Characters, correspondences and fields of values of finite groups

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A mis padres y hermanos

Contents

Resumen

Uno de los temas más importantes en teoría de grupos finitos es el estudio de la relación entre los invariantes globales y locales de un grupo. Sea G un grupo finito y p un primo. Los p-subgrupos de G son los subgrupos de G cuyo orden es una potencia de p, y los subgrupos locales de G son los normalizadores propios de p-subgrupos de G. Como paradigma de todo esto, podemos citar un teorema clásico de Frobenius, que ha inspirado muchos resultados recientes. Este teorema establece que un grupo G posee un p-complemento normal si y sólo si cada uno de sus subgrupos locales tiene un p-complemento normal.

En esta tesis, nuestra atención se centra en el estudio de caracteres de grupos finitos, tanto ordinarios (aquellos asociados a una representación sobre el cuerpo de los números complejos) como modulares (aquellos asociados a una representación sobre un cuerpo de característica p). Nos interesa especialmente la relación entre los caracteres de un grupo G y los caracteres de sus subgrupos locales. La conjetura de McKay es el ejemplo clave del tipo de problemas en el que estamos interesados. Esta conjetura es un problema central dentro de toda la teoría de representaciones y de caracteres de grupos finitos. Si G es un grupo finito y p es un primo, la conjetura de McKay afirma que tanto G como el normalizador de un p-subgrupo de Sylow de G tienen el mismo número de caracteres irreducibles de grado no divisible por p. Por tanto, predice la existencia de una biyección entre tales conjuntos de caracteres. El capítulo 3 de este trabajo trata de un caso particular de esta conjetura en el que no sólo encontraremos una biyección de tipo McKay, sino que esta biyección será *natural*.

Como hemos mencionado, nos interesa relacionar los caracteres de un grupo G con los caracteres de sus subgrupos locales. La situación es especialmente atractiva cuando podemos establecer esa relación a través de correspondencias *naturales*. Quizá, el ejemplo más representativo sea la correspondencia de Glauberman. Veamos su caso más paradigmático. Supongamos que un p-grupo P actúa sobre un grupo K cuyo orden no es divisible por p. Entonces existe una biyección *natural* [*] entre los caracteres irreducibles de K que son fijados por la acción de P, denotamos por $\mathrm{Irr}_P(K)$ a este conjunto, y los caracteres irreducibles del grupo $C = \mathbf{C}_K(P)$ de puntos fijados por la acción. De hecho, si $\chi \in \mathrm{Irr}_P(K)$, entonces

$$\chi_C = e\chi^* + p\Delta,$$

donde e es un número natural no divisible por p y Δ es o bien un carácter de C o bien es cero. Por tanto, χ^* es la única constituyente del carácter χ_C que aparece con multiplicidad no divisible entre p. Vemos que χ determina *canónicamente* a χ^* y lo mismo ocurre en sentido contrario.

La palabra *natural* ya ha aparecido en varias ocasiones en este resumen. De hecho, volverá a aparecer en distintas situaciones a lo largo de este trabajo. Por tanto, conviene que aclaremos qué entendemos cuando tildamos una biyección de *natural* o *canónica*. Para ello, usaremos palabras de I. M. Isaacs. La siguiente cita está extraída (y traducida) del importantísimo artículo de 1973 [**Isa73**] en el que I. M. Isaacs prueba la conjetura de McKay en el caso en que el orden de G es impar: «La palabra“natural” quiere decir que la correspondencia se construye a través de un algoritmo y que el resultado es independiente de cualquier elección tomada al aplicar el algoritmo».

Sea χ un carácter de G. El cuerpo de valores $\mathbb{Q}(\chi)$ de χ es la menor de las extensiones de cuerpo de $\mathbb{Q}$ que contiene todos los valores de χ. Si nos encontramos en la situación de tener una correspondencia de Glauberman y los caracteres χ y χ^* se corresponden, entonces sus cuerpos de valores coinciden $\mathbb{Q}(\chi) = \mathbb{Q}(\chi^*)$. Esto se debe a que la biyección * conmuta con la acción de automorfismos de Galois sobre caracteres. En general, esperamos que las biyecciones naturales no sean meras biyecciones entre conjuntos, es decir, que tengan propiedades adicionales. Por ejemplo, esperamos que conmuten con la acción de ciertos automorfismos de Galois y con la acción de ciertos automorfismos de grupo. En este sentido, las biyecciones naturales deben proporcionar más información que relacione la estructura global y local del grupo.

Guión de la tesis

Todos los grupos que consideramos en esta tesis son finitos. Los primeros tres capítulos de la tesis tratan sobre caracteres ordinarios, mientras que los dos últimos están dedicados al estudio de caracteres modulares, también conocidos como caracteres de Brauer. Los resultados originales que contiene esta tesis aparecen en los siguientes artículos [**NV12**], [**Val14**], [**NTV14**], [**NV15**], [**SV16**] y [**Val16**].

En el capítulo 1 exponemos la teoría de caracteres ordinaria básica que vamos a usar a lo largo de la tesis. Nuestra referencia para caracteres ordinarios es [**Isa76**]. También incluimos en este capítulo una pequeña exposición de la teoría de B_π-characteres de Isaacs que, aunque no es elemental, será usada con bastante frecuencia a lo largo de este trabajo (en los capítulos 2, 3 y 4). Finalmente, y para la comodidad del lector, presentamos al final de este primer capítulo resultados bien conocidos sobre los grupos $\mathrm{PSL}_2(q)$, puesto que estos constituyen todo el bagaje sobre grupos simples que necesitaremos en el capítulo 3.

En el capítulo 2 empieza nuestro trabajo original. Un carácter lineal λ de un grupo G no es más que un homomorfismo $G \to \mathbb{C}^\times$. Por tanto, los caracteres lineales son los más fáciles de comprender. También, un grupo es abeliano si y sólo si todos sus caracteres son lineales. Un carácter χ de G se dice monomial si está inducido a partir de algún carácter lineal. Desde esta perspectiva, los caracteres monomiales también deben ser fáciles de entender. Sin embargo, los grupos en los que todos los caracteres son monomiales son muy difíciles de entender. De hecho, varios problemas importantes sobre ellos siguen abiertos [**Nav10**]. Además, no existen demasiados resultados que garanticen que un cierto carácter es monomial. Una excepción es un bonito resultado de R. Gow [**Gow75**]: un carácter racional de grado impar en un grupo resoluble es monomial. Nosotros extendemos este resultado de Gow en los Teoremas A y B. Estos dos resultados son, por tanto, criterios para asegurar la monomialidad de un carácter, y tienen que ver con los cuerpos de valores de los caracteres así como con sus grados. Además, la conclusión del Teorema B nos permite construir una correspondencia natural de tipo global/local, hecho que mostramos en el Teorema C.

Si usamos la teoría de B_π-caracteres de Isaacs, la conclusión del Teorema B puede hacerse más fuerte. Lo bueno es que esta nueva conclusión nos permite dar una nueva forma de calcular un invariable global de un grupo de forma local. Este es el contenido de la sección 2.3.

Concluimos el capítulo 2 estudiando una conjetura de Feit [**Fei80**]. Si χ es un carácter de un grupo G, escribimos f_χ para denotar al menor natural n de forma que $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_n$ (donde $\mathbb{Q}_n$ es el cuerpo ciclotómico que resulta de adjuntar a $\mathbb{Q}$ una raíz primitiva n-ésima de la unidad).

Conjetura (Feit). *Sea G un grupo finito y sea χ un carácter irreducible (ordinario) de G. Existe un elemento $g \in G$ de forma que el orden de g es exactamente f_χ.*

Esta conjetura fue probada para grupos resolubles por G. Amit y D. Chillag en [**AC86**]. Nosotros probamos una versión global/local del teorema de Amit y Chillag en nuestro Teorema D. La prueba del Teorema D requiere el uso de la teoría de caracteres especiales de Gajendragadkar y el uso del carácter *mágico* ψ que Isaacs define en su ya mencionado artículo [**Isa73**].

En el capítulo 3 estudiamos el caso autonormalizante de la conjectura de McKay. Sea p un primo y sea P un p-subgrupo de Sylow de G. En el Teorema E probamos que si $\mathbf{N}_G(P) = P$ y p es impar, entonces existe una biyección natural entre los caracteres irreducibles de G de grado no divisible por p y los caracteres del grupo abeliano P/P'. Para probar el Teorema E está hemos necesitado la descripción del comportamiento de los caracteres de $\mathrm{PSL}_2(q)$ bajo la acción de automorfismos de cuerpo y un resultado general de extensión de caracteres que probamos en la sección 3.3. Cabe mencionar que la conclusión del Teorema E es cierta sin restricciones sobre el primo p si asumimos que el grupo G es p-resoluble.

La correspondencia natural dada en el Teorema E está descrita en términos de la restricción de caracteres de G a P. Como consecuencia de ello, conmuta con la acción de automorfismos de Galois y con la acción de cualquier automorfismo de G que estabilice a P. Estas propiedades hacen que el Teorema E tenga como corolario una caracterización de los grupos que tienen un p-subgrupo de Sylow autonormalizante para p impar. Discutiremos este corolario en la sección 3.4.

En la sección 3.5 estudiamos una extensión del Teorema E al caso en que $\mathbf{N}_G(P) = \mathbf{C}_G(P)P$. En este caso nuestra biyección natural sólo se da entre caracteres que pertenecen al bloque principal. En el Teorema F probamos que si $\mathbf{N}_G(P) = \mathbf{C}_G(P)P$ y además G es p-resoluble, entonces existe una biyección natural de tipo McKay para G (y esta biyección coincide con la dada por el Teorema E cuando $\mathbf{N}_G(P) = P$).

El capítulo 4 proporciona al lector la teoría de caracteres modular necesaria para desarrollar el capítulo 5. La primera parte contiene resultados bien conocidos sobre caracteres de Brauer, mientras que, en la segunda parte, probamos resultados más específicos. Procedemos de esta forma ya que la naturaleza del capítulo 5 es muy técnica.

La sección 4.1 es un compendio de resultados sobradamente conocidos sobre caracteres de Brauer. Nuestra referencia en este caso es [**Nav98**]. En la sección 4.2, introducimos la noción de isomorfismo central de ternas de caracteres modulares (que es análoga a la noción de isomorfismo central de ternas de caracteres ordinarios definida por G. Navarro y B. Späth en [**NS14**]) y estudiamos propiedades de las ternas de caracteres que son centralmente isomorfas. En la sección 4.3 introducimos el concepto de acción de Galois *pretendida* (cuando nos movemos en un terreno muy técnico, la traducción al castellano no es muy agradable). Estas *pretendidas* acciones de Galois servirán para compensar el hecho de que los automorfismos de Galois no actúan sobre los caracteres irreducibles de Brauer. En el capítulo 5 motivamos esta definición.

En el capítulo 5, estudiamos una versión modular de la igualdad de cardinales derivada de la correspondencia de Glauberman-Isaacs. Supongamos que un grupo A actúa sobre un grupo G y que, además, $(|G|, |A|) = 1$. Entonces, por la correspondencia de Glauberman-Isaacs se tiene que el número de caracteres irreducibles ordinarios de G que son fijados por la acción de A coincide con el número de caracteres irreducibles ordinarios del grupo $\mathbf{C}_G(A)$ de puntos fijados por la acción. K. Uno [**Uno83**] probó que lo mismo ocurre para caracteres de Brauer (con respecto al primo p) si el grupo G es p-resoluble. El siguiente problema aparece en [**Nav94**].

Conjetura. *Supongamos que un grupo A actúa sobre un grupo G y que, además, $(|G|, |A|) = 1$. Sea $C = \mathbf{C}_G(A)$. Denotemos por $\mathrm{IBr}_A(G)$ al conjunto de caracteres irreducibles de Brauer de G fijados por A y por*

$\mathrm{IBr}(C)$ *al conjunto de caracteres irreducibles de Brauer de* C*. Entonces*

$$|\mathrm{IBr}_A(G)| = |\mathrm{IBr}(C)|.$$

El resultado principal del capítulo 5 es un teorema de reducción para la conjetura anterior. Dicho de una forma más precisa, en el Teorema G probamos que si todos los grupos finitos simples satisfacen lo que llamamos la *condición inductiva de Brauer-Glauberman*, entonces la conjetura anterior se satisface para grupos cualesquiera G y A. En la última sección de este trabajo 5.7, pretendemos responder algunas preguntas naturalmente relacionadas con el tema de este último capítulo, así como plantear nuevas cuestiones.

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Introduction

One of the main topics in the Theory of Finite Groups is the study of the relationship between the global and the local invariants of a finite group. Let G be a finite group, and let p be a prime. The p-subgroups of G are the subgroups of G of order a power of p, and the local subgroups of G are the proper normalizers of the p-subgroups of G. A paradigmatic example of this is an old theorem of Frobenius, which has inspired many recent results, that asserts that a group G has a normal p-complement if and only if every local subgroup has a normal p-complement.

In this thesis we focus our attention on characters: ordinary characters, associated with representations over the field of complex numbers, and modular characters, associated with representations over fields of prime characteristic p. We are particularly interested in the relation between the characters of G and the characters of the local subgroups of G. A fundamental example of the kind of problems we are interested in is the McKay conjecture, which is nowadays at the center of the Representation and Character Theory of Finite Groups. If p is a prime and G is a finite group, then the McKay conjecture asserts that G and the normalizer of a Sylow p-subgroup of G have the same number of ordinary irreducible characters of degree not divisible by p (therefore predicting the existence of a bijection between these two character sets). Chapter 3 of this work concerns a particular case of this conjecture in which not only can we find a bijection but a *natural* correspondence.

As we said, we are interested in relating the characters of G and the characters of the local subgroups of G. The situation is especially appealing if we can establish this relation by means of *natural* correspondences. The canonical example of the type of character correspondence that interests us is the Glauberman correspondence. If a p-group P acts on a group K of order not divisible by p, then there is a *natural* bijection * between the set of irreducible ordinary characters of K fixed under the action of P, denoted $\mathrm{Irr}_P(K)$, and the irreducible characters of the subgroup $C = \mathbf{C}_K(P)$ of points fixed under the action. In fact, if $\chi \in \mathrm{Irr}_P(K)$, then the restriction χ_C of χ to C can be written as

$$\chi_C = e\chi^* + p\Delta,$$

where p does not divide e, and Δ is zero or a character of C. Hence we see that χ canonically determines χ^* and viceversa.

The word *natural* has appeared more than once in this introduction and we will talk again about *natural* or *canonical* correspondences throughout this work. We feel it is worth mentioning what we mean by that. We shall not give an exact definition of what it is, instead we quote I. M. Isaacs in his landmark paper [**Isa73**] (in which he proved the McKay conjecture for groups of odd order): «The word "natural" is intended to mean that an algorithm is given for constructing the correspondence and that the result is independent of any choices made in application of the algorithm».

Let χ be an ordinary character of a group G. The field of values $\mathbb{Q}(\chi)$ of χ is the smallest field extension of $\mathbb{Q}$ containing all the values of χ. In the Glauberman correspondence situation, if χ and χ^* are correspondents, then $\mathbb{Q}(\chi) = \mathbb{Q}(\chi^*)$. This is because * commutes with the action of Galois automorphisms. In general, we expect natural correspondences of characters to have more properties than being a mere bijection. For instance, natural correspondences are expected to commute with certain Galois actions and with the action of certain group automorphisms, and therefore they should provide additional information as relating the global and the local structure. In this sense, the benefits of having natural bijections are multiple.

We come back to the Glauberman correspondence as in the third paragraph of this introduction. At first sight, it does not seem to go from global (a finite group G) to local (a local subgroup of G), but this is only superficial. If a p-group P acts on a group K with $(|P|, |K|) = 1$, then we form the semidirect product $G = K \rtimes P$ and we notice that $\mathbf{N}_G(P) = C \times P$, where $C = \mathbf{C}_G(P)$. By using some elementary character theory, one can show that the Glauberman correspondence implies that a natural bijection exists between the set irreducible ordinary characters of G of degree not divisible by p, and the set of irreducible ordinary characters of $\mathbf{N}_G(P)$ of degree not divisible by p. Hence, the McKay conjecture in the case where the group G has a normal p-complement follows from the Glauberman correspondence.

Outline of the thesis

All the groups we will consider in this work are finite unless otherwise stated. The first three chapters of this thesis concern ordinary characters and the last two chapters are devoted to modular characters (which are also known as Brauer characters). The original results contained in this thesis appear in [**NV12**] (joint work with G. Navarro), [**Val14**], [**NTV14**] (joint work with G. Navarro and P. H. Tiep), [**NV15**] (joint work with G. Navarro), [**SV16**] (joint work with B. Späth) and [**Val16**].

Chapter 1 is an expository chapter containing the background on ordinary character theory needed for the rest of the work. Our reference is [**Isa76**]. We also include a brief exposition of Isaacs B_π-theory since this deep theory will be used in Chapters 2, 3 and 4 of the present work. Finally, for the reader's convenience we present some well-known results about groups of type $\mathrm{PSL}_2(q)$ that essentially constitute the background on simple groups needed in Chapter 3.

In Chapter 2 we start our original work. A linear character λ of a group G is just an homomorphism $G \to \mathbb{C}^\times$. Hence, linear characters are the easiest to understand. A character χ is said to be monomial if there exists a subgroup U of G and a linear character λ of U such that λ induces χ. Thus, from this perspective, monomial characters should also be easy to understand. However, groups in which every character is monomial are actually very hard to understand and still several open problems on them remain unsolved (see Section 12 of [**Nav10**]). Also, it is unfortunate that there are not many conditions known for a character to be monomial. In [**Gow75**] R. Gow proved that an odd degree rational-valued irreducible character of a solvable group is monomial. We extend Gow's result in Theorems A and B. Both results are monomiality criteria which deal with fields of values and degrees of characters. The conclusion of Theorem B allows us to construct natural correspondences of characters of global/local type in Theorem C.

We actually found that the conclusion of Theorem B could be strengthened by using Isaacs B_π-theory, and this stronger conclusion leads to a new way of computing a global invariant of a group locally. This is the content of Section 2.3.

We conclude Chapter 2 by studying a conjecture of Feit [**Fei80**]. For a character χ of a group G, we write f_χ to denote the smallest integer n such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_n$ (where $\mathbb{Q}_n$ is the cyclotomic field obtained by adjoining a primitive n-th root of unity to $\mathbb{Q}$).

Conjecture (Feit). *Let G be a finite group and let χ be an irreducible character of G. Then there exists an element $g \in G$ whose order is exactly f_χ.*

This conjecture is known to hold for solvable groups by work of G. Amit and D. Chillag [**AC86**]. We prove a global/local version of Amit-Chillag's theorem in Theorem D. The proof of Theorem D requires highly non-trivial results on solvable groups: properties of Gajendragadkar special characters and properties of the *magical* character ψ defined in the above-mentioned paper of I. M. Isaacs [**Isa73**].

In Chapter 3 we study the self-normalizing case of the McKay conjecture. Let p be a prime and let P be a Sylow p-subgroup of a group G. If $\mathbf{N}_G(P) = P$ and p is odd, then in Theorem E we prove that there exists a natural correspondence between the irreducible characters of G of degree not divisible by p and the irreducible characters of the abelian group P/P'. Among other things, the proof of Theorem E requires the description of the behavior of the character theory of $\mathrm{PSL}_2(q)$ under the action of field automorphisms (given in Section (15B) of [**IMN07**]) and a key general extension theorem that we prove in Section 3.3. We mention that the conclusion of Theorem E also holds without any restriction on p if G is assumed to be p-solvable.

The natural correspondence given by Theorem E can be entirely described in terms of the restriction of characters of G to P. Moreover, it commutes with Galois action and the action of automorphisms of G that stabilize P. These properties lead to an interesting consequence: a characterization of groups having a self-normalizing Sylow p-subgroup for odd p, that we discuss in Section 3.4.

In Section 3.5 we study an extension of Theorem E to the case where $\mathbf{N}_G(P) = \mathbf{C}_G(P)P$ and p is odd, but only between characters in the respective principal blocks. In Theorem F, we prove that if G is p-solvable and $\mathbf{N}_G(P) = \mathbf{C}_G(P)P$, then there exists a natural McKay correspondence (which in the case where $\mathbf{N}_G(P) = P$ is the one given by Theorem E).

Chapter 4 provides the modular character-theoretical background for Chapter 5. The first part provides the basic background on Brauer characters. Due to the highly technical nature of the results contained in Chapter 5, in the second part of Chapter 4 we prove more specialized results on Brauer characters.

In Section 4.1 we collect well-known results on Brauer characters. Our reference is [**Nav98**]. In Section 4.2 we introduce the notion of central isomorphism of modular character triples (which is analogous to the notion of central isomorphism of ordinary character triples defined by G. Navarro and B. Späth in [**NS14**]) and we study its properties. In Section 4.3 we introduce the concepts of fake Galois conjugate (modular) character triples and fake Galois actions. Fake Galois actions try to remedy the fact that, in general, the Galois group $\mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$ does not act on the irreducible Brauer characters of the group G (they will be key in Chapter 5).

In Chapter 5 we study a modular version of the Glauberman-Isaacs bijection. Let A act on G with $(|A|, |G|) = 1$. By the Glauberman-Isaacs correspondence, the number of irreducible ordinary characters of G fixed under the action of A equals the number of irreducible ordinary characters of the subgroup $\mathbf{C}_G(A)$ of fixed points. By work of K. Uno [**Uno83**], the same holds for p-Brauer characters whenever G is a p-solvable group. The following was asked in [**Nav94**].

Conjecture. *Suppose that a group A acts on G with $(|A|, |G|) = 1$. Write $C = \mathbf{C}_G(A)$. Also write $\mathrm{IBr}_A(G)$ to denote the set of irreducible Brauer characters of G fixed under the action of A and $\mathrm{IBr}(C)$ to denote the set of irreducible Brauer characters of C. Then*

$$|\mathrm{IBr}_A(G)| = |\mathrm{IBr}(C)|.$$

The main result of Chapter 5 is a reduction theorem for the above conjecture. More precisely, in Theorem G we prove that if every simple non-abelian group satisfies what we call the *inductive Brauer-Glauberman condition* then the above conjecture holds for all groups G and A. In the final Section 5.7 we intend to answer (and raise) some questions related to this topic.

CHAPTER 1

Preliminaries on ordinary character theory of finite groups

Unless otherwise stated all groups considered are finite.

1.1. Algebras, representations and characters

Let F be a field. Given a group G, the **group algebra** $F[G]$ of G over F consists of all the formal sums

$$\sum_{g \in G} a_g g,$$

where $a_g \in F$ for every $g \in G$. The group algebra $F[G]$ is an F-vector space in the obvious way. The elements of G can be viewed as elements of $F[G]$ via the identification of $g \in G$ with the formal sum $\sum_{x \in G} a_x x$ where $a_g = 1$ and $a_x = 0$ for $x \neq g$. In fact, the elements of G form an F-basis of $F[G]$. Define the product of two elements of G in $F[G]$ as the product in G and extends this definition of product to all $F[G]$ by linearity. Then $F[G]$ is an F-algebra.

An F**-representation** $\mathfrak{X}$ of the group G is a homomorphism

$$\mathfrak{X} : G \to \mathrm{GL}_n(F),$$

where $\mathrm{GL}_n(F)$ denotes the group of invertible matrices of size $n \times n$ over F. The positive integer n is called the **degree** of the F-representation $\mathfrak{X}$.

Let $\mathfrak{X}$ be an F-representation of G. We can extend $\mathfrak{X}$ to the group algebra $F[G]$ by linearity and we obtain an algebra homomorphism

$$F[G] \to \mathrm{Mat}_n(F),$$

where $\mathrm{Mat}_n(F)$ denotes the F-algebra of square matrices of size $n \times n$ over F. Conversely, any algebra homomorphism $F[G] \to \mathrm{Mat}_n(F)$ yields an F-representation of G by restriction to elements of G.

Two F-representations $\mathfrak{X}$ and $\mathfrak{Y}$ of G are said to be **similar** if there exists some $M \in \mathrm{GL}_n(F)$ such that $\mathfrak{X}(g) = M^{-1}\mathfrak{Y}(g)M$ for every $g \in G$.

Let $\mathfrak{X}$ be an F-representation of G. We say that $\mathfrak{X}$ is **reducible** if $\mathfrak{X}$ is similar to an F-representation $\mathfrak{Y}$ of G in block upper triangular form with at least two blocks. Note that $\mathfrak{Y}$ has the form $\begin{pmatrix} \mathfrak{Y}_1 & \mathfrak{Z} \\ 0 & \mathfrak{Y}_2 \end{pmatrix}$ and (by the product formula for matrices in block form) $\mathfrak{Y}_1$ and $\mathfrak{Y}_2$ are F-representations of G of degree strictly lower than the degree of $\mathfrak{X}$. We say that $\mathfrak{X}$ is **irreducible** if $\mathfrak{X}$ is not reducible.

An F-**character** of a group G is defined as the trace function χ of an F-representation $\mathfrak{X}$ of G (we say that $\mathfrak{X}$ **affords** χ). The trace is an invariant under similarity in $\mathrm{Mat}_n(F)$, so the following is straightforward from the definitions.

LEMMA 1.1. *Let F be a field and let G be a group.*

(a) Every F-character of G is constant on conjugacy classes of G.
(b) Similar F-representations afford the same F-character.

Let χ be an F-character of G and let $\mathfrak{X} : G \to \mathrm{GL}_n(F)$ be an F-representation affording χ. The number n is called the **degree** of χ. Notice that the degree of χ is the degree of any F-representation affording χ. We say that an F-character χ of G is **irreducible** if an F-representation affording χ is irreducible.

If $\mathfrak{X}$ and $\mathfrak{Y}$ are two F-representations of G, then

$$\mathfrak{Z}(g) = \begin{pmatrix} \mathfrak{X}(g) & 0 \\ 0 & \mathfrak{Y}(g) \end{pmatrix}$$

defines an F-representation of G. Let χ be the F-character afforded by $\mathfrak{X}$ and ψ by the F-character afforded by $\mathfrak{Y}$. Then, it is obvious that $\chi + \psi$ is the F-character afforded by $\mathfrak{Z}$. Thus, sums of characters are also characters. Moreover, a character χ is irreducible if χ cannot be written as the sum of two characters.

From now on, and unless otherwise stated, we fix $F = \mathbb{C}$. (We refer to complex characters just as characters or sometimes as ordinary characters.) We denote by $\mathrm{Irr}(G)$ the set of irreducible characters of G. The map sending every $g \in G$ to $1 \in \mathbb{C}^\times$ is a representation of G of degree one. The character afforded by this representation 1_G is the **principal character** of G. A **linear character** λ of G is a character of G of degree equal to 1. In this case, λ is a homomorphism $G \to \mathbb{C}^\times$. Of course, linear characters are irreducible.

THEOREM 1.2. *Let G be a group. The number of irreducible characters of G is equal to the number of conjugacy classes of G.*

PROOF. See Corollary 2.7 of [Isa76]. □

We have seen that similar representations afford the same character. In the complex case, it is remarkable that most of the relevant information contained in a representation can be recovered from its trace.

THEOREM 1.3. *Let G be a group, two representations $\mathfrak{X}$ and $\mathfrak{Y}$ are similar if and only if they afford the same character.*

PROOF. See Corollary 2.9 of [Isa76]. □

A **class function** on a group G is a function $\varphi : G \to \mathbb{C}$ constant on conjugacy classes. It follows from Lemma 1.1 that every character is a class function. We usually denote the set of class functions by $\mathrm{cf}(G)$. The set $\mathrm{cf}(G)$ has a structure of vector space in the natural way. It is clear that the dimension of $\mathrm{cf}(G)$ is equal to the number of conjugacy classes of G.

THEOREM 1.4. *Let G be a group. The set $\mathrm{Irr}(G)$ is a basis of $\mathrm{cf}(G)$. Moreover, every character χ of G can be expressed as a sum of irreducible characters of G.*

PROOF. See Theorem 2.8 of [**Isa76**]. □

By Theorem 1.4, if ψ is a character of G, then we can write

$$\psi = \sum_{\chi \in \mathrm{Irr}(G)} a_\chi \chi,$$

where the a_χ are non-negative integers. If $a_\chi \neq 0$, then χ is called a **constituent** of χ and a_χ is the **multiplicity** of χ as a constituent of ψ.

Let φ and θ be two characters of a group G, we define the product $\varphi\theta$ for each $g \in G$ as

$$\varphi\theta(g) = \varphi(g)\theta(g).$$

It can be proved that there exists a representation affording $\varphi\theta$ (see Theorem 4.1 and Corollary 4.2 of [**Isa76**]). Hence, products of characters are also characters.

Let $\mathrm{Cl}_G(g_1), \ldots, \mathrm{Cl}_G(g_k)$ be the conjugacy classes of G (where $g_i \in G$ are conjugacy class representatives and $g_1 = 1$) and let $\chi_1, \ldots, \chi_k$ be the irreducible characters of G (set $\chi_1 = 1_G$ the principal character of G). The $k \times k$ matrix $X(G) = (\chi_i(g_j))_{i,j=1}^k$ is known as the **character table** of G. The character table codifies fundamental information about the group. The first column of the character table is the multiset of degrees of the irreducible representations of G.

THEOREM 1.5. *Let G be a group. Then*

$$|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2.$$

PROOF. See Corollary 2.7 of [**Isa76**]. □

Hence, the first column of the character table of G determines $|G|$. Also, as a consequence of this result, we obtain that the group G is abelian if and only if every irreducible character of G is linear. The following is another fundamental relation between the degrees of irreducible characters and the order of the group.

THEOREM 1.6. *Let $\chi \in \mathrm{Irr}(G)$. Then $\chi(1)$ divides $|G|$.*

PROOF. See Theorem 3.11 of [**Isa76**]. □

We can define an inner product on the vector space $\mathrm{cf}(G)$. Let $\varphi, \theta \in \mathrm{cf}(G)$. We set

$$[\varphi, \theta] = \frac{1}{|G|} \sum_{g \in G} \varphi(g)\overline{\theta(g)} \in \mathbb{C},$$

where we denote by $\overline{\omega}$ the complex conjugate of $\omega \in \mathbb{C}$. It is easy to check that $[\ ,\ ]$ satisfies the axioms of an inner product. Indeed $[\ ,\ ]$ makes $\mathrm{cf}(G)$ into a finite dimensional Hermitian space.

By the First Orthogonality Relation (see Corollary 2.4 of [**Isa76**]), if $\chi, \psi \in \mathrm{Irr}(G)$ then

$$[\chi, \psi] = \delta_{\chi\psi},$$

where δ_{ij} is the Kronecker delta symbol. In particular, $\mathrm{Irr}(G)$ is an orthonormal basis for $\mathrm{cf}(G)$ with respect to the inner product $[\ ,\ ]$. The Second Orthogonality Relation (see Theorem 2.13 of [**Isa76**]) is a consequence of the first one and states that if $g, h \in G$, then

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(g)\overline{\chi(h)}$$

is equal to $|\mathbf{C}_G(g)|$ if g and h are conjugate, and is zero otherwise.

Let $\chi \in \mathrm{Irr}(G)$. The map $\overline{\chi}\colon G \to \mathbb{C}$ defined by $\overline{\chi}(g) = \overline{\chi(g)}$ is a class function of G. By Lemma 2.2(c) of [**Nav98**], if V is a $\mathbb{C}[G]$-module affording χ, then $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is a $\mathbb{C}[G]$-module affording $\overline{\chi}$. (The proof of Lemma 2.2(c) also applies for ordinary characters, we give this reference since, unlike in [**Isa76**], a module affording $\overline{\chi}$ is provided.) Hence $\overline{\chi} \in \mathrm{Irr}(G)$ and $\overline{\chi}(g) = \chi(g^{-1})$ for every $g \in G$. We call $\overline{\chi}$ the **complex conjugate** of χ.

Let χ be a character of G. We define the kernel of χ as

$$\ker(\chi) = \{g \in G \mid \chi(g) = \chi(1)\}.$$

LEMMA 1.7. *Let χ be a character of G and let $\mathfrak{X}$ be a representation affording χ. Then*

$$\ker(\chi) = \ker(\mathfrak{X}).$$

In particular, the kernel of χ is a normal subgroup of G.

PROOF. See Lemma 2.19 of [**Isa76**]. □

We say that the character χ is **faithful** if $\ker(\chi) = 1$.

Let $N \lhd G$ and let χ be a character of G such that $N \subseteq \ker(\chi)$. If we define $\overline{\chi}(gN) = \chi(g)$ for every $g \in G$, then $\overline{\chi}$ is a character of G/N. Conversely, if $\overline{\chi}$ is a character of G/N then the function $\chi(g) = \overline{\chi}(gN)$ is a character of G, and obviously $N \subseteq \ker(\chi)$. In both cases, χ is irreducible if and only if $\overline{\chi}$ is irreducible (see Lemma 2.22 of [**Isa76**]). Usually we shall identify χ and $\overline{\chi}$. In general, we can identify the characters of G/N with the characters of G containing N in their kernel.

It is easy to see that the linear characters of G are exactly the irreducible characters of G containing the commutator subgroup G' in their kernel. Hence we can identify the linear characters of G with the irreducible characters of the abelian group G/G' and the index $|G:G'|$ gives the number of linear characters of G. The set of linear characters of G has a group structure given by the product of characters. In fact, the group of linear characters of G is isomorphic to G/G'.

Let χ be a character of G. We can define a map $\det(\chi) : G \to \mathbb{C}^\times$ by choosing $\mathfrak{X}$ a representation that affords χ and setting

$$\det(\chi)(g) = \det(\mathfrak{X}(g)).$$

We claim that $\det(\chi)$ is a uniquely defined linear character of G. Actually if $\mathfrak{X}$ is a G-representation that affords the character χ, then $\det(\mathfrak{X}) : G \to \mathbb{C}^\times$ is a homomorphism and thus, a linear representation. For the uniqueness, just notice that two representations afford the same character if and only if they are similar, in this case both have the same determinant.

Let χ be a character of G. We write $o(\chi)$ to denote the order of $\lambda = \det(\chi)$ as an element of the group of linear characters of G. We call $o(\chi)$ the **determinantal order** of χ. Then $o(\chi) = o(\lambda) = |G : \ker(\lambda)|$.

1.2. Induction and restriction of characters

Two essential features in character theory are restriction and induction of characters. If φ is a class function of G and $H \leqslant G$, then the restricted function φ_H is obviously a class function of H. Also, if φ is a character, then φ_H is a character.

Let θ be a class function of some subgroup H of G. We define the **induced class function** $\theta^G : G \to \mathbb{C}$ of θ to G by

$$\theta^G(g) = \frac{1}{|H|} \sum_{x \in G} \dot{\theta}(xgx^{-1})$$

for every $g \in G$, where $\dot{\theta}(y) = \theta(y)$ if $y \in H$ and $\dot{\theta}(y) = 0$ otherwise. It is easy to check that θ^G is a class function of G. In fact, by Corollary 5.3 of [**Isa76**], if θ is a character of $H \leqslant G$, then the class function θ^G is also a character of G.

It is an elementary exercise to check that induction is a transitive operation on characters. As a consequence, if φ is a character of $H \leqslant G$ such that $\varphi^G \in \mathrm{Irr}(G)$, then $\varphi^S \in \mathrm{Irr}(S)$ for every $H \leqslant S \leqslant G$.

Let $H \leqslant G$ and let θ be a character of H. We can describe the kernel of the induced character θ^G as follows

$$\ker(\theta^G) = \bigcap_{x \in G} (\ker(\theta))^x = \mathrm{core}_G(\ker(\theta)),$$

where $\mathrm{core}_G(H)$ is the intersection of all the G-conjugates of H, for $H \leqslant G$. (See Lemma 5.11 of [**Isa76**]).

