STRONG CHOQUET TOPOLOGIES ON THE CLOSED LINEAR SUBSPACES OF BANACH SPACES

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In the study of Banach spaces, the development of some key properties require studying topologies on the collection of closed convex subsets of the space. The subcollection of closed linear subspaces is studied under the relative slice topology, as well as a class of topologies similar thereto. It is shown that the collection of closed linear subspaces under the slice topology is homeomorphic to the collection of their respective intersections with the closed unit ball, under the natural mapping. It is further shown that this collection under any topology in the aforementioned class of similar topologies is a strong Choquet space. Finally, a collection of category results are developed since strong Choquet spaces are also Baire spaces.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1. Discussion of the Problems, Methods and Previous Results

Three main points exist that led to the developments in this work. The first was a look at the Wijsman topology on the hyperspace of closed subsets of a metric space, and an attempt to develop a homeomorphic mapping from the closed linear subspaces of a Banach space to their respective intersections with the closed unit ball. This proved difficult and did not seem possible after some work. The second was an attempt to revive the usefulness of the Wijsman topology, namely the numerous instances in which this topology produces a Polish space, by using a similar topology called the slice topology. As proven in Beer's *Topologies on Closed and Closed Convex Sets* [2, p. 75], the slice topology remains relevant with regard to producing Polish spaces as every Banach space which has a separable dual will admit a Polish slice topology on its collection of nonvoid closed convex subsets. As a consequence, the collection of closed linear subspaces in such a Banach space would be Polish under the relative slice topology, as would the collection of their respective intersections with the closed unit ball. Ultimately, this topology serves the purpose of being conducive to the development of the aforementioned homeomorphic mapping, as is seen in Chapter 2. Another useful result was given by László Zsilinszky in the article "Polishness of the Wijsman Topology Revisited" [8]. It is the proof offered by Zsilinszky that provides the motivation for developing the lemmas and theorems leading up to the proof that the slice topology on the collection of closed linear subspaces of a Banach space is strong Choquet. In particular, the method of columnization, which is pursued by Zsilinszky before he obtains the collection of points whose closure becomes his set proving that Player NONEMPTY can win by his strategy, is crucial to the development of an appropriate strategy in the strong Choquet proof offered in Chapter

4. Moreover, Zsilinszky follows this result with a statement indicating that, in the case of the Wijsman topology on the collection of nonvoid closed subsets of a completely metrizable space, the strong Choquet property that is developed for this collection is not hereditary in all instances. This gives a need to redevelop the result for any closed (in the Wijsman topology) proper subcollection of nonvoid closed subspaces of a completely metrizable space. Hence, development of the proof in Chapter 4 is still pertinent, even though the collection of nonvoid closed convex subsets of a Banach space under the slice topology is endowed with the supremum of the Wijsman topologies on this collection associated to each equivalent renorm. Even this overall collection would need a proof to show that it is strong Choquet, if it indeed is, as it does not inherit this property from anything previously developed. As these three previous results are fundamental to understanding the present work, they shall be restated herein.

1.1.1. Wijsman Topologies

Definition 1.1. The collection $\mathrm{CL}(X)$ of nonvoid closed subsets of a metric space, X, with metric d carries the Wijsman topology provided it has a subbasis consisting of sets of two forms. The first form is $U^- = \{A \in \mathrm{CL}(X) : A \cap U \neq \emptyset\}$, where U is an open subset of X under the topology induced by d. The second form is $(X \setminus B(x,\epsilon))^+ = \{A \in \mathrm{CL}(X) : A \subset (X \setminus B(x,\epsilon))\}$, where $B(x,\epsilon) = \{y \in X : d(x,y) \leq \epsilon\}$ is the closed ϵ ball centered at x.

A note on this topology is now warranted. Beer, in his article "A Polish Topology for the Closed Subsets of a Polish Space" [1, p. 1125-1126], mentions that this topology is generally finer than the Fell topology, but coarser than both the Vietoris topology and the Hausdorff metric topology. He further gives reasons why these other topologies are too restrictive to prove fruitful in developing Polish hyperspaces beyond a quite limited pool of underlying spaces. However, he shows the beginnings of why the Wijsman topology proves fruitful in making a Polish hyperspace for any Polish space X.

Theorem 1.2. *$\langle CL(X), \tau_W \rangle$ is Polish when X is Polish.*

Unfortunately, as later authors note, this theorem, while ultimately true, was not quite proven as stated by Beer. Zsilinszky revisits this claim in his "Polishness of the Wijsman Topology Revisited" [8]. In this article, the theorem is proven as stated, but the method towards this takes on an interesting turn that we shall later take up. The main thing to note, however, is that the desire to develop a homeomorphic map ϕ defined by $\phi(A) = A \cap \{x \in X : \|x\| \leq 1\}$ for every closed linear subspace A in a Banach space X would have proven quite fruitful if it could have been realized under the Wijsman topology. Indeed, if this goal was realizable under this topology, we would have a large class of Banach spaces for which our collection of closed linear subspaces would be homeomorphic to our collection of their respective intersections with the closed unit ball and for which both of these were Polish spaces. Unfortunately, it is quite difficult, if not impossible, to establish that such a mapping is homeomorphic with the Wijsman topology. The main problem is the lack of a positive distance between the closed balls used and the closed linear subspace, which makes constructing an appropriate strategy for Player NONEMPTY in a strong Choquet game quite difficult and unnatural (not flowing naturally from our definitions and such). The other is the lack of metric independence in the topology, which makes the notion of convergence in the Wijsman topology vary somewhat from the natural idea of convergence one hopes is inherited from the underlying space. This naturally led to the seeking of a topology which preserved most, if not all, the desirability of the Wijsman topology, while achieving norm independence and also achieving the positive distance from the closed, bounded, convex subsets that are to be avoided by closed linear subspaces in the topology. The topology one gravitates towards at this point is the slice topology.

CHAPTER 2

THE FAMILY OF SLICE-LIKE TOPOLOGIES AND AN IMPORTANT HOMEOMORPHISM

2.1. Definition of Slice-Like Topologies and Preliminary Consequences for the Collection of Closed Linear Subspaces

2.1.1. Definitions

Definition 2.1. [2, p. 61] Given a normed linear space $(X, \|\cdot\|)$, we define $\mathrm{C}(X)$ to be the closed convex nonvoid subsets of X, $\mathrm{CB}(X)$ to be the bounded closed convex nonvoid subsets of X, and the slice topology on $\mathrm{C}(X)$ to be the topology generated by the subbasis composed of sets that are either of the form $V^- = \{A \in \mathrm{C}(X) : A \cap V \neq \emptyset\}$, where V is an open subset of X, or of the form

$$(X \setminus B)^{++} = \{A \in \mathrm{C}(X) : \exists \epsilon_{A,B} > 0, \{x \in X : d(x, A) < \epsilon_{A,B}\} \subseteq X \setminus B\},$$

where $B \in \mathrm{CB}(X)$. We define $S_\epsilon[A] = \{x \in X : d(x, A) < \epsilon\}$ and we define $\overline{S_\epsilon}[A] = \{x \in X : d(x, A) \leq \epsilon\}$ as a matter of convenience. We also define $X_\alpha = \{x \in X : \|x\| \leq \alpha\}$ and $B_\alpha = \{x \in X : \|x\| < \alpha\}$ when $\alpha > 0$ for the closed α-ball centered at 0_X and the open α-ball centered at 0_X, respectively.

Definition 2.2. Given a normed linear space $(X, \|\cdot\|)$, we shall first define $\mathcal{C} \subseteq \mathrm{CB}(X)$ to be a collection of subsets of X which contains all singleton sets and for which every time $B \in \mathcal{C}$ and $\epsilon > 0$, the closed convex hull of $\overline{S_\epsilon}[B]$ is also in $\mathcal{C}$. Then, we define a whole collection $\mathcal{T}$ of topologies on $\mathrm{C}(X)$ in similar fashion as the slice topology. Namely, $\mathcal{T}$ consists of all topologies $\tau_{\mathcal{C}}$ generated by a subbasis consisting of sets of the form V^-, V open in X, and of the form $(X \setminus B)^{++}$, $B \in \mathcal{C}$ for some $\mathcal{C}$ as defined above. We note that the slice topology is a member of $\mathcal{T}$ as $\tau_S = \tau_{\mathrm{CB}(X)}$.

Definition 2.3. Given a normed linear space $(X, \|\cdot\|)$, we define $\mathrm{CLS}(X)$ to be the collection of closed linear subspaces of X. When using $\mathrm{CLS}(X)$ with respect to the slice topology on $\mathrm{C}(X)$, we shall say that $\mathrm{CLS}(X)$ is given the slice topology provided it is given the topology inherited as a topological subspace of $\mathrm{C}(X)$. More generally, given a topology $\tau \in \mathcal{T}$, we shall say that $\mathrm{CLS}(X)$ is given the topology τ provided it is given the topology inherited as a topological subspace of $(\mathrm{C}(X), \tau)$.

2.1.2. Preliminary Consequences

Lemma 2.4. *$(CLS(X), \tau_{\mathcal{C}})$ is Hausdorff whenever $\tau_{\mathcal{C}} \in \mathcal{T}$.*

Proof. Let $A_1, A_2 \in \mathrm{CLS}(X)$, with $A_1 \neq A_2$. Suppose for a moment that either $A_1 \subset A_2$ or that neither A_1 nor A_2 are contained in one another. Then, there exists $x \in A_2 \setminus A_1$. We know that $d(x, A_1) > 0$. So, $A_1 \in \left(X \setminus \overline{B(x, d(x, A_1)/2)}\right)^{++}$ and $A_2 \in B(x, d(x, A_1)/2)^{-}$. However, $\left(X \setminus \overline{B(x, d(x, A_1)/2)}\right)^{++} \cap B(x, d(x, A_1)/2)^{-} = \emptyset$. Now, if we instead supposed $A_2 \subset A_1$, we would find $A_2 \in \left(X \setminus \overline{B(x, d(x, A_2)/2)}\right)^{++}$, $A_1 \in B(x, d(x, A_2)/2)^{-}$, and $\left(X \setminus \overline{B(x, d(x, A_2)/2)}\right)^{++} \cap B(x, d(x, A_2)/2)^{-} = \emptyset$. In any event, we now see that $(\mathrm{CLS}(X), \tau_{\mathcal{C}})$ is Hausdorff whenever $\tau_{\mathcal{C}} \in \mathcal{T}$. □

Lemma 2.5. *Given a closed linear subspace Y of X, $CLS(Y)$ is closed in $CLS(X)$ under any $\tau \in \mathcal{T}$.*

Proof. Since $\mathrm{CLS}(X)$ is closed within itself, we let $Y \in \mathrm{CLS}(X) \setminus \{X\}$. Notice that $\mathrm{CLS}(X) \setminus \mathrm{CLS}(Y) = \bigcup_{x \in (X \setminus Y),\ \|x\|=1} B(x, d(x, Y)/2)^{-}$ is open in τ whenever $\tau \in \mathcal{T}$. Therefore, $\mathrm{CLS}(Y)$ is a proper, nonempty, closed subset of $\mathrm{CLS}(X)$ under any such topology τ. □

Lemma 2.6. *Given a closed linear subspace Y of X, the collection $\mathcal{A} = \{A \in CLS(X) : Y \subset A\}$ is closed in $CLS(X)$ under any $\tau \in \mathcal{T}$.*

Proof. First, if Y is just the origin, then $\mathcal{A} = \mathrm{CLS}(X)$ is closed. Let us suppose that Y is at least one dimensional. Let $Z = \{y \in Y : \|y\| = 1\}$. Then, $\mathrm{CLS}(X) \backslash \mathcal{A} = \bigcup_{y \in Z} (X \setminus \{y\})^{++}$. Therefore, $\mathcal{A}$ is closed in $\mathrm{CLS}(X)$ under any $\tau \in \mathcal{T}$. □

Lemma 2.7. *The collection of finite dimensional linear subspaces of a Banach Space X are dense in $CLS(X)$ under the slice topology.*

Proof. Let $\mathbf{V} = \bigcap_{i=1}^{k} V_i^- \cap \bigcap_{j=1}^{m} (X \setminus B_j)^{++}$ be a nonempty basic open subset of $\mathrm{CLS}(X)$ in the slice topology. There exists a closed linear subspace $A \in \mathbf{V}$. Now, if $k = 0$, we merely choose $x \in A$ and let $F = \mathrm{span}(\{x\})$. On the other hand, if $k \neq 0$, we choose $x_i \in A \cap V_i$ for all $1 \leq i \leq k$ and we let $F = \mathrm{span}(\{x_1, \ldots, x_k\})$. In either case, we quickly note that $d(F, B_j) \geq d(A, B_j) > \epsilon(A, B_j) > 0$ for all $1 \leq j \leq m$ as a consequence of the fact that $F \subseteq A$. Furthermore, F was chosen so that $F \in \bigcap_{i=1}^{k} V_i^-$. Taking these two facts, we conclude that $F \in \mathbf{V}$ and $\dim(F) = k < +\infty$. Thus, the collection of finite dimensional linear subspaces of X is dense in $\mathrm{CLS}(X)$ under the slice topology. □

Lemma 2.8. *For each separable $S \in CLS(X)$, there exists a sequence $\{F_n\}_{n=0}^{+\infty} \subset CLS(X)$ such that $S = \lim_{n \to +\infty} F_n$ under slice topological limits.*

Proof. As S is separable, there exists a countable set $\{s_n\}_{n=0}^{+\infty} \subset S$ such that $S = \overline{\{s_n\}_{n=0}^{+\infty}}$. We shall let $F_n = \mathrm{span}(\{s_i\}_{i=0}^{n})$ for each $n \geq 0$. We quickly see that $S = \overline{\bigcup_{n=0}^{+\infty} F_n}$ and that both $F_n \in \mathrm{CLS}(X)$ and $\dim(F_n) = n < +\infty$ for all $n \geq 0$. Let $\mathbf{V} = \bigcap_{i=1}^{k} V_i^- \cap \bigcap_{j=1}^{m} (X \setminus B_j)^{++}$ be a basic open subset of $\mathrm{CLS}(X)$ that contains S. We quickly note that $F_n \subset S$ for each $n \geq 0$, consequently $F_n \in \bigcap_{j=1}^{m} (X \setminus B_j)^{++}$ for each $n \geq 0$. Therefore, we may restrict our focus to $\bigcap_{i=1}^{k} V_i^-$. For each $1 \leq i \leq k$, we choose $y_i \in V_i \cap S$ and we find $0 < \epsilon_i$ which satisfies $B(y_i, \epsilon_i) \subseteq V_i$. For each $1 \leq i \leq k$, there exists $n_i \geq 0$ such that $s_{n_i} \in B(y_i, \epsilon_i) \subseteq V_i$. If we let $m_0 = \max\{n_1, \ldots, n_k\}$, then $F_m \in \mathbf{V}$ for all $m \geq m_0$. Hence, $S = \lim_{n \to +\infty} F_n$ under slice topological limits. □

