}.$$

If we require that  $Ml^{t/s} \leq n$  and that

$$(124) \quad \frac{1}{2^{\epsilon l^{t/2}}} \leq \frac{1}{n^2},$$

(which, by (90), is true for  $n$  sufficiently large) then we get (the justifications for the inequalities are essentially the same as those given for Theorem 5.1):

$$(125) \quad \lambda\{\omega \in \Omega : \text{Card}(\mathcal{K}) \geq m\} \leq \binom{\lceil l^t / \beta^s \rceil}{m} \left(1 - \frac{m\beta}{l^{t/s}}\right)^n$$

$$(126) \quad \leq \frac{\left(\lceil \frac{l^t}{\beta^s} \rceil\right)^m}{m!} \left(1 - \frac{m\beta}{l^{t/s}}\right)^n$$

$$(127) \quad \leq \frac{\lceil \frac{l^t}{\beta^s} \rceil^m}{m!} \left(1 - \frac{m\beta}{l^{t/s}}\right)^{Ml^{t/s}}$$

$$(128) \quad \leq \frac{e^m \lceil \beta^{-s} l^t \rceil^m}{m^m} \left(1 - \frac{1}{l^{t/s}}\right)^{Mm\beta l^{t/s}}$$

$$(129) \quad \leq \frac{e^m \lceil \beta^{-s} l^t \rceil^m}{m^m} \left(\frac{1}{e}\right)^{Mm\beta}$$

$$(130) \quad = \left(\frac{e \lceil \beta^{-s} l^t \rceil}{m e^{M\beta}}\right)^m$$

$$(131) \quad = \left(\frac{e \lceil \beta^{-s} l^t \rceil}{\lceil \frac{\epsilon l^t}{2} \rceil e^{M\beta}}\right)^m$$

$$(132) \quad \leq \frac{1}{2^m}$$

$$(133) \quad \leq \frac{1}{2^{\epsilon l^{t/2}}}.$$

The proof is finished in exactly the same way as Theorem 5.1, taking  $n$  sufficiently large. It follows that  $\mu$  has moderated Voronoi tessellations.  $\square$

**COROLLARY 5.7.** *The measure  $\mu$  given in Theorem 5 is Mycielski-regular.*

### 5.1. Some Other Measures

Other measures can be similarly shown to be Mycielski-regular. For example, in the paper *Multifractal Decompositions of Moran Fractals* by Mauldin and Cawley [1], multifractal decompositions of the invariant set  $C$  are characterized in terms of the local behavior of a probability measure induced on  $C$  by a product measure on the coding space,  $\Theta = \{1, \dots, l\}^{\mathbb{N}}$ . This measure is defined in the following way.

For each  $\sigma \in \Theta$ , and  $k \in \mathbb{N}$ , let  $\sigma|k = (\sigma(1), \dots, \sigma(k))$ , and let  $g$  be the natural coding map of  $\Theta$  onto  $C$  defined by the condition

$$(134) \quad \{g(\sigma)\} = \bigcap_{k=1}^{\infty} J(\sigma|k),$$

( $J(\sigma|k)$  is defined similarly to the way  $J_\sigma$  was defined in the proof of Theorem 5.1).

Fix a probability vector  $(p_1, \dots, p_l)$  with each  $p_i$  positive and let  $\hat{\rho}$  be the corresponding infinite product measure on  $\Theta$ . Let  $\rho$  be the image measure on  $C$  induced by  $g$ ; that is, for  $E \subseteq X$ ,  $\rho(E) = \hat{\rho}(g^{-1}(E))$ . Then  $\rho$  can be shown to be Mycielski-regular using virtually the same proof as given in Theorem 5.4, with some modifications. In this case, let  $\zeta = 1/l^t$  and let  $\mathcal{S}_t$  be the collection of those sequences  $\sigma|k$  which are truncated at the first  $k$  such that, if  $\beta = \min p_i$ , then

$$(135) \quad \beta\zeta \leq p_{i_1} \cdots p_{i_k} \leq \zeta.$$

Each  $V_\sigma$  (with  $\sigma \in \mathcal{S}_t$ ) would have measure at most  $\zeta$ . With these modifications, one can proceed as in the proof of Theorem 5.4.

Another measure that we meet with in the same paper are probability measures  $\mu_q$  supported on a fractal  $K_{\alpha(q)}$  (see [1]). Here, these are the image under the coding map of the infinite product measure  $\hat{\mu}_q$ , on  $\Theta$ , and based on the probability vector  $(p_1^q t_1^{\beta(q)}, \dots, p_l^q t_l^{\beta(q)})$ , where  $q$  is a real number, and  $\beta(q)$  is the unique number such that

$$(136) \quad \sum_{i=1}^l p_i^q t_i^{\beta(q)} = 1.$$

The  $t_i$  are contraction ratios corresponding to contractions  $T_i$  and the  $p_i$  are as defined above. But again, the same trick works: if we let this time  $\beta = \min p_i^q t_i^{\beta(q)}$ , and  $\zeta = 1/l^t$ , and look at those sequences such that

$$(137) \quad \beta\zeta \leq p_{i_1}^q t_{i_1}^{\beta(q)} \cdots p_{i_k}^q t_{i_k}^{\beta(q)} \leq \zeta,$$

then the proof of Theorem 5.4 shows that such measures are also Mycielski-regular.

## 5.2. Future Research Possibilities

The question as to which measures are Mycielski-regular was posed for Radon probability measures. Fremlin proved it for Lebesgue measure on the unit cube. Actually, for the one-dimensional case, he has proved it for all Radon probability measures [4]. In this paper, I have shown that it is true for those self-similar measures which correspond to Hausdorff measure on an invariant subset of the unit cube. I believe it is possible, and would like to know, if it is also in fact true for conformal measures. That is, if  $T$  is a measurable endomorphism on  $X$ , and  $f$  is a non-negative function on  $X$ , then  $\mu$  is called  $f$ -conformal (see [2]) if

$$(138) \quad \mu \circ T(A) = \int_A f(x) d\mu(x),$$

where  $A$  is a measurable subset of  $X$ ,  $T(A)$  is measurable and  $T : A \rightarrow T(A)$  is invertible.

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