Frobenius reciprocity (see Lemma 5.2 of [**Isa76**]) evidences that restriction and induction of complex characters are closely related: if φ is a class function of G and θ is a class function of $H \leqslant G$, then

$$[\varphi_H, \theta] = [\varphi, \theta^G].$$

In particular, if φ and θ are irreducible, then the multiplicity of θ as constituent of φ_H is equal to the multiplicity of φ as constituent of θ^G.

Let $H \leqslant G$ and let θ be a character of H. Let $g \in G$. We can define $\theta^g(x) = \theta(x^{g^{-1}})$ for every $x \in G$. Then it is immediate that θ^g is a character of H^g and $[\theta, \theta] = [\theta^g, \theta^g]$. In particular, θ is irreducible if and only if θ^g is irreducible. Notice that if $g \in H$, then $\theta^g = \theta$ because θ is constant on conjugacy classes of H.

Mackey's formula relates the operations of induction and restriction of characters.

Lemma 1.8 (Mackey). *Let $H, K \leqslant G$. Let $\mathbb{T}$ be a set of representatives of the double cosets HgK. That is, $G = \dot{\bigcup}_{t \in \mathbb{T}} HtK$. Let θ be a character of H. Then*

$$(\theta^G)_K = \sum_{t \in \mathbb{T}} (\theta^t_{H^t \cap K})^K.$$

In particular, if $G = HK$ then $(\theta^G)_K = (\theta_{H \cap K})^K$.

Proof. This is problem 5.6 of [**Isa76**]. □

Restriction and induction of characters behave well with respect to normal subgroups. Let $N \lhd G$. If $\theta \in \mathrm{Irr}(N)$ and $g \in G$, then we have seen that $\theta^g \in \mathrm{Irr}(N)$. Hence conjugation defines a natural action of G on $\mathrm{Irr}(N)$. Let G_θ be the stabilizer of θ under this action. We call G_θ the **inertia subgroup** of θ in G. Note that $N \leqslant G_\theta \leqslant G$. Let $g \in G$. Then it follows from the definition that $G_\theta^g = G_{\theta^g}$. For $H \leqslant G$, we say that θ is H**-invariant** if $H \subseteq G_\theta$.

Theorem 1.9 (Clifford). *Let $N \lhd G$ and let $\chi \in \mathrm{Irr}(G)$. Let θ be an irreducible constituent of χ_N and denote by $\theta_1, \dots, \theta_t$ the different G-conjugates of θ in G with $\theta_1 = \theta$. Then*

$$\chi_N = e \sum_{i=1}^{t} \theta_i,$$

where $e = [\chi_N, \theta]$.

Proof. See Theorem 6.2 of [**Isa76**]. □

We establish now some notation. Let $N \lhd G$, $\theta \in \mathrm{Irr}(N)$ and $\chi \in \mathrm{Irr}(G)$. If θ is such that $[\chi_N, \theta] \neq 0$ we say that θ **lies under** χ or that χ **lies over** θ. We write $\mathrm{Irr}(G|\theta)$ to denote the set of irreducible characters of G that lie over θ and we sometimes write $\mathrm{Irr}(\chi_N)$ to denote the set of irreducible characters of N lying under χ.

By Clifford's theorem, if $N \lhd G$, $\chi \in \mathrm{Irr}(G)$ and $\theta \in \mathrm{Irr}(\chi_N)$, then the number $e = \chi(1)/\theta(1)$ is the multiplicity of θ in χ_N. If $G_\theta = G$, this number e is the degree of an *irreducible projective representation* of G/N and divides $|G/N|$. (We will talk about projective representations in Chapter 4.) More generally, we have the following.

THEOREM 1.10. *Let $N \lhd G$ and let $\theta \in \mathrm{Irr}(N)$. If $\chi \in \mathrm{Irr}(G|\theta)$ then $e = [\chi_N, \theta]$ divides $|G : N|$.*

PROOF. See Corollary 11.29 of [**Isa76**] □

The following result is fundamental and it will be often used throughout this work.

THEOREM 1.11 (Clifford Correspondence). *Let $N \lhd G$ and let θ be an irreducible character of N. Then*

(a) If $\psi \in \mathrm{Irr}(G_\theta|\theta)$ then ψ^G is irreducible.
(b) The map $\psi \mapsto \psi^G$ from $\mathrm{Irr}(G_\theta|\theta)$ onto $\mathrm{Irr}(G|\theta)$ is a bijection.
(c) Let $\chi = \psi^G$ where $\psi \in \mathrm{Irr}(G_\theta|\theta)$. Then ψ is the unique irreducible constituent of χ_{G_θ} which lies over θ.
(d) Let $\psi^G = \chi$ where $\psi \in \mathrm{Irr}(G_\theta|\theta)$. Then $[\psi_N, \theta] = [\chi_N, \theta]$.

PROOF. See Theorem 6.11 of [**Isa76**]. □

If $N \lhd G$, $\theta \in \mathrm{Irr}(N)$ and $\chi \in \mathrm{Irr}(G|\theta)$, then by the Clifford correspondence, there exists a unique $\psi \in \mathrm{Irr}(G_\theta|\theta)$ such that $\psi^G = \chi$. We say that ψ is the **Clifford correspondent** of θ and χ.

We discuss now some results on extension of characters. Let $H \leqslant G$ and $\varphi \in \mathrm{Irr}(H)$, we say that φ **extends** to G if there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi_H = \varphi$.

THEOREM 1.12 (Gallagher). *Let $N \lhd G$ and let χ be an irreducible character of G such that $\theta = \chi_N$ is irreducible. Then, the map*

$$\mathrm{Irr}(G/N) \to \mathrm{Irr}(G|\theta)$$
$$\beta \mapsto \beta\chi,$$

is a bijection.

PROOF. See Corollary 6.17 of [**Isa76**]. □

The following results give us sufficient conditions for extending irreducible characters from normal subgroups. Recall that if χ is an irreducible character of a group G, we have defined the determinantal order $o(\chi)$ of χ.

THEOREM 1.13. *Let $N \lhd G$ and $\theta \in \mathrm{Irr}(N)$ be invariant in G. Suppose that $(|G : N|, o(\theta)\theta(1)) = 1$. Then there exists a unique extension $\chi \in \mathrm{Irr}(G)$ of θ such that $(|G : N|, o(\chi)) = 1$. In fact, $o(\chi) = o(\theta)$. In particular, this holds if $(|G : N|, |N|) = 1$.*

PROOF. See Corollary 8.16 of [**Isa76**]. □

Whenever the hypotheses of Theorem 1.13 hold, we call χ the **canonical extension** of θ. We sometimes write $\chi = \widehat{\theta}$. The following are two useful extendibility criteria (which however do not guarantee the existence of a canonical extension).

THEOREM 1.14. *Let $N \lhd G$. Let $\theta \in \mathrm{Irr}(N)$ be G-invariant. Suppose that G/N is cyclic. Then θ extends to G.*

PROOF. See Corollary 11.22 of [**Isa76**]. □

THEOREM 1.15. *Let $N \lhd G$. Let $\theta \in \mathrm{Irr}(N)$ be G-invariant. Suppose that for every prime p, the character θ extends to P for every $P/N \in \mathrm{Syl}_p(G/N)$. Then θ extends to G.*

PROOF. See Corollary 11.31 of [**Isa76**]. □

1.3. Actions and characters

Let $\alpha : G \to H$ be a group isomorphism. Denote by g^α the image of $g \in G$ under α. If χ is a character of G, then the map $\chi^\alpha : H \to \mathbb{C}$ defined by $\chi^\alpha(h) = \chi(h^{\alpha^{-1}})$ for every $h \in H$ is a character of H. Moreover

$$[\chi^\alpha, \chi^\alpha] = [\chi, \chi],$$

by straightforward computations. Consequently χ^α is irreducible if and only if χ is irreducible.

We see that $\mathrm{Aut}(G)$ acts naturally on $\mathrm{Irr}(G)$. Let $\mathrm{Cl}(G)$ be the set of conjugacy classes of G. Recall that we write $\mathrm{Cl}_G(g)$ to denote the conjugacy class of the element $g \in G$. Then $\mathrm{Aut}(G)$ acts on $\mathrm{Cl}(G)$ via $\mathrm{Cl}_G(g)^\alpha = \mathrm{Cl}_G(g^\alpha)$ for every $g \in G$ and for every $\alpha \in \mathrm{Aut}(G)$. Notice that if $\alpha \in \mathrm{Aut}(G)$ and $\chi \in \mathrm{Irr}(G)$, then $\chi^\alpha(g^\alpha) = \chi(g)$ for every $g \in G$. (In the same way, if a group A acts on a group G by automorphisms, then A acts on $\mathrm{Irr}(G)$ and on $\mathrm{Cl}(G)$.)

THEOREM 1.16 (Brauer's Lemma on the character table). *Let A be a group which acts on $\mathrm{Irr}(G)$ and on $\mathrm{Cl}(G)$. Assume that*

$$\chi(g) = \chi^a(g^a),$$

for all $\chi \in \mathrm{Irr}(G)$, $g \in G$ and $a \in A$; where g^a belongs to the conjugacy class $\mathrm{Cl}_G(g)^a$. Then for each $a \in A$, the number of irreducible characters of G fixed by a is equal to the number of conjugacy classes of G fixed by a.

PROOF. See Theorem 6.32 of [**Isa76**]. □

As a consequence of Brauer's Lemma on the character table (see Lemma 13.23 of [**Isa76**]), if a cyclic group A acts on G by automorphisms, then the actions of A on $\mathrm{Irr}(G)$ and on $\mathrm{Cl}(G)$ are permutation isomorphic.

Let A and G be groups. Suppose that A acts by automorphisms on G and $(|G|, |A|) = 1$ (we will say that A **acts coprimely** on G). We write $\mathrm{Irr}_A(G)$ to denote the subset of $\mathrm{Irr}(G)$ consisting of fixed points under the action of A. Then there exists an important natural correspondence of characters between $\mathrm{Irr}_A(G)$ and $\mathrm{Irr}(\mathbf{C}_G(A))$. When A is solvable, this correspondence was constructed by G. Glauberman [**Gla68**]. If A is not a solvable group, then $|A|$ is even by the Odd Order Theorem [**FT63**]. Consequently $|G|$ is solvable of odd order. In this case, I. M. Isaacs [**Isa73**] gave a totally different construction of the desired correspondence. T. R. Wolf [**Wol78b**] proved that when both constructions apply, when A is solvable and $|G|$ is odd, they yield the same map. This is what we call the Glauberman-Isaacs correspondence (when A is solvable we refer to the map as the Glauberman correspondence).

THEOREM 1.17 (Glauberman-Isaacs correspondence). *Suppose that A acts coprimely on G. Let $C = \mathbf{C}_G(A)$. Write $\mathrm{Irr}_A(G)$ to denote the subset of $\mathrm{Irr}(G)$ consisting of fixed points under the action of A. There exists a natural correspondence*

$$\pi_{(G,A)} \colon \mathrm{Irr}_A(G) \to \mathrm{Irr}(C),$$

such that:

(a) If $B \lhd A$ and $D = \mathbf{C}_G(B)$, then $\pi_{(G,A)} = \pi_{(D,A/B)} \circ \pi_{(G,B)}$.
(b) If A is a p-group and we write $\chi^* = \pi_{(G,A)}(\chi)$ for $\chi \in \mathrm{Irr}_A(G)$, then

$$\chi_C = e\chi^* + p\Delta,$$

where p does not divide e and Δ is a character of C or zero.

PROOF. See Theorem 2.1 of [**Wol79**]. □

We consider another important example of group action on irreducible characters: Galois action. Let n be an integer divisible by $|G|$. Consider $\mathbb{Q}_n$ the n-th cyclotomic field obtained by adjoining a primitive n-th root of unity to $\mathbb{Q}$. Then $\chi(g) \in \mathbb{Q}_n$ for every $g \in G$ and for every $\chi \in \mathrm{Irr}(G)$ (using that if $M \in \mathrm{GL}_m(\mathbb{C})$ with $M^n = I_m$, then M is similar to a diagonal matrix whose entries are n-th roots of unity). Let $E \subseteq \mathbb{C}$ be the field of all algebraic numbers. Then the theory described at the beginning of this chapter can be developed over the field E instead of $\mathbb{C}$, and it works exactly as well as for $\mathbb{C}$. If we write $\mathrm{Irr}_E(G)$ to denote the irreducible E-characters of G, then $\mathrm{Irr}_E(G) = \mathrm{Irr}(G)$ (see the discussion on page 22 of [**Isa76**]).

Let $\chi \in \mathrm{Irr}(G)$. If $\sigma \in \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$, then we define the function

$$\chi^\sigma \colon G \to \mathbb{C}$$

as $\chi^\sigma(g) = \chi(g)^\sigma$ for every $g \in G$. Let $\mathfrak{X}$ be an irreducible E-representation affording χ. If $\sigma \in \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$, then by elementary field theory we can extend σ to $\hat{\sigma} \in \mathrm{Gal}(E/\mathbb{Q})$. Define $\mathfrak{X}^{\hat{\sigma}}(g) = (\mathfrak{X}(g))^{\hat{\sigma}}$ by applying $\hat{\sigma}$ to each entry of the matrix $\mathfrak{X}(g)$, for every $g \in G$. Clearly, $\mathfrak{X}^{\hat{\sigma}}$ is an E-representation of G and

$$\mathrm{trace}(\mathfrak{X}^{\hat{\sigma}}(g)) = \mathrm{trace}(\mathfrak{X}(g))^{\hat{\sigma}} = \chi(g)^{\hat{\sigma}} = \chi(g)^\sigma = \chi^\sigma(g).$$

Hence, if $\chi \in \mathrm{Irr}(G)$, then $\chi^\sigma \in \mathrm{Char}(G)$. Since

$$\begin{aligned}[\chi^\sigma, \chi^\sigma] &= \frac{1}{|G|}\sum_{g\in G} \chi(g)^\sigma \overline{\chi(g)^\sigma} \\ &= \frac{1}{|G|}\sum_{g\in G} \chi(g)^\sigma \overline{\chi(g)}^\sigma \\ &= [\chi, \chi]^\sigma = 1,\end{aligned}$$

we conclude that $\chi^\sigma \in \mathrm{Irr}(G)$. Note that $\overline{\chi(g)^\sigma} = \overline{\chi(g)}^\sigma$ for every $g \in G$ and for every $\sigma \in \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ (see Lemma 20.7 of [**Isa94**]). Hence $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ acts on $\mathrm{Irr}(G)$.

Let χ be a character of a group G, the **field of values** $\mathbb{Q}(\chi)$ of χ is the minimum field extension of $\mathbb{Q}$ containing all values of χ. Hence

$$\mathbb{Q}(\chi) = \mathbb{Q}(\chi(g) \mid g \in G).$$

LEMMA 1.18. *Let χ be an irreducible character of a group G. Let $F/\mathbb{Q}$ be an abelian Galois extension. Suppose that $\mathbb{Q}(\chi) \subseteq F$. Then χ^σ is an irreducible character of G for every $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$.*

PROOF. Let $\mathfrak{X}$ be an E-representation of G affording χ. Write $F_n = F \cap \mathbb{Q}_n$, where $n = |G|$. Since $F/\mathbb{Q}$ is an abelian Galois extension, by Theorem 18.21 of [**Isa94**] $F_n/\mathbb{Q}$ is a Galois extension. Thus $\sigma_{F_n} \in \mathrm{Gal}(F_n/\mathbb{Q})$ and we can extend σ_{F_n} to $\hat{\sigma} \in \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$. From the discussion preceding this lemma we have that $\chi^{\hat{\sigma}} = \chi^\sigma$ is irreducible. □

It is immediate to check that the Galois action on characters commutes with the action induced by group automorphisms.

A character χ of G is said to be **real** if χ only takes real values (equivalently $\mathbb{Q}(\chi) \subseteq \mathbb{R}$). It is well-known that a group of odd order has no non-principal real irreducible character.

THEOREM 1.19 (Burnside). *Let G be a group of odd order. If $\chi \in \mathrm{Irr}(G)$ is not principal, then $\overline{\chi} \neq \chi$.*

PROOF. See Problem 3.6 of [**Isa76**]. □

1.4. Basic B_π-theory

Throughout this section π is a set of primes and G is a π-separable group. We write π' to denote the complementary set of primes of π. If n is an integer, then n_π is the greatest integer whose prime factors lie in π and such that n_π divides n. If $n = n_\pi$, we say that n is a **π-number**, and if $n_\pi = 1$ then we say that n is a π'-number. If π consists of a single prime p, then we write $\pi = p$ and $\pi' = p'$.

The **π-special characters** of G where introduced by Gajendragadkar in 1979 [**Gaj79**] as the subset of $\mathrm{Irr}(G)$ consisting of characters χ with $\chi(1)$ a π-number and such that for every subnormal subgroup S of G, the determinantal order of every irreducible constituent of χ_S is a π-number. Of course, the principal character is always a π-special character. In the words of Isaacs, the π-special characters of G are those characters that *think* that G is a π-group. In fact, if G is a π-group, then every $\mathrm{Irr}(G)$ is π-special, and if G is a π'-group, the only π-special is the principal character 1_G. We collect some properties of the π-special characters. We begin with *going down* properties.

PROPOSITION 1.20. *Let G be a π-separable group. Let $M \lhd G$ and let $\chi \in \mathrm{Irr}(G)$ be π-special. Then:*

(a) Every irreducible constituent of χ_M is π-special.
(b) $\mathbf{O}_{\pi'}(G) \leqslant \ker(\chi)$.

PROOF. See Proposition 4.1 and Corollary 4.2 of [**Gaj79**]. □

The following are *going up* properties of the π-special characters.

PROPOSITION 1.21. *Let G be a π-separable group and let $N \lhd G$. Suppose that $\theta \in \mathrm{Irr}(N)$ is π-special.*

(a) If G/N is a π-group, then every $\chi \in \mathrm{Irr}(G|\theta)$ is π-special.
(b) Assume that θ is G-invariant. If G/N is a π'-group then θ^G has a unique π-special irreducible constituent $\hat{\theta}$. In fact, $\hat{\theta}$ extends θ.
(c) θ^G has a π-special constituent iff θ is K-invariant for some Hall π'-subgroup K of G.

PROOF. See Propositions 4.3 and 4.5, and Corollary 4.8 of [**Gaj79**]. □

Let H be a Hall π-subgroup of G. The π-special characters of G restrict to irreducible characters of H injectively.

PROPOSITION 1.22. *Let G be a π-separable group and let H be a Hall π-subgroup of G. Then the map $\chi \mapsto \chi_H$ is an injection from the set of π-special characters of G into the set of irreducible characters of H. Moreover, if χ is a π-special character of G, then $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_{|G|_\pi}$.*

PROOF. See Propositions 6.1 and 6.3 of [**Gaj79**]. □

The following feature about special characters is particularly surprising. Although it is not common that a product of irreducible characters remains irreducible, for special characters the following is true.

PROPOSITION 1.23. *Let G be π-separable. Let α be a π-special character of G and β a π'-special character of G. Then $\alpha\beta$ is irreducible. Moreover, if $\alpha\beta = \alpha'\beta'$ for some π-special α' and some π'-special β', then $\alpha = \alpha'$ and $\beta = \beta'$.*

PROOF. This is Proposition 7.2 of [**Gaj79**]. □

Gajendragadkar's π-special characters are the foundation of Isaacs' B_π-theory. In the important paper [**Isa84**], Isaacs defined a canonical set $B_\pi(G)$ of $\mathrm{Irr}(G)$ whose elements are called B_π-characters of G. The definition of

$B_\pi(G)$ is rather involved and we only give a few details below (for further details see Sections 4 and 5 of [**Isa84**]). The B_π-characters can be seen as a generalization of the π-special characters. In fact, the π-special characters of G are exactly those elements in $B_\pi(G)$ of π-degree (see Lemma 5.4 of [**Isa84**]) and every B_π-character is induced from some π-special character. In particular, $1_G \in B_\pi(G)$.

Perhaps, the most important application of this theory is the fact that $B_{p'}$-characters constitute a canonical lift of the so called p-Brauer characters (on which we will talk in Chapters 4 and 5) in p-solvable groups.

We say that $\chi \in \mathrm{Irr}(G)$ is **π-factorable** if there exist a π-special $\alpha \in \mathrm{Irr}(G)$ and a π'-special $\beta \in \mathrm{Irr}(G)$ such that $\chi = \alpha\beta$. Suppose that G is π-separable. For every $\chi \in \mathrm{Irr}(G)$, there exists a particular pair (W, γ) where $W \leqslant G$, $\gamma \in \mathrm{Irr}(W)$ is π-factorable and $\gamma^G = \chi$. This pair (W, γ) is determined up to G-conjugacy. Any such pair is called a **nucleus** for χ, and the character in the pair is called a **nucleus character** for χ. Let $\chi \in \mathrm{Irr}(G)$. Then χ is a **B_π-character** if some nucleus character for χ is π-special. Notice that if χ is a B_π-character, then every nucleus character is π-special, by the G-conjugacy property of the nuclei. We write $B_\pi(G)$ to denote the set of B_π-characters of G.

The following properties of B_π-characters remind of those of π-special characters.

THEOREM 1.24. *Let G be a π-separable group and let $\chi \in B_\pi(G)$. Then:*
(a) For any $N \lhd G$, the irreducible constituents of χ_N lie in $B_\pi(N)$,
(b) $\mathbf{O}_{\pi'}(G) \leqslant \ker(\chi)$, and
(c) $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_{|\pi|}$.

PROOF. See Corollaries 5.3, 7.5 and 12.1 of [**Isa84**]. □

The set $B_\pi(G)$ is closed under group automorphisms and Galois action.

THEOREM 1.25. *Let G be π-separable. Let $\chi \in B_\pi(G)$, $\alpha \in \mathrm{Aut}(G)$ and $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$. Then χ^α and χ^σ lie in $B_\pi(G)$.*

PROOF. Follows from the definition of B_π-characters in [**Isa84**]. □

We will also need some more recent results concerning the character theory of π-separable groups.

THEOREM 1.26. *Let G be a π-separable group. Let $\psi \in B_\pi(G)$ and suppose that (W, γ) is a nucleus for ψ. Then, the map $\alpha \mapsto (\alpha\gamma)^G$ is an injection from the set of π'-special characters of W into $\mathrm{Irr}(G)$. Moreover, let (W_i, γ_i) be nuclei for $\psi_i \in B_\pi(G)$ and let $\alpha_i \in \mathrm{Irr}(W_i)$ be π'-special, for $i = 1, 2$. If $(\alpha_1\gamma_1)^G = (\alpha_2\gamma_2)^G$, then the pairs (W_i, γ_i) for $i = 1, 2$ are G-conjugate.*

PROOF. See Theorems 9.1 and 9.2 of [**Nav97**]. □

In [**IN01**] the authors refer to the irreducible characters $(\alpha\gamma)^G$ given by Theorem 1.26 as the **satellites** of $\psi \in B_\pi(G)$. They note that the satellites of $\psi \in B_\pi(G)$ do only depend on ψ and that ψ is a satellite of itself. In fact, the second part of Theorem 1.26 implies that the sets of satellites of distinct members of $B_\pi(G)$ are disjoint.

Satellites will be useful for us mainly because of the following result.

THEOREM 1.27. *Let G be a π-separable group. Every $\chi \in \mathrm{Irr}(G)$ of π'-degree is a satellite of a unique $\psi \in B_\pi(G)$ of π'-degree.*

PROOF. See Theorem 3.6 of [**IN01**]. □

Therefore, in a π-separable group G, for every $\chi \in \mathrm{Irr}(G)$ of π'-degree, Theorem 1.27 guarantees the existence of a pair (W, γ), where $W \leqslant G$ and $\gamma \in \mathrm{Irr}(W)$ is π-special, and a π'-special $\alpha \in \mathrm{Irr}(W)$ such that $\chi = (\alpha\gamma)^G$.

1.5. The projective special linear group $\mathrm{PSL}_2(q)$

In Chapter 3 we will need to study certain character correspondences in groups $\mathrm{PSL}_2(3^{3^a})$ for $a \geqslant 1$. We will need to understand the action of field automorphisms on characters. We include this section in order to make this work as self-contained as possible. Throughout this section p is an odd prime. Let $q = p^f$. We write $G = \mathrm{GL}_2(q)$, $H = \mathrm{SL}_2(q)$, $Z = \mathbf{Z}(H)$ and $S = \mathrm{PSL}_2(q) = H/Z$. We write $F = \mathbb{F}_q$ to denote the Galois field of q elements, and let α be a generator of the cyclic multiplicative group $F^\times$. Then $\mathbb{F}_p$ is the prime field of F.

1.5.1. Automorphisms of $\mathrm{PSL}_2(q)$.

Let $\delta = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \in G$. Then δ is an element of order $q-1$ that acts on H as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\delta = \begin{pmatrix} a & \alpha^{-1}b \\ \alpha c & d \end{pmatrix},$$

for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H$. We notice that

$$\delta^2 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \in \mathbf{Z}(G)H,$$

and thus δ^2 acts on H as an inner automorphism (We remark that in the case where q is even, δ actually acts as an inner automorphism because every non-zero element of F is a square). We have that δ stabilizes Z. Hence δ induces an automorphism of S of order $q-1$, which we denote again by δ, via

$$(xZ)^\delta = x^\delta Z,$$

for every $x \in H$. The automorphisms in $\langle \delta \rangle \leqslant \mathrm{Aut}(S)$ are called **diagonal automorphisms**. We have δ^2 induces an inner automorphism of S. So if we embed $S \lhd \mathrm{Aut}(S)$, then we have that $\langle \delta \rangle \cap S = \langle \delta^2 \rangle$.

Let φ be the Frobenius automorphism $\varphi : F \to F$ of F given by $\varphi(a) = a^p \in F$ for every $a \in F$. We have that $\varphi \in \mathrm{Gal}(F/\mathbb{F}_p)$ generates the full Galois group $\mathrm{Gal}(F/\mathbb{F}_p)$. The Frobenius automorphism φ induces an automorphism of H by applying φ to each entry of a matrix $x \in H$ (also in G). Then φ stabilizes Z, and hence, φ defines an automorphism of S, which we denote again by φ, via

$$(xZ)^{\varphi} = x^{\varphi}Z,$$

for every $x \in G$. If we embed $S \lhd \mathrm{Aut}(S)$, we have that $\langle\varphi\rangle \cap S = 1$. To see this, suppose that some $\varphi^j \neq 1$ acts on H as the inner automorphism associated to $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H$. In particular, for every $e \in F^{\times}$, we would have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e^{p^j} & 0 \\ 0 & (e^{p^j})^{-1} \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This yields $\varphi^j(e) = e^{-1}$ for every $e \in F^{\times}$ and $a = 0 = d$. However, if we compute the action of φ^j on elements of type $\begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \in H$ for $g \in F$, we get a contradiction. The automorphisms in $\langle\varphi\rangle$ are called **field automorphisms**.

The automorphisms $\delta\varphi$ and $\varphi\delta$ of H differ by an inner automorphism of H, therefore the same holds in Aut(S). It is straightforward to check that for every $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H$

$$x^{\varphi^{-1}\delta^{-1}\varphi\delta} = \begin{pmatrix} a & (\alpha^{-1})\alpha^p b \\ \alpha(\alpha^{-1})^p c & d \end{pmatrix}.$$

Since $(\alpha^{-1})\alpha^p = \alpha^{p-1} \in \langle\alpha^2\rangle$, we have that $\varphi^{-1}\delta^{-1}\varphi\delta \in \langle\delta^2\rangle \leqslant S$. Write $\overline{\delta} = \delta S \in \mathrm{Aut}(S)/S$ and $\overline{\varphi} = \varphi S \in \mathrm{Aut}(S)/S$. Thus, $\overline{\delta}$ and $\overline{\varphi}$ commute in the outer automorphism group $\mathrm{Out}(S) = \mathrm{Aut}(S)/S$ of S.

We can embed $\mathrm{PGL}_2(q)$ into $\mathrm{Aut}(S)$. Let $\iota\colon G \to \mathrm{Aut}(S)$ be defined by $x \in G$ sends yZ to y^xZ for every $y \in H$. Then ι is a homomorphism with kernel $\mathbf{Z}(G)$. We identify $\mathrm{PGL}_2(q)$ with $\iota(G)$, so that $\mathrm{PGL}_2(q) \leqslant \mathrm{Aut}(S)$ and S is a subgroup of $\mathrm{PGL}_2(q)$ of index 2. Since $\langle\overline{\delta}\rangle \leqslant \mathrm{PGL}_2(q)$ is not contained in S we have that $\mathrm{PGL}_2(q) = S\langle\overline{\delta}\rangle$. Also $S \cap \langle\overline{\delta}\rangle = \langle\overline{\delta}^2\rangle$. In fact, there exists an automorphism γ of S such that

$$\mathrm{PGL}_2(q) = S \rtimes \langle\gamma\rangle.$$

If $q \equiv 3 \mod 4$, then let

$$\gamma = \iota\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{Aut}(S).$$

If $q \equiv 1 \mod 4$, then -1 is a square in F. Let $\varepsilon \in F$ be such that $\varepsilon^2 = -1$. Then let

$$\gamma = \iota\begin{pmatrix} 0 & -1 \\ \varepsilon & 0 \end{pmatrix} \in \mathrm{Aut}(S).$$

It is well-known that every automorphism of S is a product of an inner automorphism, a diagonal automorphism and a field automorphism (see for instance Theorem 12.5.1 of [**Car93**]). Thus

$$\mathrm{Aut}(S) = \mathrm{PGL}_2(q) \rtimes \langle \varphi \rangle = (S \rtimes \langle \gamma \rangle) \rtimes \langle \varphi \rangle.$$

In particular $|\mathrm{Aut}(S)| = fq(q^2-1)$. We have noticed that $[\overline{\delta}, \overline{\varphi}] = 1$ in $\mathrm{Out}(S)$. In fact,

$$\mathrm{Out}(S) = \langle \overline{\delta} \rangle \times \langle \overline{\varphi} \rangle.$$

Suppose that some $\delta\varphi^j$ acts on H as an inner automorphism of H. By computing the action of $\delta\varphi^j$ on elements of the form $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in H$ and $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in H$, we get a contradiction.

1.5.2. Conjugacy classes and irreducible characters of $\mathbf{PSL}_2(q)$. Let E be a quadratic extension of F, so that $|E| = q^2$. Let $\mathcal{G} = \mathrm{Gal}(E/F)$ and μ be the nontrivial element of $\mathcal{G}$. It is easy to show that μ is exactly the automorphism of E taking every $z \in E$ to $z^q \in E$. Since q is odd, we can fix $\epsilon \in F^\times$ a non-square so that $E = F[\sqrt{\epsilon}]$ is a quadratic extension of F. Notice that any $z \in E$ has the form $z = a + b\sqrt{\epsilon}$. We write $\overline{z} = a - b\sqrt{\epsilon}$, so that $\mu(z) = \overline{z}$. Now, every quadratic polynomial $r[X] \in F[X]$ factors in $E[X]$. Thus, every $x \in H$ has eigenvalues in E.

Let $x \in H$. Then there are three possibilities for the eigenvalues λ, λ^{-1} of the matrix x:

(a) $\lambda = \lambda^{-1} \in \{1, -1\}$,
(b) $\lambda \neq \lambda^{-1}$ in $F^\times$ and
(c) $\lambda \neq \lambda^{-1}$ not in $F^\times$.

Case (a): If $\lambda = \lambda^{-1} \in \{1, -1\}$ then x is H-conjugate to one of the following

$$I, -I, u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, -u, u' = \begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix}, -u'.$$

Indeed, suppose $x \neq \lambda I$ and let $\begin{pmatrix} a \\ b \end{pmatrix}$ be an eigenvector of x associated to the eigenvalue λ. We first assume $a \neq 0$. Let $y = \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \in H$, we have that $x^y \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. By matrix calculation, it follows $x^y = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$ or $x^y = \begin{pmatrix} -1 & c \\ 0 & -1 \end{pmatrix}$ for some $c \in F^\times$. Finally, conjugating by an element of the form $\begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix}$ for some $d \in F^\times$, we get that x is conjugate to $\pm u$ if c is a square and x is conjugate to $\pm u'$ if c is a non-square. In case $a = 0$, we have that $b \neq 0$. Then an analogous argument with b playing the role of a shows that x is conjugate to $\pm u$ or $\pm u'$.

Case (b): If $\lambda \neq \lambda^{-1}$ lie in $F^\times$, then the matrix $d(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ defines a conjugacy class. We have that $d(\lambda)$ is conjugate to $d(\lambda^{-1})$ and $d(\lambda)$ is not conjugate to other matrix of this form since the eigenvalues are similarity invariants. There are $\frac{1}{2}(q-3)$ such classes, the same as pairs $\{\lambda, \lambda^{-1}\} \subseteq F^\times$.

Case (c): If $\lambda \neq \lambda^{-1}$ do not lie in $F^\times$. Write $T = \{z \in E^\times \mid z\overline{z} = 1\}$. Then $\lambda \in T$. Notice that $T = \{c + d\sqrt{\epsilon} : c, d \in F \text{ and } c^2 - d^2\epsilon = 1\}$. Since the norm map $E^\times \to F^\times$ is surjective, we have that T is a cyclic group of order $q+1$. An element $c + d\sqrt{\epsilon}$ acts on E, which is a vector space over F, and can be represented as the matrix $\begin{pmatrix} c & d\epsilon \\ d & c \end{pmatrix} \in H$, with respect to the F-basis $\{1, \sqrt{\epsilon}\}$. Clearly T is isomorphic to the group of matrices of this form, so we write $T \leqslant H$ under this identification. Let $t \in T - Z$. Then t defines a non-central conjugacy class in H. We have that t is conjugate to t^{-1}. Moreover, if $t' \notin \{t, t^{-1}\}$, then the eigenvalues of t and t' are distinct, so that the matrices t and t' are not conjugate in $\mathrm{GL}_2(q^2)$. In particular, they are not conjugate in H. Thus, each pair $\{\lambda, \lambda^{-1}\} \subseteq T$ defines a conjugacy class, and there are $\frac{1}{2}(q-1)$ such classes.

Write $d = d(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$. Let t be a generator of the subgroup $T \leqslant H$. By Theorem 38.1 of [**Dor71**], the set

$$\{I, -I, u, -u, u', -u', d^l, t^m\},$$

where $1 \leqslant l \leqslant \frac{1}{2}(q-3)$ and $1 \leqslant m \leqslant \frac{1}{2}(q-1)$, is a complete set of representatives of the conjugacy classes of H. Write $e = (-1)^{\frac{q-1}{2}}$. Let ρ be a primitive $(q-1)$-th root of unity and let σ be a $(q+1)$-th primitive root of unity. By Theorem 38.1 of [**Dor71**], the character table of H is the following:

Class:	I	$-I$	u	u'	d^l	t^m
1_H	1	1	1	1	1	1
St_H	q	q	0	0	1	-1
ξ_1	$\frac{1}{2}(q+1)$	$\frac{1}{2}e(q+1)$	$\frac{1}{2}(1+\sqrt{eq})$	$\frac{1}{2}(1-\sqrt{eq})$	$(-1)^l$	0
ξ_2	$\frac{1}{2}(q+1)$	$\frac{1}{2}e(q+1)$	$\frac{1}{2}(1-\sqrt{eq})$	$\frac{1}{2}(1+\sqrt{eq})$	$(-1)^l$	0
η_1	$\frac{1}{2}(q-1)$	$-\frac{1}{2}e(q+1)$	$\frac{1}{2}(-1+\sqrt{eq})$	$\frac{1}{2}(-1-\sqrt{eq})$	0	$(-1)^{m+1}$
η_2	$\frac{1}{2}(q-1)$	$-\frac{1}{2}e(q+1)$	$\frac{1}{2}(-1-\sqrt{eq})$	$\frac{1}{2}(-1+\sqrt{eq})$	0	$(-1)^{m+1}$
χ_i	$q+1$	$(-1)^i(q+1)$	1	1	$\rho^{il}+\rho^{il}$	0
θ_j	$q-1$	$(-1)^j(q-1)$	-1	-1	0	$-(\sigma^{jm}+\sigma^{-jm})$

where $1 \leqslant i, l \leqslant \frac{1}{2}(q-3)$ and $1 \leqslant j, m \leqslant \frac{1}{2}(q-1)$. As in [**Dor71**], the columns for the classes $-u$, $-u'$ are omitted. These values can be obtained from the relations $\chi(-u) = \frac{\chi(-I)}{\chi(I)}\chi(u)$ and $\chi(-u') = \frac{\chi(-I)}{\chi(I)}\chi(u')$, for every

$\chi \in \mathrm{Irr}(H)$. Just notice that if $\chi \in \mathrm{Irr}(H)$ and $\mathfrak{X}$ is a representation affording χ, then $\mathfrak{X}(-I) = \lambda I$, where $\lambda \in \{\pm 1\}$ and $\lambda = \frac{\chi(-I)}{\chi(I)}$.