Lemma 2.9. *The collection of separable closed linear subspaces of a Banach space X is sequentially closed under slice topological limits.*

Proof. Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of separable closed linear subspaces that converges under slice topological limits with $\lim_{n\to\infty} S_n = A \in \mathrm{CLS}(X)$. If X is separable, there is nothing to show since A would also be separable and the collection of separable closed linear subspaces of X would have been the entirety of $\mathrm{CLS}(X)$. So, let us suppose X is not separable. Assume, by way of contradiction, that A is also not separable. Let $\mathcal{B} = \left\{B\left(x, \frac{1}{2}\right) : \|x\| = 1, x \in A\right\}$ be a maximal pairwise disjoint collection. Then, $\mathcal{B}$ is an uncountable collection. For each n, let $\mathcal{B}_n = \{B \in \mathcal{B} : S_n \cap B \neq \emptyset\}$. Since S_n is separable for each n, $\mathcal{B}_n$ is a countable collection. Moreover, $\bigcup_{n\in\mathbb{N}} \mathcal{B}_n \neq \mathcal{B}$ since the former is still a countable collection and the latter was noted to be uncountable. Let $B \in \mathcal{B} \setminus \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$. Then, $A \in B^-$ while $S_n \notin B^-$ for any $n \in \mathbb{N}$, a contradiction to $S_n \to A$. Hence, A must be separable. □

Lemma 2.10. *Suppose X is a normed linear space and further suppose that $V \subseteq X$ is a linear subspace. Then, V is closed if and only if $V \cap X_1$ is closed.*

Proof. The forward implication is trivial from topology since V is closed by hypothesis and X_1 is closed by definition. The reverse implication requires some proof. Suppose $A = V \cap X_1$. By hypothesis, A is closed. Let $x \in \mathrm{cl}_X(V)$ and $\{x_n\}_{n=0}^{\infty} \subset V$ such that $x_n \to x$. Now, for a moment, let us suppose $V = \{0_X\}$. If this were the case, $A = \mathrm{cl}_X(V) \cap X_1$ since $V = \mathrm{cl}_X(V)$ in such a case. This is the conclusion we seek, so we may then turn to the case where $V \neq \{0_X\}$. This means that we may presume that $x \neq 0_X$ and $\{0_X\} \cap \{x_n\}_{n=0}^{\infty} = \emptyset$ since this is the only relevant case when $V \neq \{0_X\}$. Let us then acknowledge that $y = \frac{x}{\|x\|} \in \mathrm{cl}_X(V)$ and $\left\{y_n = \frac{x_n}{\|x_n\|}\right\}_{n=0}^{\infty} \subset V$. Now, since $x_n \to x$, we know that $\|x_n\| \to \|x\|$, hence $y_n = \frac{x_n}{\|x_n\|} \to \frac{x}{\|x\|} = y$ also. However, $\{y_n\}_{n=0}^{\infty} \subset X_1$ as well, so we know $\{y_n\}_{n=0}^{\infty} \subset A$. Since A is closed by hypothesis, $y \in A$. Now, since $y \in A = V \cap X_1$, $y \in V$. Since $y = \frac{x}{\|x\|}$, $x = \|x\| \, y \in V$, by linearity of V. Thus, $V = \mathrm{cl}_X(V)$. Therefore, V is closed in X if and only if $V \cap X_1$ is a closed subset of X. □

Lemma 2.11. *Given a net $A_\lambda \to A$ in $C(X)$, $A_\lambda \in CLS(X)$ for each λ, and $x \in A$, there exists a net of values $x_\lambda \in A_\lambda$ such that $x_\lambda \to x$.*

Proof. For each λ, if $d(x, A_\lambda) > 0$, there exists an $x_\lambda \in A_\lambda$ such that $d(x, x_\lambda) \le 2d(x, A_\lambda)$, and if $d(x, A_\lambda) = 0$, then $x_\lambda = x \in A_\lambda$ and $d(x, x_\lambda) = 0 = 2 \cdot 0 = 2d(x, A_\lambda)$. In either case, for each λ we are able to obtain an $x_\lambda \in A_\lambda$ satisfying $d(x, x_\lambda) \le 2d(x, A_\lambda)$. Now, let $\epsilon > 0$. Let $V = B_{\|\cdot\|}\left(x, \frac{\epsilon}{2}\right)$. Since $A_\lambda \to A$ and $A \in V^-$, there exists λ_0 such that $A_\lambda \in V^-$ for all $\lambda \ge \lambda_0$. That is, $d(x, A_\lambda) < \frac{\epsilon}{2}$ for all $\lambda \ge \lambda_0$. But, $d(x, x_\lambda) \le 2d(x, A_\lambda) < 2 \cdot \frac{\epsilon}{2} = \epsilon$ for all $\lambda \ge \lambda_0$. Therefore, $x_\lambda \to x$. □

Lemma 2.12. *Given a net $A_\lambda \to A$ in $C(X)$, $A_\lambda \in CLS(X)$ for each λ, and $x_\lambda \to x$ with $x_\lambda \in A_\lambda$ for each λ and $x \in X$, we may conclude that $x \in A$.*

Proof. Suppose $x \notin A$. Then, we may let $\delta = \frac{d(x,A)}{2} > 0$. Note that $A \in \left(X \setminus \overline{B_{\|\cdot\|}(x, \delta)}\right)^{++}$. Since $x_\lambda \to x$, there exists λ_0 such that $d(x, x_\lambda) < \delta$ for all $\lambda \ge \lambda_0$. But, this says $A_\lambda \notin \left(X \setminus \overline{B_{\|\cdot\|}(x, \delta)}\right)^{++}$ for all $\lambda \ge \lambda_0$, contradicting the fact that $A_\lambda \to A$. So, $x \in A$. □

Lemma 2.13. *Given a net $A_\lambda \to A$ in $C(X)$, with $A_\lambda \in CLS(X)$, then $A \in CLS(X)$ also. In other words, $CLS(X)$ is a closed subset of $C(X)$.*

Proof. Let $x \in A$. Lemma 2.11 says there exists a net of values $x_\lambda \in A_\lambda$ such that $x_\lambda \to x$.

Now, suppose a net of values $x_\lambda \in A_\lambda$ converges to $x \in X$. Lemma 2.12 says that $x \in A$.

Now, let $x, y \in A$. We may obtain a net $x_\lambda \in A_\lambda$ converging to x and a net $y_\lambda \in A_\lambda$ converging to y. Note that for each λ, $x_\lambda + y_\lambda \in A_\lambda$ since A_λ is a closed linear subspace of X. But, given the aforementioned convergences, $x_\lambda + y_\lambda \to x + y$. Therefore, $x + y \in A$.

Finally, let $x \in A$ and $\alpha \in \mathbb{R}$. We may obtain a net $x_\lambda \in A_\lambda$ converging to x. Since $\alpha x_\lambda \to \alpha x$ and $\alpha x_\lambda \in A_\lambda$ for each λ, $\alpha x \in A$.

Therefore, A is a closed linear subspace of X, as desired. □

2.2. Important Geometric Equivalences in CLS (X)

Lemma 2.14. *Suppose B is a closed, bounded, convex subset of X and suppose that V is an open subset of X. Then, $(X \setminus B)^{++} = (X \setminus (-B))^{++}$ and $V^{-} = (-V)^{-}$ in $CLS(X)$.*

Proof. Let A be a closed linear subspace of X in $(X \setminus B)^{++}$. As A is a closed linear subspace,

$$d(A, x) = \inf_{y \in A} \|y - x\| = \inf_{y \in A} \|y + (-x)\| = \inf_{y \in A} \|(-y) + (-x)\|$$

for every $x \in X$. But,

$$\inf_{y \in A} \|(-y) + (-x)\| = \inf_{y \in A} \|y + x\| = \inf_{y \in A} \|y - (-x)\| = d(A, -x).$$

Thus,

$$d(A, B) = \inf_{b \in B} d(A, b) = \inf_{b \in B} d(A, -b) = \inf_{b \in -B} d(A, b) = d(A, -B).$$

As $d(A, -B) = d(A, B) > 0$, $(X \setminus B)^{++} = (X \setminus (-B))^{++}$.

Now, suppose G is a closed linear subspace of X in V^{-}. Then, using the fact that $z \in G \cap V$ if and only if $-z \in G \cap (-V)$, we get that $V^{-} = (-V)^{-}$. $\square$

Lemma 2.15. *Suppose B is a closed, bounded, convex subset of X such that $0_X \notin B$. Then, there exists an $m > 0$ which satisfies*

$$d(B, \{-\alpha b : \alpha \geq 0\}) \geq m \text{ for every } b \in B.$$

Proof. Suppose, by way of contradiction, that $z_\ell \in B$, $b_\ell \in B$, and $\alpha_\ell \geq 0$ for all $\ell \geq 0$ such that $\|z_\ell + \alpha_\ell b_\ell\| \to 0$. Since $0 \leq \frac{\|z_\ell + \alpha_\ell b_\ell\|}{1+\alpha_\ell} \leq \|z_\ell + \alpha_\ell b_\ell\|$ for all $\ell \geq 0$, $\frac{1}{1+\alpha_\ell} \|z_\ell + \alpha_\ell b_\ell\| \to 0$. But, for all $\ell \geq 0$, $\frac{1}{1+\alpha_\ell} z_\ell + \frac{\alpha_\ell}{1+\alpha_\ell} b_\ell \in B$ and $d(0_X, B) > 0$, a contradiction. Hence, there exists some $m > 0$ such that $\|z + \alpha b\| = \|z - (-\alpha b)\| \geq m$ for all $z, b \in B$ and all $\alpha > 0$. $\square$

Lemma 2.16. *Suppose B is a closed, bounded, convex subset of X such that $0_X \notin B$. Let*

$$\tilde{B} = \overline{\left\langle \left\{ \frac{x}{\|x\|} : x \in B \right\} \right\rangle}.$$

Then, $\{0_X\} \in \left(X \setminus \tilde{B}\right)^{++}$.

Proof. Let us suppose that $\sum_{i=1}^{\ell(m)} \lambda_{i,m} \frac{x_{i,m}}{\|x_{i,m}\|} \to 0$ as $m \to 0$, where $\lambda_{1,m}, \ldots, \lambda_{\ell(m),m} \geq 0$, $\sum_{i=1}^{\ell(m)} \lambda_{i,m} = 1$, and $x_{i,m} \in B$ for all $1 \leq i \leq \ell(m)$. Since $0_X \notin B$, we recognize that $0 < \|x_{i,m}\|$ for all $1 \leq i \leq \ell(m)$ and $m \geq 0$, and thus the above sequence of sums is well defined. Moreover, we will let $L = \sup_{b \in B} \|b\| < \infty$, since B is bounded. Now, for all $1 \leq i \leq \ell(m)$ and $m \geq 0$, we see that $\mu = \frac{1}{L} \leq \frac{1}{\|x_{i,m}\|} \leq \frac{1}{d(B, 0_X)} = M$. But, $0 < \mu = \mu \sum_{i=1}^{\ell(m)} \lambda_{i,m} \leq \sum_{i=1}^{\ell(m)} \lambda_{i,m} \frac{1}{\|x_{i,m}\|} \leq M \sum_{i=1}^{\ell(m)} \lambda_{i,m} = M$. Moreover,

$$\sum_{i=1}^{\ell(m)} \lambda_{i,m} \frac{x_{i,m}}{\|x_{i,m}\|} = \left(\sum_{i=1}^{\ell(m)} \lambda_{i,m} \frac{1}{\|x_{i,m}\|} \right) \frac{\sum_{i=1}^{\ell(m)} \lambda_{i,m} \frac{x_{i,m}}{\|x_{i,m}\|}}{\sum_{i=1}^{\ell(m)} \lambda_{i,m} \frac{1}{\|x_{i,m}\|}} \to 0.$$

However, due to the choices of the $\lambda_{i,m}$, we know that

$$\frac{\sum_{i=1}^{\ell(m)} \lambda_{i,m} \frac{x_{i,m}}{\|x_{i,m}\|}}{\sum_{i=1}^{\ell(m)} \lambda_{i,m} \frac{1}{\|x_{i,m}\|}} \in B,$$

by the convexity of B. Since B is closed and excludes 0_X and this collection of points cannot go to 0_X without $0_X \in B$, we must conclude that $\sum_{i=1}^{\ell(m)} \lambda_{i,m} \frac{1}{\|x_{i,m}\|} \to 0$. However, this is a contradiction since $\sum_{i=1}^{\ell(m)} \lambda_{i,m} \frac{1}{\|x_{i,m}\|} \geq \sum_{i=1}^{\ell(m)} \lambda_{i,m} \mu = \mu > 0$ for all m. That is, $d\left(0_X, \tilde{B}\right) > 0$, leaving us that $\{0_X\} \in \left(X \setminus \tilde{B}\right)^{++}$. $\square$

Lemma 2.17. *Suppose B is a closed, bounded, convex subset of X with $0_X \notin B$. Then, for every $A \in CLS(X)$ and every $\nu \geq 0$, $\nu d(A, B) = d(A, \nu B)$. Moreover, if $A \in CLS(X)$ and $0 < \mu = \frac{1}{\sup_{b \in B} \|b\|}$, $\mu d(A, B) \leq d\left(A, \tilde{B}\right)$. Finally, if $A \in CLS(X)$ and $M = \frac{1}{d(0_X, B)}$, then $M d(A, B) \geq d\left(A, \tilde{B}\right)$.*

Proof. First, $0 \cdot d(A, B) = \inf_{b \in B} \inf_{a \in A} 0 \cdot \|a - b\| = \inf_{b \in B} \inf_{a \in A} \|0 \cdot a - 0 \cdot b\| = \inf_{b \in B} \inf_{a \in A} \|a - 0 \cdot b\| = d(A, 0 \cdot B)$, because $0_X \in A$ and $0 \cdot B = \{0_X\}$.

Next, let $A \in \mathrm{CLS}(X)$ and $\nu > 0$. Then,

$$\nu d(A, B) = \inf_{b \in B} \inf_{a \in A} \nu \|a - b\| = \inf_{b \in B} \inf_{a \in A} \|\nu a - \nu b\|.$$

Also, A is linear and as $\inf\limits_{b \in B} \inf\limits_{a \in A} \|\nu a - \nu b\| = \inf\limits_{b \in B} \inf\limits_{\frac{a}{\nu} \in A} \|a - \nu b\|$, we get $\inf\limits_{b \in B} \inf\limits_{\frac{a}{\nu} \in A} \|a - \nu b\| = \inf\limits_{b \in B} \inf\limits_{a \in A} \|a - \nu b\|$. However, $\inf\limits_{b \in B} \inf\limits_{a \in A} \|a - \nu b\| = d(A, \nu B)$, so $\nu d(A, B) = d(A, \nu B)$.