Let $D = \langle d \rangle \leqslant H$. We have that D is isomorphic to $F^\times$. Let U be the subgroup of H consisting of upper unitriangular matrices. It is easy to see that $\mathbf{N}_H(U) = \{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in F^\times, b \in F\}$ and $\mathbf{N}_H(U)/U \cong D$. Every $\xi \in \mathrm{Irr}(D)$ can be viewed as a character of $\mathbf{N}_H(U)$ containing U in its kernel. Under this identification, the character ξ^H is called the **principal series character** of H associated to ξ (see Section 2.3 of [**Bon11**] for properties of the principal series characters). The character ξ^H is irreducible if and only if ξ is non-real. Moreover ξ and $\overline{\xi}$ yield the same principal series character. The characters χ_i in the above character table are principal series characters associated to some non-real $\xi \in \mathrm{Irr}(D)$. We write $\chi_i = \chi_\xi$, if χ_i comes from the pair $\{\xi, \overline{\xi}\}$. If $\xi \in \mathrm{Irr}(D)$ is real-valued, then ξ^H decomposes as the sum of two irreducible characters. By Mackey's Lemma 1.8, we have that $(1_{\mathbf{N}_H(D)})^H = 1_H + \mathrm{St}_H$, where St_H is the *Steinberg character* of H. Let ξ_0 be the only non-principal real-valued character of D, namely $\xi_0(d^l) = (-1)^l$. Then $(\xi_0)^H = \xi_1 + \xi_2$. We write $\xi_0' = \xi_1$ and $\xi_0'' = \xi_2$.

Recall $T = \langle t \rangle \leqslant H$ is cyclic of order $q+1$. Every $\eta \in \mathrm{Irr}(T)$ has associated a virtual character π_η of H (the description of π_η is more complicated than for principal series characters, see Section 4.3 of [**Bon11**] for properties of π_η). In fact, $\pi_{1_T} = \mathrm{St}_H - 1_H$ and π_η is actually a character whenever η is non-principal. The irreducible constituents of π_η, for η non-principal, are **cuspidal characters** of H. If $\eta \in \mathrm{Irr}(T)$ is non-real, then $\pi_\eta = \pi_{\overline{\eta}}$ is irreducible. The characters θ_j in the above character table are associated to pairs $\{\eta, \overline{\eta}\} \subseteq \mathrm{Irr}(T)$. We write $\theta_\eta = \theta_j$ if θ is associated to the pair $\{\eta, \overline{\eta}\}$. The characters η_1 and η_2 are the irreducible constituents of π_{η_0} the character associated to the unique real-valued irreducible character $\eta_0 \neq 1_T$ of T, namely $\eta_0(t^m) = (-1)^m$. We write $\theta_0' = \theta_1$ and $\theta_0'' = \theta_2$.

Recall that $e = 1$ if $\frac{q-1}{2}$ is even and $e = -1$ otherwise. Hence, $\xi_0(-I) = e = -\eta_0(-I)$. We write $e_\xi = \xi(-I)$ for $\xi \in \mathrm{Irr}(D)$ and $e_\eta = \eta(-1)$ for $\eta \in \mathrm{Irr}(T)$. We re-write the character table of H in this new notation:

Class:	I	$-I$	u	u'	d^l	t^m
1_H	1	1	1	1	1	1
St_H	q	q	0	0	1	-1
ξ_0'	$\frac{1}{2}(q+1)$	$\frac{e}{2}(q+1)$	$\frac{1}{2}(1+\sqrt{eq})$	$\frac{1}{2}(1-\sqrt{eq})$	$\xi_0(a^l)$	0
ξ_0''	$\frac{1}{2}(q+1)$	$\frac{e}{2}(q+1)$	$\frac{1}{2}(1-\sqrt{eq})$	$\frac{1}{2}(1+\sqrt{eq})$	$\xi_0(a^l)$	0
η_0'	$\frac{1}{2}(q-1)$	$-\frac{e}{2}(q+1)$	$\frac{1}{2}(-1+\sqrt{eq})$	$\frac{1}{2}(-1-\sqrt{eq})$	0	$-\eta_0(t^m)$
η_0''	$\frac{1}{2}(q-1)$	$-\frac{e}{2}(q+1)$	$\frac{1}{2}(-1-\sqrt{eq})$	$\frac{1}{2}(-1+\sqrt{eq})$	0	$-\eta_0(t^m)$
χ_ξ	$q+1$	$(q+1)e_\xi$	1	1	$\xi(a^l)+\overline{\xi}(a^l)$	0
θ_η	$q-1$	$(q-1)e_\eta$	-1	-1	0	$-\eta(t^m)-\overline{\eta}(t^m)$

As before $1 \leqslant i, l \leqslant \frac{1}{2}(q-3)$ and $1 \leqslant j, m \leqslant \frac{1}{2}(q-1)$ and the columns for the classes $-u$, $-u'$ are omitted. With this notation $\chi_\xi(-u) = \chi_\xi(-u') = e_\xi$ for every non-real $\xi \in \mathrm{Irr}(A)$, and $\theta_\eta(-u) = \theta_\eta(-u') = -e_\eta$ for every non-real $\eta \in \mathrm{Irr}(T)$.

The irreducible characters of S are those of H which contain Z in their kernel, so we can calculate the character table of S from the character table of H. In Chapter 3, we will be interested in the case where $q = 3^{3^a}$ for some $a \geqslant 1$, hence in the case where $q \equiv 3 \mod 4$. Assume $q \equiv 3 \mod 4$, so that $\frac{1}{2}(q-1)$ is odd and thus $e = -1$. Since -1 is not a square of F (because the subgroup of squares of $F^\times$ has order $\frac{1}{2}(q-1)$), we can fix $\epsilon = -1$. Hence, the subgroup T of H consists of matrices of the form $\begin{pmatrix} c & -d \\ d & c \end{pmatrix}$ for $c, d \in F$ with $c^2 + d^2 = 1$. Set $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in T$. Provided that $xZ = -xZ$ for every $x \in H$, it is easy to check that a complete set of representatives of the conjugacy classes of S is

$$\{IZ, uZ, u'Z, wZ, d^lZ, t^mZ\},$$

where $1 \leqslant l \leqslant \frac{1}{4}(q-3)$ and $1 \leqslant m \leqslant \frac{1}{4}(q-3)$. Notice that $w = t^{\frac{1}{4}(q+1)}$. If $\frac{1}{4}(q-3) \leqslant j \leqslant \frac{1}{2}(q-1)$, then $-t^j$ is conjugate to $-t^{-j} = t^{\frac{q+1}{2}-j}$. Thus $t^jZ = -t^jZ$ defines the same class as t^mZ, where $m = \frac{q+1}{2} - j$. Also notice that $\omega = t^{\frac{1}{4}(q+1)}$ has order two in S. The character table of S is the following:

Class:	IZ	uZ	$u'Z$	d^lZ	t^mZ
1_S	1	1	1	1	1
St_S	q	0	0	1	-1
η_0'	$1/2(q-1)$	$1/2(-1+i\sqrt{q})$	$1/2(-1-i\sqrt{q})$	0	$-\eta_0(t^m)$
η_0''	$1/2(q-1)$	$1/2(-1-i\sqrt{q})$	$1/2(-1+i\sqrt{q})$	0	$-\eta_0(t^m)$
χ_ξ	$q+1$	1	1	$\xi(a^l)+\overline{\xi}(a^l)$	0
θ_η	$q-1$	-1	-1	0	$-\eta(t^m)-\overline{\eta}(t^m)$

Here $1 \leqslant l \leqslant \frac{1}{4}(q-3)$ and $1 \leqslant m \leqslant \frac{1}{4}(q+1)$.

Consider $\varphi \in \mathrm{Aut}(S)$. We see that φ fixes the classes defined by IZ, uZ, $u'Z$ and wZ, and permutes the classes of type d^lZ and the classes of type t^mZ. In particular, φ fixes the trivial character 1_G and the Steinberg character St_G. Also notice that $(\eta_0')^\varphi$ has degree $\frac{1}{2}(q-1)$ and takes the same values as η_0' on uZ and $u'Z$. We conclude that $(\eta_0')^\varphi = \eta_0'$. Similarly $(\eta_0'')^\varphi = \eta_0''$. Furthermore, for every non-real $\xi \in \mathrm{Irr}(D)$ and every non-real $\eta \in \mathrm{Irr}(T)$

$$\chi_\xi^\varphi = \chi_{\xi^\varphi} \quad \text{and} \quad \theta_\eta^\varphi = \theta_{\eta^\varphi}.$$

Hence, we can characterize the irreducible characters of S fixed by φ in terms of the irreducible characters of D and T fixed by φ.

CHAPTER 2

Monomial characters and Feit numbers

2.1. Introduction

Let G be a finite group. There are few results guaranteeing that a single irreducible character $\chi \in \mathrm{Irr}(G)$ is monomial. Recall that χ is monomial if there exist a subgroup $U \leqslant G$ and a linear character $\lambda \in \mathrm{Irr}(U)$ such that $\lambda^G = \chi$. For instance, if G is a supersolvable group, then every irreducible character of G is monomial (see Theorem 6.22 of [**Isa76**]). However, this result depends more on the structure of the group rather than on properties of the characters themselves. An exception is a lovely result by Gow [**Gow75**] from 1975: an odd degree rational-valued irreducible character of a solvable group is monomial. The first aim in this chapter is to generalize Gow's result.

It is convenient now to define the *Feit number* of a character. If $\chi \in \mathrm{Irr}(G)$, then we have already mentioned that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_{|G|}$, so there is a smallest integer f_χ such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_{f_\chi}$. The number f_χ is called the **Feit number** of χ.

Our first original result, which generalizes Gow's result, has been published by the author in [**Val14**].

Theorem A. *Let G be a solvable group. Let $\chi \in \mathrm{Irr}(G)$. If $\chi(1)$ is odd and $(\chi(1), f_\chi) = 1$, then there exists a subgroup $U \leqslant G$ and a linear character λ of U such that $\lambda^G = \chi$. Moreover, if μ is a linear character of some subgroup $W \leqslant G$ such that $\mu^G = \chi$, then there exists some $g \in G$ such that $W = U^g$ and $\mu = \lambda^g$.*

Notice that we not only prove the monomiality of χ, but also uniqueness in the induction. The conditions $\chi(1)$ odd and G solvable are necessary as they are for Gow's original result: $\mathrm{SL}_2(3)$ has a rational-valued non-monomial character of degree 2, and A_6 has a rational-valued non-monomial character of degree 5. The new condition $(\chi(1), f_\chi) = 1$ is also necessary: Let G be `SmallGroup`$(108, 15)$. We checked with GAP that every irreducible character χ of degree 3 is not monomial and $f_\chi = 3$.

In particular, Theorem A guarantees that an odd degree irreducible character of a solvable group with values in some $\mathbb{Q}_{2^a}$ for $a \geqslant 0$ is monomial. In Theorem B below we generalize this latter statement for odd primes, although an oddness condition is still necessary. Let p be a prime. We recall that $\mathrm{Irr}_{p'}(G)$ denotes the subset of irreducible characters of G that have degree not divisible by p.

Theorem B. *Let G be a p-solvable group for some prime p. Let $P \in \mathrm{Syl}_p(G)$. Assume that $\mathbf{N}_G(P)/P$ has odd order. If $\chi \in \mathrm{Irr}_{p'}(G)$ takes values in $\mathbb{Q}_{p^a}$ for some $a \geqslant 0$, then there exist a subgroup U and a linear character λ of U with $\mathbb{Q}(\lambda) \subseteq \mathbb{Q}_{p^a}$ such that $\chi = \lambda^G$. Also, if μ is a linear character of some subgroup $W \leqslant G$ such that $\mu^G = \chi$, then $W = V^g$ and $\mu = \lambda^g$ for some $g \in G$. In particular $\mathbb{Q}(\mu) \subseteq \mathbb{Q}_{p^a}$.*

Theorem B has been published in a joint work of the author with G. Navarro [**NV12**]. Notice that the condition $|\mathbf{N}_G(P)/P|$ odd is superfluous if $p = 2$. Unfortunately, this oddness condition is necessary in general: if $p = 3$, then $\mathrm{SL}_2(3)$ has a rational-valued non-monomial character of degree 2. The p-solvability condition is also necessary. For instance, let $G = \mathrm{SL}_3(3)$ and $p = 3$. In this case, $|\mathbf{N}_G(P) : P| = 3$ for a Sylow p-subgroup P of G. The group G has a rational-valued character of degree 12 which cannot be induced from any proper subgroup of G.

By using non-trivial Isaacs B_π-theory the conclusion of Theorem B can be strengthened: such a χ is a B_p-character. We prove this fact in Section 2.4. It does not seem easy at all how to control the behavior of the normal constituents of χ without using this deep theory. We also use B_π to provide an alternative prove of Theorem A above.

Let G be a finite group, let p be a prime and let $a \geqslant 0$. How many p'-degree irreducible characters does G have with field of values contained in $\mathbb{Q}_{p^a}$? It does not seem easy at all how to answer this question in general. However, if G is p-solvable and $\mathbf{N}_G(P)/P$ has odd order, then this number can be computed locally. We write $X_{p^a}(G) = \{\chi \in \mathrm{Irr}_{p'}(G) \mid \mathbb{Q}(\chi) \subseteq \mathbb{Q}_{p^a}\}$. We prove that there exists a canonical bijection from $X_{p^a}(G)$ onto $X_{p^a}(\mathbf{N}_G(P))$. Theorem C below also appears in [**NV12**].

Theorem C. *Let G be a p-solvable group and let $P \in \mathrm{Syl}_p(G)$. Write $N = \mathbf{N}_G(P)$ and assume that N/P has odd order. Define a map*

$$\Omega : X_{p^a}(G) \to X_{p^a}(N)$$

in the following way: If $\chi \in X_{p^a}(G)$, choose a pair (U, λ) where $P \leqslant U \leqslant G$ and $\lambda \in \mathrm{Irr}(U)$ is linear with $\mathbb{Q}(\lambda) \subseteq \mathbb{Q}_{p^a}$ such that $\lambda^G = \chi$, then set $\Omega(\chi) = (\lambda_{U\cap N})^N$. Then Ω is a well-defined canonical bijection.

At the end of this chapter, we will come back to Feit numbers. The Feit number f_χ is a classic invariant in character theory that has been studied by Burnside, Blichfeldt and Brauer, among others. But it was W. Feit who following work of Blichfeldt made an astonishing conjecture that remains open until today (see for instance [**Fei80**]).

Conjecture (Feit). *Let G be a finite group and let $\chi \in \mathrm{Irr}(G)$. Then there exists an element $g \in G$ whose order is exactly f_χ.*

If G is an abelian group, then Feit's Conjecture holds for every $\chi \in \mathrm{Irr}(G)$ since G is isomorphic to $\mathrm{Irr}(G)$. In [**AC86**], G. Amit and D. Chillag proved Feit's Conjecture for solvable groups.

We prove a global/local variation (with respect to a prime p) of the Amit-Chillag theorem (for odd-degree characters of p'-degree).

THEOREM D. *Let p be a prime and let G be a solvable group. Let $\chi \in \mathrm{Irr}(G)$ of degree not divisible by p, and let $P \in \mathrm{Syl}_p(G)$. If $\chi(1)$ is odd, then there exists $g \in \mathbf{N}_G(P)/P'$ such that $o(g) = f_\chi$. In particular, the Feit number f_χ divides $|\mathbf{N}_G(P) : P'|$.*

Theorem D appears in [**Val16**]. It is unfortunate that we really need to assume that $\chi(1)$ is odd, as $G = \mathrm{GL}_2(3)$ shows us: if $\chi \in \mathrm{Irr}(G)$ is non-rational of degree 2, then $f_\chi = 8$; but the normalizer of a Sylow 3-subgroup of G has exponent 6. Also, Theorem A is not true outside solvable groups, as shown by $G = \mathrm{A}_5$, $p = 2$, and any $\chi \in \mathrm{Irr}(G)$ of degree 3 (which has $f_\chi = 5$).

This chapter is structured in the following way: In Section 2.2 we prove our two monomiality criteria, namely Theorem A and Theorem B. In Section 2.3 we study character correspondences and we prove Theorem C. In Section 2.4 we strengthen the conclusion of Theorem B by using Isaacs' B_π-theory. We also give an alternative proof of Theorem A by making use of this deep theory. The results contained in Section 2.4 appear in a joint work of the author together with G. Navarro [**NV15**]. Finally, in Section 2.5 we prove Theorem D.

2.2. Two criteria for monomiality

Let $N \lhd G$ and $\chi \in \mathrm{Irr}(G)$. Let $\theta \in \mathrm{Irr}(N)$ be a constituent of χ_N. Write $T = G_\theta$ for the inertia subgroup of θ. By Clifford's correspondence there is a unique $\psi \in \mathrm{Irr}(T|\theta)$ such that $\psi^G = \chi$. Of course, $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\psi)$, but in general these two fields are not equal. The **semi-inertia subgroup of** θ is defined as

$$T^* = \{g \in G \mid \theta^g = \theta^\sigma \text{ for some } \sigma \in \mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})\}.$$

Since $T \leqslant T^*$, then $\eta = \psi^{T^*} \in \mathrm{Irr}(T^*|\theta)$ also induces χ. Moreover $\mathbb{Q}(\eta) = \mathbb{Q}(\chi)$. Due to this fact, when dealing with character fields and normal subgroups, the semi-inertia group is a useful tool.

LEMMA 2.1. *Let $N \lhd G$ and $\chi \in \mathrm{Irr}(G)$. Let $\theta \in \mathrm{Irr}(N)$ be a constituent of χ_N. Write T and T^* for the inertia and the semi-inertia groups of θ. If $\psi \in \mathrm{Irr}(T|\theta)$ is the Clifford correspondent of χ, then $\mathbb{Q}(\psi^{T^*}) = \mathbb{Q}(\chi)$.*

PROOF. See Lemma 2.2 of [**NT10**]. □

We do not need more preparation in order to prove Theorem A, which we restate below. We recall that if $H \leqslant G$, then

$$\mathrm{core}_G(H) = \bigcap_{g \in G} H^g.$$

Theorem 2.2. *Let G be a solvable group. Let $\chi \in \mathrm{Irr}(G)$. If $\chi(1)$ is odd and $(\chi(1), f_\chi) = 1$, then there exists a subgroup $U \leqslant G$ and a linear character λ of U such that $\lambda^G = \chi$. Moreover, if μ is a linear character of some subgroup $W \leqslant G$ such that $\mu^G = \chi$, then there exists some $g \in G$ such that $W = U^g$ and $\mu = \lambda^g$.*

Proof. First, we prove by induction on $|G|$ that χ is monomial. We prove it in a series of steps.

Step 1. We may assume χ is faithful and that there are no proper subgroups $H < G$ and $\psi \in \mathrm{Irr}(H)$ such that $\psi^G = \chi$ and $\mathbb{Q}(\psi) = \mathbb{Q}(\chi)$.

Let $K = \ker(\chi)$. If $K > 1$, all the hypotheses hold in G/K so by induction we are done. Assume there exists $H < G$ and $\psi \in \mathrm{Irr}(H)$ with $\mathbb{Q}(\psi) = \mathbb{Q}(\chi)$ such that $\psi^G = \chi$. Then the degree of ψ divides the degree of χ and $f_\psi = f_\chi$. By induction hypothesis ψ is monomial, and thus χ is monomial.

Step 2. $\mathbf{F}(G) = \prod_{p \nmid \chi(1)} \mathbf{O}_p(G)$.

Let p be a prime. Suppose that p divides $\chi(1)$. In particular p is odd, and p does not divide f_χ. Let M be a normal p-subgroup of G. Since $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_{f_\chi}$ we have that $\mathbb{Q}_{|M|} \cap \mathbb{Q}(\chi) = \mathbb{Q}$. Hence χ_M is rational-valued. If $\xi \in \mathrm{Irr}(M)$, then $[\chi_M, \xi] = [\chi_M, \bar{\xi}]$. Since $\chi(1)$ is odd, there exists a real irreducible constituent ξ of χ_M. Since $|M|$ is odd, we have that $\xi = 1_M$, by Theorem 1.19. By Step 1, we know that χ is faithful and we conclude $M = 1$.

Step 3. $F = \mathbf{F}(G)$ is abelian.

Let M be a normal p-subgroup of G, where p does not divide $\chi(1)$. It then follows that the irreducible constituents of χ_M are linear. Let $\lambda \in \mathrm{Irr}(M)$ be under χ. We have that $M' \leqslant \ker(\lambda^g) = \ker(\lambda)^g$ for every $g \in G$. Then $M' \leqslant \mathrm{core}_G(\ker(\lambda)) \leqslant \ker(\chi) = 1$, so that M is abelian. Hence F is abelian by Step 2.

Step 4. Let $N \lhd G$ and let $\theta \in \mathrm{Irr}(N)$ be under χ. Let $g \in G$. Then $\theta^g = \theta^\sigma$ for some $\sigma \in \mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})$. Also θ is faithful.

Let $T = G_\theta$ be the stabilizer of θ in G, and write T^* for the semi-inertia subgroup of θ. Recall $T^* = \{g \in G \mid \theta^g = \theta^\sigma \text{ for some } \sigma \in \mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})\}$. By Lemma 2.1, if $\psi \in \mathrm{Irr}(T|\theta)$ is the Clifford correspondent of χ, then $\eta = \psi^{T^*} \in \mathrm{Irr}(T^*)$ induces χ and $\mathbb{Q}(\eta) = \mathbb{Q}(\chi)$. By Step 1, we have that $T^* = G$, and so every G-conjugate of θ is actually a Galois conjugate. Thus $\ker(\theta^g) = \ker(\theta)$ for every $g \in G$. It follows that $\ker(\theta) \lhd G$ and $\ker(\theta)$ is contained in $\ker(\chi)$ by Clifford's theorem. So θ is faithful by Step 1.

Final Step. If $\lambda \in \mathrm{Irr}(F)$ is under χ, then $\lambda^G = \chi$.

Let $\lambda \in \mathrm{Irr}(F)$ be under χ. If $y \in G$ is such that $\lambda^y = \lambda$, then we have $[x, y] \in \ker(\lambda)$ for every $x \in F$. By Step 4, λ is faithful, so the element y centralizes F. Since F is self-centralizing, see 6.1.4 of [**KS04**], necessarily $y \in F$. We have proved $G_\lambda = F$. Thus λ^G is irreducible and thus $\lambda^G = \chi$. This finishes the proof that χ is monomial.

Now, we work by induction on $|G|$ to show that if U and V are subgroups of G and $\lambda \in \mathrm{Irr}(U)$ and $\mu \in \mathrm{Irr}(V)$ are linear such that $\lambda^G = \chi = \mu^G$, then there is some $g \in G$ such that $V = U^g$ and $\mu = \lambda^g$. Since $K = \ker(\chi) \leqslant \mathrm{core}_G(\ker(\lambda)) \bigcap \mathrm{core}_G(\ker(\mu))$ we may assume that χ is faithful, for if $K > 1$ then we can work in G/K. If p is a prime not dividing $\chi(1)$, then $\mathbf{O}_p(G)$ is contained in both U and V, because $|G : U| = \chi(1) = |G : V|$. Let $F = \mathbf{F}(G)$. By Step 2 (for which we only required that χ is faithful), we have that

$$F = \prod_{p \nmid \chi(1)} \mathbf{O}_p(G) \leqslant U \cap V\,.$$

Now λ_F and μ_F lie under χ, so that $\mu_F = (\lambda_F)^g$ for some $g \in G$ by Clifford's theorem. We may assume that $\mu_F = \nu = \lambda_F$, by replacing the pair (U, λ) by some G-conjugate. Thus U and V are contained in $T = G_\nu$ and also in T^*, the semi-inertia subgroup of ν. Since λ^G and μ^G are irreducible, also λ^T and μ^T are irreducible. By uniqueness of the Clifford correspondent, we deduce that $\lambda^T = \mu^T$. In particular $\lambda^{T^*} = \mu^{T^*} = \psi \in \mathrm{Irr}(T^*|\nu)$. We know that $\mathbb{Q}(\psi) = \mathbb{Q}(\chi)$, again using Lemma 2.1. If $T^* < G$, then the result follows by induction. Hence, we may assume $T^* = G$. In particular, arguing as in the first part of the proof, we conclude that $\nu^G = \chi$. This implies that $U = F = V$ and the theorem is proven. □

Under the hypothesis of Theorem 2.2, it can also be proved that in fact χ is **supermonomial**, that is, that every character inducing χ is monomial. The arguments are the same as in the proof of Theorem 2.2.

Let G be solvable. Let $\chi \in \mathrm{Irr}(G)$ be an odd degree character. If $f_\chi = 2^a$ for some $a \geqslant 0$, then Theorem 2.2 guarantees that the character χ is monomial. Let p be a prime. We can prove a p-version of the latter statement for p-solvable groups, alas an oddness condition is still necessary. This is our Theorem B mentioned in the introduction, which appears in this section as Theorem 2.6.

The following Lemma follows from an standard argument and will be often used along this work.

Lemma 2.3. *Let $N \lhd G$ and $P \in \mathrm{Syl}_p(G)$. Let $\chi \in \mathrm{Irr}(G)$ have p'-degree. Then there is a P-invariant $\theta \in \mathrm{Irr}(N)$ under χ, and any two of them are $\mathbf{N}_G(P)$-conjugate. In particular, if $\mathbf{N}_G(PN) = PN$, then θ is unique.*

Proof. Let $\theta_1 \in \mathrm{Irr}(N)$ be under χ, let $T_1 = G_{\theta_1}$ be the stabilizer of θ_1 in G and let $\psi_1 \in \mathrm{Irr}(T_1|\theta_1)$ be the Clifford correspondent of χ over θ_1. Since χ has p'-degree, we have that $|G : T_1|$ is not divisible by p, and then $P^{h^{-1}} \leqslant T_1$ for some $h \in G$. Then $P \leqslant T = G_\theta$, where $\theta = (\theta_1)^h \in \mathrm{Irr}(N)$. Also, if $\eta \in \mathrm{Irr}(N)$ is also P-invariant under χ, then by Clifford's theorem we have that $\eta^g = \theta$ for some $g \in G$. Then $P, P^g \leqslant T$, and thus $P^{gt} = P$ for some $t \in T$ by Sylow Theory. Now $\eta^{gt} = \theta^t = \theta$, and hence η and θ are $\mathbf{N}_G(P)$-conjugate. The second part easily follows. □

Remark 2.4. Let π be a set of primes, let G be a π-separable group, let H be a Hall π-subgroup of G and let $N \lhd G$. If $\chi \in \mathrm{Irr}(G)$ has π'-degree, then there is some H-invariant $\theta \in \mathrm{Irr}(N)$ under χ and any two of them are $\mathbf{N}_G(H)$-conjugate. The argument is analogous to the one given in the proof of Lemma 2.3

The following easy argument will be used sometimes in this chapter.

Lemma 2.5. *Let P be a p-group. Suppose that P acts coprimely on a group N and that $\mathbf{C}_N(P)$ has odd order. Let $\theta \in \mathrm{Irr}(N)$ be P-invariant. If θ is real, then $\theta = 1$.*

Proof. let $\theta \in \mathrm{Irr}(N)$ be By the Glauberman correspondence (see Theorem 1.17) with respect to the coprime action of the p-group P on N, we have a natural bijection

$$^*\colon \mathrm{Irr}_P(N) \to \mathrm{Irr}(\mathbf{C}_N(P))\,,$$

where $\mathrm{Irr}_P(N)$ is the set of P-invariant irreducible characters of N. Since θ is real, then also θ^* is real. But $\mathbf{C}_N(P)$ has odd order by hypothesis. Thus $\theta^* = 1$ by Theorem 1.19 on real characters of groups of odd order, and therefore $\theta = 1_N$ since * is bijective and $(1_N)^* = 1_{\mathbf{C}_N(P)}$. □

Now, we can prove Theorem B.

Theorem 2.6. *Let G be a p-solvable group for some prime p. Let $P \in \mathrm{Syl}_p(G)$. Assume that $\mathbf{N}_G(P)/P$ has odd order. If $\chi \in \mathrm{Irr}_{p'}(G)$ takes values in $\mathbb{Q}_{p^a}$, then there exist a subgroup U and linear character λ of U with $\mathbb{Q}(\lambda) \subseteq \mathbb{Q}_{p^a}$ such that $\chi = \lambda^G$. Also, if there exist $J \leqslant G$ and $\psi \in \mathrm{Irr}(J)$ with $\psi^G = \chi$, then $\psi = \tau^J$ for some linear character τ of a subgroup $W \leqslant J$, $W = U^g$ and $\tau = \lambda^g$ for some $g \in G$. In particular $\mathbb{Q}(\psi) \subseteq \mathbb{Q}_{p^a}$.*

Proof. We first show the existence of U and λ. We argue by induction on $|G|$. Of course, we may assume that G is non-abelian, so $G' > 1$.

Step 1. If $N \lhd G$, $\theta \in \mathrm{Irr}(N)$, $g \in G$ and $\sigma \in \mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})$, then we have that $(\theta^\sigma)^g = (\theta^g)^\sigma$. In particular, the stabilizer of θ in G is the stabilizer of θ^σ in G.

This immediately follows from the corresponding definitions.

Step 2. Suppose that $N \lhd G$ and let $\theta \in \mathrm{Irr}(N)$ be P-invariant under χ. If the complex conjugate $\bar\theta$ of θ is also an irreducible constituent of χ_N, then $\theta = \bar\theta$.

By Step *1*, we have that $\bar\theta$ is also P-invariant, and therefore there exists $g \in \mathbf{N}_G(P)$ such that $\bar\theta = \theta^g$, by Lemma 2.3. Now, g^2 fixes θ (also using Step *1*). Now since $\mathbf{N}_G(P)/P$ has odd order by hypothesis, we conclude that $\langle gP \rangle = \langle g^2 P \rangle$. Therefore g fixes θ and $\theta = \bar\theta$ is real.

Step 3. We have that $N = \mathbf{O}_{p'}(G) \leqslant \ker(\chi)$.

By Lemma 2.3, let $\theta \in \mathrm{Irr}(N)$ be P-invariant under χ, let T be the stabilizer of θ in G and let $\psi \in \mathrm{Irr}(T)$ be the Clifford correspondent of χ over θ. We prove now that θ is real. By Step *2*, it suffices to show that $\bar\theta$ is under χ. Notice that $\mathbb{Q}(\chi_N) \subseteq \mathbb{Q}_{p^a} \cap \mathbb{Q}_{|N|} = \mathbb{Q}$. In particular, χ_N is real-valued and so $\overline{\theta}$ also lies under χ. Notice that $\mathbf{C}_N(P) = \mathbf{N}_N(P) \leqslant \mathbf{N}_G(P)/P$ has odd order by hypothesis. By Lemma 2.5, we have that $\theta = 1_N$. Hence $\mathbf{O}_{p'}(G) \leqslant \ker(\chi)$, as claimed.

Step 4. The character χ is faithful. In particular $N = 1$.

Let $L = \ker(\chi)$. Then $\mathbf{N}_{G/L}(PL/L)/(PL/L) \cong \mathbf{N}_G(P)/P\mathbf{N}_L(P)$ has odd order. If $L > 1$, then the existence of suitable U and λ is readily obtained by applying the inductive hypothesis in G/L.

Step 5. G is not a p-group.

Otherwise, since p does not divide $\chi(1)$, we would have χ is linear. In this case, there is nothing to prove.

Step 6. $M = \mathbf{O}_p(G) > 1$ is abelian.

Let $\nu \in \mathrm{Irr}(M)$ be under χ. Since χ has p'-degree, then we have that ν is linear. Thus M' is contained in the kernel of every G-conjugate of ν. Since χ is faithful, we deduce that M is abelian. By Step *4*, $\mathbf{O}_{p'}(G) = 1$ so $M > 1$.

Step 7. Let K be a minimal normal subgroup of G contained in G'. Let $\mu \in \mathrm{Irr}(K)$ be P-invariant under χ. Then $\mathbb{Q}(\mu) \subseteq \mathbb{Q}_{p^a}$.

By Lemma 2.3, there exists a P-invariant $\mu \in \mathrm{Irr}(K)$ lying under χ. By Step *4* K is an elementary abelian p-group. Hence μ is linear and $\mathbb{Q}(\mu) \subseteq \mathbb{Q}_p$. If $a \geqslant 1$, then $\mathbb{Q}(\mu) \subseteq \mathbb{Q}_{p^a}$. Otherwise, χ is rational. Hence $\overline{\mu}$ lies under χ. By Step *2* μ is real, hence rational.

Step 8. Let I be the stabilizer of μ in G. Then $I < G$ and μ extends to I.

Let Q/K be s Sylow q-subgroup of I/K. If $p \neq q$, then μ extends to Q because K is a p-group. If $q = p$, then I is a p-group. Since χ has p'-degree, then χ_Q has some linear constituent, and hence μ extends to Q. It follows that μ extends to I. Since μ is linear and χ faithful, then $K \nsubseteq I'$, and thus $I < G$.

Final Step. Let ψ be the Clifford correspondent of χ with respect to μ. Since both χ and μ have values in $\mathbb{Q}_{p^a}$ (use Step *7*), then also ψ has values in $\mathbb{Q}_{p^a}$. Since $\psi(1)$ divides $\chi(1)$, we have that $\psi(1)$ is a p'-number. Also I contains P and the oddness condition still holds in I. By Step *8*, we have that $I < G$, so by induction hypothesis, there exists $U \leqslant I$ and a linear character λ of U with values in $\mathbb{Q}_{p^a}$ such that $\psi = \lambda^I$. Also $\lambda^G = \chi$.

Now, suppose that $J \leqslant G$ and $\varphi \in \mathrm{Irr}(J)$ with $\psi^G = \chi$. Then $|G : J|$ is a p'-number and so $K \leqslant J$. Thus φ lies over some G-conjugate of μ. We may replace the pair (J, φ) by some G-conjugate and assume that φ actually lies over μ. Let $S = I \cap J = J_\mu$ and let $\eta \in \mathrm{Irr}(S)$ be the Clifford correspondent of φ with respect to μ. Then $\varphi = \eta^J$, so that $\eta^G = \varphi^G = \chi$. This implies that η^I is irreducible, lies over μ and induces χ. By the uniqueness of the Clifford correspondent $\eta^I = \psi$. Since the character $\psi = \lambda^I$ and λ is a linear character of $U \leqslant I$ with values in $\mathbb{Q}_{p^a}$, by the inductive hypothesis applied in I, we get that $\eta = \tau^S$, where τ is a linear character of a subgroup $W \leqslant I$ such that the pairs (U, λ) and (W, τ) are conjugate in I. □

Remark 2.7. Notice that we have actually proven that the character χ as in Theorem 2.6 is supermonomial.

Remark 2.8. Let χ be as in Theorem 2.6. Then $f_\chi = p^b$, where $b \geqslant 0$. Suppose that $\chi = \psi^G$ for some $\psi \in \mathrm{Irr}(H)$ and $H \leqslant G$. By Theorem 2.6, $\mathbb{Q}(\psi) \subseteq \mathbb{Q}_{p^b}$, so that f_ψ divides p^b. By the induction formula, we have that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\psi)$. This implies that f_χ divides f_ψ. Hence $f_\psi = f_\chi$.

Remark 2.9. Let π be a set of primes, let G be a π-separable group and let H be a Hall π-subgroup of G. We can mimic the proof Theorem 2.6 with π and H playing the role of p and P to get a π-version of this result (use Remark 2.4 and Hall theory in π-separable groups).

2.3. Certain character correspondences

Let G be a finite group, let p be a prime and let $a \geqslant 0$. How many p'-degree irreducible characters does G have with field of values contained in $\mathbb{Q}_{p^a}$? It does not seem easy at all how to answer this question in general. However, if G is p-solvable and $\mathbf{N}_G(P)/P$ has odd order, then this number can be computed locally. We write $X_{p^a}(G) = \{\chi \in \mathrm{Irr}_{p'}(G) \mid \mathbb{Q}(\chi) \subseteq \mathbb{Q}_{p^a}\}$. We are going to prove that there exists a canonical bijection from $X_{p^a}(G)$ onto $X_{p^a}(\mathbf{N}_G(P))$. The fact that there exists such a canonical bijection follows by using the natural correspondences $\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$ constructed by Isaacs (in the case where $p = 2$) and Turull (in the case where $|\mathbf{N}_G(P)|$ is odd) (see [**Isa73**] and [**Tur08**] for these highly non-trivial theorems). By using the fact that $X_{p^a}(G)$ consists of monomial characters, see Theorem 2.6, and the main result of [**Isa90**] it is easier to construct a canonical bijection $X_{p^a}(G) \to X_{p^a}(\mathbf{N}_G(P))$, as we are going to show.

Lemma 2.10. *Let $U \leqslant G$ and let p be a prime. Let λ be a linear character of U such that $\chi = \lambda^G \in \mathrm{Irr}(G)$. Let P be a Sylow p-subgroup of U and write $N = \mathbf{N}_G(P)$. Then*

(a) $(\lambda_{U\cap N})^N$ is irreducible.
(b) Suppose $\mu^G = \varphi \in \mathrm{Irr}(G)$ for some linear character μ of a subgroup $L \leqslant G$ which contains P. If $(\lambda_{U\cap N})^N = (\mu_{L\cap N})^N$ then $\chi = \varphi$.

Proof. See the proof of Lemma 2.3 of [**Isa90**]. □

Let G be p-solvable and let $P \in \mathrm{Syl}_p(G)$. Suppose that $\mathbf{N}_G(P)/P$ has odd order. If $\chi \in X_{p^a}(G)$, then by Theorem 2.6 there exist $U \leqslant G$ and a linear $\mathrm{Irr}(U)$ with $\mathbb{Q}(\lambda) \subseteq \mathbb{Q}_{p^a}$ such that $\lambda^G = \chi$. Write $N = \mathbf{N}_G(P)$ and $\varphi = (\lambda_{N\cap U})^N$. Then Lemma 2.10 guarantees $\varphi \in \mathrm{Irr}(N)$. Clearly $\varphi \in X_{p^a}(N)$. In order to see that the map $X_{p^a}(G) \to X_{p^a}(N)$ given by $\chi \mapsto \psi$ is a well-defined bijection we need some preliminary results.

Let G be p-solvable. We recall how to extend a linear character λ from a Sylow p-subgroup of G to a subgroup M in a maximal way guaranteeing that an extension of λ to M induces an irreducible character of G.

Theorem 2.11. *Let G be a p-solvable group and let $P \in \mathrm{Syl}_p(G)$. Suppose that $\lambda \in \mathrm{Irr}(P)$ is linear, then there exists a unique subgroup M of G containing P and maximal such that λ extends to M. Moreover, if $\mu \in \mathrm{Irr}(M)$ extends λ and $o(\mu)$ is a power of p, then $\mu^G \in \mathrm{Irr}(G)$.*

Proof. For the first part see Corollary 2.2 of [**IN08**]. The second part is Theorem 3.3 of [**IN08**] □

We will also need the following Lemma.

Lemma 2.12. *Let G be a group and let p be a prime. Let $P \in \mathrm{Syl}_p(G)$ and assume that $P \triangleleft G$ and G/P has odd order . If $\lambda \in \mathrm{Irr}(P)$ is linear and $\mathbb{Q}(\lambda) \subseteq \mathbb{Q}_{p^a}$, then there exists a unique $\chi \in \mathrm{Irr}(G)$ over λ with $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_{p^a}$.*

Proof. First note that every $\psi \in \mathrm{Irr}(G)$ lying over λ has p'-degree. By Lemma 1.13, let $\widehat{\lambda}$ be the canonical extension of λ to $T = G_\lambda$. We have that $o(\widehat{\lambda}) = o(\lambda)$. Hence $\widehat{\lambda}^G \in X_{p^a}(G)$. Now, suppose $\psi \in X_{p^a}(G)$ lies over λ. Then $\psi = \tau^G$ for some $\tau \in \mathrm{Irr}(T|\lambda)$. Using Theorem 1.12 $\tau = \beta\widehat{\lambda}$ for some $\beta \in \mathrm{Irr}(T/P)$. Since T/P has odd order, by Theorem 1.19 if β is real then $\beta = 1$. Therefore, if β is real, then $\psi = \chi$. Assume β is not real and let $\sigma \in \mathrm{Gal}(\mathbb{Q}(\beta)/\mathbb{Q})$ be the complex conjugation. Since $\mathbb{Q}(\beta) \subseteq \mathbb{Q}_{|T/P|}$ and $|T/P|$ is not divisible by p, we can extend σ to an element of $\mathrm{Gal}(\mathbb{Q}_{p^a}(\beta)/\mathbb{Q}_{p^a})$, by the natural irrationalities theorem of Galois theory. We have, $\overline{\beta} = \beta^\sigma \neq \beta$. Since $\psi^\sigma = \psi$ and $\lambda^\sigma = \lambda$, it must be $\tau^\sigma = \beta^\sigma\widehat{\lambda}^\sigma = \overline{\beta}\widehat{\lambda} = \beta\widehat{\lambda} = \tau$. Using Theorem 1.12 again, $\overline{\beta}\widehat{\lambda} = \beta\widehat{\lambda}$ implies $\overline{\beta} = \beta$, a contradiction. □

We are ready to prove Theorem C, which we restate below.

Theorem 2.13. *Let G be a p-solvable group and let $P \in \mathrm{Syl}_p(G)$. Write $N = \mathbf{N}_G(P)$ and assume that N/P has odd order. Define a map*

$$\Omega\colon X_{p^a}(G) \to X_{p^a}(N)$$

in the following way: If $\chi \in X_{p^a}(G)$, choose a pair (U, λ) where $P \leqslant U \leqslant G$ and $\lambda \in \mathrm{Irr}(U)$ linear with $\mathbb{Q}(\lambda) \subseteq \mathbb{Q}_{p^a}$ such that $\lambda^G = \chi$, then set $\Omega(\chi) = (\lambda_{U\cap N})^N$. Then Ω is a bijection.

Proof. Let $\chi \in X_{p^a}(G)$, by Theorem 2.6 we know that there exists a pair (U, λ) where $U \leqslant G$, λ is a linear character of U with $\mathbb{Q}(\lambda) \subseteq \mathbb{Q}_{p^a}$ and $\lambda^G = \chi$. As $\chi(1) = |G : U|$ is a p'-number, U contains a Sylow p-subgroup of G. We may assume $P \leqslant U$ maybe replacing (U, λ) by a conjugate pair. By Lemma 2.10(a), we have that $\varphi = (\lambda_{U\cap N})^N \in \mathrm{Irr}(N)$. Also $(\lambda_{U\cap N})^N(1) = |N : U \cap N|$ is a p'-number, and $\mathbb{Q}(\varphi) \subseteq \mathbb{Q}(\lambda) \subseteq \mathbb{Q}_{p^a}$. Then, we get $\varphi \in X_{p^a}(N)$. In this situation, we will say that φ arises from

χ. Using Lemma 2.10(b) we know that φ cannot arise from any $\psi \in X_{p^a}(G)$ different from χ. We have seen that in case Ω is well-defined, it is injective.

In order to prove that Ω defines a bijection, we need to show that Ω is well defined and surjective. First, we show that φ is the only element that arises from χ. Suppose that (W, ν) is another pair such that $P \leqslant W \leqslant G$, ν is linear and $\nu^G = \chi$. Then by Theorem 2.6, we have that $W = U^g$ and $\nu = \lambda^g$ for some $g \in G$. Thus, $P, P^g \leqslant W$. By Sylow Theory, there exists $t \in W$ such that $P = P^{gt}$. Also $W = U^{gt}$ and $\nu = \lambda^{gt}$. Hence we may assume $g \in N$. Then, it suffices to show that $(\lambda^g_{U^g \cap N})^N = (\lambda_{U \cap N})^N$. Write $\mu = \lambda^g_{U^g \cap N}$. For every $n \in N$ we have that

$$\begin{aligned}
\mu^N(n) &= \frac{1}{|U \cap N|} \sum_{x \in N} \dot{\mu}(xnx^{-1}) \\
&= \frac{1}{|U \cap N|} \sum_{x \in N} \dot{\lambda}_{U \cap N}(gxn(gx)^{-1}) \\
&= \frac{1}{|U \cap N|} \sum_{y \in N} \dot{\lambda}_{U \cap N}(yny^{-1}) \\
&= (\lambda_{U \cap N})^N(n).
\end{aligned}$$

Finally, we prove that Ω is surjective. Let $\theta \in X_{p^a}(N)$, let $\lambda \in \mathrm{Irr}(P)$ be under θ. Then, λ is linear and $o(\lambda) = |P : \ker(\lambda)|$. By Theorem 2.11 there exists $M \leqslant G$ containing P maximal such that λ extends to M. Furthermore, $M = PU$ where P normalizes U and $P \cap U = \ker(\lambda)$. Then, we can choose the unique extension ν of λ with $o(\nu) = o(\lambda)$. Again using Theorem 2.11, we have that $\nu^G \in \mathrm{Irr}_{p'}(G)$. We need to show that $\mathbb{Q}(\nu) \subseteq \mathbb{Q}_{p^a}$. It suffices to see that $\mathbb{Q}(\lambda) \subseteq \mathbb{Q}_{p^a}$. We distinguish the cases where $a = 0$ and $a > 0$.

Case $a = 0$. In this case θ is rational. Then both λ and $\overline{\lambda}$ are under θ. By Clifford's theorem $\overline{\lambda} = \lambda^g$ for some $g \in N$, and we see that g normalizes T the stabilizer of λ in N for the two actions commute. Then, $\lambda = \overline{\lambda^g} = (\overline{\lambda})^g = \lambda^{g^2}$, $g^2 \in T$. But $\mathbf{N}_N(T)/T$ has odd order and thus $g \in T$. Hence λ takes real values, since λ is linear we conclude λ is rational.

Case $a > 0$. We have that $\mathbb{Q}(\lambda) \subseteq \mathbb{Q}_{o(\lambda)}$ where $o(\lambda) = p^b$. Suppose $p^b > p^a$ and let $1 \neq \sigma \in \mathrm{Gal}(\mathbb{Q}_{p^b}/\mathbb{Q}_{p^a})$. Since

$$|\mathrm{Gal}(\mathbb{Q}_{p^b}/\mathbb{Q}_{p^a})| = \frac{|\mathrm{Gal}(\mathbb{Q}_{p^b}/\mathbb{Q})|}{|\mathrm{Gal}(\mathbb{Q}_{p^a}/\mathbb{Q})|} = \frac{p^{b-1}(p-1)}{p^{a-1}(p-1)} = p^{b-a},$$

σ has p-power order. Since $\chi^\sigma = \chi$, by Clifford's Theorem, $\lambda^\sigma = \lambda^g$ for some $g \in G$ that normalizes T, the stabilizer of λ in N. We know that the action by the automorphisms of G and Galois action on $\mathrm{Irr}(N)$ commute, then $g^{o(\sigma)} \in T$. Since $|\mathbf{N}_N(T)/T|$ is a p'-number, it must be that $g \in T$, and consequently $\lambda^\sigma = \lambda$. Now, we know that $(\nu_{M \cap N})^N \in X_{p^a}(N)$. If we show that $(\nu_{M \cap N})^N = \theta$ we will be done. This is true since by Lemma 2.12 there is a unique element of $X_{p^a}(N)$ over λ. □

The bijection Ω given by Theorem 2.13 is completely canonical. In particular, let α be an automorphism of G that fixes P. Let $\chi \in X_{p^a}(G)$ and $\lambda \in \mathrm{Irr}(U)$ such that $\Omega(\chi) = (\lambda_{U\cap N})^N$. Then we have that $\Omega(\chi^\alpha) = \Omega(\chi)^\alpha$ because $(\lambda^\alpha_{U^\alpha \cap N})^N = ((\lambda_{U\cap N})^N)^\alpha$.

2.4. Certain monomial characters and their subnormal constituents

As promised in the introduction, we use Isaacs B_π-theory of π-separable groups (see Section 1.4) to strengthen the conclusion of Theorem B. The main feature of the use of this theory is that in the situation of Theorem B we can guarantee that such χ is not only monomial but also any subnormal constituent of χ is monomial, see Theorem 2.15 below. The results contained in this section have been published in a joint work of the author with G. Navarro in [**NV15**].

We refer the reader to Section 1.4 for the definition and first properties of the Isaacs B_π-characters.

Lemma 2.14. *Suppose that G is a p-solvable group. Let $P \in \mathrm{Syl}_p(G)$, and assume that $\mathbf{N}_G(P)/P$ has odd order. If $\alpha \in \mathrm{Irr}(G)$ is p'-special and real, then α is the trivial character.*

Proof. We argue by induction on $|G|$. Let $N = \mathbf{O}_p(G)$. By Proposition 1.20, we have that $N \leqslant \ker(\alpha)$. If $N > 1$, then we apply induction in G/N. Otherwise, let $K = \mathbf{O}_{p'}(G)$. Then $K > 1$. By Lemma 2.3, there is some P-invariant $\theta \in \mathrm{Irr}(K)$ under α, and any two of them are $\mathbf{N}_G(P)$-conjugate (since α is p'-special it has in particular p'-degree). Since α is real, then $\bar\theta$ is also under α, and therefore there is $g \in \mathbf{N}_G(P)$ such that $\bar\theta = \theta^g$. Now g^2 fixes θ, and since $\mathbf{N}_G(P)/P$ has odd order, we see that $\bar\theta = \theta$. Also, the fact that $\mathbf{N}_G(P)/P$ has odd order implies that that $\mathbf{C}_K(P)$ has odd order. By Lemma 2.5, we have that $\theta = 1_K$. Thus $K \leqslant \ker(\alpha)$, and we apply induction in G/K. □

We are ready to prove the main result of this section.

Theorem 2.15. *Let p be a prime, let G be a p-solvable group, and let $P \in \mathrm{Syl}_p(G)$. Let $\chi \in \mathrm{Irr}(G)$ be such that p does not divide $\chi(1)$ and such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_{p^a}$ for some $a \geqslant 0$. If $|\mathbf{N}_G(P)/P|$ is odd, then $\chi \in B_p(G)$. In particular, if $N \lhd\lhd G$ and θ is an irreducible constituent of χ_N, then θ is monomial.*

Proof. By Theorem 1.27, there exists a subgroup $P \leqslant W \leqslant G$ and a p-special linear character $\lambda \in \mathrm{Irr}(W)$, such that: $\psi = \lambda^G \in \mathrm{Irr}(G)$ is a B_p-character, and (W, λ) is a nucleus of ψ. Also, there is a p'-special character $\alpha \in \mathrm{Irr}(W)$ such that $\chi = (\lambda\alpha)^G$. By Theorem 1.26, the pair $(W, \lambda\alpha)$ is unique up to G-conjugacy. Now, let $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q}_{|G|_p})$ be the unique Galois automorphism that complex conjugates the p'-roots of unity and fixes p-power roots of unity. Since χ and λ are fixed by σ, then we deduce that there is $g \in G$ such that $(W^g, \lambda^g\alpha^g) = (W, \lambda\alpha^\sigma)$. Hence $\alpha^g = \alpha^\sigma$ by Proposition 1.23. Since $P, P^g \leqslant W$, then $P^{gw} = P$ for some $w \in W$, and we may assume that $g \in \mathbf{N}_G(P)$. Also, $\alpha^{g^2} = \alpha$, and therefore since $\mathbf{N}_G(P)/P$ has odd order, we see that $\alpha^\sigma = \alpha$. Now, let H be a p-complement of W. Then

$$\bar{\alpha}_H = \overline{\alpha_H} = (\alpha^\sigma)_H = \alpha_H$$

and we deduce that $\bar{\alpha} = \alpha$, by using Proposition 1.22. Since $\mathbf{N}_W(P)/P$ has odd order, by Lemma (2.1), we have that $\alpha = 1_W$. Thus $\chi = \psi \in B_p(G)$ and χ is monomial. Now, to prove the second part of the theorem, use that subnormal constituents of B_p-characters are B_p-characters by Theorem 1.24, and the second part of Theorem 2.2 of [**CN08**] , that asserts that B_p-characters of p'-degree are monomial. □

We obtain the following consequence, in which a global invariant of a finite group is calculated locally.

Corollary 2.16. *Let p be a prime, let G be a p-solvable group, and let $P \in \mathrm{Syl}_p(G)$. Assume that $\mathbf{N}_G(P)/P$ has odd order. Then the number of irreducible characters χ of G such that $\chi(1)$ is not divisible by p and $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_{|G|_p}$ is the number of orbits of the natural action of $\mathbf{N}_G(P)$ on P/P'.*

Proof. Recall the notation from the previous section. If a is a non-negative integer we write $X_{p^a}(G) = \{\chi \in \mathrm{Irr}_{p'}(G) \mid \mathbb{Q}(\chi) \subseteq \mathbb{Q}_{p^a}\}$. Let $|P| = p^a$. By Theorem 2.15, we have that $X_{p^a}(G) = B_p(G) \cap \mathrm{Irr}_{p'}(G)$ and $X_{p^a}(N) = B_p(N) \cap \mathrm{Irr}_{p'}(N)$. By Theorem 2.2 and Corollary 2.3 of [**CN08**], we have that $|X_{p^a}(G)| = |X_{p^a}(N)|$. We will show that $|X_{p^a}(N)|$ is equal to the number of orbits under the natural action of $\mathbf{N}_G(P)$ on $\mathrm{Irr}(P/P')$. Let Λ be a complete set of representatives of the $\mathbf{N}_G(P)$ orbits on P/P'. If $\theta \in X_{p^a}(G)$, then by Clifford's theorem θ lies over a unique $\lambda \in \Lambda$. Hence $|X_{p^a}| \leqslant |\Lambda|$. Now, suppose that $\theta, \theta_1 \in X_{p^a}(N)$ lie over the same $\lambda \in \Lambda$. Then there exist $\psi, \psi_1 \in \mathrm{Irr}(N_\lambda|\lambda)$ such that $\theta = \psi^N$ and $\theta_1 = (\psi_1)^G$. Notice that ψ and ψ_1 have p'-degree by the induction formula. By Theorem 1.13 λ extends canonically to N_λ and by Theorem 1.12, we have that $\psi = \beta\hat{\lambda}$ and $\theta_1 = \beta_1\hat{\lambda}$, where $\beta, \beta_1 \in \mathrm{Irr}_{p'}(N_\lambda/P)$ and $\hat{\lambda}$ is the canonical extension of λ to N_λ. Since $\mathbb{Q}(\theta)$, $\mathbb{Q}(\theta_1)$ and $\mathbb{Q}(\lambda)$ are contained in $\mathbb{Q}_{p^a}$, then also $\mathbb{Q}(\psi)$ and $\mathbb{Q}(\psi_1)$ are contained in $\mathbb{Q}_{p^a}$. Hence β and β_1 are rational valued. Since by assumption N/P is odd, by Burnside's theorem 1.19 we have that $\beta = \beta_1$ and therefore $\theta = \theta_1$. □

As we have already mentioned, we can shorten the proof of our Theorem 2.2 if we are willing to use Isaacs B_π-theory.

Theorem 2.17. *Suppose that G is solvable. Suppose that $\chi \in \mathrm{Irr}(G)$ has odd degree. Suppose that $(\chi(1), f_\chi) = 1$. Then χ is monomial.*

Proof. Let π be the set of primes (possibly empty) dividing f_χ. Then χ has π'-degree. By Theorem 1.27, there exist a pair (W, γ), where $W \leqslant G$

and $\gamma \in \mathrm{Irr}(W)$ is π-special, and a π'-special character $\alpha \in \mathrm{Irr}(W)$ such $\chi = (\alpha\gamma)^G$. Also, $\gamma^G = \psi \in B_\pi(G)$ and (W, γ) is a nucleus for ψ. Also by Theorem 1.26, the pair $(W, \gamma\alpha)$ is unique up to G-conjugacy. Notice that $\chi(1) = |G : W|\alpha(1)\gamma(1)$ implies that $\gamma(1) = 1$ and W contains a full Sylow 2-subgroup of G.

We have that

$$\mathbb{Q}(\chi) \subseteq \mathbb{Q}_{f_\chi} \subseteq \mathbb{Q}_{|G|_\pi}.$$

Since γ is π-special, $\mathbb{Q}(\gamma) \subseteq \mathbb{Q}_{|G|_\pi}$. Now, let $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|_{\pi'}}/\mathbb{Q})$. We may extend σ to some $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q}_{|G|_\pi})$. Then σ fixes χ and γ. Thus $(W, \gamma\alpha^\sigma) = (W, \gamma\alpha)^g$ for some $g \in G$. In particular, $g \in \mathbf{N}_G(W)$. By Proposition 1.23, we have that $\gamma = \gamma^g$ and and $\alpha^\sigma = \alpha^g$. Let $\beta = \alpha^{\mathbf{N}_G(W)}$. Now, although β is not necessarily irreducible we have that $\mathbb{Q}(\beta) \subseteq \mathbb{Q}_{|G|_{\pi'}}$. Moreover, since $\alpha^\sigma = \alpha^g$, we have that β is fixed by σ and therefore β is rational valued (of odd) degree.

Now $\alpha^{\mathbf{N}_G(W)} = \beta = \overline{\beta} = (\bar{\alpha})^{\mathbf{N}_G(W)}$. It follows that some irreducible constituent ψ of β lies over α and $\bar{\alpha}$. Hence $\bar{\alpha} = \alpha^g$ for some $g \in \mathbf{N}_G(W)$. Since $\mathbf{N}_G(W)_\alpha = W$, we have that $g^2 \in W$, but $|\mathbf{N}_G(W)/W|$ odd implies that $g \in W$. Then α is real, and by Gow's theorem it is monomial. Hence χ is monomial. □

2.5. Feit's Conjecture and p'-degree characters

Our aim in this section is to prove Theorem D of the introduction. We need a little preparation in order to do that. We will use Gajendragadkar special characters (we refer the reader to Section 1.4). The following result will help us to control fields of values under certain circumstances.

Lemma 2.18. *Let G be a finite group, let q be a prime and let ζ be a primitive q-th root of unity. Suppose that G is q-solvable and $\chi \in \mathrm{Irr}(G)$ is q-special. If $\chi \neq 1$, then $\zeta \in \mathbb{Q}(\chi)$.*

Proof. Let Q be a Sylow q-subgroup of G. By Lemma 1.22, we have that $\psi \mapsto \psi_Q$ is an injection from the set of q-special characters of G into the set $\mathrm{Irr}(Q)$. In particular $\mathbb{Q}(\chi) = \mathbb{Q}(\chi_Q)$. Of course $\chi_Q \neq 1$. Thus, we may assume that G is a q-group. We also may assume that χ is faithful by modding out by $\ker(\chi)$. Choose $x \in \mathbf{Z}(G)$ of order q. We have that $\chi_{\langle x\rangle} = \chi(1)\lambda$, where $\lambda \in \mathrm{Irr}(\langle x\rangle)$ is faithful. Hence $\lambda(x^i) = \zeta$ for some integer i. In particular $\zeta \in \mathbb{Q}(\chi)$. □

The following elementary observation is stated as a Lemma for the reader's convenience.

Lemma 2.19. *Suppose that λ is a linear character of a finite group, and let $P \in \mathrm{Syl}_p(G)$. Let $\mathbf{N}_G(P) \subseteq H \leqslant G$ and let $\nu = \lambda_H$. Then $o(\lambda) = o(\nu)$.*

Proof. If $\lambda = 1_G$, then there is nothing to prove. We may assume λ is non-principal and hence $G' < G$. We have that $P \subseteq PG' \lhd G$. By the

Frattini argument, we have that $G = G'\mathbf{N}_G(P) = G'H$. Since $G' \subseteq \ker(\lambda)$ and $\ker(\nu) = \ker(\lambda) \cap H$, the result follows. □

The proof of Theorem D requires the use of a *magical* character; the canonical character associated to a *character five* defined by Isaacs in [**Isa73**]. We summarize the properties of ψ below.

Let $L \subseteq K \lhd G$ with $L \lhd G$ and K/L abelian. Let $\theta \in \mathrm{Irr}(K)$ and $\varphi \in \mathrm{Irr}(\theta_L)$. Suppose that θ is the unique irreducible constituent of φ^K (in this case we say that φ is **fully ramified** with respect to K/L or equivalently that θ is **fully ramified** with respect to K/L) and φ is G-invariant. Then we say that $(G, K, L, \theta, \varphi)$ is a **character five**.

Theorem 2.20. *Let $(G, K, L, \theta, \varphi)$ be a character five. Suppose that K/L is a q-group for some odd prime q. Then there exist a character ψ of G with $K \subseteq \ker(\psi)$ and a subgroup $U \leqslant G$ such that*

(a) $UK = G$ and $U \cap K = L$;
(b) $\psi(g) \neq 0$ for every $g \in G$, $\psi(1)^2 = |K : L|$ and the determinantal order of ψ is a power of q;
(c) if $K \subseteq W \leqslant G$, then the equation $\xi_W = \psi_W \xi_0$ for $\xi \in \mathrm{Irr}(W|\theta)$ and $\xi_0 \in \mathrm{Irr}(W \cap U|\varphi)$ defines a one-to-one correspondence between these two sets; and
(d) if $K \subseteq W \leqslant G$, then $\xi \in \mathrm{Irr}(W|\theta)$ and $\xi_0 \in \mathrm{Irr}(W \cap U|\varphi)$ correspond in the sense of (c) if and only if $\xi_0^G = \overline{\psi}_W \xi$, where $\overline{\psi}$ denotes the complex conjugate of ψ.
(e) If K/L is elementary abelian, then $\mathbb{Q}(\psi) \subseteq \mathbb{Q}_q$.

Proof. For parts (a), (b) and (c) see Theorem 3.1 of [**Nav02**]. Part (d) follows from Corollary 9.2 of [**Isa73**] (since the complement U provided by [**Nav02**] is "good" not only for G/L but also for every W/L where $K \subseteq W \leqslant G$). For part (e), by Theorem 9.1 of [**Isa73**] and the discussion at the end of the page 619 of [**Isa73**], the values of the character ψ are $\mathbb{Q}$-linear combinations of products of values of the bilinear multiplicative symplectic form $\ll , \gg_\varphi \colon K \times K \to \mathbb{C}^\times$ associated to φ (defined at the beginning of Section 2 of [**Isa73**]). The values of $\ll , \gg_\varphi$ are values of linear characters of cyclic subgroups of K/L. Since K/L is q-elementary abelian, we do obtain that $\mathbb{Q}(\psi) \subseteq F$. □

We can prove Theorem D, which we restate here.

Theorem 2.21. *Let p be a prime and let G be a finite solvable group. Let $\chi \in \mathrm{Irr}(G)$ of degree not divisible by p, and let $P \in \mathrm{Syl}_p(G)$. If $\chi(1)$ is odd, then there exists $g \in \mathbf{N}_G(P)/P'$ such that $o(g) = f_\chi$. In particular, the Feit number f_χ divides $|\mathbf{N}_G(P) : P'|$.*

Proof. By the Amit-Chillag theorem [**AC86**], we may assume that p divides $|G|$. We proceed by induction on $|G|$.

Let $N \lhd G$. If $\theta \in \mathrm{Irr}(N)$ is P-invariant and lies under χ, then we may assume that θ is G-invariant. Let $\psi \in \mathrm{Irr}(G_\theta|\theta)$ be the Clifford correspondent of χ. By the character formula for induction, $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\psi)$ and $\chi(1) = |G : G_\theta|\psi(1)$. Thus the character ψ satisfies the hypotheses of the theorem in G_θ and f_χ divides f_ψ. If $G_\theta < G$, then by induction there exists some $g \in \mathbf{N}_{G_\theta}(P)/P' \leqslant \mathbf{N}_G(P)/P'$ (notice that the P-invariance of θ implies $P \subseteq G_\theta$) such that $o(g) = f_\psi$. Hence, some power of g has order f_χ and we may assume $G_\theta = G$.

We claim that we may assume that χ is primitive. Otherwise, suppose that χ is induced from $\psi \in \mathrm{Irr}(H)$ for some $H < G$. In particular, p does not divide $|G : H|$ and so H contains some Sylow p-subgroup of G, which we may assume is P. Again by the character formula for induction, the degree $\psi(1)$ is an odd p'-number and f_χ divides f_ψ. By induction there is $g \in \mathbf{N}_H(P)/P' \leqslant \mathbf{N}_G(P)/P'$ such that $o(g) = f_\psi$. Thus some power of g has order f_χ, as claimed.

By Theorem 2.6 of [**Isa81**] the primitive character χ factorizes as a product

$$\chi = \prod_q \chi_q,$$

where the χ_q are q-special characters of G for distinct primes q. Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q}(\chi))$. Then

$$\prod_q \chi_q^\sigma = \prod_q \chi_q.$$

By using the uniqueness of the product of special characters, see Proposition 1.23, we conclude that $\chi_q^\sigma = \chi_q$ for every q. Hence f_{χ_q} divides f_χ for every q, and since the f_{χ_q}'s are coprime also $\prod_q f_{\chi_q}$ divides f_χ. Notice that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\chi_q \mid q) \subseteq \mathbb{Q}_{\prod_q f_{\chi_q}}$ by elementary Galois theory. This implies the equality $f_\chi = \prod_q f_{\chi_q}$.

Now, consider $K = \mathbf{O}^{p',p}(G) < G$. Notice that $PK = \mathbf{O}^p(G) \lhd G$. By the Frattini argument $G = PK\mathbf{N}_G(P) = K\mathbf{N}_G(P)$. If $K = 1$, then $P \lhd G$ and we are done in this case. We may assume that $K > 1$. Let K/L be a chief factor of G. Then K/L is an abelian p'-group. If $H = \mathbf{N}_G(P)L$, then $G = KH$ and $K \cap H = L$, by a standard group theoretical argument. Furthermore, all the complements of K in G are G-conjugate to H. Finally, notice that $\mathbf{C}_{K/L}(P) = 1$ using that $H \cap K = L$.

We claim that for every q, there exists some q-special $\chi_q^* \in \mathrm{Irr}(H)$ such that $f_{\chi_q^*} = f_{\chi_q}$ and $\chi_q^*(1)$ is an odd p'-number.

If $q \in \{2, p\}$, then $\lambda = \chi_q$ is linear (because χ has odd p'-degree). Let $\lambda^* = \lambda_H$. Then λ^* is q-special (since λ is linear and q-special, this is straightforward from the definition) and $f_{\lambda^*} = f_\lambda$ by Lemma 2.19.

Let $q \neq p$ be an odd prime and write $\eta = \chi_q$. We work to find some $\eta^* \in \mathrm{Irr}(H)$ of odd p'-degree with $f_{\eta^*} = f_\eta$. By Lemma 2.3, let $\theta \in \mathrm{Irr}(K)$ be some P-invariant constituent of η_K and let $\varphi \in \mathrm{Irr}(L)$ be some P-invariant constituent of η_L. By the second paragraph of the proof, we know that both

θ and φ are G-invariant and hence φ lies under θ. By Theorem 6.18 of [**Isa76**] one of the following holds:

(a) $\theta_L = \sum_{i=1}^{t} \varphi_i$, where the $\varphi_i \in \mathrm{Irr}(L)$ are distinct and $t = |K : L|$,
(b) $\theta_L \in \mathrm{Irr}(L)$, or
(c) $\theta_L = e\varphi$, where $\varphi \in \mathrm{Irr}(L)$ and $e^2 = |K : L|$.

Notice that the situation described in (a) cannot occur here, because φ is G-invariant.

In the case described in (b), we have $\varphi = \theta_L \in \mathrm{Irr}(L)$. Then restriction defines a bijection between the set of irreducible characters of G lying over θ and the set of irreducible characters of H lying over φ (by Corollary (4.2) of [**Isa86**]). Write $\xi = \eta_H$. By Theorem A of [**Isa86**], we know that ξ is q-special. We claim that $\mathbb{Q}(\eta) = \mathbb{Q}(\xi)$. Clearly, $\mathbb{Q}(\xi) \subseteq \mathbb{Q}(\eta)$. If $\sigma \in \mathrm{Gal}(\mathbb{Q}(\eta)/\mathbb{Q}(\xi))$, then notice that φ is σ-invariant because ξ_L is a multiple of φ. Now, φ is P-invariant, and because $\mathbf{C}_{K/L}(P) = 1$, there is a unique P-invariant character over φ (by Problem 13.10 of [**Isa76**]). By uniqueness, we deduce that $\theta^\sigma = \theta$. Now, η^σ lies over θ and restricts to ξ, so we deduce that $\eta^\sigma = \eta$, by the uniqueness in the restriction. Thus $\mathbb{Q}(\eta) = \mathbb{Q}(\xi)$. We write $\eta^* = \xi$.

Finally, we consider the situation described in (c). Since θ_L is not irreducible, then $|K : L|$ is not a q'-group, by Theorem 1.10. Hence K/L is q-elementary abelian and e is a power of q. By Theorem 2.20 (and using that all the complements of K/L in G/L are conjugate), there exists a (not necessarily irreducible) character ψ of G such that:

(i) ψ contains K in its kernel, $\psi(g) \neq 0$ for every $g \in G$, $\psi(1) = e$ and the determinantal order of ψ is a power of q.
(ii) if $K \subseteq W \leqslant G$ and $\xi \in \mathrm{Irr}(W|\theta)$, then $\xi_{W \cap H} = \psi_{W\cap H}\xi_0$ for a unique irreducible character ξ_0 of $W \cap H$.
(iii) The values of ψ lie on $\mathbb{Q}_q$.