We now develop the second part of the conclusion. Note that $\mu = \frac{1}{\sup\limits_{b \in B} \|b\|} > 0$, since B is bounded. Also, let $x_1, \ldots, x_\ell \in B$ and $\lambda_i \geq 0$ for all $1 \leq i \leq \ell$, where $\sum\limits_{i=1}^{\ell} \lambda_i = 1$. By the proof of Lemma 2.16, $0 < \mu = \mu \sum\limits_{i=1}^{\ell} \lambda_i \leq \sum\limits_{i=1}^{\ell} \frac{\lambda_i}{\|x_i\|}$ and $\frac{\sum_{i=1}^{\ell} \frac{\lambda_i x_i}{\|x_i\|}}{\sum_{i=1}^{\ell} \frac{\lambda_i}{\|x_i\|}} \in B$. Also, from earlier in this proof, $\mu d(A, B) \leq \mu d\left(A, \frac{\sum_{i=1}^{\ell} \frac{\lambda_i x_i}{\|x_i\|}}{\sum_{i=1}^{\ell} \frac{\lambda_i}{\|x_i\|}}\right) \leq \sum\limits_{i=1}^{\ell} \frac{\lambda_i}{\|x_i\|} d\left(A, \frac{\sum_{i=1}^{\ell} \frac{\lambda_i x_i}{\|x_i\|}}{\sum_{i=1}^{\ell} \frac{\lambda_i}{\|x_i\|}}\right) = d\left(A, \sum\limits_{i=1}^{\ell} \frac{\lambda_i x_i}{\|x_i\|}\right)$. That is, $\mu d(A, B) \leq d\left(A, \tilde{B}\right)$.

Finally, we develop the last part of the conclusion. Note that $M = \frac{1}{d(0_X, B)} < +\infty$, since $0_X \notin B$. Now, noting that $\frac{1}{\|b\|} \leq M$ for all $b \in B$, we get $Md(A, B) = d(A, MB) = \inf\limits_{b \in B} d(A, Mb) = \inf\limits_{b \in B} d\left(A, M\|b\| \frac{b}{\|b\|}\right) = \inf\limits_{b \in B} M\|b\| d\left(A, \frac{b}{\|b\|}\right) \geq \inf\limits_{b \in B} d\left(A, \frac{b}{\|b\|}\right)$, since $1 \leq M\|b\|$ for all $b \in B$. However, for every $b \in B$, $\frac{b}{\|b\|} \in \tilde{B}$, by definition. So, $\inf\limits_{b \in B} d\left(A, \frac{b}{\|b\|}\right) \geq d\left(A, \tilde{B}\right)$. That is, $Md(A, B) \geq d\left(A, \tilde{B}\right)$. □

Lemma 2.18. *Suppose B is a closed, bounded, convex subset of X with $0_X \notin B$. Then, $(X \setminus B)^{++} = \left(X \setminus \tilde{B}\right)^{++}$ in CLS(X).*

Proof. Suppose $A \in \mathrm{CLS}(X)$ and $A \in (X \setminus B)^{++}$. That is, $d(A, B) > \epsilon(A, B) > 0$. By Lemma 2.17, we have that $0 < \mu\epsilon(A, B) < \mu d(A, B) \leq d\left(A, \tilde{B}\right)$ for a $\mu > 0$. So, $A \in \left(X \setminus \tilde{B}\right)^{++}$.

Now, suppose $G \in \mathrm{CLS}(X)$ and $G \in \left(X \setminus \tilde{B}\right)^{++}$. That is, $d\left(G, \tilde{B}\right) > \epsilon\left(G, \tilde{B}\right) > 0$. Again, by Lemma 2.17, we have that $0 < \epsilon\left(G, \tilde{B}\right) < d\left(G, \tilde{B}\right) \leq Md(G, B)$ for an $M > 0$. So, $G \in (X \setminus B)^{++}$.

Therefore, $(X \setminus B)^{++} = \left(X \setminus \tilde{B}\right)^{++}$, as desired. □

Lemma 2.19. *Suppose V is an open subset of X which satisfies $0_X \notin V$. Then $V^- = \tilde{V}_n^-$, where*

$$\tilde{V}_n = \left(\bigcup_{x\in V}\left\{\alpha x : \alpha > 0 \;\&\; 1 - \frac{1}{2^{n+1}} < \alpha\|x\| < 1 + \frac{1}{2^{n+1}}\right\}\right),$$

in $CLS(X)$.

Proof. First, let $U = \bigcup_{\alpha>0} \alpha V$ and $W = \left\{x \in X : 1 - \frac{1}{2^{n+1}} < \|x\| < 1 - \frac{1}{2^{n+1}}\right\}$. Note that $\tilde{V}_n = U \cap W$. Since U and W are both open sets, $\tilde{V}_n$ must also be open.

Next, suppose $A \in \mathrm{CLS}(X)$ and satisfies $A \in V^-$. Note that $A \neq \{0_X\}$. Then, let $z \in A \cap V$ and note that $z \neq 0_X$. We then see that

$$\frac{z}{\|z\|} \in \bigcup_{x\in V}\left\{\alpha x : \alpha > 0 \;\&\; 1 - \frac{1}{2^{n+1}} < \alpha\|x\| < 1 + \frac{1}{2^{n+1}}\right\} \cap A.$$

Thus,

$$A \in \left(\bigcup_{x\in V}\left\{\alpha x : \alpha > 0 \;\&\; 1 - \frac{1}{2^{n+1}} < \alpha\|x\| < 1 + \frac{1}{2^{n+1}}\right\}\right)^-.$$

Whence, $V^- \subset \left(\bigcup_{x\in V}\left\{\alpha x : \alpha > 0 \;\&\; 1 - \frac{1}{2^{n+1}} < \alpha\|x\| < 1 + \frac{1}{2^{n+1}}\right\}\right)^-$.

Finally, suppose $G \in \mathrm{CLS}(X)$ that satisfies

$$G \in \left(\bigcup_{x\in V}\left\{\alpha x : \alpha > 0 \;\&\; 1 - \frac{1}{2^{n+1}} < \alpha\|x\| < 1 + \frac{1}{2^{n+1}}\right\}\right)^-.$$

Then, there is a $y \in G \cap \tilde{V}_n$. So, there exists an $x \in V$ and an $\alpha > 0$ such that $\alpha x = y$. Moreover, $\{\beta y : \beta \in \mathbb{R}\} \subset G$. Therefore, $x \in G$. Consequently,

$$\left(\bigcup_{x\in V}\left\{\alpha x : \alpha > 0 \;\&\; 1 - \frac{1}{2^{n+1}} < \alpha\|x\| < 1 + \frac{1}{2^{n+1}}\right\}\right)^- \subset V^-.$$

Whence,

$$V^- = \left(\bigcup_{x\in V}\left\{\alpha x : \alpha > 0 \;\&\; 1 - \frac{1}{2^{n+1}} < \alpha\|x\| < 1 + \frac{1}{2^{n+1}}\right\}\right)^-.$$

□

Lemma 2.20. *Suppose $u \in X$ with $\|u\| = 1$ and $0 < \delta < 1$. Then, the diameter of $\left\{\alpha z : \alpha > 0 \;\&\; z \in B_{\|\cdot\|}(u, \delta) \;\&\; 1 - \delta < \alpha\|z\| < 1 + \delta\right\}$ is at most $\frac{2\delta^2+6\delta}{1-\delta}$.*

Proof. Let $A = \{z \in X : 1-\delta < \|z\| < 1+\delta\}$. Note that if $y \in B_{\|\cdot\|}(u,\delta)$, then $\|u-y\| < \delta$. So, $-\delta < \|y\| - \|u\| < \delta$. That is, $1-\delta < \|y\| < 1+\delta$. In other words, $B_{\|\cdot\|}(u,\delta) \subset A$.

Now, let $\lambda \geq 0$, $y \in B_{\|\cdot\|}(u,\delta)$ such that $\lambda y \in A$. This gives $1-\delta < \|\lambda y\| < 1+\delta$ and $1-\delta < \|y\| < 1+\delta$ simultaneously. That is, $\frac{1-\delta}{1+\delta} < \frac{1-\delta}{\|y\|} < \lambda < \frac{1+\delta}{\|y\|} < \frac{1+\delta}{1-\delta}$. Furthermore, $\|\lambda y - u\| \leq \|\lambda y - y\| + \|y-u\| < |1-\lambda|\,\|y\| + \delta$.

If $\lambda \geq 1$, $|\lambda - 1| = \lambda - 1 \leq \frac{1+\delta}{1-\delta} - 1 = \frac{2\delta}{1-\delta}$. If $0 \leq \lambda \leq 1$, $|\lambda - 1| = 1-\lambda \leq 1 - \frac{1-\delta}{1+\delta} = \frac{2\delta}{1+\delta} \leq \frac{2\delta}{1-\delta}$.

So, summarizing, $\|\lambda y - u\| < \frac{2\delta}{1-\delta}\|y\| + \delta \leq \frac{2\delta(1+\delta)}{1-\delta} + \delta = \frac{\delta^2+3\delta}{1-\delta}$. That is, the diameter of

$$\left\{\alpha z : \alpha > 0 \ \&\ z \in B_{\|\cdot\|}(u,\delta)\ \&\ 1-\delta < \alpha\|z\| < 1+\delta\right\}$$

is at most $\frac{2\delta^2+6\delta}{1-\delta}$. □

2.3. The Homeomorphism of CLS(X) with Its Restriction to the Closed Unit Ball

2.3.1. The Difficulty Faced in Nonreflexive Banach Spaces

Theorem 2.21. *Let* $\psi = \left(1-\frac{1}{n}\right) \in \ell^\infty$. *For all* $f \in \ell^1$, $\|f\|_1 \leq 1$ *implies* $\left|\int f\cdot\psi\right| = \left|\sum_{n=1}^{\infty}\left(f_n - \frac{f_n}{n}\right)\right| < 1$. *Therefore, the nonempty, closed, bounded convex set*

$$A = \left\{f \in \ell^1 : \int f\cdot\psi = 1\right\} \cap X_2$$

is disjoint from X_1, *where* $X = \ell^1$, *but* $d(X_1, A) = 0$.

Proof. To start, let us examine the set $B = \left\{f \in \ell^1 : \int f\cdot\psi = 1\right\}$ in ℓ^1. Given $f_1, f_2 \in B$, and $\lambda \in \mathbb{R}$, $\int(\lambda f_1 + (1-\lambda) f_2)\cdot\psi = \left(\lambda\int f_1\cdot\psi\right) + \left((1-\lambda)\int f_2\cdot\psi\right) = \lambda + 1 - \lambda = 1$. Therefore, B is a convex subset of ℓ^1. Moreover, let (f_n) be a sequence of points in B that converges to the element $f \in \ell^1$. Since $f_n \to f$, we obtain $\int(f_n - f)\cdot\psi = \left(\int f_n\cdot\psi\right) - \left(\int f\cdot\psi\right) \to 1 - \int f\cdot\psi = 0$. Therefore, B is closed in ℓ^1. As a closed, convex subset of ℓ^1, and letting $X = \ell^1$, we see that the intersection of B with the closed 2-ball, $A = X_2 \cap B$, is a closed, bounded, convex subset of ℓ^1.

Now, suppose $f \in X_1$. Then, we will note $f = (a_n)$, a sequence of real numbers for which $\sum_{n=1}^{\infty}|a_n| \leq 1$. Let n_0 be the first positive integer for which $a_{n_0} \neq 0$. We see that $\int\|f\cdot\psi\| =$

$\sum_{n=1}^{\infty} \frac{n-1}{n} |a_n| \leq \sum_{n=1}^{\infty} |a_n| - \frac{1}{n_0} |a_{n_0}| < \|f\|_1 \leq 1$. Therefore, $\left| \sum_{n=1}^{\infty} \frac{n-1}{n} a_n \right| \leq \sum_{n=1}^{\infty} \frac{n-1}{n} |a_n| < 1$. That is, $X_1 \cap A \subset X_1 \cap B = \emptyset$.

Lastly, we shall create a sequence (f_n) in A for which $d(f_n, X_1) \to 0$. For each positive integer N, we let $f_{N,n} = 0$ whenever $n \neq N+1$, but we let $f_{N,N+1} = \frac{N+1}{N}$. Now, $\|f_n\|_1 = \frac{n+1}{n}$ for each positive integer n. So, $d(f_n, X_1) \to 0$. What remains is to show this sequence lies in A. Indeed, $\sum_{n=1}^{\infty} \frac{n-1}{n} f_{N,n} = \frac{N}{N+1} \frac{N+1}{N} = 1$ for each positive integer N. Therefore, A is a closed, bounded, convex subset of ℓ^1 which is disjoint from the closed unit ball, but nevertheless has a distance of 0 from the closed unit ball. □

2.3.2. The Correct View of Subbasic Open Sets Based on Closed Bounded Convex Sets

LEMMA 2.22. *If X is a Banach space, then the map $\phi : CLS(X) \to \{V \cap X_1 : V \in CLS(X)\}$, defined by $\phi(V) = V \cap X_1$, has the property that*

$$\phi^{-1}\left[(X \setminus B)^{++} \cap \phi[CLS(X)]\right] = \bigcup_{n=1}^{\infty} \left(\left(X \setminus \left(B \cap X_{1+\frac{1}{n}} \right) \right)^{++} \cap CLS(X) \right),$$

when $B \in CB(X)$. Consequently, $\phi^{-1}\left[(X \setminus B)^{++} \cap \phi[CLS(X)]\right]$ is an open set in $CLS(X)$ for every $B \in CB(X)$.

PROOF. First, we need to show that $(X \setminus B)^{++} \cap \phi[\mathrm{CLS}(X)]$, or $(X \setminus B)^{++}$ in $\phi[\mathrm{CLS}(X)]$, is identical to $\left(X \setminus \left(B \cap X_{1+\frac{1}{n}} \right) \right)^{++}$ in $\phi[\mathrm{CLS}(X)]$ for every positive integer n. Indeed, given $A \in (X \setminus B)^{++}$, $d\left(B \cap X_{1+\frac{1}{n}}, A\right) \geq d(B, A) > 0$. Moreover, given a closed linear subspace $A \in \left(X \setminus \left(B \cap X_{1+\frac{1}{n}} \right) \right)^{++}$, we get both $d\left(B \cap X_{1+\frac{1}{n}}, A\right) > 0$ and $d\left(B \setminus X_{1+\frac{1}{n}}, A\right) \geq d\left(B \setminus X_{1+\frac{1}{n}}, X_1\right) \geq \frac{1}{n} > \frac{1}{2n} > 0$, so

$$d(B, A) \geq \min\left\{ d\left(B \cap X_{1+\frac{1}{n}}, A\right), d\left(B \setminus X_{1+\frac{1}{n}}, A\right) \right\} > 0.$$

Thus, the above mentioned equality holds for every positive integer n.

Next, we need to show that $A \in (X \setminus B)^{++}$ in $\phi[\mathrm{CLS}(X)]$ implies there exists a positive integer n such that $V = \overline{\mathrm{span}}(A) \in \left(X \setminus \left(B \cap X_{1+\frac{1}{n}} \right) \right)^{++}$ in $\mathrm{CLS}(X)$. Indeed, let us suppose this was not the case and derive a contradiction. So, we are supposing that $A \in (X \setminus B)^{++}$ in $\phi[\mathrm{CLS}(X)]$, that $V = \overline{\mathrm{span}}(A)$, and that $V \notin \left(X \setminus \left(B \cap X_{1+\frac{1}{n}} \right) \right)^{++}$ in

CLS (X) for any positive integer n. There would then exist a sequence $\{v_{m,n}\}_{m=1}^{\infty}$ and a sequence $\{b_{m,n}\}_{m=1}^{\infty}$ for each positive integer n where $d(v_{m,n}, b_{m,n}) \to 0$ as $m \to \infty$ and where $1 \leq \|b_{m,n}\| \leq 1 + \frac{1}{n}$ for every positive integer n. This would then say that the limit $\lim_{n\to\infty} \lim_{m\to\infty} \left\| \frac{v_{m,n}}{\|v_{m,n}\|} - b_{m,n} \right\| = 0$, but that contradicts our earlier statement that $A \in (X \setminus B)^{++}$ in $\phi[\text{CLS}(X)]$. Hence, we may conclude that $A \in (X \setminus B)^{++}$ in $\phi[\text{CLS}(X)]$ implies there exists a positive integer n such that $V = \overline{\text{span}}(A) \in \left(X \setminus \left(B \cap X_{1+\frac{1}{n}}\right)\right)^{++}$ in CLS (X).

Finally, we may note that $V \in \left(X \setminus \left(B \cap X_{1+\frac{1}{n}}\right)\right)^{++}$ in CLS (X) implies the inequality $d\left(V \cap X_1, B \cap X_{1+\frac{1}{n}}\right) \geq d\left(V, B \cap X_{1+\frac{1}{n}}\right) > 0$, so $(V \cap X_1) \in (X \setminus B)^{++}$ in $\phi[\text{CLS}(X)]$ by the first equality we mentioned in this proof.

Consequently, we have shown that

$$\phi^{-1}\left[(X \setminus B)^{++} \cap \phi[\text{CLS}(X)]\right] = \bigcup_{n=1}^{\infty} \left(\left(X \setminus \left(B \cap X_{1+\frac{1}{n}}\right)\right)^{++} \cap \text{CLS}(X)\right)$$

is an open set in CLS (X). □

2.3.3. The Homeomorphism Result

THEOREM 2.23. *If X is a Banach space, the map $\phi : CLS(X) \to \{V \cap X_1 : V \in CLS(X)\}$, defined by $\phi(V) = V \cap X_1$, is a continuous map.*

PROOF. According to Lemma 2.22, $\phi^{-1}\left[(X \setminus B)^{++}\right]$ is open in CLS (X) for every $B \in$ CB (X). That leaves checking our other subbasic open sets to see if their inverse image is open.

Let $V \in \tau_{\|\cdot\|}$. Also, let $B_1 = \{x \in X : \|x\| < 1\}$. Now, note that $\{A : \phi(A) \cap V \neq \emptyset\} = \{A : \phi(A) \cap V \cap X_1 \neq \emptyset\}$ because $\phi(A) \cap V \cap X_1 = A \cap V \cap X_1$ for all $A \in$ CLS (X). Moreover, the set containment $\{A : \phi(A) \cap V \cap X_1 \neq \emptyset\} \supseteq \{A : \phi(A) \cap V \cap B_1 \neq \emptyset\}$ occurs because $X_1 \supseteq B_1$. For the moment, let $A \in$ CLS (X) which satisfies $\phi(A) \cap V \cap X_1 \neq \emptyset$. So, there exists $x \in A \cap V \cap X_1$. Now, if $x = 0_X$, then $x \in A \cap V \cap B_1$ already. Otherwise, there exists $0 < \delta < \frac{\|x\|}{2}$ which satisfies $B(x, \delta) \subseteq V$. But, if we let $0 < \lambda = 1 - \frac{\delta}{2} < 1$, then $\|\lambda x - x\| = \left\|\left(1 - \frac{\delta}{2} - 1\right) x\right\| = \frac{\delta}{2}\|x\| \leq \frac{\delta}{2} < \delta$. That is, $\lambda x \in B(x, \delta)$. However, $\|\lambda x\| = \lambda \|x\| \leq \lambda < 1$, so $\lambda x \in B_1$. Moreover, $\lambda x \in A$ since $x \in A \in$ CLS (X). Therefore,

$\{A : \phi(A) \cap V \cap X_1 \neq \emptyset\} = \{A : \phi(A) \cap V \cap B_1 \neq \emptyset\}$. Finally, we may make note of the equality $\{A : \phi(A) \cap V \cap B_1 \neq \emptyset\} = \{A : A \cap V \cap B_1 \neq \emptyset\}$ because $\phi(A) \cap V \cap B_1 = A \cap V \cap B_1$ for all $A \in \mathrm{CLS}(X)$. That is, $\phi^{-1}(V^-) = (V \cap B_1)^-$. Hence, $\phi^{-1}(V^-)$ is a subbasic open subset of $\mathrm{CLS}(X)$.

Now, given any basic open set $\mathbf{V} = \bigcap_{i=1}^{k} V_i^- \cap \bigcap_{j=1}^{m} (X \setminus C_j)^{++}$, we get that $\phi^{-1}(\mathbf{V}) = \bigcap_{i=1}^{k} \phi^{-1}(V_i^-) \cap \bigcap_{j=1}^{m} \phi^{-1}\left((X \setminus C_j)^{++}\right)$, which is also a basic open set in the slice topology. Whence, ϕ is continuous. □

Theorem 2.24. *The map $\phi : CLS(X) \to \{V \cap X_1 : V \in CLS(X)\}$, defined by $\phi(V) = V \cap X_1$, is a bijection.*

Proof. To see that ϕ is one-to-one, we need only note the following:

Let $A, B \in \phi(\mathrm{CLS}(X))$. By Lemma 2.10, we note that $\mathrm{span}(A)$ and $\mathrm{span}(B)$ are elements of $CLS(X)$. Now, suppose that $A = B$. This says that $\mathrm{span}(A) = \mathrm{span}(B)$. Whence, ϕ is $1-1$.

To see that ϕ is onto, we need only note that by the definitions of $\{V \cap X_1 : V \in \mathrm{CLS}(X)\}$ and ϕ, we will get the desired equation $\phi[\mathrm{CLS}(X)] = \{V \cap X_1 : V \in \mathrm{CLS}(X)\}$.

Therefore, ϕ is a bijection. □

Theorem 2.25. *The map $\phi : CLS(X) \to \{V \cap X_1 : V \in CLS(X)\}$, defined by $\phi(V) = V \cap X_1$, is an open map.*

Proof. Let $C \in \mathrm{CB}(X)$. By Lemma 2.18, we know that $(X \setminus C)^{++} = \left(X \setminus \tilde{C}\right)^{++}$. By the same arguments that led to the result of Lemma 2.18, we could also obtain that, given $\tilde{C}' = \overline{\left\langle\left\{\frac{x}{2\|x\|} : x \in C\right\}\right\rangle}$, $(X \setminus C)^{++} = \left(X \setminus \tilde{C}'\right)^{++}$. Also, we see that $\phi\left[\left(X \setminus \tilde{C}'\right)^{++}\right] = \left\{\phi(A) : A \in \mathrm{CLS}(X) \text{ and } d\left(A, \tilde{C}'\right) > \epsilon\left(A, \tilde{C}'\right) > 0\right\}$, which is contained in the collection $\left\{\phi(A) : A \in \mathrm{CLS}(X) \text{ and } d\left(\phi(A), \tilde{C}'\right) > \epsilon\left(\phi(A), \tilde{C}'\right) > 0\right\}$. By way of a contradiction, let us suppose that the reverse set inclusion is not true. Then, there is an $A \in \mathrm{CLS}(X)$ for which $A \notin \left(X \setminus \tilde{C}'\right)^{++} = (X \setminus C)^{++}$ while $d\left(\phi(A), \tilde{C}'\right) > 0$. But, this would yield a sequence $\{a_n\}_{n\in\mathbb{N}} \subset A$ with $\|a_n\| > 1$ and a sequence $\{c_n\}_{n\in\mathbb{N}} \subset \tilde{C}'$ such that $\|a_n - c_n\| \to 0$.

However, for all $n \in \mathbb{N}$, $\|c_n\| \leq \frac{1}{2}$. As $\|a_n - c_n\| \geq |\|a_n\| - \|c_n\|| > 1 - \frac{1}{2} = \frac{1}{2} > 0$ for all $n \in \mathbb{N}$, $\|a_n - c_n\| \not\to 0$, contradicting the fact that $A \notin \left(X \setminus \tilde{C}'\right)^{++} = (X \setminus C)^{++}$. Therefore, we get the equation

$$\phi\left[\left(X \setminus \tilde{C}'\right)^{++}\right] = \left\{\phi(A) : A \in \mathrm{CLS}(X) \text{ and } d\left(\phi(A), \tilde{C}'\right) > \epsilon\left(\phi(A), \tilde{C}'\right) > 0\right\}.$$

We see that since the latter set is by definition a subbasic open subset of $\phi[\mathrm{CLS}(X)]$, the former must be subbasic open.

Let $V \in \tau_{\|\cdot\|}$ such that $0_X \notin V$, for otherwise $V^- = \mathrm{CLS}(X)$ and $\phi[V^-] = \phi[\mathrm{CLS}(X)]$ would be an open subset of $\phi[\mathrm{CLS}(X)]$ under the subspace topology it inherits from $C(X)$ under the slice topology. Also, let $B_1 = \{x \in X : \|x\| < 1\}$. By Lemma 2.19, we know that $V^- = \tilde{V}_n^-$ for any $n \in \mathbb{N}$. However, the same proofs that led to the result of Lemma 2.19 also lead to $V^- = \left(\tilde{V}_n'\right)^-$ where $\tilde{V}_n' = \bigcup_{x \in V} \left\{\alpha x : \alpha > 0 \ \&\ \frac{1}{2} - \frac{1}{2^{n+2}} < \alpha\|x\| < \frac{1}{2} + \frac{1}{2^{n+2}}\right\}$ for any $n \in \mathbb{N}$. Now, $\phi\left[\left(\tilde{V}_n'\right)^-\right] = \left\{\phi(A) : A \in \mathrm{CLS}(X) \text{ and } A \cap \tilde{V}_n' \neq \emptyset\right\}$. However, since $\tilde{V}_n' \subset B_1$, we now have $A \cap \tilde{V}_n' \neq \emptyset$ if and only if $(A \cap X_1) \cap \tilde{V}_n' \neq \emptyset$ whenever $A \in \mathrm{CLS}(X)$. That is, $\phi(A) \cap \tilde{V}_n' \neq \emptyset$ if and only if $A \in \left(\tilde{V}_n'\right)^-$. Since the set $\{\phi(A) : A \in \mathrm{CLS}(X)$ and $\phi(A) \cap \tilde{V}_n' \neq \emptyset\Big\}$ is the definition for the other type of subbasic open subsets of $\phi[\mathrm{CLS}(X)]$ and is the same as the set $\phi\left[\left(\tilde{V}_n'\right)^-\right]$, we have that the latter is subbasic open.

Finally, we shall let $\mathbf{V} = \bigcap_{i=1}^{k}\left(\tilde{V}_i'\right)^- \cap \bigcap_{j=1}^{m}\left(X \setminus \tilde{B}_j'\right)^{++}$, an arbitrary basic open subset of $\mathrm{CLS}(X)$ under the slice topology. Now, $\phi[\mathbf{V}] = \bigcap_{i=1}^{k}\phi\left[\left(\tilde{V}_i'\right)^-\right] \cap \bigcap_{j=1}^{m}\phi\left[\left(X \setminus \tilde{B}_j'\right)^{++}\right]$ since ϕ is a bijection by Theorem 2.24, but in particular because ϕ is $1-1$. Now, that tells us that $\phi[\mathbf{V}]$ is a basic open subset of $\phi[\mathrm{CLS}(X)]$. Thus, ϕ is an open map. □

THEOREM 2.26. *If X is a Banach space, the map $\phi : CLS(X) \to \{V \cap X_1 : V \in CLS(X)\}$, defined by $\phi(V) = V \cap X_1$, is a homeomorphism.*

PROOF. This follows from Theorems 2.23, 2.24, and 2.25. □

CHAPTER 3

KNOWN RESULTS REGARDING STRONG CHOQUET TOPOLOGIES

3.1. Definition

Definition 3.1. [5, p. 44-45] Given a topological space (X, τ), a strong Choquet game is a two-player non-cooperative game in which we shall call the first player Player EMPTY and the second player Player NONEMPTY. The game is played so that Player EMPTY begins with a pair $(x_0, V_0) \in (X \times \tau)$ which satisfies $x_0 \in V_0$ and Player NONEMPTY follows by playing a set $U_0 \in \tau$ that satisfies $x_0 \in U_0 \subseteq V_0$. Furthermore, for each positive integer n, Player EMPTY must play a pair $(x_n, V_n) \in (X \times \tau)$ which satisfies $x_n \in V_n \subseteq U_{n-1}$ and Player NONEMPTY follows this up by playing a set $U_n \in \tau$ that satisfies $x_n \in U_n \subseteq V_n$. The game is won by Player EMPTY if $\bigcap_{n=0}^{\infty} U_n = \bigcap_{n=0}^{\infty} V_n = \emptyset$. Otherwise, the game is won by Player NONEMPTY.

Definition 3.2. [5, p. 44-45] A topological space (X, τ) is a strong Choquet space if there exists a strategy by which Player NONEMPTY is assured to win every strong Choquet game in which this strategy is employed.

Theorem 3.3. *Given a topological space (X, τ) and a basis $\mathcal{B}$ for τ, the existence of a winning strategy for Player NONEMPTY using basis elements is equivalent to the existence of a winning strategy for Player NONEMPTY using open sets in general.*

Proof. For the forward implication, suppose σ is a winning strategy for Player NONEMPTY using only basis elements. Suppose (x_0, V_0) is played by Player EMPTY. Player NONEMPTY will play $C_0 = \sigma((x_0, B_0))$, where $x_0 \in B_0 \subset V_0$ for a basic open set B_0. Suppose on move n, Player EMPTY makes a move (x_n, V_n) where V_n is an open set that is not necessarily basic open. Player NONEMPTY will play $C_n = \sigma((x_0, B_0), \ldots, (x_n, B_n))$ where $B_0, \ldots, B_{n-1} \in \mathcal{B}$

are the basic open sets used in the previous moves by Player NONEMPTY and where B_n is a basic open set obeying $x_n \in B_n \subset V_n$. In this way, Player NONEMPTY will win the game.