In particular, $\eta_H = \psi\eta_0$, so that $\eta_0 \in \mathrm{Irr}(H|\varphi)$ (where we are viewing ψ as a character of H). We claim that η_0 is q-special. First notice that $\eta_0(1) = \eta(1)/e$ is a power of q. Now, we want to show that whenever S is a subnormal subgroup of H, the irreducible consituents of $(\eta_0)_S$ have determinantal order a power of q. Since $(\eta_0)_L$ is a multiple of φ, which is q-special, we only need to control the irreducible constituents of $(\eta_0)_S$ when $L \subseteq S \lhd \lhd H$, by using Proposition 1.20. We have that $K \subseteq SK \lhd \lhd G$. Write

$$\eta_{SK} = a_1\gamma_1 + \cdots + a_r\gamma_r,$$

where the $\gamma_i \in \mathrm{Irr}(SK)$ are q-special because η is q-special and $a_i \in \mathbb{N}_0$. By using the property (ii) of ψ, we have that $\eta_S = \psi_S(\eta_0)_S$ also decomposes as

$$\begin{aligned}\eta_S &= a_1\psi_S(\gamma_1)_0 + \cdots + a_r\psi_S(\gamma_r)_0\\ &= \psi_S(a_1(\gamma_1)_0 + \cdots + a_r(\gamma_r)_0).\end{aligned}$$

Since ψ never vanishes on G, we conclude that $(\eta_0)_S = a_1(\gamma_1)_0+\cdots+a_r(\gamma_r)_0$. It suffices to see that $o((\gamma_i)_0)$ is a power of q for every γ_i constituent of η_{SK}. Just notice that

$$\begin{aligned}\det((\gamma_i)_S) &= \det(\psi_S(\gamma_i)_0)\\ &= \det(\psi_S)^{(\gamma_i)_0(1)}\det((\gamma_i)_0)^e.\end{aligned}$$

Since $o(\psi)$, $o(\gamma_i)$, $\gamma_i(1)$ and e are powers of q, we easily conclude that also the determinantal order of $(\gamma_i)_0$ is a power of q. This proves that η_0 is q-special. We claim that $\mathbb{Q}(\eta) = \mathbb{Q}(\eta_0)$ so that the two Feit numbers are the same. Let ζ be a primitive q-th root of unity and write $F = \mathbb{Q}(\zeta)$. Then the values of ψ lie in F. We next see that η and η_0 are non-principal. This is obvious because θ and φ are fully ramified. Suppose that $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|}/F)$ stabilizes η. Then

$$\psi\eta_0 = \psi^\sigma\eta_0^\sigma = \psi\eta_0^\sigma.$$

Using that ψ is never zero, we conclude that $\eta_0^\sigma = \eta_0$. Now, by part (d) of Theorem 2.20, we have that ξ and ξ_0 correspond (as in part (c) of Theorem 2.20) if and only if $(\xi_0)^G = \psi\xi$. Hence, if $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|}/F)$ and $\eta_0^\sigma = \eta_0$, then $\overline{\psi}\eta = (\eta_0)^G = (\eta_0^\sigma)^G = \overline{\psi}\eta^\sigma$ (because also $\mathbb{Q}(\overline{\psi}) \subseteq F$). This implies again that $\eta^\sigma = \eta$. By Galois theory, we have that $F(\eta) = F(\eta_0)$. By Lemma 2.18, this implies $\mathbb{Q}(\eta) = \mathbb{Q}(\eta_0)$. We set $\eta^* = \eta_0$. The claim follows.

Now, we define $\chi^* = \prod_q {\chi_q}^*$ which has odd p'-degree. The character χ^* is irreducible by Proposition 1.23. Also $f_{\chi^*} = \prod_q \chi_q^*$ as in the fourth paragraph of this proof. Hence

$$f_{\chi^*} = \prod_q f_{{\chi_q}^*} = \prod_q f_{\chi_q} = f_\chi.$$

By the inductive hypothesis, there exists $g \in \mathbf{N}_H(P)/P' \leqslant \mathbf{N}_G(P)/P'$ such that $o(g) = f_{\chi^*}$ and we are done. □

CHAPTER 3

McKay natural correspondences of characters

3.1. Introduction

Let p be a prime. Recall that for a finite group G, we denote by $\mathrm{Irr}_{p'}(G)$ the set of the irreducible characters $\chi \in \mathrm{Irr}(G)$ of G which have degree $\chi(1)$ not divisible by p. The McKay conjecture, one of the main problems in the Representation Theory of Finite Groups, asserts that $|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(P))|$, where $P \in \mathrm{Syl}_p(G)$. It is suspected that in general no choice-free correspondence can exist between $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(\mathbf{N}_G(P))$, at least there does not exist one commuting with Galois action. For instance, if $G = \mathrm{GL}_2(3)$, $p = 3$ and $P \in \mathrm{Syl}_p(G)$, then all the irreducible characters of $\mathbf{N}_G(P)$ are rational-valued, while G has characters of degree 2 which are not rational-valued.

A key case to consider is when $\mathbf{N}_G(P) = P$. If $\mathbf{N}_G(P) = P$, then McKay conjecture predicts the existence of a bijection between $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}(P/P')$. We prove that much more is happening for p odd.

Theorem E. *Let G be a group, let p be an odd prime and let $P \in \mathrm{Syl}_p(G)$. Suppose that $P = \mathbf{N}_G(P)$. If $\chi \in \mathrm{Irr}_{p'}(G)$, then*

$$\chi_P = \chi^* + \Delta\,,$$

where $\chi^* \in \mathrm{Irr}(P)$ is linear and Δ is either zero or Δ is a character whose irreducible constituents have all degree divisible by p. Furthemore, the map $\chi \mapsto \chi^*$ is a natural bijection $\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}(P/P')$.

Theorem E was proved for p-solvable groups in [**Nav07**] for any prime p, although it was suspected long time ago that it should hold in general for odd primes. For $p = 2$, Theorem E is not true, the symmetric group $\mathfrak{S}_5$ provides a counterexample. However, E. Giannelli [**Gia16**] found canonical bijections for $p = 2$ and $\mathfrak{S}_n$. (This bijection cannot be described by restriction unless $n = 2^k$.) We also mention that in [**GKNT16**] the authors provide a different canonical bijection for $p = 2$ and $\mathfrak{S}_n$ as well as for $\mathrm{GL}_n(q)$ and $\mathrm{GU}_n(q)$ (for q odd).

We will also work in greater generality. Instead of assuming that $P \in \mathrm{Syl}_p(G)$ is self-normalizing, we assume that $\mathbf{N}_G(P) = \mathbf{C}_G(P)P$ (namely $\mathbf{N}_G(P)$ is p-decomposable). We prove that a bijection as in Theorem E does exist but at the moment only between characters in the principal blocks (again for p odd). Recall that a character $\chi \in \mathrm{Irr}(G)$ lies in the principal (p-)block $B_0(G)$ of G if $\sum_{x \in G^0} \chi(x) \neq 0$, where G^0 is the set of elements of

G of order not divisible by p. If we further assume that G is p-solvable, then we can prove the following (with no restriction on p).

THEOREM F. *Let G be a finite p-solvable group, and let $P \in \mathrm{Syl}_p(G)$. Suppose that $\mathbf{N}_G(P) = P\mathbf{C}_G(P)$, and let $N = \mathbf{O}_{p'}(G)$. Let $\mathrm{Irr}_P(N)$ be the set of P-invariant characters $\theta \in \mathrm{Irr}(N)$. Then, for every $\theta \in \mathrm{Irr}_P(N)$ and $\lambda \in \mathrm{Irr}(P/P')$ linear, there is a canonically defined character*

$$\lambda \star \theta \in \mathrm{Irr}_{p'}(G)\,.$$

Furthermore, the map

$$\mathrm{Irr}(P/P') \times \mathrm{Irr}_P(N) \to \mathrm{Irr}_{p'}(G)$$

given by $(\lambda, \theta) \mapsto \lambda \star \theta$ is a bijection. As a consequence, if $\theta^* \in \mathrm{Irr}(\mathbf{C}_N(P))$ is the Glauberman correspondent of $\theta \in \mathrm{Irr}_P(N)$ (see Theorem 1.17), then the map

$$\lambda \times \theta^* \mapsto \lambda \star \theta$$

is a natural bijection $\mathrm{Irr}_{p'}(\mathbf{N}_G(P)) \to \mathrm{Irr}_{p'}(G)$. Also, if $\theta = 1_N$ and $\lambda \in \mathrm{Irr}(P/P')$, then $\lambda \times \theta^*$ is the unique linear constituent of $(\lambda \star \theta)_{\mathbf{N}_G(P)}$.

This chapter is structured in the following way: We begin by studying some character correspondences in the groups $\mathrm{PSL}_2(3^{3^a})$ for $a \geqslant 1$ in Section 3.2. There is a good reason for that: these groups appear as the only non-abelian composition factors of groups with a self-normalzing Sylow p-subgroup for odd p (see [**GMN04**]). In Section 3.3 we prove the following extension theorem which is key to prove Theorem E.

THEOREM. *Let $N \lhd G$. Let p be an odd prime and let $P \in \mathrm{Syl}_p(G)$. Assume that $\mathbf{N}_G(P) = P$. If $\chi \in \mathrm{Irr}_{p'}(G)$ and $\theta \in \mathrm{Irr}(N)$ lies under χ, then θ extends to G_θ.*

In Section 3.4 we prove Theorem E and we present an application to characterize groups with self-normalizing Sylow p-subgroups for odd p. In Section 3.5, we consider the case where the normalizer of the Sylow p-subgroup is p-decomposable.

We will start by proving a key group theoretical result, which extends a classical work of J. Thompson (see Theorem 3.14 of [**Isa08**]).

THEOREM. *Let G be a group, let p be a prime, and let $P \in \mathrm{Syl}_p(G)$. Suppose that $\mathbf{N}_G(P) = P \times X$. If p is odd or G is p-solvable, then $X \leqslant \mathbf{O}_{p'}(G)$. In particular, if $\mathbf{N}_G(P) = P\mathbf{C}_G(P)$, then $\mathbf{O}_{p'}(\mathbf{N}_G(P)) \leqslant \mathbf{O}_{p'}(G)$.*

After that we prove there exist character correspondences as in Theorem E between characters in the principal blocks. In the last section, we prove Theorem F.

All the results contained in this chapter, unless otherwise stated, appear in a joint work of the author together with G. Navarro and P. H. Tiep [**NTV14**].

3.2. Character correspondences associated with $\mathrm{PSL}_2(3^{3^a})$

By the main result of [**GMN04**], if a group G has a self-normalizing Sylow p-subgroup for some odd prime p, then either G is solvable or $p = 3$ and G has a composition factor of type $\mathrm{PSL}_2(3^{3^a})$ where $a \geqslant 1$. In the case where G is not solvable, it is easy to show that all non-abelian composition factors of G are of type $\mathrm{PSL}_2(3^{3^a})$ with $a \geqslant 1$.

LEMMA 3.1. *Let G be a group. Let p be an odd prime. Suppose that $\mathbf{N}_G(P) = P$ for some $P \in \mathrm{Syl}_p(G)$. If K/L is a non-abelian composition factor of G, then K/L is of type $\mathrm{PSL}_2(3^{3^a})$ with $a \geqslant 1$.*

PROOF. By Corollary 1.2 of [**GMN04**], the claim holds when G is simple. We proceed by induction on $|G|$. Let N be a minimal normal subgroup of G. Then $1 < |N| < |G|$. Since both $K \cap N$ and L are normal subgroups of K, we have that $L \leqslant (K \cap N)L \lhd K$. However, K/L simple yields $(K \cap N)L = L$ or $(K \cap N)L = K$.

Suppose $(K \cap N)L = L$. Then $K/L \cong KN/KL$ is a nonabelian composition factor of G/N. Since $\mathbf{N}_{G/N}(PN/N) = \mathbf{N}_G(P)N/N = PN/N$, by induction hypothesis K/L is of type $\mathrm{PSL}_2(3^{3^a})$ with $a \geqslant 1$.

Suppose that $(K \cap N)L = K$. Then $K/L \cong (K \cap N)/(L \cap N)$ is a non-abelian composition factor of $G_0 = PN$. Therefore N is a non-abelian minimal normal subgroup. We can write $N = T_1 \times \cdots \times T_r$, where the T_i's are non-abelian simple groups transitively permuted by G and consequently all isomorphic. Also, the T_i's are the composition factors of N up to isomorphism by the Jordan-Hölder Theorem. Thus $K/L \cong T_i$ for some i. Since $\mathbf{N}_{G_0}(P) = P$, if $G_0 < G$, then we are done by induction. We may assume that $G = PN$. By the main result of [**GMN04**], some composition factor of G, thus some composition factor T_j of N is of type $\mathrm{PSL}_2(3^{3^a})$ with $a \geqslant 1$. Hence K/L is also of this type and we are done. □

Due to these facts, the groups of type $\mathrm{PSL}_2(3^{3^a})$ with $a \geqslant 1$ play an important role in the proof of our Theorem E. The description of the behavior of the character theory of groups of type $\mathrm{PSL}_2(q)$ under the action of automorphisms was accomplished in Section (15B) of [**IMN07**]. In order to make this chapter as self-contained as possible, this section is devoted to describe how does the character theory of $\mathrm{PSL}_2(3^{3^a})$ behave under the action of their field automorphisms using the background of Section 1.5. (We will use results contained in Section 1.5 without specific reference).

First we need the following result about some special actions of p-groups on groups of type $\mathrm{PSL}_2(p^{p^a})$.

LEMMA 3.2. *Let $S = \mathrm{PSL}_2(q)$, where $q = p^{p^a}$, for some odd prime p and $a \geqslant 1$. Embed $S \lhd \mathrm{Aut}(S) = A$. Let $P \leqslant A$ be a p-subgroup such that $P \cap S = Q \in \mathrm{Syl}_p(S)$. Suppose that P acts on S with $\mathbf{C}_{\mathbf{N}_S(Q)/Q}(P) = 1$. Then $p = 3$ and $P \in \mathrm{Syl}_p(A)$.*

Proof. We proceed in a series of steps.

Step 1. We may assume that Q is the Sylow p-subgroup Q_1 of S induced from upper unitriangular matrices.

We have that $Q = Q_1^s$ for some $s \in S$. Then $Q_1 = P^{s^{-1}} \cap S \lhd P^{s^{-1}} = P_1$ and P_1 acts on S with $\mathbf{C}_{\mathbf{N}_S(Q_1)/Q_1}(P_1) = (\mathbf{C}_{\mathbf{N}_S(Q)/Q}(P))^{s^{-1}} = 1$.

Step 2. We describe the Sylow p-subgroups of A and $\mathbf{N}_A(Q)$.

Write $F = \mathbb{F}_q$. Let φ be the Frobenius automorphism of F. Then, φ induces an automorphism of S and $\langle\varphi\rangle \leqslant A$ has order $f = p^a$, see Section 1.5.1. By Step *1*, the group $\langle\varphi\rangle$ normalizes Q, and so $Q\langle\varphi\rangle$ is a subgroup of A of order $p^a \cdot q$ (we are using that $\langle\varphi\rangle \cap S = 1$). Hence $Q\langle\varphi\rangle \in \mathrm{Syl}_p(A)$, because $|A| = p^a q(q^2-1)$. Write $H = \mathbf{N}_A(Q)$. We claim:

(a) $\mathrm{Syl}_p(A) = \{(Q\langle\varphi\rangle)^s \mid s \in S\}$,
(b) $\mathrm{Syl}_p(H) = \{(Q\langle\varphi\rangle)^t \mid t \in \mathbf{N}_S(Q)\}$.

We first prove (a). Since A/S is abelian, we have that $S\langle\varphi\rangle \lhd A$ and $Q\langle\varphi\rangle$ is a Sylow p-subgroup of $S\langle\varphi\rangle$. By Frattini's argument (see 3.2.7 of [**KS04**] for instance)

$$A = S\langle\varphi\rangle\mathbf{N}_A(Q\langle\varphi\rangle) = S\mathbf{N}_A(Q\langle\varphi\rangle),$$

hence the description in (a) follows. To prove (b), we proceed analogously. Since $Q \in \mathrm{Syl}_p(S)$ and $S \lhd A$, we have that $A = S\mathbf{N}_A(Q) = SH$ by Frattini's argument. In particular, $H/\mathbf{N}_S(Q)$ is abelian. We have that $\langle\varphi\rangle$ normalizes $\mathbf{N}_S(Q)$. Hence $\mathbf{N}_S(Q)\langle\varphi\rangle \lhd H$ and using Frattini's argument one more time, we conclude

$$H = \mathbf{N}_S(Q)\langle\varphi\rangle\mathbf{N}_H(Q\langle\varphi\rangle) = \mathbf{N}_S(Q)\mathbf{N}_H(Q\langle\varphi\rangle).$$

Step 3. We may assume $P \leqslant Q\langle\varphi\rangle$. In particular $P = Q\langle\varphi^e\rangle$ for some integer e.

Since $P \leqslant \mathbf{N}_A(Q)$ is a p-subgroup, then $P \leqslant (Q\langle\varphi\rangle)^t$ for some $t \in \mathbf{N}_S(Q)$, by Step *2*. Let $s = t^{-1}$. We have $Q^s = Q \leqslant P^s \leqslant Q\langle\varphi\rangle$. We may replace P by P^s because P^s acts on S stabilizing Q and

$$\mathbf{C}_{\mathbf{N}_S(Q)/Q}(P^s) = (\mathbf{C}_{\mathbf{N}_S(Q)/Q}(P))^s = 1.$$

Thus, we may assume $P \leqslant Q\langle\varphi\rangle$. Hence $P = P \cap Q\langle\varphi\rangle = Q(P \cap \langle\varphi\rangle) = Q\langle\varphi^e\rangle$, for some integer e.

Final step. We conclude $P = Q\langle\varphi\rangle$ is a full Sylow p-subgroup of A.

By Step *3*, we know that $P = Q\langle\varphi^e\rangle$ for some integer e. Consider φ^e both as a Galois automorphism of F and as an automorphism of S. Suppose that φ^e fixes an element $a \in F^\times$. Let $d(a) = \mathrm{diag}(a, a^{-1}) \in \mathbf{N}_S(Q)$. By easy matrix computations, for every $x = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in Q$ and for every integer k, we have that

$$d(a)^{-1}d(a)^{x\varphi^{ek}} = \begin{pmatrix} 1 & b^{p^{ek}}(a-a^{-1}) \\ 0 & 1 \end{pmatrix} \in Q.$$

This proves that $d(a) \in \mathbf{C}_{\mathbf{N}_S(Q)/Q}(P)$. By hypothesis $\mathbf{C}_{\mathbf{N}_S(Q)/Q}(P) = 1$, so that $d(a) \in Q$. This means that $d(a)$ is the identity of S. Thus $a = \pm 1 \in F$. Hence, the subfield of F fixed by φ^e is exactly $\mathbb{F}_3$. Since $\varphi^e \in \mathrm{Gal}(F/\mathbb{F}_p)$, we conclude that $p = 3$ and $\langle \varphi^e \rangle = \langle \varphi \rangle$, by Galois Theory. □

Let $S = \mathrm{PSL}_2(q)$, where $q = 3^{3^a}$ for some $a \geqslant 1$. We shall also need the following result about the automorphisms of S that centralize a Sylow 3-subgroup of S.

LEMMA 3.3. *Let $S = \mathrm{PSL}_2(q)$, where $q = 3^{3^a}$ for some $a \geqslant 1$. Write $A = \mathrm{Aut}(S)$. Let $P \in \mathrm{Syl}_3(A)$. Write $Q = P \cap S \in \mathrm{Syl}_3(S)$. Suppose that $Y \leqslant A$ is a $3'$-subgroup that centralizes Q. Then $Y = 1$.*

PROOF. We may assume that Q is the Sylow 3-subgroup of S induced from upper unitriangular matrices. Since $A = \mathrm{PGL}_2(q) \rtimes \langle \varphi \rangle$, where $\langle \varphi \rangle$ is the group of field automorphisms of S with $o(\varphi) = 3^a$, and Y is a $3'$-subgroup of A, we have that $Y \leqslant \mathrm{PGL}_2(q)$. The result follows since $\mathbf{C}_{\mathrm{PGL}_2(q)}(Q) = Q$. □

LEMMA 3.4. *Let G be a group and let $S \leqslant G$. Suppose that $S = \mathrm{PSL}_2(q)$, where $q = 3^{3^a}$ and $a \geqslant 1$. Write $p = 3$ and let P be a p-subgroup of G such that $Q = P \cap S \in \mathrm{Syl}_p(S)$. Suppose further that P acts on S with $\mathbf{C}_{\mathbf{N}_S(Q)/Q}(P) = 1$. If $\alpha \in \mathrm{Irr}_{p'}(S)$ is P-invariant, then*

$$\alpha_Q = \alpha^* + \Delta,$$

where α^* is P-invariant and no irreducible constituent of the character Δ is P-invariant. Moreover, the map $\alpha \mapsto \alpha^*$ defines a bijection between the set of irreducible character of S of p'-degree fixed by P and the set of irreducible characters of Q fixed by P.

PROOF. We proceed in a series of steps. Let $F = \mathbb{F}_q$. We write φ to denote the Frobenius automorphism of the field F.

Step 1. We may assume Q is the Sylow p-subgroup Q_1 of S induced from upper unitriangular matrices.

We have that $Q^s = Q_1$ for some $s \in S$. Then $Q_1 = P^s \cap S \lhd P^s = P_1$ and P_1 acts on S with $\mathbf{C}_{\mathbf{N}_S(Q_1)/Q_1}(P_1) = (\mathbf{C}_{\mathbf{N}_S(Q)/Q}(P))^s = 1$. Also $\mathrm{Irr}_{P_1}(Q_1) = (\mathrm{Irr}_P(Q))^s$.

Step 2. We may work in $\mathrm{Aut}(S)$ *and we may assume that $P = Q\langle \varphi \rangle$, where we identify φ with the field automorphism of S corresponding to φ as in Section 1.5.*

Write $A = \mathrm{Aut}(S)$. Let $\gamma\colon P \to A$ be the homomorphism defined by conjugation by P. Write $P_1 = \gamma(P) \leqslant A$. Embed $S \lhd \mathrm{Aut}(S)$. Then $Q \leqslant P_1$. Since the actions of P and P_1 on S are equivalent, we have that $\mathbf{C}_{\mathbf{N}_S(Q)/Q}(P_1) = 1$. By Lemma 3.2, we have that P_1 is a Sylow p-subgroup of A. Since $P_1 \leqslant \mathbf{N}_A(Q)$, we have that $P_1 = (Q\langle \varphi \rangle)^t$ for some $t \in \mathbf{N}_S(Q)$ (proceed as in Step *2* of the proof of Lemma 3.2). Hence $Q \leqslant P_2 = (P_1)^{t^{-1}} =$

$Q\langle\varphi\rangle$. We notice that

$$\mathbf{C}_{\mathbf{N}_S(Q)/Q}(P_2) = (\mathbf{C}_{\mathbf{N}_S(Q)/Q}(P_1))^{t^{-1}} = 1,$$

$\mathrm{Irr}_{\langle\varphi\rangle}(Q) = \mathrm{Irr}_{P_2}(Q) = \mathrm{Irr}_{P_1}(Q)$ and $\mathrm{Irr}_{\langle\varphi\rangle}(S) = \mathrm{Irr}_P(S)$. This proves the claim.

Step 3. We describe the set $\mathrm{Irr}_{\langle\varphi\rangle}(S)$.

We have that $q \equiv 3 \mod 4$, for $a \geqslant 1$. The character table of S has been given in Section 1.5. We keep the notation of Section 1.5 throughout this proof. Also in Section 1.5, it is is shown that $\{1_S, St_S, \eta_0', \eta_0''\} \subseteq \mathrm{Irr}_{\langle\varphi\rangle}(S)$. By Lemma 15.1 of [**IMN07**], we have that $\mathrm{Irr}_{\langle\varphi\rangle}(S) = \{1_S, \mathrm{St}_S, \eta_0', \eta_0''\}$. l

The only character of this set with degree divisible by p is the Steinberg character St_S. Of course $(1_S)_Q = 1_Q$ has the desired form. Hence, in order to prove the statement, we need to understand how does $\alpha \in \{\eta_0', \eta_0''\}$ restrict to Q.

Step 4. We compute $(\eta_0')_Q$ *and* $(\eta_0'')_Q$.

We denote by Tr the trace map $\mathrm{Tr}_{F/\mathbb{F}_3} \colon F \to \mathbb{F}_3$ associated to the field extension $F/\mathbb{F}_3$. The trace map is $\mathbb{F}_3$-linear and $\mathrm{Tr}(b) = b + b^p + \cdots + b^{p^a-1}$ for every $b \in F$, by Corollary 23.11 of [**Isa94**]. Fix ϵ a primitive cubic root of unity. For every $b \in F$, we define a homomorphism $\lambda_b \colon Q \to \mathbb{C}^\times$ as follows: if $c \in F$, then $\lambda_b \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = \epsilon^{\mathrm{Tr}(bc)}$. We have that

$$\mathrm{Irr}(Q) = \{\lambda_b \mid b \in F\}.$$

The automorphism φ acts on Q. Hence φ acts on $\mathrm{Irr}(Q)$. In fact $\lambda_b^\varphi = \lambda_{\varphi(b)}$ for every $b \in F$. By Theorem 1.16, the number of elements of Q fixed by φ is equal to the number of irreducible characters of Q fixed by φ. Since the subfield of F fixed by φ is the prime field $\mathbb{F}_3$, we have that φ fixes exactly three elements of Q. It is clear that $\mathrm{Irr}_{\langle\varphi\rangle}(Q) = \{1_Q, \lambda_1, \lambda_{-1}\}$.

We denote by U the subgroup of $F^\times$ consisting of the squares of $F^\times$. The subgroup U has index 2 in $F^\times$. Since -1 is not a square in F, every $b \in F^\times$ is either in U or there exists an $u \in U$ such that $b = -u$. From the character table of S we see that

$$(\eta_0')_Q + (\eta_0'')_Q = \rho_Q - 1_Q,$$

where ρ_Q is the regular character of Q (we recall ρ_Q has degree $|Q| = q$ and ρ_Q vanishes on every nontrivial element of Q). Thus

$$\rho_Q = \sum_{b \in F} \lambda_b = 1_Q + (\eta_0')_Q + (\eta_0'')_Q.$$

We conclude that $(\eta_0')_Q$ is the sum of $\frac{1}{2}(q-1)$ nontrivial λ_b's and $(\eta_0'')_Q$ is exactly the sum of the $\frac{1}{2}(q-1)$ nontrivial λ_b's that do not appear in $(\eta_0')_Q$. The normalizer of Q in S is

$$\mathbf{N}_S(Q) = \{\begin{pmatrix} c & b \\ 0 & c^{-1} \end{pmatrix} \mid c \in F^\times, b \in F\} = \bigcup_{c \in F^\times} \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} Q.$$

Let $b \in F$. Write $d(c) = \mathrm{diag}(c, c^{-1}) \in H$, where $c \in F^\times$. It is straightforward to check that

$$\lambda_b^{d(c)} = \lambda_{c^2 b}.$$

Thus the action of $\mathbf{N}_S(Q)$ on $\mathrm{Irr}(Q)$ decomposes $\mathrm{Irr}(Q)$ into three orbits as follows

$$\mathrm{Irr}(Q) = \{1_Q\} \cup \{\lambda_b \mid b \in U\} \cup \{\lambda_{-b} \mid b \in U\}.$$

Hence, if λ_b is a consituent of $(\eta'_0)_Q$, then

$$(\eta'_0)_Q = \sum_{u \in U} \lambda_{ub}$$

is exactly the sum over all squares of F if b is a square or the sum over all the non-squares of F if b is a non-square. In particular, we have that λ_e is a constituent of $(\eta'_0)_Q$ if and only if λ_{-e} is a constituent of $(\eta''_0)_Q$ for $e \in \{-1, 1\}$. This finishes the proof of the statement. □

REMARK 3.5. Under the hypothesis of Lemma 3.2, we have that

$$\mathrm{Irr}_P(S) = \{1_S, \mathrm{St}_S, \eta', \eta''\},$$

where St_S is the Steinberg character of S and η' and η'' are the two irreducible cuspidal characters of S of degree $\frac{1}{2}(q-1)$, by Lemma 15.1 of [**IMN07**].

3.3. An extension theorem

In this section we prove a non-trivial extension result which is key to proving Theorem E.

THEOREM 3.6. *Let $N \lhd G$ and let p be an odd prime. Let $P \in \mathrm{Syl}_p(G)$ and suppose that $\mathbf{N}_G(P) = P$. Let $\chi \in \mathrm{Irr}_{p'}(G)$. If $\theta \in \mathrm{Irr}(N)$ lies under χ, then θ extends to G_θ.*

The proof of Theorem 3.6 we present here is totally different from the one we originally gave in [**NTV14**]. This new shorter and cleaner proof is based upon the ideas contained in [**NT16**]. In [**NT16**] the authors were interested in the case $p = 2$, but their argument becomes easier for odd primes.

We begin with an elementary extension result.

LEMMA 3.7. *Let N be a normal p-subgroup of G. Let $\chi \in \mathrm{Irr}_{p'}(G)$ and let $\theta \in \mathrm{Irr}(N)$ be under χ. Then θ extends to G_θ.*

PROOF. We may assume that θ is G-invariant since the Clifford correspondent $\psi \in \mathrm{Irr}(G_\theta|\theta)$ of χ has p'-degree. Let $P \in \mathrm{Syl}_p(G)$. Then $N \leqslant P$. Since χ has p'-degree, some irreducible consituent μ of χ_P has p'-degree. In particular, μ is linear and lies over θ, hence $\mu_N = \theta$. If $Q \in \mathrm{Syl}_q(G)$ for some prime $q \neq p$, we have that θ also extends to NQ by Corollary 6.20 of [**Isa76**]. According to Corollary 11.31 of [**Isa76**] θ extends to G. □

We shall also need an extension result from [**NT16**], which is not hard to prove.

LEMMA 3.8. *Let $N \lhd G$. Suppose that $N = S_1 \times \cdots \times S_t$ is the direct product of subgroups which are transitively permuted by G by conjugation. Write $S = S_1$ and view $S/\mathbf{Z}(S) \lhd A = \mathrm{Aut}(S)$. Let $\theta = \theta_1 \times .. \times \theta_t \in \mathrm{Irr}(N)$ be G-invariant, where $\theta_i \in \mathrm{Irr}(S_i)$ and $\theta_1 \in \mathrm{Irr}(S/\mathbf{Z}(S))$. If θ_1 extends to A_{θ_1}, then θ extends to G.*

PROOF. This is Lemma 2.8 of [**NT16**]. □

We are ready to prove the main result of this section.

PROOF OF THEOREM 3.6. Choose (G, N) a counterexample of minimal $|G| + |N|$. We may assume that $G = G_\theta$ by minimality of (G, N) since the Clifford correspondent $\psi \in \mathrm{Irr}(G_\theta|\theta)$ of χ has p'-degree and $|G : G_\theta|$ is a p'-number. Thus $\chi_N = e\theta$ for some $e \geqslant 1$.

Suppose $M \lhd G$ and $M < N$. Then

$$\chi_M = f \sum_{i=1}^{t} \tau^{x_i},$$

where $\tau \in \mathrm{Irr}(M)$ and $\{\tau^{x_1}, ..., \tau^{x_t}\}$ is a G-orbit. Also

$$\chi_M = e\theta_M = ef' \sum_{i=1}^{r} \tau^{n_i},$$

where the sum now is over an N-orbit. Write $I = G_\tau$. Then

$$|G : I| = t = r = |N : N \cap I|,$$

so $G = NI$. By Lemma 2.3 some irreducible constituent of χ_M is P-invariant, let us say τ, so $P \leqslant I$. Let $\rho \in \mathrm{Irr}(N \cap I|\tau)$ be the Clifford correspondent of θ. Since both τ and θ are I-invariant, also ρ is I-invariant. Now, let $\psi \in \mathrm{Irr}(I)$ be under χ and over ρ. In particular, ψ lies over τ and then it must be the Clifford correspondent of χ over τ. Hence $\psi^G = \chi$ and $\psi \in \mathrm{Irr}_{p'}(I)$. By minimality of (G, N), the character ρ extends to some $\mu \in \mathrm{Irr}(I)$. Notice that $I = G_\rho$, so that $\mu^G \in \mathrm{Irr}(G)$ and $(\mu^G)_N = (\mu_{N\cap I})^N = \rho^N = \theta$, a contradiction.

Hence, we may assume that N is a minimal normal subgroup of G. If N is a p-group, then Lemma 3.7 yields a contradiction. If we suppose that N is a p'-group, then the hypothesis $\mathbf{N}_G(P) = P$ implies $\mathbf{C}_N(P) = 1$ by coprime action. By the Glauberman correspondence (see Theorem 1.17) $\theta = 1_N$, which obviously extends to G, a contradiction. We may hence assume that $N = S_1 \times \cdots \times S_k$ is the direct product of some simple non-abelian groups $\{S_1, ..., S_k\}$ of order divisible by p which are transitively permuted by G. Write $\theta = \theta_1 \times \cdots \times \theta_k \in \mathrm{Irr}(N)$ with $\theta_i \in \mathrm{Irr}(S_i)$. Write $S = S_1$ and $S_i = S^{x_i}$ for some $x_i \in G$ for $i = 2, \ldots, k$. By Lemma 3.1, S is isomorphic to $\mathrm{PSL}_2(3^{3^a})$ for some $a \geqslant 1$. View $S \lhd A = \mathrm{Aut}(S)$. In Section 1.5.1 we have seen that $|A/S| = 2 \cdot 3^a$. Notice that $o(\theta_1) = 1$ because S_1 has no non-principal linear character. If $Q/S \in \mathrm{Syl}_3(A_{\theta_1}/S)$, then θ_1 extends to Q by Lemma 1.13. If $P/S \in \mathrm{Syl}_2(A_{\theta_1}/S)$ it must be cyclic or trivial, in any case, θ_1 extends to P by Theorem 1.14. By Theorem 1.15, we conclude that θ_1 extends to A_θ. By Lemma 3.8, we see that θ extends to G, contradicting the choice of G as a minimal counterexample. □

3.4. The self-normalizing case

In this section we prove Theorem E. We also present a (perhaps surprising) consequence of Theorem E: a characterization of groups having a self-normalizing Sylow p-subgroup in terms of the decomposition of a certain permutation character (for odd primes). We shall need the following result from [**NTT07**].

LEMMA 3.9. *Suppose that a finite p-group P acts on a finite group G, stabilizing $N \lhd G$. Suppose that $Q/N \in \mathrm{Syl}_p(G/N)$ is P-invariant, and assume that $G/N = T_1/N \times \cdots \times T_r/N$, where the T_i's are permuted by P. Let $Q_i = Q \cap T_i$, and let P_i be the stabilizer of T_i in P. If $\mathbf{C}_{\mathbf{N}_G(Q)/Q}(P) = 1$, then $\mathbf{C}_{\mathbf{N}_{T_i}(Q_i)/Q_i}(P_i) = 1$.*

PROOF. Apply Lemma (4.1) of [**NTT07**] to each of the orbits defined by Q on $\{T_1, \ldots, T_r\}$. □

The only way we have found to prove Theorem E is to use a strong induction over normal subgroups, and Theorem 3.6 is key in this inductive process. The following result is a *relative to normal subgroups* version of Theorem E.

THEOREM 3.10. *Let G be a finite group, p an odd prime, $P \in \mathrm{Syl}_p(G)$, and suppose that $P = \mathbf{N}_G(P)$. Let $L \lhd G$. Let $\chi \in \mathrm{Irr}_{p'}(G)$. Then*

$$\chi_{LP} = \chi^* + \Delta,$$

where $\chi^* \in \mathrm{Irr}_{p'}(LP)$ and either Δ is zero, or Δ is a character of LP whose irreducible constituents have all degree divisible by p.

PROOF. Let G be a counterexample with $|G| \cdot |G/L|$ smallest possible.