Now, to see the reverse implication, let σ be a winning strategy for Player NONEMPTY using open sets in general. Suppose Player EMPTY plays (x_0, B_0) where B_0 is a basic open set. Player NONEMPTY first derives $U_0 = \sigma((x_0, B_0))$, an open set that may not be basic open, and then uses a basic open set C_0 such that $x_0 \in C_0 \subset U_0$. Suppose on move n, Player EMPTY plays (x_n, B_n), where B_n is basic open. Then, Player NONEMPTY will first get $U_n = \sigma((x_0, B_0), \ldots, (x_n, B_n))$ and then play a basic open set C_n such that $x_n \in C_n \subset U_n$. Note that $\emptyset \neq \bigcap_{n=0}^{\infty} U_n = \bigcap_{n=0}^{\infty} B_n = \bigcap_{n=0}^{\infty} C_n$. Therefore, Player NONEMPTY will win based on this variant on strategy σ. □

3.2. Strong Choquet Subsets of Strong Choquet Spaces

Theorem 3.4. *Whenever (X, τ) is a strong Choquet space and $O \in \tau$, O is also a strong Choquet space under the subspace topology.*

Proof. Let Γ be a strong Choquet game on O as follows. Player EMPTY picks (x_0, V_0) from $O \times \tau_O$. Player NONEMPTY treats this game as a strong Choquet game on X from this point onward and uses the winning strategy σ available, since all open subsets of O are open under X. Thus, $\Gamma = (x_0, V_0), \sigma(x_0, V_0), (x_1, V_1), \sigma(x_1, V_1), \ldots, (x_n, V_n), \sigma(x_n, V_n), \ldots$ is our strong Choquet game over X and O simultaneously and Player NONEMPTY must win over X, so Player NONEMPTY wins over O. □

Theorem 3.5. *Whenever (X, τ) is a strong Choquet space and Y is a G_δ subset of X, Y is also a strong Choquet space under the subspace topology.*

Proof. Let Y be a G_δ subset of X. Given that $Y = \bigcap_{n=0}^{\infty} Z_n$ with Z_n open in X for all n, for each m we will let $Y_m = \bigcap_{n=0}^{m} Z_n$. Finally, we are ready to engage in a strong Choquet game with Y. First, Player EMPTY chooses (x_0, V_0) from (Y, τ_Y) and Player NONEMPTY treats this choice as (x_0, V_0'), where $V_0' \in \tau_{Y_0}$ satisfying $V_0' \cap Y = V_0$, and subsequently chooses $U_0' = \sigma\left(x_0, V_0'\right)$ using the winning strategy over X and lets $U_0 = Y \cap U_0'$. Now, Player EMPTY always makes a valid choice (x_n, V_n) in our strong Choquet game over Y and then Player NONEMPTY treats this choice as some $V_n' \in \tau_{Y_n}$ satisfying $x_n \in V_n' \subseteq U_{n-1}'$, makes the choice $U_n' = \sigma\left(x_n, V_n'\right)$ and then lets $U_n = Y \cap U_n'$. Given this, we may conclude that Player NONEMPTY wins the concurrent strong Choquet game that has developed over X, leaving $\emptyset \neq \bigcap_{n=0}^{\infty} U_n' \subseteq Y$. But, $\bigcap_{n=0}^{\infty} V_n = \bigcap_{n=0}^{\infty} U_n = \bigcap_{n=0}^{\infty} U_n' \neq \emptyset$. Therefore, Player NONEMPTY wins the strong Choquet game by this strategy, confirming that Y is a strong Choquet space. □

3.3. The Strong Choquet Spaces Compared to Choquet Spaces and Baire Spaces

DEFINITION 3.6. [5, p. 43-44] Given a topological space (X, τ), a Choquet Game is a two-player non-cooperative game in which we shall call the first player Player EMPTY and the second player Player NONEMPTY. The game is played so that Player EMPTY begins with a nonempty set $V_0 \in \tau$ and Player NONEMPTY follows by playing a set $U_0 \in \tau$ that satisfies $U_0 \subseteq V_0$. Furthermore, for each positive integer n, Player EMPTY must play a nonempty set $V_n \in \tau$ which satisfies $V_n \subseteq U_{n-1}$ and Player NONEMPTY follows this up by playing a set $U_n \in \tau$ that satisfies $U_n \subseteq V_n$. The game is won by Player EMPTY if $\bigcap_{n=0}^{\infty} U_n = \bigcap_{n=0}^{\infty} V_n = \emptyset$. Otherwise, the game is won by Player NONEMPTY.

DEFINITION 3.7. [5, p. 43-44] A topological space (X, τ) is a Choquet space if there exists a strategy by which Player NONEMPTY is assured to win every Choquet Game in which this strategy is employed.

THEOREM 3.8. *Every strong Choquet space is a Choquet space.*

PROOF. Suppose (X, τ) is a strong Choquet space. There is a strategy σ that Player NONEMPTY may employ and be guaranteed to win. For every nonempty $V \in \tau$, let us pick $x_V \in V$. Now, let us suppose we are going to play a Choquet game. Player EMPTY will start us off by playing a nonempty $V_0 \in \tau$. Now, Player NONEMPTY notices that $x_{V_0} \in V_0$ and employs σ to obtain $U_0 = \sigma\left(\left(x_{V_0}, V_0\right)\right)$. This set satisfies $U_0 \subset V_0$.

At each stage n, Player EMPTY plays a nonempty set $V_n \in \tau$ satisfying $V_n \subset U_{n-1}$ and Player NONEMPTY notes that $x_{V_n} \in V_n$ to employ σ on $(x_{V_0}, V_0), \ldots, (x_{V_n}, V_n)$ to obtain $U_n = \sigma((x_{V_0}, V_0), \ldots, (x_{V_n}, V_n))$. Since σ guarantees a win by Player NONEMPTY on the apparently concurrent strong Choquet game $(x_{V_0}, V_0), U_0, (x_{V_1}, V_1), U_1, \ldots$, we know that $\bigcap_{n=0}^{\infty} U_n \neq \emptyset$. Therefore, the strategy σ' defined by $\sigma'(V_0, \ldots, V_n) = \sigma((x_{V_0}, V_0), \ldots, (x_{V_n}, V_n))$ for any $V_0, \ldots, V_n$ from a partial run of a Choquet game on (X, τ) guarantees that Player NONEMPTY will win. Whence, strong Choquet spaces are Choquet spaces. □

Theorem 3.9. [5, p. 43-44] [7] *Every Choquet space is a Baire space.*

Proof. Let (X, τ) be a Choquet space. Let $\{W_0, \ldots, W_n, \ldots\}$ be a countable collection of dense open subsets of X. Let O be a nonempty open subset of X. We begin to play a Choquet game. Player EMPTY begins by playing $V_0 = W_0 \cap O$, which we know to be both nonempty and open. Player NONEMPTY shall use their strategy σ to obtain U_0. Since $\emptyset \neq U_0 \subset V_0 \subset O$, Player EMPTY is able to play $V_1 = U_0 \cap W_1$, which is nonempty (W_1 is dense and U_0 is open), open (both U_0 and W_1 are open), and a subset of U_0. Player NONEMPTY proceeds by using σ to obtain U_1. Continuing in this manner, Player EMPTY shall play $V_n = U_{n-1} \cap W_n$, which is valid because it is nonempty (U_{n-1} is open and W_n is dense), open (U_{n-1} and W_n are open), and a subset of U_{n-1}. Likewise, Player NONEMPTY proceeds to use σ to obtain U_n. Taking the full run of the game, we find that this is a Choquet game in which Player NONEMPTY has used σ at each stage. Therefore, we know that $\emptyset \neq \bigcap_{n=0}^{\infty} U_n = \bigcap_{n=1}^{\infty} (U_{n-1} \cap W_n) \cap (W_1 \cap O) \subset \left(\bigcap_{n=0}^{\infty} W_n \right) \cap O$. So, the intersection $W = \bigcap_{n=0}^{\infty} W_n$ is dense in X. This being true of every such countable intersection of dense open subsets of X, we know that (X, τ) is a Baire space. □

Corollary 3.10. *Every strong Choquet space is a Baire space*

Proof. By transitivity via Theorem 3.8 and Theorem 3.9. □

3.4. Topological and Metric Spaces that are Strong Choquet

THEOREM 3.11. *Given a complete metric space (X, d), the space (X, τ_d) is strong Choquet.*

PROOF. Given any move (x_n, V_n) by Player EMPTY in a strong Choquet game, let Player NONEMPTY play $U_n = B_d\left(x_n, \frac{d(x_n, X \setminus V_n)}{2^n}\right) = \sigma\left(\left(x_0, V_0\right), \ldots, \left(x_n, V_n\right)\right)$. Since (X, d) is a complete metric space, $V_{n+1} \subset U_n \subset \overline{U_n} \subset V_n$ for each $n \geq 0$, and $\operatorname{diam}(U_n) \to 0$, we know that $\bigcap_{n=0}^{\infty} U_n = \bigcap_{n=0}^{\infty} \overline{U_n} = \{x\}$ for some $x \in X$. Therefore, σ is a winning strategy for Player NONEMPTY. Therefore, (X, τ_d) is a strong Choquet space. □

THEOREM 3.12. *Every compact Hausdorff space (X, τ) is strong Choquet.*

PROOF. Let $x \in V \in \tau$. There exists a compact neighborhood K for which $x \in \operatorname{int}(K) \subset K \subset V$. Now, let us start a strong Choquet game. For any move (x_n, V_n) made by Player EMPTY, Player NONEMPTY will retort with the interior of a compact neighborhood $x_n \in \operatorname{int}(K_n) \subset K_n \subset V_n$. In other words, $U_n = \operatorname{int}(K_n)$. Given a full run of the game with Player NONEMPTY following this strategy, we find $\bigcap_{n=0}^{\infty} U_n = \bigcap_{n=0}^{\infty} K_n \neq \emptyset$. Therefore, (X, τ) is a strong Choquet space. □

COROLLARY 3.13. *Every locally compact Hausdorff space is strong Choquet.*

PROOF. Same proof as in Theorem 3.12. □

3.5. Weakness of Strong Choquet

REMARK 3.14. [4, p. 201] The space $\mathbb{R}$ with the topology formed from the subbasis consisting of the open intervals and of the set of irrationals is a strong Choquet space. Moreover, $\mathbb{Q}$ is a closed subset of $\mathbb{R}$ in this topology, but the topology it inherits as a topological subspace is the usual topology on $\mathbb{Q}$, which is not Baire.

The fact that $\mathbb{R}$ is strong Choquet in this topology, which was mentioned in this remark, is proven by Debs in another of his papers [3, p. 31-32] in more generality. Furthermore, the other comments in the remark are quite obvious after a moments thought.

CHAPTER 4

IMPORTANT ESTIMATES AND THE STRONG CHOQUET PROPERTY FOR SLICE-LIKE TOPOLOGIES

4.1. Important Estimates Needed

LEMMA 4.1. *Suppose* $v = (v_1, \ldots, v_n) \in X^n$ *such that* $v_1, \ldots, v_n$ *are linearly independent. There exists a* $c > 0$ *such that for every* $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ *and for every* $v' = \left(v_1', \ldots, v_n'\right) \in X^n$, $c\|\lambda\| \leq \left\|\sum_{i=1}^{n} \lambda_i v_i\right\|$ *and* $\left\|\sum_{i=1}^{n} \lambda_i v_i'\right\| \geq \left(c - \left(\sum_{i=1}^{n} \left\|v_i - v_i'\right\|^2\right)^{\frac{1}{2}}\right)\|\lambda\|$.

PROOF. First, the function $f : \mathbb{R}^n \to \mathbb{R}$, defined by $f(\lambda) = \left\|\sum_{i=1}^{n} \lambda_i v_i\right\|$ for all $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, is a continuous function. Since the set $\Lambda = \{\lambda \in \mathbb{R}^n : \|\lambda\| = 1\}$ is compact in $\mathbb{R}^n$, $c = \min_{\lambda \in \Lambda} f(\lambda)$ is well-defined. Moreover, the fact that $v_1, \ldots, v_n$ are linearly independent implies that $c > 0$. Because $\left\|\sum_{i=1}^{n} \frac{\lambda_i}{\|\lambda\|} v_i\right\| \geq c$ for all $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \setminus \{0_{\mathbb{R}^n}\}$, we also get that $\left\|\sum_{i=1}^{n} \lambda_i v_i\right\| \geq c\|\lambda\|$ for all $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$. We now let $\lambda \in \mathbb{R}^n$ and $v' = \left(v_1', \ldots, v_n'\right) \in X^n$. Notice that $c\|\lambda\| \leq f(\lambda) = \left\|\sum_{i=1}^{n} \lambda_i v_i\right\| \leq \sum_{i=1}^{n} |\lambda_i| \left\|v_i - v_i'\right\| + \left\|\sum_{i=1}^{n} \lambda_i v_i'\right\| \leq \|\lambda\| \left(\sum_{i=1}^{n} \left\|v_i - v_i'\right\|^2\right)^{\frac{1}{2}} + \left\|\sum_{i=1}^{n} \lambda_i v_i'\right\|$ by the Schwarz inequality. We finally get the inequality $\left(c - \left(\sum_{i=1}^{n} \left\|v_i - v_i'\right\|^2\right)^{\frac{1}{2}}\right)\|\lambda\| \leq \left\|\sum_{i=1}^{n} \lambda_i v_i'\right\|$, as desired. $\square$

COROLLARY 4.2. *Suppose* $v = (v_1, \ldots, v_n) \in X^n$ *such that* $v_1, \ldots, v_n$ *are linearly independent. Also, suppose* $B_1, \ldots, B_m$ *are bounded subsets of* X*. Then, given* $M = \sup_{1 \leq j \leq m} \sup_{b \in B_j} \|b\| + 1$, *there exists a* $M \geq c > 0$ *such that for every* $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ *and for every* $v' = \left(v_1', \ldots, v_n'\right) \in X^n$, *if* $\|\lambda\| \geq \frac{2M+2}{c}$ *and for every* $1 \leq i \leq n$, $\left\|v_i - v_i'\right\| < \frac{c}{2\sqrt{n}}$, *then* $d\left(\sum_{i=1}^{n} \lambda_i v_i', B_j\right) \geq 2 > 1 > 0$ *for each* $1 \leq j \leq m$.