(a) By Lemma 2.3, let $\theta \in \mathrm{Irr}(L)$ be P-invariant under χ. Let $T = G_\theta \geqslant LP$, and let $\psi \in \mathrm{Irr}(T|\theta)$ be the Clifford correspondent of χ over θ. Assume that $T < G$. By the choice of G, we have that

$$\psi_{LP} = \psi^* + \Delta\,,$$

where ψ^* has p'-degree and the irreducible constituents of Δ have degree divisible by p. Let $\mathbb{T}$ be a transversal for the double cosets of T and P in G. We may assume $1 \in \mathbb{T}$ Write

$$G = \bigcup_{x \in \mathbb{T}} TxP$$

Then, by Mackey's Lemma 1.8, we have that

$$\chi_{LP} = (\psi^G)_{LP} = \psi_{LP} + \sum_{1 \neq x \in \mathbb{T}} ((\psi^x)_{T^x \cap LP})^{LP} .$$

Let $\alpha \in \mathrm{Irr}_P(L)$ be an irreducible constituent of $((\psi^x)_{T^x \cap LP})^{LP}$. Suppose that α has degree not divisible by p. Hence $\alpha_L \in \mathrm{Irr}(L)$. Thus the irreducible character $\alpha_{T^x \cap LP}$ lies under ψ^x. However $(\psi^x)_L = d\theta^x$ for some $d \geqslant 1$, so we conclude that $\theta^x = \alpha_L$. Hence θ^x is P-invariant. By Lemma 2.3, we have that $\theta^{xy} = \theta$ for some $y \in P$ and therefore $x \in T$. But this is impossible since $x \neq 1$ lies in $\mathbb{T}$.

We may assume then that θ is G-invariant. By Theorem 3.6, we have that θ has an extension $\tilde{\theta} \in \mathrm{Irr}(G)$. By Gallagher Theorem 1.12, we have that $\chi = \beta\tilde{\theta}$, for some $\beta \in \mathrm{Irr}(G/L)$. Now, if $L \neq 1$, then the theorem holds for G/L, whence we have that β_{PL} is the sum of a p'-degree irreducible character β^* of PL/L (and hence linear) plus some character Δ of PL/L such that all of its irreducible constituents have degree divisible by p. Then

$$\chi_{LP} = (\beta^*)\tilde{\theta}_{LP} + \Delta\tilde{\theta}_{LP} ,$$

and using Gallagher Theorem 1.12, we see that we are done again. Hence $L = 1$.

(b) Suppose now that K is a minimal normal subgroup of G. Assume first that K is a p-group. Then $|G/K| < |G| = |G/L|$, and since $KP = P$, then the theorem is proved.

Assume next that K is a p'-group. Since $\mathbf{N}_G(P) = P$, we have that $\mathbf{C}_K(P) = 1$. Since $\theta \in \mathrm{Irr}(K)$ is P-invariant, we necessarily have that $\theta = 1_K$ by the Glauberman correspondence (see Theorem 1.17). But in this case, $K \leqslant \ker(\chi)$, and we can work in the group G/K.

Hence, G has no abelian normal subgroup. In particular, $F(G) = 1$. We have $E = F^*(G) = E(G) \geqslant \mathbf{C}_G(E(G))$ (see Theorem 6.5.8 of [**KS04**]). Since $\mathbf{C}_G(E) = \mathbf{Z}(E) \lhd G$, it follows that $\mathbf{Z}(E) = 1$ and so E is product of subnormal nonabelian simple subgroups. By the main result of [**GMN04**], $p = 3$ and G has a composition factor of type $\mathrm{PSL}_2(3^{3^a})$. By Lemma 3.1, $E = S_1 \times \ldots \times S_n$ is a direct product of non-abelian simple groups $S_i \cong \mathrm{PSL}_2(q_i)$, where $q_i = 3^{3^{a_i}}$ with $a_i \geqslant 1$.

(c) Let $Q = P \cap E \in \mathrm{Syl}_p(E)$ and write $Q = Q_1 \times \ldots \times Q_n$ with $Q_i \in \mathrm{Syl}_p(S_i)$. Since P is self-normalizing in EP, by part (ii) of Lemma 2.1 of [**NTT07**], we have that $\mathbf{C}_{\mathbf{N}_G(Q)/Q}(P) = 1$. This in turn implies, by Lemma 3.9, that $\mathbf{C}_{\mathbf{N}_{S_i}(Q_i)/Q_i}(P_i) = 1$ for $P_i = P_{S_i}$. By Lemma 3.2, it follows that P_i must induce the full subgroup $C_{3^{a_i}}$ of field automorphisms of S_i. By Remark 3.5, we see that the P_i-invariant irreducible characters of S_i are $\alpha_i = 1_{S_i}$, the Steinberg character St_{S_i} of degree q_i and the two cuspidal characters η_i' and η_i'' of degree $\frac{1}{2}(q_i - 1)$. Furthermore for each $\alpha \in \{\alpha_i, \eta_i', \eta_i''\}$, P_i fixes a unique irreducible constituent of α_{Q_i}, appearing with multiplicity one, by Lemma 3.4. We denote by α^* this constituent, so that the map * defines a bijection from the P_i-invariant irreducible characters of p'-degree of S_i onto the P_i-invariant irreducible characters of Q_i, again this is Lemma 3.4.

Since the theorem holds for (G, E), $\chi_{EP} = \chi^* + \Delta$, where $\chi^* \in \mathrm{Irr}_{p'}(EP)$ and all irreducible constituents of Δ have degree divisible by p. In particular, $\theta = (\chi^*)_E$ is irreducible. Write

$$\theta = \theta_1 \times \cdots \times \theta_n$$

with $\theta_i \in \mathrm{Irr}_{p'}(S_i)$. Since θ is P-invariant, it follows that θ_i is P_i-invariant of p'-degree, and so $\theta_i \in \{\alpha_i, \eta_i', \eta_i''\}$. As mentioned above,

$$(\theta_i)_{Q_i} = \theta_i^* + \delta_i,$$

where $\theta_i^* \in \mathrm{Irr}(Q_i)$ is P_i-invariant, and δ_i is a sum of non-P_i-invariant irreducible characters of Q_i. Setting

$$\tilde{\theta} = \theta_1^* \times \cdots \times \theta_n^*,$$

we see that each irreducible constituent of $\theta_Q - \tilde{\theta}$ is non-P-invariant and so must lie under an irreducible character of P of degree divisible by p. But p does not divides $\chi^*(1)$. Hence $(\chi^*)_P$ contains a linear constituent which must be unique and lies above $\tilde{\theta}$. Denote this constituent by θ^*. We have shown that every irreducible constituent of $(\chi^*)_P - \theta^*$ is of degree divisible by p, whereas $\theta^*(1) = 1$.

(d) It remains to show that every irreducible constituent of Δ_P has degree divisible by p. Assume the contrary: Δ_P contains a linear constituent λ, and write

$$\lambda_Q = \lambda_1 \times \cdots \times \lambda_n$$

with $\lambda_i \in \mathrm{Irr}(Q_i)$. Let $\gamma \in \mathrm{Irr}(EP)$ be an irreducible constituent of Δ that contains λ upon restriction to P. Also, let

$$\beta = \beta_1 \times \cdots \times \beta_n \in \mathrm{Irr}(E)$$

be lying under γ and above λ_Q. Since $E \lhd G$, we have that β is G-conjugate to θ. Note that the β_i's are P_i-invariant of degree not divisible by p, thus $\beta_i \in \{\alpha_i, \eta_i', \eta_i''\}$. As shown in (c), the restriction $(\beta_i)_{Q_i}$ contains a unique P_i-invariant irreducible constituent β_i^*, of multiplicity one. Denoting

$$\tilde{\beta} = \beta_1^* \times \cdots \times \beta_n^*,$$

we see that no irreducible constituent of $\beta_Q - \tilde{\beta}$ can be invariant under P. But λ_Q lies under β and is P-invariant. Hence $\lambda_Q = \tilde{\beta}$ and $\lambda_i = \beta_i^*$. Now we consider two cases.

Case 1: β is not P-invariant. In this case, there is some $g \in P$ such that $\beta^g \neq \beta$. Then β^g lies above $(\lambda_Q)^g = \lambda_Q$ and under γ. Writing $\beta^g = \beta_1' \times \ldots \times \beta_n'$ with $\beta_i' \in \{\alpha_i, \eta_i', \eta_i''\}$ for β^g is G-conjugate to θ and so the β_i' are P_i-invariant. Arguing as above, we see that $\beta_i^* = \lambda_i = (\beta_i')^*$. By Lemma 3.4, the map $\alpha \mapsto \alpha^*$ is a bijection. It follows that $\beta_i = \beta_i'$ and so $\beta = \beta^g$, a contradiction.

Case 2: β is P-invariant. Then, by Theorem 3.6, β extends to $\hat{\beta} \in \mathrm{Irr}(EP)$. Since γ lies above β and p divides $\gamma(1)$, we have that $\gamma = \hat{\beta}\mu$, where $\mu \in \mathrm{Irr}(P/Q)$ is considered as a character of EP/E and p divides $\mu(1)$. Certainly, $\mu\lambda$ is irreducible over P and non-linear. On the other hand, as shown above, every irreducible constituent of

$$(\gamma_P - \mu\lambda)_Q = \mu_Q\hat{\beta}_Q - \mu_Q\tilde{\beta} = \mu(1)\cdot(\beta_Q - \tilde{\beta})$$

is non-P-invariant and so must lie under an irreducible character of P of degree divisible by p. Thus the degree of every irreducible constituent of $\gamma_P - \mu\lambda$ is divisible by p. Consequently, λ cannot be a constituent of γ_P, a contradiction. □

Theorem E is now a corollary of Theorem 3.10.

Corollary 3.11. *Let G be a finite group, let p be an odd prime and let $P \in \mathrm{Syl}_p(G)$. Suppose that $P = \mathbf{N}_G(P)$. If $\chi \in \mathrm{Irr}_{p'}(G)$, then*

$$\chi_P = \chi^* + \Delta\,,$$

where $\chi^* \in \mathrm{Irr}(P)$ is linear and Δ is either zero or Δ is a character whose irrreducible constituents have all degree divisible by p. Furthemore, the map $\chi \mapsto \chi^*$ is a natural bijection $\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}(P/P')$.

Proof. The first part follows from considering $L = 1$ in Theorem 3.10. Let $\lambda \in \mathrm{Irr}(P/P')$. Then

$$\lambda^G = a_1\chi_1 + \cdots + a_n\chi_n,$$

where the a_i's are natural numbers and $\chi_i \in \mathrm{Irr}(G)$. Since $\lambda^G(1) = |G : P|$, some χ_i must have degree not divisible by p. So $\chi_i \in \mathrm{Irr}_{p'}(G)$. By Theorem 3.10, $(\chi_i)_P = (\chi_i)^* + \Delta$, where $(\chi_i)^*$ is linear and no irreducible constituent of Δ is linear. However, λ is a linear constituent of $(\chi_i)_P$, and so $(\chi_i)^* = \lambda$. Thus, * is surjective. By the main theorem of [**GMN04**] and Theorem A of [**IMN07**], we have that $|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}(P/P')|$. Hence, the natural map * defines a bijection. □

We finish this section with an application of Theorem E. We obtain the following characterization of groups having a self-normalizing Sylow p-subgroup, for an odd prime p.

Corollary 3.12. *Let G be a finite group, let p be odd, and let $P \in \mathrm{Syl}_p(G)$. Then $\mathbf{N}_G(P) = P$ if and only if*

$$(1_P)^G = 1_G + \Xi\,,$$

where Ξ is either zero or a character whose irreducible constituents all have degree divisible by p.

Proof. One implication follows from Corollary 3.11. Assume now that

$$(1_P)^G = 1_G + \Xi\,,$$

where Ξ is either zero or a character whose irreducible constituents all have degree divisible by p, but $N = \mathbf{N}_G(P) > P$. Then there exists a non-principal character $\gamma \in \mathrm{Irr}(N/P)$, which can be viewed as a character of N. Since γ has p'-degree (because N/P is a p'-group), it follows that γ^G possesses an irreducible constituent $\chi \in \mathrm{Irr}_{p'}(G)$. Now, χ lies over $\gamma \neq 1_N$ and therefore $\chi \neq 1_G$ lies over 1_P, a contradiction. □

It is remarkable that Corollary 3.12 gives the exact opposite of a result by G. Malle and G. Navarro in [**MN12**]: a finite group G has a normal Sylow p-subgroup if and only if all irreducible constituents of $(1_P)^G$ have degree not divisible by p.

The conclusion of corollary 3.12 is false for $p = 2$, as shown by $G = \mathfrak{S}_5$: in this case $(1_P)^G$ contains the trivial character of G and an irreducible character of degree 5.

3.5. The p-decomposable Sylow normalizer case

Let N be a group and let p be a prime. We say that N is p-decomposable if $N = P \times X$, where $P \in \mathrm{Syl}_p(N)$. Suppose that the group G has a p-decomposable Sylow p-normalizer for an odd prime p (by Schur-Zassenhaus' theorem [**Isa08**, Thm. 3.5] this is equivalent to $\mathbf{N}_G(P) = \mathbf{C}_G(P)P$ for $P \in \mathrm{Syl}_p(G)$). Then we can prove the following.

Theorem 3.13. *Let G be a finite group, let p be an odd prime, and let $P \in \mathrm{Syl}_p(G)$. Suppose that $\mathbf{N}_G(P) = P\mathbf{C}_G(P)$. If $\chi \in \mathrm{Irr}_{p'}(G)$ lies in the principal block, then*

$$\chi_{\mathbf{N}_G(P)} = \chi^* + \Delta\,,$$

where $\chi^* \in \mathrm{Irr}(\mathbf{N}_G(P))$ is linear in the principal block, and Δ is either zero or Δ is a character whose irreducible constituents all have degree divisible by p. Furthermore, the map $\chi \mapsto \chi^*$ is a bijection

$$\mathrm{Irr}_{p'}(B_0(G)) \to \mathrm{Irr}_{p'}(B_0(\mathbf{N}_G(P))),$$

where $\mathrm{Irr}_{p'}(B_0(G))$ is the set of irreducible characters in the principal block of G of degree not divisible by p.

Theorem 3.13 is the main result of this section. We begin by proving Theorem 3.14 below. Theorem 3.14 extends a classical result by J. Thompson (see Theorem 3.14 of [**Isa08**]) and it will be key for us later in this section.

Theorem 3.14. *Let G be a group, let p be a prime, and let $P \in \mathrm{Syl}_p(G)$. Suppose that $\mathbf{N}_G(P) = P \times X$. If p is odd or G is p-solvable, then $X \leqslant \mathbf{O}_{p'}(G)$. In particular, if p is odd or G is p-solvable and also $\mathbf{N}_G(P) = P\mathbf{C}_G(P)$, then $\mathbf{O}_{p'}(\mathbf{N}_G(P)) \leqslant \mathbf{O}_{p'}(G)$.*

Proof. We argue by induction on $|G|$. If $L \lhd G$, then

$$\mathbf{N}_{G/L}(PL/L) = \mathbf{N}_G(P)L/L = PL/L \times XL/L\,.$$

Hence, if $L > 1$, then we have that $XL/L \leqslant \mathbf{O}_{p'}(G/L)$. In particular, we may assume that $\mathbf{O}_{p'}(G) = 1$. Now, suppose that $N = \mathbf{O}_p(G) > 1$. Then $XN/N \leqslant \mathbf{O}_{p'}(G/N)$ implies that $X \leqslant \mathbf{O}_{pp'}(G) = M$. Since $[X, P] = 1$, then $[X, N] = 1$. By Schur-Zassenhaus theorem [**Isa08**, Thm. 3.5], N has a complement in H in M, similarly $\mathbf{Z}(N)$ has a complement K in $\mathbf{C}_M(N)$ which must be contained in some conjugate H^g of H. Since $K \lhd H^g$ it follows that $K \lhd M$, so $K \subseteq \mathbf{O}_{p'}(M)$. However, using that $\mathbf{O}_{p'}(G) = 1$, we have that $\mathbf{C}_M(N) = \mathbf{Z}(N) \times \mathbf{O}_{p'}(M) = \mathbf{Z}(N)$, and we conclude that $X = 1$, in this case. Hence, we may assume that G is not p-solvable, and that p is odd.

Now, let N be a minimal normal subgroup of G. By [**GMN04**], we have that $N = S_1 \times \cdots \times S_k$, where $\{S_1, \ldots, S_k\}$ are transitively permuted by G and $S_1 = S = \mathrm{PSL}_2(3^{3^a})$. Now $P \cap N = (P \cap S_1) \times \cdots \times (P \cap S_k)$. Fix some index i. Since $[P, X] = 1$, then $[Q_i, X] = 1$, where $1 < Q_i = P \cap S_i \in \mathrm{Syl}_3(S_i)$. Now, if $x \in X$, then we have that $(S_i)^x = S_j$ for some j. However $Q_i^x \leqslant S_i^x \cap S_i = S_j \cap S_i$, so we conclude that $X \leqslant \mathbf{N}_G(S_i)$ for all i with $[X, Q_i] = 1$. Let $Y_i = X\mathbf{C}_G(S_i)/\mathbf{C}_G(S_i)$, which is a $3'$-subgroup of $\mathrm{Aut}(S_i)$ centralizing the Sylow 3-subgroup Q_i of S_i. By Lemma 1.5, $Y_i = 1$, whence $X \leqslant \mathbf{C}_G(S_i)$ for all i. Thus $X \leqslant \mathbf{C}_G(N)$ for every minimal normal subgroup. Since $\mathbf{F}(G) = 1$, then $\mathbf{F}^*(G) = \mathbf{E}(G) = E$. Since $\mathbf{Z}(E) = 1$, then E is semisimple and $\mathbf{C}_G(E) = 1$ by Theorems 9.7 and 9.8 of [**Isa08**]. Now E is a direct product of non-abelian simple groups K_i and the normal closure of K_i is a minimal normal subgroup of G (by Lemma 9.17 of [**Isa08**], for instance), and we conclude that $X \leqslant \mathbf{C}_G(E) = 1$, as desired.

Finally, since $\mathbf{C}_G(P) = \mathbf{Z}(P) \times \mathbf{O}_{p'}(\mathbf{N}_G(P))$ (by the Schur-Zassenhaus theorem [**Isa08**, Thm. 3.5]), it follows that if $\mathbf{N}_G(P) = P\mathbf{C}_G(P)$, then

$$\mathbf{N}_G(P) = P \times \mathbf{O}_{p'}(\mathbf{N}_G(P)),$$

and we apply the first part of the theorem. □

Note that the conclusion of Theorem 3.14 is not true for $p = 2$: If $G = \mathrm{E}_6(11)$ and $P \in \mathrm{Syl}_2(G)$, then $\mathbf{N}_G(P) = P \times C_5$, cf. [**KM03**, Theorem 6(c)].

Let G be a group and let p be a prime. We denote by G^0 the set of p-regular elements of G (those elements whose order is not divisible by p). For the purpose of this exposition, we define the irreducible characters in the principal (p)-block of G as the set of $\chi \in \mathrm{Irr}(G)$ satisfying

$$\sum_{x \in G^0} \chi(x) \neq 0.$$

We write $\mathrm{Irr}(B_0(G))$ to denote the set of the irreducible characters of G that lie in the principal block of G. (See Theorem 3.19 of [**Nav98**]).

Theorem 3.15. *Let $N = \mathbf{O}_{p'}(G)$. Suppose that $\chi \in \mathrm{Irr}(G)$ lies in the principal block of G, then $N \subseteq \ker(\chi)$ and the corresponding $\overline{\chi} \in \mathrm{Irr}(G/N)$ lies in the principal block of G/N.*

Proof. We notice that $x \in G$ is p-regular if and only if $Nx \in G/N$ is p-regular. Hence, we can write

$$G^0 = Nx_1 \cup \ldots \cup Nx_t$$

as a disjoint union (the p-regular classes of G in N correspond to the class of $N1$ in G/N). Let $\mathfrak{X}$ be a representation affording χ. We can extend $\mathfrak{X}$ by linearity to a representation $\mathbb{C}G \to \mathrm{Mat}(n, \mathbb{C})$ which we denote again by $\mathfrak{X}$. We also denote by χ the trace $\mathbb{C}G \to \mathbb{C}$ of this representation. Notice that $\chi(\sum_{g \in G} a_g g) = \sum_{g \in G} a_g \chi(g)$ for every $\sum_{g \in G} a_g g \in \mathbb{C}G$. If $X \subseteq G$, then we write $\hat{X} = \sum_{x \in X} x \in \mathbb{C}G$. By hypothesis, we have that $\chi(\hat{G^0}) \neq 0$. Hence, the matrix $\mathfrak{X}(\hat{G^0}) = \mathfrak{X}(\hat{N})\mathfrak{X}(x_1) + \cdots + \mathfrak{X}(\hat{N})\mathfrak{X}(x_t)$ is non-zero. In particular, $\mathfrak{X}(\hat{N}) \neq 0$. Notice that for every $g \in G$

$$\mathfrak{X}(\hat{N})\mathfrak{X}(g) = \mathfrak{X}(\hat{N}g) = \mathfrak{X}(g\hat{N}) = \mathfrak{X}(g)\mathfrak{X}(\hat{N}),$$

so that $\mathfrak{X}(\hat{N})$ is scalar. We conclude that the trace of $\mathfrak{X}(\hat{N})$ is non-zero, this is, $0 \neq \sum_{n \in N} \chi(n) = |N|[\chi_N, 1_N]$. Consequently $N \subseteq \ker(\chi)$. The following observation

$$\sum_{x \in G^0} \chi(x) = |N| \sum_{j=1}^{t} \chi(x_j) = |N| \sum_{j=1}^{t} \overline{\chi}(Nx_j) = |N| \sum_{x \in (G/N)^0} \overline{\chi}(x)$$

proves the second statement of the theorem. □

The proof of the following lemma is a straightforward consequence of Theorem 3.15.

Lemma 3.16. *Suppose that N is a normal subgroup of H, with $N \leqslant \mathbf{O}_{p'}(H)$. Suppose that $H = NU$ for some $U \leqslant H$. Then restriction defines a bijection between the characters in the principal block of H and the characters in the principal block of U.*

As we prove below, in the case where $\mathbf{N}_G(P) = P$, every character of p'-degree of G lies in the principal block.

Lemma 3.17. *Let G be a group and let p be a prime. Suppose that $\mathbf{N}_G(P) = P$ where $P \in \mathrm{Syl}_p(G)$. Let $\chi \in \mathrm{Irr}(G)$. Then*

$$\sum_{x \in G^0} \chi(x) \equiv \chi(1) \mod p.$$

In particular, if $\chi \in \mathrm{Irr}(G)$ has p'-degree, then χ lies in the principal block $B_0(G)$ of G.

Proof. Let P act on G^0 by conjugation. The set of fixed points under this action is $G^0 \cap \mathbf{C}_G(P) = 1$ by the assumption $\mathbf{N}_G(P) = P$. Since the orbits corresponding to non-trivial elements of G^0 have length divisible by p we have that

$$\sum_{x \in G^0} \chi(x) \equiv \chi(1) \mod p.$$

The second statement follows from the definition of $\mathrm{Irr}(B_0(G))$. □

We can finally prove Theorem 3.13. The key to do that is to reduce the proof to the self-normalizing case and then apply Theorem E. This reduction is possible fundamentally thanks to Theorem 3.14.

Proof of Theorem 3.13. Let G be a counterexample to the first part of the theorem with $|G|$ as small as possible. Let $N = \mathbf{O}_{p'}(G)$. Using Theorem 3.14 we can write $\mathbf{N}_G(P) = P \times X$, where $X \leqslant N$. Write $\bar{G} = G/N$ and use the bar convention. Hence $\bar{P} = PN/N \in \mathrm{Syl}_p(\bar{G})$ and $\mathbf{N}_{\bar{G}}(\bar{P}) = \bar{P} \times \bar{X} = \bar{P}$, by elementary group theory. By Lemma 3.15, $N \subseteq \ker(\chi)$. If $N > 1$, then considering χ as a character of $\bar{G}$, by inductive hypothesis we have that

$$\chi_{\mathbf{N}_{\bar{G}}(\bar{P})} = \chi_{\bar{P}} = \chi^* + \Delta,$$

where the character χ^* is an irreducible character of p'-degree lying in the principal block of $\bar{P} = PN/N$ and Δ is either zero or a character of PN/N such that every irreducible constituent of Δ has degree divisible by p. Now, Lemma 3.16 applies and we are done in this case. Hence, we may assume $N = 1$. In particular, $\mathbf{N}_G(P) = P$ and the first part of the statement follows from Theorem 3.10.

Now, we prove that our map $\chi \mapsto \chi^*$ is a bijection. By Theorem 3.14 we have that $\mathbf{O}_{p'}(\mathbf{N}_G(P)) \leqslant \mathbf{O}_{p'}(G)$ and by Theorem 3.15 we have that $\mathbf{O}_{p'}(G)$ is contained in the kernel of every $\chi \in \mathrm{Irr}(B_0(G))$. Modding out by $\mathbf{O}_{p'}(G)$, we may assume that $\mathbf{O}_{p'}(G) = 1$ and so $\mathbf{N}_G(P) = P$. By Lemma 3.17 every irreducible character of p'-degree of G lies in $\mathrm{Irr}(B_0(G))$. We have that $\mathrm{Irr}_{p'}(B_0(G)) = \mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(B_0(P)) = \mathrm{Irr}_{p'}(P)$. By Lemma 3.1 all the non-abelian composition factors of G are of type $\mathrm{PSL}_2(3^{3^a})$ with $a \geqslant 1$. We know by Theorem A of [**IMN07**] that the McKay conjecture holds for G. Hence $|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(P)| = |\mathrm{Irr}(P/P')|$. Now, if $\lambda \in \mathrm{Irr}(P)$ is linear, then some irreducible constituent χ of λ^G has p'-degree. Now, χ_P contains λ and by the first part of the proof it must be $\chi^* = \lambda$. Our map $\chi \mapsto \chi^*$ is surjective, and therefore injective. □

It is entirely possible that, under the hypotheses of Theorem 3.13, a natural correspondence exists between *all* the characters in $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(\mathbf{N}_G(P))$ (not only the characters in the principal blocks). We have been able to find it for p-solvable groups, see coming Section 3.6.

3.6. The p-solvable case

We finish this chapter by proving Theorem F. Our correspondence in Theorem F extends the Glauberman correspondence (see Theorem 1.17) and also the correspondence in Theorem 3.13. We shall use B_π-theory, for which we refer the reader to Section 1.4.

LEMMA 3.18. *Suppose that $L \lhd G$, $P \in \mathrm{Syl}_p(G)$ and $\mathbf{N}_{G/L}(PL/L) = PL/L$. Assume that G/L is p-solvable. Let $\theta \in \mathrm{Irr}(L)$ be P-invariant and p'-special. Then there exists a unique $\hat{\theta} \in \mathrm{Irr}(G|\theta)$ such that $\hat{\theta}$ is p'-special.*

PROOF. We argue by induction on $|G : L|$. Let K/L be a chief factor of G, and notice that G/K has self-normalizing Sylow p-subgroups, by elementary group theory. Assume first that K/L is a p-group, and let $\eta \in \mathrm{Irr}(K|\theta)$ be the unique p'-special character lying over θ, by using part (b) of Proposition 1.21. By uniqueness, η is P-invariant, and by induction, there is a unique p'-special character $\hat{\eta} \in \mathrm{Irr}(G)$ that lies over η (and therefore over θ). Now, if γ is any other p'-special character of G lying over θ and $\psi \in \mathrm{Irr}(K)$ lies under γ and over θ, we have that ψ is p'-special by part (a) of Proposition 1.20, and therefore $\psi = \eta$, by uniqueness. But in this case, $\gamma = \hat{\eta}$, by using the inductive hypothesis.

Suppose finally that K/L is a p'-group. Then $\mathbf{C}_{K/L}(PL/L) = 1$ using that PL/L is self-normalizing and coprime action. Hence, by Problem (13.10) of [**Isa76**], there exists a unique P-invariant $\tau \in \mathrm{Irr}(K|\theta)$. Also, τ is p'-special by part (a) of Proposition 1.21. Since $|G : K| < |G : L|$, by induction there exists a unique p'-special character $\hat{\tau}$ of G lying over τ (and therefore over θ). Suppose now that $\gamma \in \mathrm{Irr}(G)$ is any other p'-special character lying over θ. By Lemma 2.3, let $\varphi \in \mathrm{Irr}(K)$ be P-invariant under γ, and, by Theorem (13.27) of [**Isa76**], let $\rho \in \mathrm{Irr}(L)$ be P-invariant under φ. Then ρ and θ are P-invariant lying under γ, so $\rho = \theta$ by Lemma 2.3. Then $\varphi = \tau$ by the uniqueness of τ, and hence $\gamma = \hat{\tau}$ by induction. □

We translate the statement of Theorem 1.26 for p-solvable groups.

REMARK 3.19. Let G be a p-solvable group and let $P \in \mathrm{Syl}_p(G)$. For $\chi \in \mathrm{Irr}(G)$, the following are equivalent:

(a) χ is a satellite of some $\psi \in B_p(G)$ of p'-degree.
(b) There exists a linear character λ of P and a p'-special character $\alpha \in \mathrm{Irr}(W)$, where W is the maximal subgroup of G to which λ extends, such that $\chi = (\alpha\hat{\lambda})^G$, where $\hat{\lambda}$ is the unique extension of λ to W with p-power order.

PROOF. Assume that χ is a satellite of $\psi \in B_p(G)$ of degree not divisible by p. Then there exists a nucleus (W, γ) for ψ and a p'-special character $\alpha \in \mathrm{Irr}(W)$ such that $\chi = (\alpha\gamma)^G$. Notice that $\chi(1) = |G : W|\alpha(1)\gamma(1)$ is a p'-number. Hence, we have that $\gamma(1) = 1$ and W contain a full Sylow p-subgroup of G. We may assume $P \leqslant W$, by conjugating the pairs (W, γ) and (W, α) with an element of G. Let $\lambda = \gamma_P \in \mathrm{Irr}(P)$. Then γ is the unique extension of λ to W with p-power order. The fact that $\gamma^G \in \mathrm{Irr}(G)$ guarantees that W is maximal with the property that λ extends to W.

To prove the converse, just notice that $(\hat{\lambda})^G \in B_p(G)$ by Theorem 2.2 of [**IN01**]. □

Recall that whenever a group A acts on the irreducible characters $\mathrm{Irr}(G)$ of a group G, we write $\mathrm{Irr}_A(G)$ to denote the set of fixed characters of G under the action of A. We can prove Theorem F, which we restate here.

THEOREM 3.20. *Let G be a p-solvable group, and let $P \in \mathrm{Syl}_p(G)$. Suppose that $\mathbf{N}_G(P) = P\mathbf{C}_G(P)$, and let $L = \mathbf{O}_{p'}(G)$. Then for every $\theta \in \mathrm{Irr}_P(L)$ and $\lambda \in \mathrm{Irr}(P/P')$ linear, there is a canonically defined*

$$\lambda \star \theta \in \mathrm{Irr}_{p'}(G).$$

Furthermore, the map

$$\mathrm{Irr}(P/P') \times \mathrm{Irr}_P(L) \to \mathrm{Irr}_{p'}(G)$$

given by $(\lambda, \theta) \mapsto \lambda \star \theta$ is a bijection. As a consequence, if $\theta^* \in \mathrm{Irr}(\mathbf{C}_L(P))$ is the Glauberman correspondent of $\theta \in \mathrm{Irr}_P(L)$ (see Theorem 1.17), then the map

$$\lambda \times \theta^* \mapsto \lambda \star \theta$$

is a natural bijection $\mathrm{Irr}_{p'}(\mathbf{N}_G(P)) \to \mathrm{Irr}_{p'}(G)$. Also, if $\theta = 1_L$ and $\lambda \in \mathrm{Irr}(P/P')$, then $\lambda \times \theta^*$ is the unique linear constituent of $(\lambda \star \theta)_{\mathbf{N}_G(P)}$.

PROOF. By using Theorem 3.14, write $\mathbf{N}_G(P) = P \times X$, where $X = \mathbf{C}_L(P)$. Let $\lambda \in \mathrm{Irr}(P)$ be linear and let $\theta \in \mathrm{Irr}_P(L)$. Since $P \cap L = 1$, we trivially have that λ extends to PL. By Theorem 2.11, there exists a maximal subgroup $P \leqslant W \leqslant G$ such that λ extends to W. Hence $PL \leqslant W$. Now, by elementary character theory, let $\hat{\lambda} \in \mathrm{Irr}(W)$ be the unique linear character of p-power order that extends λ. Since $\mathbf{N}_{W/L}(PL/L) = PL/L$, by Lemma 3.18, there exists a unique p'-special $\hat{\theta} \in \mathrm{Irr}(W)$ lying over θ. By Theorem 2.2 of [**IN01**], the pair $(W, \hat{\lambda})$ is a nucleus for $(\hat{\lambda})^G \in \mathrm{Irr}(G)$. Thus, by Theorem 1.26 we have that

$$\lambda \star \theta = (\hat{\theta}\hat{\lambda})^G \in \mathrm{Irr}(G)\,.$$

Notice that $\lambda \star \theta$ has p'-degree, because $\hat{\theta}$ has p'-degree and $|G : W|$ is not divisible by p. (We notice for the record that $(\lambda \star \theta)_W$ contains $\hat{\theta}\hat{\lambda}$, and therefore, when restricted to L, we have that $(\lambda \star \theta)$ lies over θ. It is not in general true that $\lambda \star \theta$ lies over λ, on the other hand.)

We have now defined a map

$$\mathrm{Irr}(P/P') \times \mathrm{Irr}_P(L) \to \mathrm{Irr}_{p'}(G)$$

given by $(\lambda, \theta) \mapsto \lambda \star \theta$.

Next we show that our map is surjective. Let $\chi \in \mathrm{Irr}_{p'}(G)$. By Theorem 1.27 (see also Remark 3.19), we have that there exist a linear character $\delta \in \mathrm{Irr}(P)$ and a p'-special character $\alpha \in \mathrm{Irr}(U)$, where U is the maximal subgroup of G to which δ extends, such that

$$\chi = (\hat{\delta}\alpha)^G\,,$$

where the order of $\hat{\delta}$ is a p-power and $\hat{\delta}$ extends δ. Now, α_L contains a (unique) P-invariant character $\mu \in \mathrm{Irr}_P(L)$ by Lemma 2.3, and it follows that α is the unique p'-special character of U lying over μ by Lemma 3.18. It follows then that $\chi = \delta \star \mu$, and therefore, that our map is surjective.

Recall that the Glauberman correspondence provides a natural bijection

$$\mathrm{Irr}_P(L) \to \mathrm{Irr}(\mathbf{C}_L(P)) .$$

Since the McKay conjecture is true for p-solvable groups (see for instance [**IMN07**]) we have that

$$|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(P))| = |\mathrm{Irr}(P/P')||\mathrm{Irr}(\mathbf{C}_L(P))| = |\mathrm{Irr}(P/P')||\mathrm{Irr}_P(L)| .$$

It then follows that our map is bijective.

In the case where $\theta = 1_L$, for every $\lambda \in \mathrm{Irr}(P/P')$, we have that $\lambda \star \theta = (\hat{\lambda})^G$, where $\hat{\lambda}$ is a p-special extension of λ to a subgroup $W \leqslant G$ with the property that λ does not extend to any subgroup of G properly containing W. Let $\mathbb{T}$ be a set of representatives of the double cosets of $\mathbf{N}_G(P)$ and W in G with $1 \in \mathbb{T}$. By Mackey Lemma 1.8, we have that

$$(\hat{\lambda}^G)_{\mathbf{N}_G(P)} = \lambda \times 1 + \sum_{1 \neq t \in \mathbb{T}} ((\hat{\lambda}^t)_{\mathbf{N}_G(P) \cap W^t})^{\mathbf{N}_G(P)}.$$

By the first part of the proof $((\hat{\lambda}^t)_{\mathbf{N}_G(P) \cap W^t})^{\mathbf{N}_G(P)} \in \mathrm{Irr}(\mathbf{N}_G(P))$, and so $\lambda \times 1$ is the unique linear constituent of $(\lambda \star \theta)_{\mathbf{N}_G(P)}$. □

CHAPTER 4

Preliminaries on modular character theory of finite groups

The first part of this chapter is the modular version of Chapter 1. Our aim is to collect the basic results on (p-)Brauer characters that will be used later in this work, namely in Chapter 5. However, since the results contained in Chapter 5 are of a more technical nature (than the rest of results of this work), in the second part of this chapter we shall develop some new techniques on Brauer characters.