Proof. First, obtain $c_1 > 0$ from Lemma 4.1. Then, using $c = \min\{c_1, M\}$, we obtain

$$\left\|\sum_{i=1}^{n} \lambda_i v_i'\right\| \geq \left(c - \left(\sum_{i=1}^{n} \left\|v_i - v_i'\right\|^2\right)^{\frac{1}{2}}\right) \|\lambda\| \geq \left(c - \sqrt{n\frac{c^2}{4n}}\right) \|\lambda\| .$$

But, $\left(c - \sqrt{n\frac{c^2}{4n}}\right) \|\lambda\| = \frac{c}{2} \|\lambda\|$ and $\|\lambda\| \geq \frac{2M+2}{c}$. So, we then obtain $\left\|\sum_{i=1}^{n} \lambda_i v_i'\right\| \geq M + 1$. That is, $d\left(\sum_{i=1}^{n} \lambda_i v_i', B_j\right) \geq \left|\left\|\sum_{i=1}^{n} \lambda_i v_i'\right\| - (M-1)\right| \geq 2 > 1 > 0$ for each $1 \leq j \leq m$, as desired. □

Corollary 4.3. *Suppose $v = (v_1, \ldots, v_n) \in X^n$ such that $v_1, \ldots, v_n$ are linearly independent. Then, there exists a $c > 0$ such that for every $v' = \left(v_1', \ldots, v_n'\right) \in X^n$, if for every $1 \leq i \leq n$, $\left\|v_i - v_i'\right\| < \frac{c}{2\sqrt{n}}$, then $v_1', \ldots, v_n'$ are linearly independent.*

Proof. Let $\lambda \in \mathbb{R}^n$ and obtain $c > 0$ by Lemma 4.1. Then, $\left\|\sum_{i=1}^{n} \lambda_i v_i'\right\| \geq \frac{c}{2} \|\lambda\|$. If we suppose that $\|\lambda\| > 0$, then $\left\|\sum_{i=1}^{n} \lambda_i v_i'\right\| \geq \frac{c}{2} \|\lambda\| > 0$. That is, $v_1', \ldots, v_n'$ are linearly independent. □

Lemma 4.4. *Suppose $v = (v_1, \ldots, v_n) \in X^n$ and $V_1, \ldots, V_n$ are open subsets of X for which $v_i \in V_i$ for all $1 \leq i \leq n$. Then, there exists a set $L \subset \{1, \ldots, n\}$ such that $\{v_i : i \in L\}$ is linearly independent with a span equal to the span of $\{v_1, \ldots, v_n\}$. Furthermore, there exists $\epsilon > 0$ such that $\bigcap_{i \in L} B(v_i, \epsilon)^- \subset \bigcap_{i=1}^{n} V_i^-$.*

Proof. To take care of a trivial case, suppose $0_X \in V_j$ for some $1 \leq j \leq n$. Since $0_X \in A$ for all $A \in \mathrm{CLS}(X)$, we note that $X^- = V_j^- = \mathrm{CLS}(X)$. This says that $\bigcap_{i=1}^{j-1} V_i^- \cap \bigcap_{i=j+1}^{n} V_i^- = \bigcap_{i=1}^{n} V_i^-$ for every $1 \leq j \leq n$ satisfying $0_X \in V_j$. Without loss of generality, we shall assume $0_X \notin V_i$ for all $1 \leq i \leq n$.

Let $\tilde{v}_1 = v_1$ and for each $1 \leq i \leq n-1$, let $[v_{i+1}] \in X/\mathrm{span}\{v_1, \ldots, v_i\}$. Then, let $\tilde{v}_{i+1} = 0_X$ if $[v_{i+1}] = [0_X]$, but let $\tilde{v}_{i+1} = v_{i+1}$ otherwise, for $1 \leq i \leq n-1$. Furthermore, let $L = \{i \in \{1, \ldots, n\} : \tilde{v}_i \neq 0\}$, let $\Lambda = \{1, \ldots, n\} \setminus L$, and let $\ell = |L|$. Then, $\{v_i : i \in L\}$ is linearly independent and has a span equal to that of $\{v_1, \ldots, v_n\}$.

Let $j \in \Lambda$. Let $\iota : \{1, \ldots, \ell\} \to L$ be an increasing bijection. Let $\lambda^{(j)} = \left(\lambda_1^{(j)}, \ldots, \lambda_\ell^{(j)}\right) \in \mathbb{R}^\ell$ such that $0_X \neq v_j = \sum_{i=1}^{\ell} \lambda_i^{(j)} v_{\iota_i}$. Then, because of the aforementioned linear independence and our assumption that $0_X \notin V_j$, $\left\|\lambda^{(j)}\right\| > 0$. Let $0 < \epsilon_j = \frac{d(v_j, X \setminus V_j)}{2\left\|\lambda^{(j)}\right\|\sqrt{\ell}} < +\infty$. Then, so long as $\left\|v_{\iota_i} - v'_{\iota_i}\right\| < \epsilon_j$ for all $1 \leq i \leq \ell$, $\left\|v_j - \sum_{i=1}^{\ell} \lambda_i^{(j)} v'_{\iota_i}\right\| = \left\|\sum_{i=1}^{\ell} \lambda_i^{(j)} \left(v_{\iota_i} - v'_{\iota_i}\right)\right\| \leq \sum_{i=1}^{\ell} \left|\lambda_i^{(j)}\right| \left\|v_{\iota_i} - v'_{\iota_i}\right\| \leq \left\|\lambda^{(j)}\right\| \left(\sum_{i=1}^{\ell} \left\|v_{\iota_i} - v'_{\iota_i}\right\|^2\right)^{\frac{1}{2}}$, which is strictly less than $\left\|\lambda^{(j)}\right\| \left(\ell \left(\epsilon_j\right)^2\right)^{\frac{1}{2}}$, which equals $\frac{d(v_j, X \setminus V_j)}{2}$. Now, suppose $A \in \bigcap_{i \in L} B\left(v_i, \epsilon_j\right)^-$. This says that we have $v_i^- \in A \cap B\left(v_i, \epsilon_j\right)$ for all $i \in L$. By the above inequalities, this implies that $\sum_{i=1}^{\ell} \lambda_i^{(j)} v'_{\iota_i} \in V_j$. So, $\emptyset \neq A \cap V_j$. Thus, $A \in V_j^-$. That is, $\bigcap_{i \in L} B\left(v_i, \epsilon_j\right)^- \subset V_j^-$. Moreover, if we let $\min\left\{\min_{i \in L} \frac{d(v_i, X \setminus V_i)}{2}, \min_{j \in \Lambda} \epsilon_j\right\}$ be denoted by ϵ, then $0 < \epsilon$ and $\bigcap_{i \in L} B\left(v_i, \epsilon\right)^- \subset \bigcap_{i=1}^{n} V_i^-$, as desired. □

Lemma 4.5. *Suppose A is a closed linear subspace of X, $v = (v_1, \ldots, v_n)$ with $v_1, \ldots, v_n \in X$ linearly independent, $V_1, \ldots, V_n$ are open subsets of X, and $B_1, \ldots, B_m$ are closed, bounded, convex subsets of X satisfying $A \in \bigcap_{1 \leq i \leq n} V_i^- \cap \bigcap_{1 \leq j \leq m} \left(X \setminus B_j\right)^{++}$ and $v_i \in A \cap V_i$ for each $1 \leq i \leq n$. Suppose further that $M = \max_{1 \leq j \leq m} \sup_{b \in B_j} \|b\| + 1$ and $c = \min\{M, c_1\}$, where $c_1 = \min_{\|\lambda\|=1} \left\|\sum_{i=1}^{n} \lambda_i v_i\right\|$ as in Lemma 4.1. Then, there exists an $\epsilon > 0$ such that given $w = (w_1, \ldots, w_n) \in X^n$ and $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, $\|v_i - w_i\| < \epsilon$ for all $1 \leq i \leq n$ implies $\inf_{\|\lambda\| \leq \frac{2M+2}{c}} \min_{1 \leq j \leq m} d\left(\sum_{i=1}^{n} \lambda_i w_i, B_j\right) > \min_{1 \leq j \leq m} \frac{\epsilon(A, B_j)}{2} > 0$.*

Proof. Let $L = \left\{\lambda \in \mathbb{R}^n : \|\lambda\| \leq \frac{2M+2}{c}\right\}$. Let $f : \mathbb{R}^n \times X^n \to \mathbb{R}$ be defined by $f(\lambda, w) = \min_{1 \leq j \leq m} d\left(\sum_{i=1}^{n} \lambda_i w_i, B_j\right)$. Note that f is a continuous function as $(\lambda, w) \mapsto \sum_{i=1}^{n} \lambda_i w_i$ is continuous via X being a Banach space and both distance function and the minimum function are continuous. Also, note that the span of $v_1, \ldots, v_n$ is contained in A. Then, $\inf_{\lambda \in L} f(\lambda, v) \geq \min_{1 \leq j \leq m} d\left(A, B_j\right) > 0$. Furthermore, if $\lambda \in L$, then $\sum_{i=1}^{n} |\lambda_i| \leq D < +\infty$ for some $D > 0$ depending only on $\frac{2M+2}{c}$.

Let $1 \leq j_0 \leq m$ such that $\epsilon(A, B_{j_0}) \leq \epsilon(A, B_j)$ for all $1 \leq j \leq m$. Let $\epsilon = \frac{\epsilon\left(A, B_{j_0}\right)}{2Dn}$. Then, $\epsilon > 0$ and if $\|v_i - w_i\| < \epsilon$ for each $1 \leq i \leq n$, we have $\frac{\epsilon\left(A, B_{j_0}\right)}{2} = \epsilon(A, B_{j_0}) - \frac{\epsilon\left(A, B_{j_0}\right)}{2} \leq \epsilon(A, B_{j_0}) - D\frac{\epsilon\left(A, B_{j_0}\right)}{2D} \leq \epsilon(A, B_{j_0}) - \left(\sum_{i=1}^{n} |\lambda_i|\right) \frac{n\epsilon\left(A, B_{j_0}\right)}{2Dn} < d\left(\sum_{i=1}^{n} \lambda_i v_i, B_{j_0}\right) - \sum_{i=1}^{n} \|\lambda_i (w_i - v_i)\| \leq d\left(\sum_{i=1}^{n} \lambda_i v_i, B_{j_0}\right) - \left\|\sum_{i=1}^{n} \lambda_i (w_i - v_i)\right\| \leq \min_{1 \leq j \leq m} d\left(\sum_{i=1}^{n} \lambda_i w_i, B_j\right)$ for all $\lambda \in L$. That is, $\inf_{\lambda \in L} f(\lambda, w) > \min_{1 \leq j \leq m} \frac{\epsilon(A, B_j)}{2} > 0$ whenever $\|w_i - v_i\| < \epsilon$ for all $1 \leq i \leq n$. $\square$

4.2. The Strong Choquet Result

THEOREM 4.6. *Suppose $\mathcal{C} \subseteq CB(X)$ satisfies the conditions laid out in Definition 2.2. Then the space $CLS(X)$ with the topology $\tau_{\mathcal{C}}$ is strong Choquet.*

PROOF. Given A_n, a closed linear subspace of X, and $A_n \in \mathbf{V}_n$, a basic open subset of $\mathrm{CLS}(X)$ under $\tau_{\mathcal{C}}$, we obtain $\mathbf{U}_n$ in the following manner

Initially, we must tend to three trivial cases. First, suppose Player EMPTY plays X as the choice of closed linear subspace of X and an open set of the form $\mathbf{V} = \bigcap_{i=1}^{k} V_i^-$. Player NONEMPTY would respond with $\mathbf{U} = \mathbf{V}$. Should Player EMPTY continue in such a manner through the rest of the game, X will still be in the intersection, so the game is trivially won by Player NONEMPTY.

For the second trivial case, should Player EMPTY play a proper closed linear subspace of X, say A, and an open set of the form $\mathbf{V} = \bigcap_{i=1}^{k} (V_i)^-$ with $A \in \mathbf{V}$, then there exists a point $x \notin A$. Since A is closed, $0 < d(x, A)$. Let $B_1 = \{x\}$, a closed, bounded, convex subset of X satisfying $A \in (X \setminus B_1)^{++}$. For each $i \in [1, k] \cap \mathbb{N}$, fix $v_i \in V_i \cap A$. Let $v = (v_1, \ldots, v_k) \in X^k$. We will now appeal to the algorithm used in the proof of Lemma 4.4 to obtain ϵ_1, $L \subset \{1, \ldots, k\}$, and $\ell = |L|$ such that $\{v_i : i \in L\}$ is linearly independent and $\bigcap_{i \in L} B(v_i, \epsilon_1)^- \subset \bigcap_{i=1}^{k} (V_i)^-$. Next, we let $\iota : \{1, \ldots, \ell\} \to L$ be a one-to-one, increasing, onto function. Letting $u = (v_{\iota_1}, \ldots, v_{\iota_\ell})$ and $M = \|x\| + 1$, we now appeal to Corollary 4.2 to obtain $M \geq c > 0$ such that $d\left(\sum_{i=1}^{\ell} \lambda_i v'_{\iota_i}, x\right) \geq 2 > 1 > 0$ whenever $\left\|v'_{\iota_i} - v_{\iota_i}\right\| < \frac{c}{2\sqrt{\ell}}$ for each $1 \leq i \leq \ell$ and $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathbb{R}^\ell$ satisfies $\|\lambda\| \geq \frac{2M+2}{c}$. Note that Corollary 4.3 tells us that whenever $v'_{\iota_i} \in B\left(v_{\iota_i}, \frac{c}{2\sqrt{\ell}}\right)$ for each $1 \leq i \leq \ell$, we know that $v'_{\iota_1}, \ldots, v'_{\iota_\ell}$ are linearly independent also. Finally, we appeal to Lemma 4.5 to obtain ϵ_2 such that whenever $\lambda \in \mathbb{R}^\ell$ satisfies $\|\lambda\| \leq \frac{2M+2}{c}$ and $v_{\iota_i}^{'} \in B(v_{\iota_i}, \epsilon_2)$ for all $1 \leq i \leq \ell$, then $d\left(\sum_{i=1}^{\ell} \lambda_i v_{\iota_i}^{'}, x\right) > \frac{\epsilon(A,B_1)}{2} > 0$. Letting $\epsilon = \frac{1}{2}\min\left\{1, \epsilon_1, \frac{c}{2\sqrt{\ell}}, \epsilon_2\right\} > 0$, we now have that whenever $v_{\iota_i}^{'} \in B(v_{\iota_i}, \epsilon)$ for every $1 \leq i \leq \ell$, then $\inf_{\lambda \in \mathbb{R}^\ell} d\left(\sum_{i=1}^{\ell} \lambda_i v_{\iota_i}^{'}, x\right) > \min\left\{\frac{1}{2}, \frac{\epsilon(A,B_1)}{2}\right\} > 0$. We shall let $\mu = \min\left\{\frac{1}{2}, \frac{\epsilon}{2}, \frac{\epsilon(A,B_1)}{2}\right\} > 0$. We let $\mathbf{U} = \bigcap_{i \in L} B(v_i, \epsilon)^- \cap \left(X \setminus \overline{S}_\mu[B_1]\right)^{++}$. We further let $U_i = B(v_{\iota_i}, \epsilon)$ for all $1 \leq i \leq \ell$ and $D_1 = \overline{S}_\mu[B_1]$. We note that $A \in \mathbf{U} \subset \mathbf{V}$. Moreover, for all future moves, Player EMPTY is now forced to have $k_n > 0$ and $m_n > 0$, leaving us with nontrivial cases from move $n = 0$ onward.

For the third and final trivial case, should Player EMPTY play A a proper closed linear subspace of X and an open subset of the form $\mathbf{V} = \bigcap_{j=1}^{m} (X \setminus B_j)^{++}$ with $A \in \mathbf{V}$, then there exists a point $v \in A \setminus \{0_X\}$ with $\|v\| = 1$. Letting $0 < \delta = \min\left\{\frac{1}{2}, \min_{1 \leq j \leq m} \frac{\epsilon(A,B_j)}{2}\right\}$, we let $V = B(v, \delta)$ and $v_1 = v$. We may forgo appealing to Lemma 4.4 since $\{v\}$ is a trivially linearly independent set of vectors in X. We may directly obtain $L = \{1\}$, $\ell = 1$, and $\epsilon_1 = \delta$. Now, let $M = \max_{1 \leq j \leq m} \sup_{b \in B_j} \|b\| + 1$ and $u = v_1 = v$. We appeal to Corollary 4.2 to obtain $M \geq c > 0$ such that $d\left(\lambda_1 v_1^{'}, B_j\right) \geq 2 > 1 > 0$ whenever $1 \leq j \leq m$, $\left\|v_1^{'} - v_1\right\| < \frac{c}{2}$, and $\lambda_1 \in \mathbb{R}$ satisfies $|\lambda_1| \geq \frac{2M+2}{c}$. Obviously, $\left\{v_1^{'}\right\}$ is a linearly independent set of vectors in X. So, we may forgo appealing to Corollary 4.3. We now appeal to Lemma 4.5 to obtain ϵ_2 such that whenever $\lambda_1 \in \mathbb{R}$ satisfies $|\lambda_1| \leq \frac{2M+2}{c}$ and whenever $v_1^{'} \in B(v_1, \epsilon_2)$, we have $\min_{1 \leq j \leq m} d\left(\lambda_1 v_1^{'}, B_j\right) > \min_{1 \leq j \leq m} \frac{\epsilon(A,B_j)}{2} > 0$. We let $\epsilon = \frac{1}{2}\min\left\{1, \epsilon_1, \frac{c}{2}, \epsilon_2\right\}$. Now, whenever $v_1^{'} \in B(v_1, \epsilon)$ and $1 \leq j \leq m$, we have that $\inf_{\lambda_1 \in \mathbb{R}} d\left(\lambda_1 v_1^{'}, B_j\right) > \min\left\{\frac{1}{2}, \min_{1 \leq j' \leq m} \frac{\epsilon\left(A, B_{j'}\right)}{2}\right\} > 0$. We let $\mu = \min\left\{\frac{1}{2}, \frac{\epsilon}{2}, \min_{1 \leq j' \leq m} \frac{\epsilon\left(A, B_{j'}\right)}{2}\right\} > 0$, $\mathbf{U} = B(v, \epsilon)^- \cap \bigcap_{j=1}^{m} \left(X \setminus \overline{S}_\mu[B_j]\right)^{++}$, $U_1 = B(v, \epsilon)$, and $D_j = \overline{S}_\mu[B_j]$ for all $1 \leq j \leq m$. Note $A \in \mathbf{U} \subset \mathbf{V}$. Moreover, for all future moves, Player EMPTY is now forced to have $k_n > 0$ and $m_n > 0$, leaving us with nontrivial cases from move $n = 0$ onward.

Having attended to the three trivial cases, all other initial choices by Player EMPTY shall be considered nontrivial, and will thus be considered as the 0^{th} move of Player EMPTY.

For $n = 0$, we first note that $\mathbf{V}_0 = \bigcap_{i=1}^{k_0} (V_i^0)^- \cap \bigcap_{j=1}^{m_0} \left(X \setminus B_j^0\right)^{++}$. Subject to the caveats above, we may assume $k_0 > 0$ and $m_0 > 0$, allowing us to go about finding $\mathbf{U}_0$. We fix $v_i^0 \in V_i^0 \cap A_0$ for each i. We then let $v^{(0)} = \left(v_1^0, \ldots, v_{k_0}^0\right) \in X^{k_0}$ and appeal to the algorithm used in the proof of Lemma 4.4 to obtain $\epsilon_{0,1}$, $L_0 \subset \{1, \ldots, k_0\}$, and $\ell_0 = |L_0|$ such that $\{v_i^0 : i \in L_0\}$ is linearly independent and $\bigcap_{i \in L_0} B\left(v_i^0, \epsilon_{0,1}\right)^- \subset \bigcap_{i=1}^{k_0} (V_i^0)^-$. Next, we let $\iota^0 : \{1, \ldots, \ell_0\} \to L_0$ be one-to-one, onto, and strictly increasing. We let $u_0 = \left(v_{\iota_1^0}^0, \ldots, v_{\iota_{\ell_0}^0}^0\right)$ and $M_0 = \max_{1 \le j \le m_0} \sup_{b \in B_j^0} \|b\| + 1$ in order to appeal to Corollary 4.2 and obtain $M_0 \ge c_0 > 0$ such that $d\left(\sum_{i=1}^{\ell_0} \lambda_i v'_{\iota_i^0}, B_j^0\right) \ge 2 > 1 > 0$ whenever $1 \le j \le m_0$, $\left\|v'_{\iota_i^0} - v_{\iota_i^0}^0\right\| < \frac{c_0}{2\sqrt{\ell_0}}$, and $\lambda = (\lambda_1, \ldots, \lambda_{\ell_0}) \in \mathbb{R}^{\ell_0}$ satisfies $\|\lambda\| \ge \frac{2M_0+2}{c_0}$. Note, Corollary 4.3 tells us that whenever $v'_{\iota_i^0} \in B\left(v_{\iota_i^0}^0, \frac{c_0}{2\sqrt{\ell_0}}\right)$ for every $1 \le i \le \ell_0$, we know $v'_{\iota_1^0}, \ldots, v'_{\iota_{\ell_0}^0}$ are linearly independent also. Finally, we appeal to Lemma 4.5 to obtain $\epsilon_{0,2}$ such that whenever $\lambda \in \mathbb{R}^{\ell_0}$ satisfies $\|\lambda\| \le \frac{2M_0+2}{c_0}$ and $v'_{\iota_i^0} \in B\left(v_{\iota_i^0}^0, \epsilon_{0,2}\right)$ for all $1 \le i \le \ell_0$, then $\min_{1 \le j \le m_0} d\left(\sum_{i=1}^{\ell_0} \lambda_i v'_{\iota_i^0}, B_j^0\right) > \min_{1 \le j \le m_0} \frac{\epsilon\left(A_0, B_j^0\right)}{2} > 0$. Letting $\epsilon_0 = \frac{1}{2} \min\left\{\epsilon_{0,1}, \frac{c_0}{2\sqrt{\ell_0}}, \epsilon_{0,2}\right\}$, we now have that whenever $v'_{\iota_i^0} \in B\left(v_{\iota_i^0}^0, \epsilon_0\right)$ for every $1 \le i \le \ell_0$ and whenever $1 \le j \le m_0$, $\inf_{\lambda \in \mathbb{R}^{\ell_0}} d\left(\sum_{i=1}^{\ell_0} \lambda_i v'_{\iota_i^0}, B_j^0\right) > \min\left\{\frac{1}{2}, \min_{1 \le j' \le m_0} \frac{\epsilon\left(A_0, B_{j'}^0\right)}{2}\right\} > 0$. We shall let $\mu_0 = \min\left\{\frac{1}{2}, \frac{\epsilon_0}{2}, \min_{1 \le j' \le m_0} \frac{\epsilon\left(A_0, B_{j'}^0\right)}{2}\right\} > 0$. We let $\mathbf{U}_0 = \bigcap_{i \in L_0} B\left(v_i^0, \epsilon_0\right)^- \cap \bigcap_{1 \le j \le m_0} \left(X \setminus \overline{S}_{\mu_0}\left[B_j^0\right]\right)^{++}$. We further let $U_i^0 = B\left(v_{\iota_i^0}^0, \epsilon_0\right)$, where $1 \le i \le \ell_0$ and $D_j^0 = \overline{S}_{\mu_0}\left[B_j^0\right]$ for all $1 \le j \le m_0$. Note that $A_0 \in \mathbf{U}_0$.

For $n > 0$, we start by fixing $v_i^n \in U_i^{n-1} \cap A_n$, for each $1 \le i \le \ell_{n-1}$ and then fixing $v_i^n \in V_{i-\ell_{n-1}}^n \cap A_n$ for each $\ell_{n-1} + 1 \le i \le \ell_{n-1} + k_n$. We let $v^{(n)} = \left(v_1^n, \ldots, v_{\ell_{n-1}+k_n}^n\right)$. We remember that, without loss of generality, we may and shall assume that $\left\{D_j^{n-1} : 1 \le j \le m_{n-1}\right\} \subset \left\{B_j^n : 1 \le j \le m_n\right\}$. As before, we start by appealing to the algorithm used in the proof of Lemma 4.4 to obtain $\epsilon_{n,1}$, $L_n \subset \{1, \ldots, \ell_{n-1} + k_n\}$, and $\ell_n = |L_n|$ such that $\{v_i^n : i \in L_n\}$ is linearly independent and $\bigcap_{i \in L_n} B\left(v_i^n, \epsilon_{n,1}\right)^- \subset \bigcap_{i=1}^{\ell_{n-1}} \left(U_i^{n-1}\right)^- \cap \bigcap_{i=1}^{k_n} (V_i^n)^- \subset \bigcap_{i=1}^{k_n} (V_i^n)^-$. Note, by using the algorithm in the proof of the aforementioned Lemma, we have that $\{1, \ldots, \ell_{n-1}\} \subset$

L_n. Next, we let $\iota^n : \{1, \ldots, \ell_n\} \to L_n$ be a one-to-one, onto, and strictly increasing function. We let $u_n = \left(v^n_{\iota^n_1}, \ldots, v^n_{\iota^n_{\ell_n}}\right)$ and $M_n = \max_{1 \le j \le m_n} \sup_{b \in B^n_j} \|b\| + 1$ in order to appeal to Corollary 4.2 and obtain $M_n \ge c_n > 0$ such that $d\left(\sum_{i=1}^{\ell_n} \lambda_i v'_{\iota^n_i}, B^n_j\right) \ge 1 > \frac{1}{2} > 0$ whenever $1 \le j \le m_n$, $\left\|v'_{\iota^n_i} - v^n_{\iota^n_i}\right\| < \frac{c_n}{2\sqrt{\ell_n}}$, and $\lambda = (\lambda_1, \ldots, \lambda_{\ell_n}) \in \mathbb{R}^{\ell_n}$ satisfies $\|\lambda\| \ge \frac{2M_n+2}{c_n}$. Note, Corollary 4.3 tells us that whenever $v'_{\iota^n_i} \in B\left(v^n_{\iota^n_i}, \frac{c_n}{2\sqrt{\ell_n}}\right)$ for every $1 \le i \le \ell_n$, we know $v'_{\iota^n_1}, \ldots, v'_{\iota^n_{\ell_n}}$ are linearly independent also. Finally, we appeal to Lemma 4.5 to obtain $\epsilon_{n,2}$ such that whenever $\lambda \in \mathbb{R}^{\ell_n}$ satisfies $\|\lambda\| \le \frac{2M_n+2}{c_n}$ and $v'_{\iota^n_i} \in B\left(v^n_{\iota^n_i}, \epsilon_{n,2}\right)$ for all $1 \le i \le \ell_n$, then $\min_{1 \le j \le m_n} d\left(\sum_{i=1}^{\ell_n} \lambda_i v'_{\iota^n_i}, B^n_j\right) > \min_{1 \le j \le m_n} \frac{\epsilon\left(A_n, B^n_j\right)}{2} > 0$. Letting $\epsilon_n = \frac{1}{2} \min\left\{\epsilon_{n-1}, \epsilon_{n,1}, \frac{c_n}{2\sqrt{\ell_n}}, \epsilon_{n,2}\right\}$, we now have that whenever $v'_{\iota^n_i} \in B\left(v^n_{\iota^n_i}, \epsilon_n\right)$ for every $1 \le i \le \ell_n$ and whenever $1 \le j \le m_n$, $\inf_{\lambda \in \mathbb{R}^{\ell_n}} d\left(\sum_{i=1}^{\ell_n} \lambda_i v'_{\iota^n_i}, B^n_j\right) > \min\left\{\frac{\mu_{n-1}}{2}, \min_{1 \le j' \le m_n} \frac{\epsilon\left(A_n, B^n_{j'}\right)}{2}\right\} > 0$. We shall let $\mu_n = \min\left\{\frac{\mu_{n-1}}{2}, \frac{\epsilon_n}{2}, \min_{1 \le j \le m_n} \frac{\epsilon\left(A_n, B^n_j\right)}{2}\right\} > 0$. We let $\mathbf{U}_n = \bigcap_{i \in L_n} B\left(v^n_i, \epsilon_n\right)^- \cap \bigcap_{1 \le j \le m_n} \left(X \setminus \overline{S}_{\mu_n}\left[B^n_j\right]\right)^{++}$. We further let $U^n_i = B\left(v^n_{\iota^n_i}, \epsilon_n\right)$, where $1 \le i \le \ell_n$ and $D^n_j = \overline{S}_{\mu_n}\left[B^n_j\right]$ for all $1 \le j \le m_n$. Note that $A_n \in \mathbf{U}_n$ and note that $v^n_i \in U^n_i \subset \overline{U}^n_i \subset U^{n-1}_i$ for all $1 \le i \le \ell_{n-1}$.

Using the strategy just defined, we shall take a run of the strong Choquet game on $\mathrm{CLS}(X)$. We shall immediately take stock of the moves made by Player NONEMPTY, i.e., the $\mathbf{U}_n$ sets. Notice that given a fixed n_0 and a fixed $\ell_{n_0-1} < i \le \ell_{n_0}$, where $\ell_{-1} = 0$, $U^{n_0}_i \supset \overline{U^{n_0+1}_i} \supset U^{n_0+1}_i \supset \cdots$ and $0 < \epsilon_n \le \frac{\epsilon_0}{2^n}$ and $\frac{\epsilon_0}{2^n} \to 0$. That is, $\operatorname{diam}\left(U^{n_0+\alpha}_i\right) \to 0$ as $\alpha \to +\infty$. Therefore, we may let $\{x_i\} = \bigcap_{\alpha > -1} U^{n_0+\alpha}_i$. Now, let $\beta > 0$. Then, we note that $A^\beta = \operatorname{span}\{x_i : 0 \le i \le \ell_\beta\}$ obeys $d\left(A^\beta, B^n_j\right) > \mu_n > 0$ for all $0 \le n$ and $1 \le j \le m_n$. So, $d\left(\overline{\operatorname{span}}\{x_i : i \in \mathbb{N}\}, B^n_j\right) \ge \mu_n > \frac{\mu_n}{2} > 0$ for all $0 \le n$ and $1 \le j \le m_n$. In other words, $\overline{\operatorname{span}}\{x_i : i \in \mathbb{N}\} \in \bigcap_{0 \le n,\ 1 \le j \le m_n} \left(X \setminus B^n_j\right)^{++} = \bigcap_{0 \le n,\ 1 \le j \le m_n} \left(X \setminus D^n_j\right)^{++}$. All that remains to show, then, is that $\overline{\operatorname{span}}\{x_i : i \in \mathbb{N}\} \in \bigcap_{0 \le n,\ 1 \le i \le \ell_n} \left(U^n_i\right)^-$. Indeed, for each $i \in \mathbb{N}$, we recall that $x_i = \lim_{\alpha \to +\infty} v^{n_i+\alpha}_i$ where $\ell_{n_i-1} < i \le \ell_{n_i}$. So, $x_{i'} \in \overline{\operatorname{span}}\{x_i : i \in \mathbb{N}\} \cap U^{n_{i'}+\alpha}_{i'}$ for all $\alpha \ge 0$ and for all $i' \in \mathbb{N}$ where $\ell_{n_{i'}-1} < i' \le \ell_{n_{i'}}$. That is, $\overline{\operatorname{span}}\{x_i : i \in \mathbb{N}\} \in \bigcap_{0 \le n,\ 1 \le i \le \ell_n} \left(U^n_i\right)^-$.