In Section 4.1, we introduce Brauer characters and we focus on properties that Brauer characters share with ordinary characters. In Section 4.2, we study a modular version of the notion of *central isomorphism of character triples* firstly defined in [**NS14**] and later developed in [**Spä16**]. In Section 4.3, we introduce the concept of *fake Galois action* on the set of Brauer irreducible characters $\mathrm{IBr}(N)$ of a group N. This latter definition will help us to avoid some difficulties in Chapter 5 originated from the fact that, in general, the Galois group $\mathrm{Gal}(\mathbb{Q}_{|N|}/\mathbb{Q})$ does not act on $\mathrm{IBr}(N)$. (See Section 5.4 for the motivation of the definition of fake Galois action.)

4.1. Brauer characters (as characters)

We fix a prime p and a maximal ideal $\mathcal{M}$ in the ring $\mathbf{R}$ of algebraic integers with $p \in \mathcal{M}$. We let $F = \mathbf{R}/\mathcal{M}$ and write ${}^*\colon \mathbf{R} \to F$ to denote the natural ring homomorphism. This homomorphism can be extended to $S = \{r/s \mid r \in \mathbf{R}, s \in \mathbf{R}\backslash\mathcal{M}\}$ by

$$(r/s)^* = r^*(s^*)^{-1},$$

for every $r \in \mathbf{R}$ and $s \in \mathbf{R}\backslash\mathcal{M}$. Let $\mathbf{U} \subseteq \mathbf{R}$ be the multiplicative group of roots of unity of order not divisible by p, so that $\mathbf{U} = \{\xi \in \mathbb{C} \mid \xi^k = 1 \text{ for some integer } k \text{ not divisible by } p\}$.

Lemma 4.1. *The restriction of* * *to* $\mathbf{U}$ *defines an isomorphism* $\mathbf{U} \to F^\times$ *of multiplicative groups. Also* F *is an algebraically closed field of characteristic* p*.*

Proof. This is Lemma 2.1 of [**Nav98**]. □

Let $g \in G$. We say that x is **p-regular** if p does not divide $o(g)$. We recall that G^0 denotes the subset of p-regular elements of G. Suppose that $\mathfrak{X}\colon G \to \mathrm{GL}_n(F)$ is an F-representation of the group G. If $g \in G^0$, then by Lemma 4.1, the eigenvalues of the matrix $\mathfrak{X}(g)$ are $\xi_1^*, \dots, \xi_n^* \in F^\times$ for uniquely determined $\xi_1, \ldots, \xi_n \in \mathbf{U}$ (because F is algebraically closed). We define $\varphi\colon G^0 \to \mathbb{C}$ by $\varphi(g) = \xi_1 + \cdots + \xi_n$. Then we say that φ is the **Brauer character** afforded by $\mathfrak{X}$. (Brauer characters are also called **modular characters**.) The **degree** of φ is $\varphi(1)$, which is the degree of any F-representation affording φ. Notice that similar F-representations afford the same Brauer characters and Brauer characters are constant on conjugacy classes.

We say that φ is **irreducible** if an F-representation $\mathfrak{X}$ of G affording φ is irreducible. We denote by $\mathrm{IBr}(G)$ the set of irreducible Brauer characters of G. Unlike ordinary characters, the degrees of the irreducible Brauer characters do not divide, in general, the order of the group ($\mathrm{PSL}_2(7)$ for $p = 7$ has an irreducible Brauer character of degree 5).

We define the field of values $\mathbb{Q}(\varphi)$ of $\varphi \in \mathrm{IBr}(G)$ as $\mathbb{Q}(\varphi(g) \mid g \in G^0)$. Notice that $\mathbb{Q}(\varphi) \subseteq \mathbb{Q}_{|G|_{p'}} \subseteq \mathbb{Q}_{|G|}$.

Write $\mathrm{cf}(G^0)$ to denote the $\mathbb{C}$-vector space of **class functions** on G^0 (functions $\theta\colon G^0 \to \mathbb{C}$ constant on conjugacy classes of G^0). Of course the dimension of $\mathrm{cf}(G^0)$ is equal to the number of conjugacy classes of p-regular elements of G. Brauer characters are class functions on G^0.

If $H \leqslant G$ and φ is a Brauer character of G, then we denote by φ_H the restriction of φ to H^0. The map φ_H is a Brauer character of H.

As happens with ordinary characters, Brauer characters are nonnegative integer linear combination of irreducible Brauer characters.

THEOREM 4.2. *Let G be a group. Then $\mathrm{IBr}(G)$ is a basis of $\mathrm{cf}(G^0)$. Moreover, $\psi \in \mathrm{cf}(G^0)$ is a Brauer character of G if and only if*

$$\psi = \sum_{\varphi \in \mathrm{IBr}(G)} a_\varphi \varphi,$$

where $a_\varphi \in \mathbb{N}$.

PROOF. See Corollary 2.10 and Theorem 2.3 of [**Nav98**]. □

The nonnegative integer a_φ in the decomposition of ψ in Theorem 4.2 is called the **multiplicity** of φ in ψ. If $a_\varphi \neq 0$, then we call φ a **constituent** of ψ.

By Theorem 4.2, the number $|\mathrm{IBr}(G)|$ equals the number of conjugacy classes of p-regular elements of G. Also, a Brauer character φ is irreducible if and only if φ cannot be written as $\alpha + \beta$ for Brauer characters α and β.

As a consequence of Theorem 4.2, if $\varphi \in \mathrm{IBr}(G)$, then any two irreducible representations affording φ are similar. Hence, φ uniquely determines an F-representation of G up to similarity.

If $\chi \in \mathrm{Irr}(G)$, we denote by χ^0 the restriction of χ to G^0. By Corollary 2.9 of [**Nav98**], χ^0 is a Brauer character of G. Hence

$$\chi^0 = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\varphi} \varphi,$$

for suitable nonnegative integers $d_{\chi\varphi}$. The nonnegative integers $d_{\chi\varphi}$ in the above decomposition are called the **decomposition numbers** of χ.

The study of Brauer characters is more interesting when p divides $|G|$ because of the following.

THEOREM 4.3. *Let G be a group. If p does not divide $|G|$, then* $\mathrm{IBr}(G) = \mathrm{Irr}(G)$.

PROOF. This is Theorem 2.12 of [**Nav98**]. □

As in the ordinary case, if φ and θ are Brauer characters of G, then the product $\varphi\theta$ defined by

$$\varphi\theta(g) = \varphi(g)\theta(g)$$

for every $g \in G^0$ is a Brauer character of G (see Theorem 2.23 of [**Nav98**]).

Let $\varphi \in \mathrm{IBr}(G)$ and let $n = \varphi(1)$. Since φ uniquely determines an irreducible F-representation $\mathfrak{X}$ up to similarity, then $\ker(\mathfrak{X})$ is uniquely determined by φ. We can define the kernel of φ as $\ker(\varphi) = \{g \in G \mid \mathfrak{X}(g) = I_n\}$. As in the ordinary case, if $N \lhd G$, one can identify the irreducible Brauer characters of G containing N in their kernel and the irreducible Brauer characters of the quotient group G/N. Usually we identify these sets and we think of $\mathrm{IBr}(G/N)$ as a subset of $\mathrm{IBr}(G)$.

We recall that $\mathbf{O}_p(G)$ is the maximal normal p-subgroup of G.

LEMMA 4.4. *Let G be a group. If $\varphi \in \mathrm{IBr}(G)$, then $\mathbf{O}_p(G) \subseteq \ker(\varphi)$. In particular, we can identify* $\mathrm{IBr}(G/\mathbf{O}_p(G))$ *with* $\mathrm{IBr}(G)$.

PROOF. See Lemma 2.32 of [**Nav98**]. □

Let $\varphi \in \mathrm{IBr}(G)$. The fact that φ uniquely determines up to similarity an F-representation $\mathfrak{X}$ of G allows us to define the determinantal order $o(\varphi)$ of φ as in the ordinary case. The determinant $\det(\mathfrak{X})\colon G \to F^\times$ of $\mathfrak{X}$ is a homomorphism. We write $o(\varphi)$ to denote the smallest positive integer k such that $\det(\mathfrak{X})^k = 1$.

As for ordinary characters, every isomorphism of groups $\alpha\colon G \to H$ defines a bijection between $\mathrm{IBr}(G)$ and $\mathrm{IBr}(H)$, by defining for $\varphi \in \mathrm{IBr}(G)$ the map φ^α given by $\varphi^\alpha(g^\alpha) = \varphi(g)$ for every $g \in G^0$. In particular, if $N \lhd G$, then G acts on $\mathrm{IBr}(N)$ via the action of conjugation of G on N.

Induction and restriction are essential features also in modular character theory. We have already noticed that the restriction of a Brauer character to a subgroup is again a Brauer character. Let $H \leqslant G$ and let $\theta \in \mathrm{cf}(H^0)$. We define for every $x \in G^0$, the function

$$\theta^G(x) = \frac{1}{|H|}\sum_{g\in G}\dot{\theta}(gxg^{-1}),$$

where $\dot{\theta}(y) = \theta(y)$ if $y \in H^0$ and 0 otherwise. Then $\theta^G \in \mathrm{cf}(G^0)$. In fact, we have the following.

Theorem 4.5. *Let $H \leqslant G$ and let φ be a Brauer character of H. Then φ^G is a Brauer character of G.*

Proof. This is Theorem 8.2 of [**Nav98**]. □

As for ordinary characters, if $H \leqslant G$ and $\theta \in \mathrm{IBr}(H)$, we write $\mathrm{IBr}(G|\theta) = \{\varphi \in \mathrm{IBr}(G) \mid \varphi \text{ is a constituent of } \theta^G\}$ (this set is non-empty by Corollary 8.3 of [**Nav98**]). If $\varphi \in \mathrm{IBr}(G|\theta)$ we will say that φ **lies over** θ or that θ **lies under** φ. If $\varphi \in \mathrm{IBr}(G)$, then we write $\mathrm{IBr}(\varphi_H)$ to denote the set of irreducible Brauer characters which are constituents of the Brauer character φ_H.

The main difference between ordinary and modular characters with respect to the induction-restriction process is that in the modular case we do not have Frobenius reciprocity. For $H \leqslant G$, this means that if $\theta \in \mathrm{IBr}(H)$ and $\varphi \in \mathrm{IBr}(G)$, then the multiplicity of φ in θ^G is not necessarily equal to the multiplicity of θ in φ_H. In fact, it can happen that φ is a constituent of θ^G but θ is not a constituent of φ_H and viceversa (see the discussion after Corollary 8.3 of [**Nav98**] for these extreme examples).

Despite this fact, restriction-induction techniques with respect to normal subgroups work exactly as well as for ordinary characters.

Theorem 4.6. *Let $N \lhd G$. If $\theta \in \mathrm{IBr}(N)$ and $\varphi \in \mathrm{IBr}(G)$, then φ is a constituent of θ^G iff θ is a constituent of φ_N. In this case,*

$$\varphi_N = e \sum_{i=1}^{t} \theta^{x_i},$$

for some $e \geqslant 1$, where $x_1 = 1$ and $\theta^{x_1}, \ldots, \theta^{x_t}$ are the distinct G-conjugates of θ.

Proof. This is Corollary 8.7 of [**Nav98**]. □

Let $N \lhd G$. If $\theta \in \mathrm{IBr}(N)$, then we write $G_\theta = \{g \in G \mid \theta^g = \theta\}$ to denote the **inertia group** of θ in G. For $H \leqslant G$, we say that θ is H**-invariant** if $H \subseteq G_\theta$.

Theorem 4.7 (Clifford correspondence). *Let $N \lhd G$ and let $\theta \in \mathrm{IBr}(G)$. Then the map $\psi \mapsto \psi^G$ is a bijection from $\mathrm{Irr}(G_\theta|\theta)$ onto $\mathrm{IBr}(G|\theta)$.*

Proof. This is Theorem 8.9 of [**Nav98**]. □

In view of Theorem 4.6 and Theorem 4.7, we see that there is a Clifford theory for modular characters. Also, Gallagher's Theorem works for modular characters.

Theorem 4.8. *Let $N \lhd G$ and let $\varphi \in \mathrm{IBr}(G)$. If $\varphi_N = \theta \in \mathrm{IBr}(N)$, then the characters $\beta\varphi$ for $\beta \in \mathrm{IBr}(G/N)$ are irreducible, distinct for distinct β and they are all the irreducible constituents of θ^G.*

Proof. See Corollary 8.10 of [**Nav98**]. □

Next, we collect some extendibility criteria. If $H \leqslant G$ and $\theta \in \mathrm{IBr}(H)$, then we say that θ **extends** if there is $\varphi \in \mathrm{IBr}(G)$ such that $\varphi_H = \theta$.

THEOREM 4.9 (Green). *Let $N \lhd G$ and let $\theta \in \mathrm{IBr}(N)$. If G/N is a p-group, then there exists a unique $\varphi \in \mathrm{IBr}(G|\theta)$. Furthermore, φ_N is the sum of the distinct G-conjugates of θ. In particular, if θ is G-invariant then $\varphi_N = \theta$.*

PROOF. See Theorem 8.11 of [**Nav98**]. □

The following criteria are modular analogues of well-known results on ordinary characters.

THEOREM 4.10. *Let $N \lhd G$. Suppose that G/N is cyclic. If $\theta \in \mathrm{IBr}(N)$ is G-invariant, then θ extends to G.*

PROOF. See Theorem 8.12 of [**Nav98**]. □

THEOREM 4.11. *Let $N \lhd G$ and let $\theta \in \mathrm{IBr}(N)$ be G-invariant. If θ extends to Q for every $Q/N \in \mathrm{Syl}_q(G/N)$ and for every prime $q \neq p$, then θ extends to G.*

PROOF. See Theorem 8.29 of [**Nav98**]. □

THEOREM 4.12. *Let $N \lhd G$ and let $\theta \in \mathrm{IBr}(N)$ be G-invariant. If one of the following holds:*

(a) $(|N|, |G:N|) = 1$, *or*
(b) $(o(\theta)\theta(1), |G:N|) = 1$,

then θ extends to G.

PROOF. See Theorem 8.13 and Theorem 8.23 of [**Nav98**]. □

Notice that part (a) does not follow from part (b) since, as we said, the degrees of irreducible Brauer characters do not divide in general the order of the group.

4.2. Isomorphisms of modular character triples

In this section we start by introducing the theory of projective representations we will later need. We will follow Chapter 8 of [**Nav98**] since we are interested in the modular case, but this theory works exactly the same both in the ordinary and the modular case (a reference for the ordinary case is Chapter 11 of [**Isa76**]). After that, we will introduce the notion of centrally isomorphic modular character triples, which is a modular analogue of the notion of centrally isomorphic character triples introduced in [**NS14**] for ordinary character triples. (Hence the proofs of most of the results concerning the construction of centrally isomorphic modular character triples will follow from arguments contained in [**NS14**] and [**Spä16**].)

If $N \lhd G$ and $\theta \in \mathrm{IBr}(N)$ is G-invariant, then the triple (G, N, θ) is called a **modular character triple**. Every modular character triple (G, N, θ) has associated an (F-)projective representation $\mathcal{P}$ (up to similarity and up to product by some function $\mu\colon G \to F^\times$) satisfying certain properties. (See Theorem 4.13 and Remark 4.15 below.)

A **projective representation** $\mathcal{P}$ of G is a map $\mathcal{P}\colon G \to \mathrm{GL}_n(F)$ satisfying that for every $g, g' \in G$

$$\mathcal{P}(g)\mathcal{P}(g') = \alpha(g,g')\mathcal{P}(gg'),$$

where $\alpha(g,g') \in F^\times$. We say that n is the **degree** of $\mathcal{P}$. This definition yields a function $\alpha\colon G \times G \to F^\times$, which is the **factor set** (see page 164 of [**Nav98**]) associated to $\mathcal{P}$.

The notions of similarity and irreducibility of projective representations are analogous to those on representations.

Theorem 4.13. *Let (G,N,θ) be a modular character triple. Let $\mathfrak{X}$ be an F-representation affording the Brauer character θ. Then, there exists a projective representation $\mathcal{P}$ of G, such that $\mathcal{P}_N = \mathfrak{X}$ and the factor set α associated to $\mathcal{P}$ satisfies*

$$\alpha(g,n) = 1 = \alpha(n,g),$$

for every $g \in G$ and $n \in N$. Moreover, if $\mathcal{Q}$ is another such projective representation, then there exists a map $\mu\colon G \to F^\times$ with $\mu(1) = 1$ which is constant on N-cosets in G and such that $\mathcal{Q} = \mu\mathcal{P}$.

Proof. See Theorem 8.14 of [**Nav98**]. □

The factor set α associated to $\mathcal{P}$ in Theorem 4.13 can be seen as a map defined on $G/N \times G/N$ (see the remarks after Theorem 8.14 of [**Nav98**]).

Definition 4.14. Let (G,N,θ) be a modular character triple. We say that a projective representation $\mathcal{P}$ of G is **associated with** θ if:

(i) $\mathcal{P}_N$ is a representation affording θ, and
(ii) the factor set α of $\mathcal{P}$ satisfies

$$\alpha(g,n) = 1 = \alpha(n,g)$$

for every $g \in G$ and $n \in N$.

Notice that if (G,N,θ) is a modular character triple and $\mathcal{P}$ is a projective representation of G associated to θ, then condition (ii) in Definition 4.14 implies that for every $g \in G$ and $n \in N$

$$\mathcal{P}(gn) = \mathcal{P}(g)\mathcal{P}(n) \quad \text{and} \quad \mathcal{P}(ng) = \mathcal{P}(n)\mathcal{P}(g).$$

Remark 4.15. Let (G,N,θ) be a modular character triple. By Theorem 4.13, projective representations of G associated with θ do always exist. In fact, it is easy to prove that if $\mathcal{P}$ and $\mathcal{Q}$ are two projective representations of G associated with θ, then there exists a map $\mu\colon G \to F^\times$ with $\mu(1) = 1$ which is constant on cosets of N and such that $\mathcal{Q}$ is similar to $\mu\mathcal{P}$. Let α be the factor set of $\mathcal{P}$. Then the factor set β of $\mathcal{Q}$ is given by

$$\beta(g,g') = \frac{\mu(g)\mu(g')}{\mu(gg')}\alpha(g,g')$$

for every $g, g' \in G$. (This latter fact follows from straightforward calculations.)

We can study the Clifford theory of a character triple (G, N, θ) via projective representations associated to θ and projective representations of G/N. We will view the projective representations of G/N as projective representations $\mathcal{Q}$ of G satisfying $\mathcal{Q}(gn) = \mathcal{Q}(g)$ for every $g \in G$ and $n \in N$.

Let $\mathcal{P}$ be a projective representation associated with θ with factor set α (recall we can consider α as a function defined on $G/N \times G/N$). Let $N \leqslant J \leqslant G$ and let γ denote the restriction of the factor set α^{-1} to $J/N \times J/N$. By Theorem 8.16 and Theorem 8.18 of [**Nav98**], if $\mathrm{Proj}_F(J/N, \gamma)$ is a set of representatives of the similarity classes of irreducible projective F-representations of J/N with factor set γ, then

$$\mathrm{Rep}_F(J, \theta) = \{Q \otimes \mathcal{P}_J \mid Q \in \mathrm{Proj}_F(J/N, \gamma)\}$$

is a set of representatives of similarity classes of representations affording a Brauer character in $\mathrm{IBr}(J|\theta)$.

Definition 4.16. Let (G, N, θ) be a modular character triple. We denote by $\mathrm{Br}(G|\theta)$ the set of Brauer characters χ of G such that χ_N is a multiple of θ. Hence this is the set of nonnegative integer linear combinations of $\mathrm{IBr}(G|\theta)$. Let (Γ, M, φ) be another modular character triple and suppose that $\tau\colon G/N \to \Gamma/M$ is an isomorphism of groups. For every $N \leqslant J \leqslant G$, write $J^\tau/N = \tau(J/N)$ and suppose that there exists a map $\sigma_J\colon \mathrm{Br}(J|\theta) \to \mathrm{Br}(J^\tau|\varphi)$ such that σ_J yields a bijection $\mathrm{IBr}(J|\theta) \to \mathrm{IBr}(J^\tau|\varphi)$. Suppose further that for every $N \leqslant K \leqslant J \leqslant G$ and for every $\chi, \psi \in \mathrm{Br}(J|\theta)$ the following hold:

(a) $\sigma_J(\chi + \psi) = \sigma_J(\chi) + \sigma_J(\psi)$.
(b) $\sigma_K(\chi_K) = \sigma_J(\chi)_K$.
(c) $\sigma_K(\chi\beta) = \sigma_K(\chi)\beta^\tau$, for every $\beta \in \mathrm{IBr}(J/N)$.

Then we say that $(\sigma, \tau)\colon (G, N, \theta) \to (\Gamma, M, \varphi)$ is an **isomorphism** of modular character triples.

To define an isomorphism (σ, τ) of modular character triples it is enough to give for $N \leqslant J \leqslant G$ bijections

$$\sigma_J\colon \mathrm{IBr}(J|\theta) \to \mathrm{IBr}(J^\tau|\varphi),$$

extend these maps using (a) and check that conditions (b) and (c) hold for every $\chi, \psi \in \mathrm{IBr}(J|\theta)$.

We strengthen Definition 4.16. Let $N \leqslant J \leqslant G$. For $\psi \in \mathrm{IBr}(J|\theta)$ and $\overline{g} = gN \in G/N$, we define $\psi^{\overline{g}} \in \mathrm{IBr}(J^g|\theta)$ by

$$\psi^{\overline{g}}(x^g) = \psi(x) \text{ for every } x \in J.$$

Note that this is well-defined. We say that a modular character triple isomorphism $(\sigma, \tau)\colon (G, N, \theta) \to (\Gamma, M, \varphi)$ is **strong** if

$$(\sigma_J(\psi))^{\tau(\overline{g})} = \sigma_{J^g}(\psi^{\overline{g}}),$$

for all $\overline{g} \in G/N$, all groups J with $N \leqslant J \leqslant G$ and all $\psi \in \mathrm{IBr}(J|\theta)$.

From now on, we will work with strong isomorphisms of character triples. Isomorphisms of modular character triples define an equivalence relation on modular character triples.

We collect below some examples of (strong) modular character triple isomorphisms.

Lemma 4.17. *Let (G, N, θ) be a modular character triple.*

(a) If $\alpha\colon G \to H$ is an isomorphism of groups, then (G, N, θ) is strongly isomorphic to (H, M, φ) where $M = \alpha(N)$ and $\varphi \in \mathrm{IBr}(M)$ is the character defined by $\varphi(\alpha(n)) = \theta(n)$ for every $n \in N^0$.
(b) If $M \lhd G$ and $M \leqslant \ker(\theta)$, then (G, N, θ) and $(G/M, N/M, \overline{\theta})$ are strongly isomorphic, where $\overline{\theta}(nM) = \theta(n)$ for every $n \in N^0$.
(c) Suppose that $\mu\colon G \to H$ is an epimorphism and that $K = \ker(\mu) \leqslant \ker(\theta)$. Then (G, N, θ) and (H, M, φ) are strongly isomorphic, where $M = \mu(N)$ and $\varphi \in \mathrm{Irr}(M)$ is the unique character of M with $\varphi(\mu(n)) = \theta(n)$ for every $n \in N^0$.
(d) Suppose that there exists some $\eta \in \mathrm{IBr}(G)$ such that $\eta_N\theta = \varphi \in \mathrm{IBr}(N)$. Then (G, N, θ) and (G, N, φ) are strongly isomorphic.

Proof. Parts (a) and (b) are particular cases of part (c). To prove (c) mimic the proof of Lemma 11.26 of [**Isa76**]. It is easy to check that the isomorphism obtained this way is strong. Part (d) is Theorem 8.26 of [**Nav98**]. As before, it is straightforward to check that the isomorphism given by this proof is strong. □

The following result explains how to construct (strong) isomorphisms of modular character triples via projective representations.

Theorem 4.18. *Let (G, N, θ) and (H, M, θ') be modular character triples satisfying the following assumptions:*

(i) $G = NH$ and $M = N \cap H$,
(ii) there exist projective representations $\mathcal{P}$ and $\mathcal{P}'$ of G and H associated to θ and θ', respectively, whose factor sets α and α' coincide via the natural isomorphism $\tau\colon G/N \to H/M$.

For $N \leqslant J \leqslant G$, write $\gamma = (\alpha^{-1})_{J/N \times J/N}$. If $\psi \in \mathrm{IBr}(J|\theta)$ is afforded by $\mathcal{Q} \otimes \mathcal{P}_J$, where $\mathcal{Q} \in \mathrm{Proj}_F(J/N, \gamma)$, then let $\sigma_J(\psi)$ be the Brauer character afforded by the irreducible representation $\mathcal{Q} \otimes \mathcal{P}'_{J\cap H}$. Then the map $\sigma_J\colon \mathrm{IBr}(J|\theta) \to \mathrm{IBr}(J \cap H|\theta')$ is a bijection. These bijections σ_J together with τ give a strong isomorphism

$$(\sigma, \tau)\colon (G, N, \theta) \to (H, M, \theta')$$

of modular character triples.

Proof. See the proof of Theorem 3.2 of [**NS14**]. The fact that (σ, τ) is strong follows from straightforward calculations. □

In the situation of Theorem 4.18, we say that (σ,τ) is an **isomorphism of modular character triples given by $\mathcal{P}$ and $\mathcal{P}'$.**

We are finally ready to define when two modular character triples are centrally isomorphic.

Definition 4.19. Let (G,N,θ) and (H,M,θ') be modular character triples satisfying the following conditions:

(i) $G = NH$, $M = N \cap H$ and $\mathbf{C}_G(N) \leqslant H$.

(ii) There exist a projective representation $\mathcal{P}$ of G associated to θ with factor set α and a projective representation $\mathcal{P}'$ of H associated to θ' with factor set α' such that

(ii.1) $\alpha|_{H\times H} = \alpha'$, and

(ii.2) for every $c \in \mathbf{C}_G(N)$ the scalar matrices $\mathcal{P}(c)$ and $\mathcal{P}'(c)$ are associated with the same scalar (notice that $\mathcal{P}(c)$ and $\mathcal{P}'(c)$ are scalar by Schur's Lemma [**Isa76**, Lem. 1.5]).

Let (σ,τ) be the isomorphism of character triples given by $\mathcal{P}$ and $\mathcal{P}'$ as in Theorem 4.18. Then we call (σ,τ) a **central isomorphism of modular character triples**, and we write

$$(G,N,\theta) \succ_{Br,c} (H,M,\theta').$$

By the proof of Lemma 3.3 of [**NS14**], condition (ii.2) in Definition 4.19 is equivalent to

$$\mathrm{IBr}(\psi_{\mathbf{C}_J(N)}) = \mathrm{IBr}(\sigma_J(\psi)_{\mathbf{C}_J(N)}),$$

for every $\psi \in \mathrm{IBr}(J|\theta)$ and $N \leqslant J \leqslant G$. Definition 4.19 is a modular analogue of the relation $\sim_c$ defined in [**NS14**]. In particular, the fact that $\succ_{Br,c}$ defines an order relation on the set of modular character triples and is thereby transitive follows from the proof of Lemma 3.8 of [**NS14**].

Remark 4.20. Suppose that $(G,N,\theta) \succ_{Br,c} (H,M,\theta')$. Then $\mathbf{Z}(N) \subseteq M$. The condition (ii.2) in Definition 4.19 implies that θ and θ' lie over the same $\lambda \in \mathrm{IBr}(\mathbf{Z}(N))$.

We are interested in studying how to construct new centrally isomorphic modular character triples from given ones. In order to do that, we shall frequently use the following.

Lemma 4.21. *Let (G,N,θ) and (H,M,θ') be modular character triples with*

$$(G,N,\theta) \succ_{Br,c} (H,M,\theta').$$

(a) If $\mathcal{P}$ is any projective representation of G associated to θ with factor set α, then there exists a projective representation $\mathcal{P}'$ of H associated to θ' with factor set α' such that

(a.1) $\alpha|_{H\times H} = \alpha'$, and

(a.2) for every $c \in \mathbf{C}_G(N)$ the scalar matrices $\mathcal{P}(c)$ and $\mathcal{P}'(c)$ are associated with the same scalar.

(b) If $\mathcal{P}'$ is any projective representation of H associated to θ' with factor set α', then there exists a projective representation $\mathcal{P}$ of G associated to θ with factor set α such that

(b.1) $\alpha|_{H\times H} = \alpha'$, and

(b.2) for every $c \in \mathbf{C}_G(N)$ the scalar matrices $\mathcal{P}(c)$ and $\mathcal{P}'(c)$ are associated with the same scalar.

PROOF. We first prove part (a). Let $\mathcal{Q}$ and $\mathcal{Q}'$ be projective representations as in Definition 4.19 giving $(G, N, \theta) \succ_{Br,c} (H, M, \theta')$. In the situation of (a), by Remark 4.15, there exists a map $\mu\colon G \to F^\times$ and a matrix $M \in \mathrm{GL}_{\theta(1)}(F)$ such that $\mathcal{P} = M^{-1}\mu\mathcal{Q}M$. Let $\mathcal{P}' = \mu\mathcal{Q}'$. Then $\mathcal{P}'$ is a projective representation of H associated to θ' whose factor set certainly satisfies (a.2). By the second part of Remark 4.15, the factor set of $\mathcal{P}'$ also satisfies (a.1).

The proof of part (b) is analogous. □

The following nearly trivial observation on centrally isomorphic modular character triples will be useful later.

LEMMA 4.22. *Suppose that $(G, N, \theta) \succ_{Br,c} (H, M, \theta')$. Let Γ with $N \leqslant G \leqslant \Gamma$ and $x \in \Gamma$. Then $(G^x, N^x, \theta^x) \succ_{Br,c} (H^x, M^x, (\theta')^x)$.*

PROOF. According to Definition 4.19 one can obtain projective representations giving $(G^x, N^x, \theta^x) \succ_{Br,c} (H^x, M^x, (\theta')^x)$, by using the isomorphism $x\colon G \to G^x$ given by $g \mapsto g^x$, from those projective representations giving $(G, N, \theta) \succ_{Br,c} (H, M, \theta')$. □

We analyze the behavior of centrally isomorphic modular character triples with respect to certain quotients.

LEMMA 4.23. *Suppose that $(G, N, \theta) \succ_{Br,c} (H, M, \theta')$. Let $\epsilon\colon G \to G_1$ be an epimorphism. Write $N_1 = \epsilon(N)$, $H_1 = \epsilon(H)$ and $M_1 = \epsilon(M)$. Suppose that $Z = \ker(\epsilon) \leqslant \ker(\theta) \cap \ker(\theta')$ and $\epsilon(\mathbf{C}_G(N)/Z) = \mathbf{C}_{G_1}(N_1)$. Then*

$$(G_1, N_1, \theta_1) \succ_{Br,c} (H_1, M_1, \theta_1'),$$

where $\theta_1 \in \mathrm{IBr}(N_1)$ is such that $\theta = \theta_1 \circ \epsilon$ and $\theta_1' \in \mathrm{IBr}(M_1)$ is such that $\theta' = \theta_1' \circ \epsilon$.

PROOF. See the proof of Corollary 4.5 of [**NS14**]. (Note that there the stronger assumption $Z \leqslant \mathbf{Z}(G)$ in [**NS14**] is only used for the block-theoretic statements that are not relevant in our context.) □

Let G_i be finite groups for $i = 1, 2$. Of course $(G_1 \times G_2)^0 = G_1^0 \times G_2^0$. Recall that $\mathrm{IBr}(G_1 \times G_2) = \{\theta_1 \times \theta_2 \mid \theta_i \in \mathrm{IBr}(G_i)\}$. (See Theorem 8.21 of [**Nav98**].) The following lemma tells us how to construct centrally isomorphic modular character triples using direct and semi-direct products.

If G is a group and m is an integer, then we denote by G^m the external direct product of m copies of G. The group $\mathfrak{S}_m$ acts naturally on G^m by $(g_1, \ldots, g_m)^\sigma = (g_{\sigma^{-1}(1)}, \ldots, g_{\sigma^{-1}(m)})$.

THEOREM 4.24. *Suppose that* $(G, N, \theta_i) \succ_{Br,c} (H, M, \theta'_i)$*, for* $1 \leqslant i \leqslant m$*. Let* $\theta = \theta_1 \times \cdots \times \theta_m \in \mathrm{Irr}(N^m)$*, and* $\theta' = \theta'_1 \times \cdots \times \theta'_m \in \mathrm{Irr}(H^m)$*. Suppose that* $A \leqslant \mathfrak{S}_m$ *stabilizes* θ *and* θ'*. Then*

$$(G^m \rtimes A, N^m, \theta) \succ_{Br,c} (H^m \rtimes A, M^m, \theta').$$

PROOF. If $\sigma \in \mathfrak{S}_m$, then $\theta^\sigma = \theta_{\sigma^{-1}(1)} \times \cdots \times \theta_{\sigma^{-1}(m)}$, (evaluate θ^σ on elements of the form $(1, \ldots, n, \ldots, 1)$ for $n \in N$). Therefore, σ fixes θ if and only if $\theta_{\sigma(i)} = \theta_i$ for every $i \in \{1, \ldots, m\}$.

We may certainly assume that $N \neq 1$. Notice that $(n, 1, \ldots, 1)^\sigma = (n, 1, \ldots, 1)$ for $1 \neq n \in N$ if and only if $\sigma(1) = 1$. Using this for every $i \in \{1, \ldots, m\}$, we check that $\mathbf{C}_{G^m A}(N^m) = \mathbf{C}_G(N)^m \subseteq H^m$.

Let $V_i = F^{\theta_i(1)}$ for each $i \in \{1, \ldots, m\}$. Write $V = V_1 \otimes \cdots \otimes V_m$. Notice that if $\sigma \in A$, then $V_i = V_{\sigma(i)}$ for every $i \in \{1, \ldots, m\}$. Let $\sigma \in A$. We define a linear map $\tilde{\sigma}\colon V \to V$ by $\tilde{\sigma}(v_1 \otimes \cdots \otimes v_m) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}$ for $v_i \in V_i$, and extending linearly to all tensors. This defines a group homomorphism $A \to \mathrm{GL}(V)$. Let $\mathcal{R}(\sigma) \in \mathrm{GL}_n(F)$ be the matrix associated to $\tilde{\sigma}$, so that

$$\mathcal{R}(\sigma)(M_1 \otimes \cdots \otimes M_m)\mathcal{R}(\sigma)^{-1} = M_{\sigma(1)} \otimes \cdots \otimes M_{\sigma(m)}$$

for matrices $M_i \in \mathrm{GL}_{\theta_i(1)}(F)$.