Therefore, $\overline{\operatorname{span}}\{x_i : i \in \mathbb{N}\} \in \bigcap_{n\geq 0} \mathbf{U}_n$. Whence, the strategy used by Player NONEMPTY was a winning strategy and $\operatorname{CLS}(X)$ under $\tau_{\mathcal{C}}$ is strong Choquet. $\square$

COROLLARY 4.7. *The space CLS(X) under the slice topology is strong Choquet.*

PROOF. Since the slice topology on $\operatorname{CLS}(X)$ is in the collection $\mathcal{T}$ of topologies on $\operatorname{C}(X)$ defined in Definition 2.2, Theorem 4.6 yields the result. $\square$

4.2.1. Category Results

COROLLARY 4.8. *Given an infinite dimensional Banach space X, the collection of finite dimensional closed linear subspaces of X is a meager F_σ in CLS(X) under the slice topology.*

PROOF. First, let $\mathcal{F} = \{A \in \operatorname{CLS}(X) : \dim(A) < +\infty\}$. Also, for each positive integer n, let $\mathcal{F}_n = \{A \in \mathcal{F} : \dim(A) \leq n\}$. Note that $\mathcal{F} = \bigcup_{n>0} \mathcal{F}_n$. All that remains to show now is that $\mathcal{F}_n$ is both closed and nowhere dense for each positive integer n. We shall start by establishing the former property. To do this, we shall let

$$K_n = \big\{(x_1, \ldots, x_{n+1}) \in X^{n+1} : x_1, \ldots, x_{n+1} \text{ are linearly independent}\big\}$$

for each positive integer n. Now, we let n be a positive integer. Then, $\operatorname{CLS}(X) \setminus \mathcal{F}_n = \bigcup_{x=(x_1,\ldots,x_{n+1})\in K_n} \bigcap_{i=1}^{n+1} B(x_i, \epsilon_x)^-$, where ϵ_x is obtained by Corollary 4.3 for each $x \in X^{n+1}$. That is, $\mathcal{F}_n$ is closed. All that now remains is to prove the latter property.

By way of contradiction, let us suppose n is a positive integer that satisfies $\operatorname{int}(\mathcal{F}_n) \neq \emptyset$. Let $A \in \operatorname{int}(\mathcal{F}_n)$. Then, there exists $\mathbf{V} = \bigcap_{i=1}^{k} V_i^- \cap \bigcap_{j=1}^{m} (X \setminus B_j)^{++} \subseteq \operatorname{int}(\mathcal{F}_n)$, where $V_1, \ldots, V_k \in \tau_X$ and $B_1, \ldots, B_m \in \operatorname{CB}(X)$, that satisfies $A \in \mathbf{V}$. Note that A is a circled closed convex subset of X and that we just stated B_j is a closed bounded convex subset of X for each $1 \leq j \leq m$. By Kelley & Namioka [6, p. 118-119], since $0_X \notin \overline{A - B_j}$ for each $1 \leq j \leq m$, by virtue of $A \in (X \setminus B_j)^{++}$, then there exists f_j, a continuous linear functional that satisfies $\sup_{\alpha\in A} |f_j(\alpha)| < \inf_{\beta\in B^j} |f_j(\beta)|$. As $A \in \operatorname{CLS}(X)$ and $B_j \in \operatorname{CB}(X)$ for all $1 \leq j \leq m$, we note that $\sup_{\alpha\in A} |f_j(\alpha)| < +\infty$. Thus, for all $1 \leq j \leq m$, for all $\lambda \in \mathbb{R}$, and for all $\alpha \in A$, $|f_j(\lambda\alpha)| = |\lambda| |f_j(\alpha)| < \inf_{\beta\in B_j} |f_j(\beta)|$, and thus $|f_j(\alpha)| = 0$ for all $\alpha \in A$. That is,

$A \subseteq \ker(f_j)$ for each $1 \leq j \leq m$. Now, let $K = \bigcap_{j=1}^{m} \ker(f_j)$. Clearly, $A \subseteq K$. Let us also note that for each $1 \leq j \leq m$, $\ker(f_j)$ is a closed linear subspace of X. Therefore, $K \in \mathrm{CLS}(X)$. Moreover, for each $1 \leq j \leq m$, $\operatorname{codim}(\ker(f_j)) = 1$ since f_j is a continuous linear functional. As K is the finite intersection of these kernels, $1 \leq \operatorname{codim}(K) \leq m < +\infty$. Therefore, $\dim(K)$ is not finite. However, we again appeal to Kelley & Namioka [6, p. 118-119] to obtain that $0_X \notin \overline{K - B_j}$ for each $1 \leq j \leq m$. That is, there exists $0 < \epsilon(K, B_j) < d(K, B_j)$ such that $S_{\epsilon(K,B_j)}[K] \subseteq X \setminus B_j$. That is, $K \in \bigcap_{j=1}^{m} (X \setminus B_j)^{++}$. Moreover, as $A \subseteq K$ and $A \in \bigcap_{i=1}^{k} V_i^-$, then $K \in \bigcap_{i=1}^{k} V_i^-$. That is, $K \in \mathbf{V} \subseteq \operatorname{int}(\mathcal{F}_n)$. However, this contradictorily states that K is finite dimensional. Therefore, $\operatorname{int}(\mathcal{F}_n) = \emptyset$ for all positive integers n. Therefore, for each positive integer n, we have shown that $\mathcal{F}_n$ is a closed nowhere dense subset of $\mathrm{CLS}(X)$ under the slice topology. Whence, $\mathcal{F}$ is a meager F_σ under the slice topology. □

Corollary 4.9. *The collection of closed linear subspaces of an infinite dimensional Banach space X that have infinite dimension are a dense G_δ subset of CLS(X) under the slice topology. Moreover, this collection forms a strong Choquet space under the slice topology.*

Proof. This is the complement of the collection from Corollary 4.8. Hence, it is a dense G_δ subset of $\mathrm{CLS}(X)$ under the slice topology. Consequently, Corollary 4.7 and Theorem 3.5 tell us that this collection is strong Choquet under the slice topology. □

Corollary 4.10. *The collection of closed linear subspaces of an infinite dimensional Banach space X that have infinite co-dimension is a dense G_δ subset of CLS(X) under the slice topology. Moreover, this collection forms a strong Choquet space under the slice topology.*

Proof. First, let n be a fixed positive integer. Let $S_{x_1,\ldots,x_n} = (X_1 \setminus B_1) \cap \operatorname{span}(x_1, \ldots, x_n)$ for $x_1, \ldots, x_n$ linearly independent vectors of X. Notice that each $S_{x_1,\ldots,x_n}$ is the unit sphere of an n dimensional linear subspace of X, and is thus compact. Let

$$C_{x_1,\ldots,x_n} = \left\{ \operatorname{span}(x_1, \ldots, x_n) \cap \overline{B(x, 1/2)} : x \in S_{x_1,\ldots,x_n} \right\},$$

a cover in $\operatorname{span}(x_1, \ldots, x_n)$ of the compact $S_{x_1,\ldots,x_n}$, and let $F_{x_1,\ldots,x_n}$ be a finite subcover. Then, as $F_{x_1,\ldots,x_n} \subset \mathrm{CB}(X)$ is finite, $\bigcap_{B \in F_{x_1,\ldots,x_n}} (X \setminus B)^{++}$ is a basic open subset of $\mathrm{CLS}(X)$. Moreover, we shall let $\mathrm{CD}_n = \bigcup_{x_1,\ldots,x_n \in X \text{ lin. indep.}} \bigcap_{B \in F_{x_1,\ldots,x_n}} (X \setminus B)^{++}$, which is an open subset of $\mathrm{CLS}(X)$. Notice that CD_n is the collection of all closed linear subspaces of X which are a positive distance from some $S_{x_1,\ldots,x_n}$, where $x_1, \ldots, x_n$ are linearly independent vectors of X. In other words, CD_n is the collection of co-dimension n or greater closed linear subspaces of X. Since CD_n contains the finite dimensional linear subspaces of X, CD_n is dense in $\mathrm{CLS}(X)$ under the slice topology according to Lemma 2.7. Moreover, the construction of CD_n is clearly open. Now, let $\mathrm{CD}_\infty = \bigcap_{n=1}^{\infty} \mathrm{CD}_n$. By Corollary 4.7 and Theorem 3.5, we know not only that CD_∞ is a dense G_δ subset of $\mathrm{CLS}(X)$ under the slice topology, which shows it to be co-meager, but also that CD_∞ is itself a strong Choquet space. □

Corollary 4.11. *The collection of closed linear subspaces of an infinite dimensional Banach space X that have both infinite dimension and infinite co-dimension is a dense G_δ subset of CLS(X) under the slice topology. Moreover, this collection forms a strong Choquet space under the slice topology.*

Proof. Let the collection from Corollary 4.9 be labelled I and let the collection from Corollary 4.10 be labelled IC for the moment. Note that the collection referenced in the present corollary is $I \cap IC$, a dense G_δ subset of $\mathrm{CLS}(X)$. Hence, Corollary 4.7 and Theorem 3.5 again tell us that this collection $I \cap IC$ is a strong Choquet space under the slice topology. □

Corollary 4.12. *Given a proper, closed linear subspace Y of a Banach space X, CLS(Y) is a closed, nowhere-dense subset of CLS(X) under the slice topology.*

Proof. We first note that $\mathrm{CLS}(Y)$ is closed by Lemma 2.5. To see that $\mathrm{CLS}(Y)$ has an empty interior, we shall assume it has a nonempty interior and derive a contradiction. Suppose $\mathbf{V} = \bigcap_{i=1}^{n} V_i^- \cap \bigcap_{j=1}^{m} (X \setminus B_j)^{++}$, where V_i open in X and $B_j \in \mathrm{CB}(X)$ for each i and j, satisfies $\emptyset \neq \mathbf{V} \subset \operatorname{int}(\mathrm{CLS}(Y))$. Let $y_i \in V_i \cap Y$ for each i. Then, by Lemma 4.4, there exists a set $L \subset \{1 \ldots, n\}$ such that $\{y_i : i \in L\}$ is linearly independent with a span equal to that of $\{y_1, \ldots, y_n\}$ and there exists an $\epsilon > 0$ such that $\bigcap_{i \in L} B(y_i, \epsilon)^- \subset \bigcap_{i=1}^{n} V_i^-$. Furthermore, the results from Lemma 4.1 through Lemma 4.5 produce an $\epsilon' > 0$ such that, so long as $\|x_i - y_i\| < \epsilon'$ for each $i \in L$, then the span of $\{x_i : i \in L\}$ is in $\mathbf{V}$. Let $i_1 \in L$ be minimal. There is an $x_{i_1} \in B(y_i, \epsilon')$ such that $x_{i_1} \notin Y$. Moreover, if $\ell = |L|$ and $i_1 < i_2 < \cdots < i_\ell$ for $\{i_1, \ldots, i_\ell\} = L$, then if A is the span of $\{x_{i_1}, y_{i_2}, \ldots, y_{i_\ell}\}$, A is not in $\mathrm{CLS}(Y)$. However, $A \in \mathbf{V}$, contradicting the fact that $\mathbf{V} \subset \mathrm{int}(\mathrm{CLS}(Y)) \subset \mathrm{CLS}(Y)$. The only remaining options are $\mathbf{V} = \bigcap_{i=1}^{n} V_i^-$ for some V_i open in X, which is immediately contradicted by noting there is an $x_1 \in V_1 \setminus Y$ and the span of $\{x_1, y_2, \ldots, y_n\}$ is not in $\mathrm{CLS}(Y)$ but is in $\mathbf{V}$, or $\mathbf{V} = \bigcap_{j=1}^{m} (X \setminus B_j)^{++}$ for some $B_j \in \mathrm{CB}(X)$, which is contradicted by the existence of $x \in B(y, \epsilon) \setminus Y$ that yields the span of x is in $\mathrm{CLS}(X) \setminus \mathrm{CLS}(Y)$ and in $\mathbf{V}$ for sufficiently small $\epsilon > 0$. Ergo, $\mathrm{int}(\mathrm{CLS}(Y)) = \emptyset$. □

Corollary 4.13. *Given a proper, infinite dimensional, closed linear subspace Y of an infinite dimensional Banach space X, $\mathcal{A} = \{A \in CLS(X) : Y \subset A\}$ is a closed, nowhere-dense subset of $CLS(X)$ under the slice topology.*

Proof. Suppose, by way of contradiction, that $\mathbf{V} = \bigcap_{i=1}^{n} V_i^- \cap \bigcap_{j=1}^{m} (X \setminus B_j)^{++}$, where V_i open in X and $B_j \in \mathrm{CB}(X)$ for each i and j, satisfies $\emptyset \neq \mathbf{V} \subset \mathrm{int}(\mathcal{A})$. Then, let $a_i \in V_i$ for each i such that the span of $\{a_1, \ldots, a_n\}$, call it A, is in $\bigcap_{j=1}^{m} (X \setminus B_j)^{++}$. So, $A \in \mathbf{V} \setminus \mathcal{A}$, a contradiction. Suppose $\mathbf{V} = \bigcap_{i=1}^{n} V_i^-$, for some V_i open in X, is in $\mathrm{int}(\mathcal{A})$. Again, let $a_i \in V_i$ for each i such that the span of $\{a_1, \ldots, a_n\}$, call it A, is in $\bigcap_{j=1}^{m} (X \setminus B_j)^{++}$. So, $A \in \mathbf{V} \setminus \mathcal{A}$, a contradiction. Lastly, suppose $\mathbf{V} = \bigcap_{j=1}^{m} (X \setminus B_j)^{++}$, for some $B_j \in \mathrm{CB}(X)$, satisfies $\mathbf{V} \subset \mathrm{int}(\mathcal{A})$. Then, there exists $x \in X$ such that $A = \mathrm{span}(x)$ is in $\mathbf{V}$. Moreover, A is finite dimensional, so it misses $\mathcal{A}$, a contradiction. Having exhausted all the possible cases, we know that $\mathcal{A}$ is nowhere-dense in $\mathrm{CLS}(X)$ under the slice topology. Furthermore, Lemma 2.6 says that $\mathcal{A}$ is closed in $\mathrm{CLS}(X)$. □

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