By hypothesis, we have projective representations $\mathcal{P}_i$ and $\mathcal{P}'_i$ associated with θ_i and θ'_i giving $(G, N, \theta_i) \succ_{Br,c} (H, M, \theta_i)$. Next, we define a projective representation $\mathcal{P}$ of $G^m \rtimes A$ associated with θ. Set

$$\mathcal{P}((g_1, \ldots, g_m)\sigma) = (\mathcal{P}_1(g_1) \otimes \cdots \otimes \mathcal{P}_m(g_m))\, \mathcal{R}(\sigma),$$

for $g_i \in G$ and $\sigma \in A$. It is easy to check that the factor set α of $\mathcal{P}$ satisfies

$$\alpha((g_1, \ldots, g_m)\sigma, (g'_1, \ldots, g'_m)\tau) = \prod_{i=1}^{m} \alpha_i(g_i, g'_{\sigma(i)}),$$

for $g_i, g'_i \in G$ and $\sigma, \tau \in A$, where α_i is the factor set of $\mathcal{P}_i$ for each $i \in \{1, \ldots, m\}$. Analogously, we construct a projective representation $\mathcal{P}'$ of $H^m \rtimes A$ associated with θ'. Let α'_i be the factor set of $\mathcal{P}'_i$ for each $i \in \{1, \ldots, m\}$. By construction, the factor set α' of $\mathcal{P}'$ satisfies

$$\alpha'((h_1, \ldots, h_m)\sigma, (h'_1, \ldots, h'_m)\tau) = \prod_{i=1}^{m} \alpha'_i(h_i, h'_{\sigma(i)}),$$

for $h_i, h'_i \in H$ and $\sigma, \tau \in A$. Since α_i and α'_i agree on $H \times H$, we have that α and α' satisfy Definition 4.19(ii.1). Recall that $\mathbf{C}_{G^m \rtimes A}(N^m) = \mathbf{C}_G(N)^m$, hence $\mathcal{P}$ and $\mathcal{P}'$ trivially satisfy Definition 4.19(ii.2). □

Let $G = G_1$ be a group, and let $\alpha_i\colon G \to G_i$ be a group isomorphism for each $i \in \{1, \ldots, m\}$, with $\alpha_1 = \mathrm{id}_G$. Let $\tilde{G} = G_1 \times \cdots \times G_m$. Then the symmetric group $\mathfrak{S}_m$ acts on $\tilde{G}$ in the following way:

$$(x_1^{\alpha_1}, \ldots, x_m^{\alpha_m})^\sigma = ((x_{\sigma^{-1}(1)})^{\alpha_1}, \ldots, (x_{\sigma^{-1}(m)})^{\alpha_m}).$$

Corollary 4.25. *With the previous notation, for each $i \in \{1, \dots, m\}$, assume that $(G_i, N_i, \theta_i) \succ_{Br,c} (H_i, M_i, \theta_i')$, where $H_i = H^{\alpha_i}$, $N_i = N^{\alpha_i}$ and $M_i = M^{\alpha_i}$, for some subgroups N, M and H of G. Write $\tilde{H} = H_1 \times \cdots \times H_m$, $\tilde{N} = N_1 \times \cdots \times N_m$, and $\tilde{M} = M_1 \times \cdots \times M_m$. Write $\tilde{\theta} = \theta_1 \times \cdots \times \theta_m$, and $\tilde{\theta}' = \theta_1' \times \cdots \times \theta_m'$. Suppose $(\mathfrak{S}_m)_{\tilde{\theta}} = (\mathfrak{S}_m)_{\tilde{\theta}'}$. Then*

$$(\tilde{G} \rtimes (\mathfrak{S}_m)_{\tilde{\theta}}, \tilde{N}, \tilde{\theta}) \succ_{Br,c} (\tilde{H} \rtimes (\mathfrak{S}_m)_{\tilde{\theta}}, \tilde{M}, \tilde{\theta}').$$

Proof. Define $\alpha\colon G^m \rtimes (\mathfrak{S}_m)_{\tilde{\theta}} \to \tilde{G} \rtimes (\mathfrak{S}_m)_{\tilde{\theta}}$ by $\alpha((g_1, \dots, g_m)\sigma) = (g_1^{\alpha_1}, \dots, g_m^{\alpha_m})\sigma$. Then α is an isomorphism. Write $\beta = \alpha^{-1}$, $\theta = (\tilde{\theta})^\beta$, $\theta' = (\tilde{\theta}')^\beta$. By Theorem 4.24 we have that

$$(G^m \rtimes (\mathfrak{S}_m)_\theta, N^m, \theta) \succ_{Br,c} (H^m \rtimes (\mathfrak{S}_m)_\theta, M^m, \theta').$$

Let $\mathcal{P}$ and $\mathcal{P}'$ be projective representations giving the above central isomorphism. Then it is easy to check that $\mathcal{P}^\alpha$ and $(\mathcal{P}')^\alpha$ give the desired central isomorphism of modular character triples. □

The following key result is deeper than the others mentioned in this section so far. It is basically a modular version of Theorem 5.3 of [**Spä16**] without taking into account p-blocks. The proof we present is the modular version of the one given in [**NS16**].

Theorem 4.26. *Let $(G, N, \theta) \succ_{Br,c} (H, M, \theta')$. Suppose that $N \lhd \hat{G}$ and $\hat{G}/\mathbf{C}_{\hat{G}}(N)$ is equal to $G/\mathbf{C}_G(N)$ as a subgroup of $\mathrm{Aut}(N)$. Let $\hat{H} \leqslant \hat{G}$ such that $\hat{H} \geqslant \mathbf{C}_{\hat{G}}(N)$, and $\hat{H}/\mathbf{C}_{\hat{G}}(N)$ and $H/\mathbf{C}_G(N)$ are equal as subgroups of $\mathrm{Aut}(N)$. Then*

$$(\hat{G}, N, \theta) \succ_{Br,c} (\hat{H}, M, \theta').$$

Proof. By assumption we have that $\hat{H} = \{x \in \hat{G} \mid \text{for some } h \in H,\ n^x = n^h \text{ for every } n \in N\}$.

We start by proving that $\hat{G} = N\hat{H}$. Let $x \in \hat{G}$. Then there exists some $g \in G$ such that $m^x = m^g$ for every $m \in N$. Since $g = hn$ for some $h \in H$ and $n \in N$, we conclude $xn^{-1} \in \hat{H}$. Therefore $\hat{G} = N\hat{H}$. Now, we prove that $N \cap \hat{H} = M = N \cap H$. The inclusion $N \cap H \subseteq N \cap \hat{H}$ is obvious. Assume now that $m \in N \cap \hat{H}$. Hence there is some $h \in H$ such that $n^m = n^h$ for every $n \in N$. Hence $mh^{-1} \in \mathbf{C}_G(N) \subseteq H$. Thus $n \in H$ and $n \in N \cap H = M$. Therefore $N \cap \hat{H} \subseteq M$.

The map $\tau\colon G/\mathbf{C}_G(N) \to \hat{G}/\mathbf{C}_{\hat{G}}(N)$ given by $\tau(x\mathbf{C}_G(N)) = y\mathbf{C}_{\hat{G}}(N)$ if and only if x and y induce the same conjugation automorphism of N is a group isomorphism sending $N\mathbf{C}_G(N)/\mathbf{C}_G(N)$ onto $N\mathbf{C}_{\hat{G}}(N)/\mathbf{C}_{\hat{G}}(N)$. In fact, $\tau(n\mathbf{C}_G(N)) = n\mathbf{C}_{\hat{G}}(N)$.

Notice that θ is $\hat{G}$-invariant. This is derived from the fact that θ is G-invariant since every element of $\hat{G}$ acts on N as an element of G. Similarly θ' is $\hat{H}$-invariant.

By assumption $(G, N, \theta) \succ_{Br,c} (H, M, \theta')$, so there exist projective representations $\mathcal{P}$ of G associated with θ with factor set α and $\mathcal{P}'$ of H associated with θ' with factor set α' such that $\alpha'(h_1, h_2) = \alpha(h_1, h_2)$ for every $h_1, h_2 \in H$. Also, there is a map $\nu\colon \mathbf{C}_G(N) \to F^\times$ such that $\mathcal{P}(c) = \nu(c)I_{\theta(1)}$ and $\mathcal{P}'(c) = \nu(c)I_{\theta'(1)}$ for every $c \in \mathbf{C}_G(N)$. Notice that if $\lambda \in \mathrm{IBr}(\mathbf{Z}(N))$ lies under θ and θ', then $\nu_{\mathbf{Z}(N)}$ affords λ. Also $\nu(cz) = \nu(c)\nu(z)$ for every $c \in \mathbf{C}_G(N)$ and $z \in \mathbf{Z}(N)$.

In order to construct projective representations $\hat{\mathcal{P}}$ of $\hat{G}$ associated to θ and $\hat{\mathcal{P}}'$ of $\hat{H}$ associated to θ', we need some ingredients: appropriate transversals $\mathbb{T}$ and $\hat{\mathbb{T}}$ of the $N\mathbf{C}_G(N)$-cosets in N and the $N\mathbf{C}_{\hat{G}}(N)$-cosets in $\hat{G}$, and a function $\hat{\nu}\colon \mathbf{C}_{\hat{G}}(N) \to F^\times$ playing the role of ν for $\mathcal{P}$ and $\mathcal{P}'$.

Let $\mathbb{T} \subseteq H$ be a complete set of representatives of cosets of $N\mathbf{C}_G(N)$ in G with $1 \in \mathbb{T}$ (we can choose such $\mathbb{T}$ because of $G = NH$). Hence every $g \in G$ can be written as tcn for some $t \in \mathbb{T}$, $c \in \mathbf{C}_G(N)$ and $n \in N$. Notice that $tcn = t_1c_1n_1$ if and only if there is some $z \in \mathbf{Z}(N)$ such that $t = t_1$, $c = c_1z$, and $n = z^{-1}n_1$. By elementary group theory, observe that $\mathbb{T}$ is also a complete set of representatives of cosets of $M\mathbf{C}_G(N)$ in H.

Since $\hat{H}/\mathbf{C}_{\hat{G}}(N)$ and $H/\mathbf{C}_G(N)$ are equal as subgroups of $\mathrm{Aut}(N)$, for every $t \in \mathbb{T}$ we can choose $\hat{t} \in \hat{H}$ such that $m^t = m^{\hat{t}}$ for every $m \in N$. Also we can set $\hat{1} = 1$. By definition, we have that $\tau(t\mathbf{C}_G(N)) = \hat{t}\mathbf{C}_{\hat{G}}(N)$. Using that τ is a group isomorphism, we have that $\hat{\mathbb{T}} = \{\hat{t} \mid t \in \mathbb{T}\}$ is a complete set of representatives of right cosets of $N\mathbf{C}_{\hat{G}}(N)$ in $\hat{G}$. Also since $\mathbb{T}$ is a complete set of representatives of cosets of $M\mathbf{C}_G(N)$ in H, we have that $\hat{\mathbb{T}}$ is a complete set of representatives of cosets of $M\mathbf{C}_{\hat{G}}(N)$ in $\hat{H}$.

Let $\hat{\nu}\colon \mathbf{C}_{\hat{G}}(N) \to F^\times$ be any function such that $\hat{\nu}(cz) = \hat{\nu}(c)\nu(z)$ for every $c \in \mathbf{C}_{\hat{G}}(N)$ and $z \in \mathbf{Z}(N)$. (For instance, write $\mathbf{C}_{\hat{G}}(N) = \bigcup_{j=1}^t x_j\mathbf{Z}(N)$, and define $\hat{\nu}(x_jz) = \nu(z)$.)

We define functions

$$\hat{\mathcal{P}}\colon \hat{G} \to \mathrm{GL}_{\theta(1)}(F) \quad \text{and} \quad \hat{\mathcal{P}}'\colon \hat{H} \to \mathrm{GL}_{\theta'(1)}(F)$$

by

$$\hat{\mathcal{P}}(\hat{t}nc) = \mathcal{P}(t)\mathcal{P}(n)\hat{\nu}(c) \quad \text{and} \quad \hat{\mathcal{P}}'(\hat{t}mc) = \mathcal{P}'(t)\mathcal{P}'(m)\hat{\nu}(c),$$

where $t \in \mathbb{T}$, $n \in N$, $c \in \mathbf{C}_{\hat{G}}(N)$ and $m \in M$.

Notice that if $\hat{t}nc = \hat{t}_1n_1c_1$, then $t = t_1$, $c = c_1z$ and $n = z^{-1}n_1$ for some $z \in \mathbf{Z}(N)$, by a previous argument. Then

$$\hat{\mathcal{P}}(n_1c_1) = \mathcal{P}(n_1)\hat{\nu}(c_1) = \mathcal{P}(n)\nu(z)\hat{\nu}(z^{-1}c) = \mathcal{P}(n)\hat{\nu}(c) = \hat{\mathcal{P}}(nc),$$

using our defining property for $\hat{\nu}$. Hence $\hat{\mathcal{P}}$ is well-defined. Similarly, one proves that $\hat{\mathcal{P}}'$ is a well-defined function. Notice that $\hat{\mathcal{P}}(n) = \mathcal{P}(n)$ and $\hat{\mathcal{P}}'(m) = \mathcal{P}'(m)$ for every $n \in N$ and $m \in M$.

We want to show that $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}'$ define projective representations of $\hat{G}$ and $\hat{H}$, associated with θ and θ', respectively, with factor sets $\hat{\alpha}$ and $\hat{\alpha}'$

coinciding on $\hat{H} \times \hat{H}$, and such that for $x \in \mathbf{C}_{\hat{G}}(N)$ they are associated with the same scalar. The latter part is obvious by the definition of $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}'$. Let β be the factor set associated with $\mathcal{P}$ or with $\mathcal{P}'$. We recall that $\beta(h,n) = 1 = \beta(n,h)$ for every $h \in H$ and $n \in N$ and, in particular, $\beta(h, h^{-1}nh) = 1$.

Let $x \in \hat{G}$ and let $n \in N$. We prove that $\hat{\mathcal{P}}(xn) = \hat{\mathcal{P}}(x)\hat{\mathcal{P}}(n)$. Write $x = \hat{t}mc$, where $m \in N$ and $c \in \mathbf{C}_{\hat{G}}(N)$. Then $xn = \hat{t}(mn)c$ and

$$\hat{\mathcal{P}}(xn) = \hat{\mathcal{P}}(\hat{t}(mn)c) = \mathcal{P}(t)\mathcal{P}(mn)\hat{\nu}(c) = \mathcal{P}(t)\mathcal{P}(m)\hat{\nu}(c)\mathcal{P}(n) = \hat{\mathcal{P}}(x)\hat{\mathcal{P}}(n)\,.$$

Next, we prove that $\hat{\mathcal{P}}(nx) = \hat{\mathcal{P}}(n)\hat{\mathcal{P}}(x)$. By the comment in the previous paragraph about the factor set of $\mathcal{P}$, we have that $\mathcal{P}(n)\mathcal{P}(t) = \mathcal{P}(t)\mathcal{P}(t^{-1}nt)$. Then

$$\begin{aligned}\hat{\mathcal{P}}(nx) = \hat{\mathcal{P}}(n\hat{t}mc) &= \hat{\mathcal{P}}(\hat{t}\hat{t}^{-1}n\hat{t}mc) = \mathcal{P}(t)\mathcal{P}(\hat{t}^{-1}n\hat{t})\mathcal{P}(m)\hat{\nu}(c)\\ &= \mathcal{P}(t)\mathcal{P}(t^{-1}nt)\mathcal{P}(m)\hat{\nu}(c) = \mathcal{P}(n)\mathcal{P}(t)\mathcal{P}(m)\hat{\nu}(c)\\ &= \hat{\mathcal{P}}(n)\hat{\mathcal{P}}(x).\end{aligned}$$

Next we show that $\hat{\mathcal{P}}$ is a projective representation of $\hat{G}$. Suppose that $t_1, t_2, t_3 \in \mathbb{T}$, $c_1, c_2, c_3 \in \mathbf{C}_{\hat{G}}(N)$, and $n_1, n_2, n_3 \in N$ are such that

$$(\hat{t}_1 n_1 c_1)(\hat{t}_2 n_2 c_2) = \hat{t}_3 n_3 c_3.$$

Notice that

$$\tau(t_1 n_1 t_2 n_2 \mathbf{C}_G(N)) = \widehat{t_1} n_1 \widehat{t_2} n_2 \mathbf{C}_{\hat{G}}(N) = \widehat{t_3} n_3 \mathbf{C}_{\hat{G}}(N) = \tau(t_3 n_3 \mathbf{C}_G(N))\,.$$

Thus $t_1 n_1 t_2 n_2 = t_3 n_3 c$, for some $c \in \mathbf{C}_G(N)$. Then

$$\begin{aligned}\hat{\mathcal{P}}(\hat{t}_1 n_1 c_1)\hat{\mathcal{P}}(\hat{t}_2 n_2 c_2) &= \mathcal{P}(t_1 n_1)\hat{\nu}(c_1)\mathcal{P}(t_2 n_2)\hat{\nu}(c_2)\\ &= \mathcal{P}(t_1 n_1 t_2 n_2)\alpha(t_1, t_2)\hat{\nu}(c_1)\hat{\nu}(c_2)\\ &= \mathcal{P}(t_3 n_3 c)\alpha(t_1, t_2)\hat{\nu}(c_1)\hat{\nu}(c_2)\\ &= \mathcal{P}(t_3 n_3)\mathcal{P}(c)\alpha(t_3, c)^{-1}\alpha(t_1, t_2)\hat{\nu}(c_1)\hat{\nu}(c_2)\\ &= \mathcal{P}(t_3 n_3)\hat{\nu}(c_3)\mu(c)\alpha(t_3, c)^{-1}\alpha(t_1, t_2)\hat{\nu}(c_1)\hat{\nu}(c_2)\hat{\nu}(c_3)^{-1}.\end{aligned}$$

This implies that $\hat{\mathcal{P}}$ is a projective representation of $\hat{G}$ with factor set

$$\hat{\alpha}(\hat{t}_1 n_1 c_1, \hat{t}_2 n_2 c_2) = \mu(c)\alpha(t_3, c)^{-1}\alpha(t_1, t_2)\hat{\nu}(c_1)\hat{\nu}(c_2)\hat{\nu}(c_3)^{-1},$$

where c is any element of $\mathbf{C}_G(N)$ satisfying the equation $t_1 n_1 t_2 n_2 = t_3 n_3 c$. The same argument, substituting elements in N by elements in M, shows that $\hat{\mathcal{P}}'$ is a projective representation of $\hat{H}$ with factor set

$$\hat{\alpha}'(\hat{t}_1 m_1 c_1, \hat{t}_2 m_2 c_2) = \mu(c)\alpha(t_3, c)^{-1}\alpha'(t_1, t_2)\hat{\nu}(c_1)\hat{\nu}(c_2)\hat{\nu}(c)^{-1},$$

where $c \in \mathbf{C}_G(N)$ satisfies $t_1 m_1 t_2 m_2 = t_3 m_3 c$. It is now clear that both factor sets coincide on $\hat{H} \times \hat{H}$. This finishes the proof. □

Next, we discuss the Brauer characters of a central product of groups and their relation with centrally isomorphic modular character triples.

Lemma 4.27. *Let $N \lhd G$ and T_i with $N \leqslant T_i \leqslant G$ be such that $G/N = T_1/N \times \cdots \times T_k/N$. Suppose that $[T_i, T_j] = 1$ for every $i \neq j$. Given $\theta \in \mathrm{IBr}(N)$ and $\varphi_i \in \mathrm{IBr}(T_i|\theta)$, there is a unique $\chi = \varphi_1 \cdot \ldots \cdot \varphi_k \in \mathrm{IBr}(G|\theta)$ such that χ_{T_i} is a multiple of φ_i. Moreover, the map*

$$\mathrm{IBr}(T_1|\theta) \times \cdots \times \mathrm{IBr}(T_k|\theta) \to \mathrm{IBr}(G|\theta)$$
$$(\varphi_1, \ldots, \varphi_k) \mapsto \varphi_1 \cdot \ldots \cdot \varphi_k$$

is a natural bijection.

Proof. This is a natural adaptation of Lemma 5.1 of [**IMN07**] to Brauer characters. □

In the situation described in Lemma 4.27, if $\varphi_1 \in \mathrm{IBr}(T_1|\theta)$,…,$\varphi_k \in \mathrm{IBr}(T_k|\theta)$, we refer to $\varphi_1 \cdot \ldots \cdot \varphi_k \in \mathrm{IBr}(G|\theta)$ as the **dot product** of φ_1, … ,φ_k.

Lemma 4.28. *Let $N \lhd H \leqslant G$. Suppose that $(H, N, \theta) \succ_{Br,c} (K, M, \theta')$. Let $Z \lhd G$ be an abelian group such that $Z \leqslant \mathbf{C}_G(N)$ and $Z \cap N = Z \cap M$. Then*

$$(HZ, NZ, \theta \cdot \nu) \succ_{Br,c} (KZ, MZ, \theta' \cdot \nu)$$

for every $\nu \in \mathrm{IBr}_H(Z|\lambda)$ where $\lambda \in \mathrm{IBr}(\theta_{Z \cap M})$.

Proof. See the proof of Proposition 3.9(b) of [**NS14**] together with Theorem 4.26. □

The following method for constructing projective representations from representations given in [**NS14**] will be useful later in this work. More precisely, it will be useful to control the values of certain projective representations in Section 4.3 (see the proof of Lemma 4.34).

We recall that if $\epsilon\colon \hat{G} \to G$ is an epimorphism with $\ker(\epsilon) = Z$, then a Z-**section** $\mathrm{rep}\colon G \to \hat{G}$ of ϵ is a map such that $\epsilon \circ \mathrm{rep} = \mathrm{id}_G$ and $\mathrm{rep}(1) = 1$.

Theorem 4.29. *Let (G, N, θ) be a modular character triple. There exists a finite group $\hat{G}$, an epimorphism $\epsilon\colon \hat{G} \to G$ with cyclic kernel $Z \leqslant \mathbf{Z}(\hat{G})$ and a Z-section $\mathrm{rep}\colon G \to \hat{G}$ satisfying:*

(a) $\hat{N} = N_1 \times Z = \epsilon^{-1}(N)$, $N_1 \cong N$ via $\delta = \epsilon|_{N_1}$ and $N_1 \lhd \hat{G}$. The action of $\hat{G}$ on $\hat{N}$ coincides with the action of G on N via ϵ.

(b) The character $\theta_1 = \theta^{\delta^{-1}} \in \mathrm{IBr}(N_1)$ extends to $\hat{G}$. The Z-section $\mathrm{rep}\colon G \to \hat{G}$ satisfies $\mathrm{rep}(n) \in N_1$, $\mathrm{rep}(ng) = \mathrm{rep}(n)\mathrm{rep}(g)$ and $\mathrm{rep}(gn) = \mathrm{rep}(g)\mathrm{rep}(n)$ for every $n \in N$ and $g \in G$.

(c) $\epsilon(\mathbf{C}_{\hat{G}}(\hat{N})) = \mathbf{C}_G(N)$.

In particular, if $\mathcal{D}$ is a representation of $\hat{G}$ such that $\mathcal{D}|_{N_1}$ affords θ_1, then the map $\mathcal{P}$ defined for every $g \in G$ by

$$\mathcal{P}(g) = \mathcal{D}(\mathrm{rep}(g))$$

is a projective representation of G associated to θ.

Proof. See the proof of Theorem 4.1 of [**NS14**]. □

4.3. Fake Galois action on Brauer characters

Perhaps, the most important application of Isaacs' B_π-theory (see Section 1.4) relies on the fact that $B_{p'}$-characters constitute a canonical lift of Brauer characters of p-solvable groups.

THEOREM 4.30 (Isaacs). *Let G be a p-solvable group. Then, restriction to p-elements yields a bijection from $B_{p'}(G)$ onto $\mathrm{IBr}(G)$. In particular, the Galois group $\mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$ acts on $\mathrm{IBr}(G)$.*

PROOF. For the first part see Corollary 10.3 of [**Isa84**]. Then the latter statement follows directly from Theorem 1.25. □

However, in general, $\mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$ does not act on $\mathrm{IBr}(G)$. The group $G = \mathrm{SL}_2(7)$ and the prime $p = 7$ exemplify this, as pointed out to us by P. H. Tiep. Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q}_{|G|_{2'}})$ be the automorphism sending a 16-th root of unity ξ to ξ^3. By using GAP, we see that if $\varphi \in \mathrm{Irr}(G)$ has degree 6, then φ^σ is not irreducible.

Let N be a group and let m be a positive integer. If θ is a class function of N (or of N^0), then $\theta^{(m)}(n) = \theta(n^m)$ is a class function of N (respectively of N^0). If $(|N|, m) = 1$ and $\theta \in \mathrm{Irr}(N)$, then $\theta^{(m)} = \theta^\sigma$ for a certain $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|N|}/\mathbb{Q})$ and so $\theta^{(m)} \in \mathrm{Irr}(N)$. It is no longer true that $\theta^{(m)} \in \mathrm{IBr}(N)$ if we consider $\theta \in \mathrm{IBr}(N)$. (We refer to the above example for $m = 211$.)

The aim of the final part of this works is to reduce Conjecture 5.1 to a problem on quasi-simple groups. In order to do that, we will need that for every quasi-simple group X, for every integer m with $(|X|, m) = 1$ and for every $\varphi \in \mathrm{IBr}(X)$, there exists some $\varphi' \in \mathrm{IBr}(X)$ that in some sense *behaves* like $\varphi^{(m)}$. This is what we call a *fake m-th Galois conjugate* of φ. Let us state this definition clearly. The results of this section are part of an original joint work with B. Späth [**SV16**].

Let $\mathbf{V} = \{\xi \in \mathbb{C} \mid o(\xi) = n \text{ for some natural } n\} \leqslant \mathbb{C}^\times$. Recall that $\mathbf{U}$ is the subgroup of p'-roots of unity of $\mathbb{C}$. Hence $\mathbf{U} \leqslant \mathbf{V}$. For a fixed positive integer m, let π be the set of primes dividing m. Define $\sigma_m \colon \mathbf{V} \to \mathbf{V}$ by

$$\sigma_m(\xi) = \xi^{\sigma_m} = \xi_\pi \xi_{\pi'}^m,$$

for every root of unity $\xi \in \mathbf{V}$, where ξ_π and $\xi_{\pi'}$ are respectively the π-part and the π'-part of ξ. Notice that σ_m is an automorphism of $\mathbf{V}$ and restriction of σ_m to elements of $\mathbf{U}$ defines an automorphism of $\mathbf{U}$ which we denote again by σ_m. Let $\omega_m \colon F^\times \to F^\times$ be the group homomorphism that σ_m induces via $* \colon \mathbf{U} \to F^\times$. We denote by ζ^{ω_m} the image of $\zeta \in F^\times$ under ω_m.

Moreover, for any positive integer n, by elementary Galois Theory, we have that σ_m defines a Galois automorphism of $\mathbb{Q}_n$ (which we denote again by $\sigma_m \in \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$) that sends every n-th root of unity $\xi \in \mathbb{Q}_n$ to ξ^{σ_m}.

DEFINITION 4.31. Let (G, N, φ) and (G, N, φ') be modular character triples and let m be a positive integer coprime to $|N|$. We write

$$(G, N, \varphi)^{(m)} \approx (G, N, \varphi'),$$

if there exist projective representations $\mathcal{P}$ and $\mathcal{P}'$ of G associated to φ and φ' such that:

(i) for every $x, y \in G$ the factor sets α and α' of $\mathcal{P}$ and $\mathcal{P}'$ satisfy
$$\alpha(x,y)^{\omega_m} = \alpha'(x,y),$$
and

(ii) for every $c \in \mathbf{C}_G(N)$ the scalar matrices $\mathcal{P}(c)$ and $\mathcal{P}'(c)$ are associated with scalars ζ and ζ^{ω_m}.

In the above situation we may say that φ' is a **fake m-th Galois conjugate** of φ with respect to $N \lhd G$ and that the triples (G, N, φ) and (G, N, φ') are **fake m-th Galois conjugate**.

Remark 4.32. Let (G, N, φ) be a modular character triple, and let m be a positive integer coprime to $|N|$.

(a) If φ is linear, then $\varphi^{\sigma_m} = \varphi^m \in \mathrm{IBr}(N)$.
(b) In general $\varphi^{\sigma_m} = \varphi^{(m)}$ is an integer linear combination of $\mathrm{IBr}(N)$.
(c) Let $\varphi' \in \mathrm{IBr}(N)$ and $\lambda \in \mathrm{IBr}(\varphi|_{\mathbf{Z}(N)})$. Then $(N, N, \varphi)^{(m)} \approx (N, N, \varphi')$ if and only if $\mathrm{IBr}(\varphi'|_{\mathbf{Z}(N)}) = \{\lambda^m\}$.

Proof. For part (a), notice that $\mathbb{Q}(\varphi) \subseteq \mathbb{Q}_{|N|}$, then $\varphi^{\sigma_m}(n) = \varphi(n)^{\sigma_m} = \varphi(n)^m$ for every $n \in N^0$, by the definition of σ_m. Of course, $\varphi^m \in \mathrm{IBr}(N)$, in this case. For part (b), let $n \in N^0$. Since $\varphi_{\langle n \rangle} = \lambda_1 + \cdots + \lambda_t$, where $\lambda_i \in \mathrm{IBr}(\langle n \rangle)$ are linear, we have that
$$\begin{aligned}\varphi(n)^{\sigma_m} &= \lambda_1(n)^{\sigma_m} + \cdots + \lambda_t(n)^{\sigma_m}\\ &= \lambda_1(n^m) + \cdots + \lambda_t(n^m)\\ &= \varphi(n^m) = \varphi^{(m)}(n).\end{aligned}$$
The class function $\varphi^{(m)}$ is an integer linear combination of $\mathrm{IBr}(N)$ by Problem 2.11 of [**Nav98**]. Part (c) follows from the definition of fake m-th Galois conjugate modular character triples. □

We shall give an alternative reformulation of Definition 4.31 in Lemma 4.35. This new way of defining fake Galois conjugate modular character triples will make it easier to work with them later on. We first need to show that for a modular character triple (G, N, φ) there always exists a projective representation associated to φ with particular properties.

Lemma 4.33. *Let $N \lhd G$ and $\chi \in \mathrm{IBr}(G)$ with $\chi_N \in \mathrm{IBr}(N)$. Then $\mathbf{C}_G(N)' \leqslant \ker(\chi)$.*

Proof. Let $\mathcal{D}$ be a representation affording χ and $c \in \mathbf{C}_G(N)$. Then $\mathcal{D}(c)$ commutes with the irreducible representation $\mathcal{D}_N$. By Schur's Lemma [**Isa76**, Lem. 1.5], this implies that $\mathcal{D}(c) = \lambda(c)I$ is a scalar matrix. The map $\lambda\colon \mathbf{C}_G(N) \to F^\times$ given by $\mathcal{D}(c) = \lambda(c)I$ is a homomorphism. Hence $\mathbf{C}_G(N)/\ker(\lambda)$ is an abelian p'-group. This proves the statement. □

Lemma 4.34. *Let (G, N, φ) be a modular character triple, m an integer coprime to $|N|$ and $(F^{\times})_{m'}$ the subgroup in $F^{\times}$ of elements of order coprime to m. Then, there exists a projective representation $\mathcal{P}$ of G associated to φ with factor set α such that:*

(i) $\alpha(g, g') \in (F^{\times})_{m'}$ for every $g, g' \in G$, and

(ii) for every $c \in \mathbf{C}_G(N)$, $\mathcal{P}(c)$ is the scalar matrix associated to some $\xi \in (F^{\times})_{m'}$.

Proof. Let ρ be the set of primes dividing m different from p, so that $p \notin \rho$. (Then $\rho = \pi - \{p\}$ with the notation of this section.)

Suppose that φ extends to some $\chi \in \mathrm{IBr}(G)$. We claim that there exists an extension ψ of φ to G such that every ρ-element of $\mathbf{C}_G(N)$ lies in $\ker(\psi)$.

By Lemma 4.33, we may assume that $\mathbf{C}_G(N)' = 1$. Hence $\mathbf{C}_G(N)$ is abelian and the Hall ρ-subgroup C of $\mathbf{C}_G(N)$ satisfies $C \lhd G$. Then $N \cap C = 1$ and $NC \cong N \times C$. We want to prove that there exists an extension of φ to G containing C in its kernel. It suffices to prove that the character $\overline{\varphi} \in \mathrm{IBr}(NC/C)$ given by φ (more precisely $\overline{\varphi}(ncC) = \varphi(n)$ for every $n \in N$ and $c \in C$) extends to G/C.

Let q be any prime. If $q \in \rho$ and $Q/N \in \mathrm{Syl}_q(G/N)$, then $\varphi \in \mathrm{IBr}(NC/C)$ extends to QC/C because of $(q, |NC : C|) = 1$ by Theorem 4.12. If $q \notin \rho$ and $Q/N \in \mathrm{Syl}_q(G/N)$ we have that $Q \cap C = 1$. Since $C \lhd G$, then $\chi_Q \in \mathrm{IBr}(Q)$ defines an irreducible Brauer character of $QC/C \cong Q$ which extends $\overline{\varphi}$. By Theorem 4.11, this implies that $\overline{\varphi}$ extends to G/C, and the claim follows.

Now, by Theorem 4.29, there exists a central extension $\epsilon\colon \hat{G} \to G$ of G with finite cyclic kernel Z and a Z-section $\mathrm{rep}\colon G \to \hat{G}$ of ϵ such that:

(a) $\hat{N} = N_1 \times Z = \epsilon^{-1}(N)$, the groups N_1 and N are isomorphic via $\delta = \epsilon|_{N_1}$ and $N_1 \lhd \hat{G}$. Moreover, the action of $\hat{G}$ on $\hat{N}$ coincides with the action of G on N via ϵ.

(b) $\varphi_1 = \varphi^{\delta^{-1}} \in \mathrm{IBr}(N_1)$ extends to $\hat{G}$. The Z-section $\mathrm{rep}\colon G \to \hat{G}$ satisfies $\mathrm{rep}(n) \in N_1$, $\mathrm{rep}(ng) = \mathrm{rep}(n)\mathrm{rep}(g)$ and $\mathrm{rep}(g)\mathrm{rep}(n) = \mathrm{rep}(gn)$ for every $n \in N$ and $g \in G$.

(c) $\epsilon(\mathbf{C}_{\hat{G}}(\hat{N})) = \mathbf{C}_G(N)$.

According to (b) the character φ_1 extends to $\hat{G}$. By the first part of the proof applied to $\hat{G}$, there is an extension $\chi_1 \in \mathrm{IBr}(\hat{G})$ such that every ρ-element of $\mathbf{C}_{\hat{G}}(\hat{N})$ lies in $\ker(\chi_1)$. Let $\mathcal{D}$ be a representation affording χ_1 and let $\mathcal{P}\colon G \to \mathrm{GL}_{\varphi(1)}(F)$ be defined by

$$\mathcal{P}(g) = \mathcal{D}(\mathrm{rep}(g)) \text{ for every } g \in G.$$

For every $g, g' \in G$ we obtain

$$\mathcal{P}(g)\mathcal{P}(g') = \mathcal{D}(z_{g,g'})\mathcal{P}(gg'),$$

where $z_{g,g'} \in Z \leqslant \mathbf{Z}(\hat{G})$ is given by $\mathrm{rep}(g)\mathrm{rep}(g') = z_{g,g'}\mathrm{rep}(gg')$. Since $\mathcal{D}(z_{g,g'})$ is a scalar matrix, $\mathcal{P}$ is a projective representation of G associated to φ with factor set $\alpha\colon G \times G \to F^\times$ defined by $\alpha(g,g')I = \mathcal{D}(z_{g,g'})$ for every $g, g' \in G$. Since $\mathcal{D}(c) = I_{\varphi_1(1)}$ for every ρ-element $c \in \mathbf{C}_{\hat{G}}(N)$, it is straightforward to check that $\mathcal{P}$ satisfies the required properties. □

As a consequence of Lemma 4.34, we can reformulate Definition 4.31 in the following (easier to handle) way.

Lemma 4.35. *Let (G, N, φ) and (G, N, φ') be modular character triples, and let m be coprime to $|N|$. Then the following are equivalent:*

(a) $(G, N, \varphi)^{(m)} \approx (G, N, \varphi')$,

(b) there exist projective representations $\mathcal{P}$ and $\mathcal{P}'$ of G associated to φ and φ' satisfying the properties (i) and (ii) in 4.34 and such that

(b.1) the factor sets α and α' of $\mathcal{P}$ and $\mathcal{P}'$ satisfy

$$\alpha(g,g')^m = \alpha'(g,g') \text{ for every } g, g' \in G,$$

and

(b.2) for every $c \in \mathbf{C}_G(N)$, the scalar matrices $\mathcal{P}(c)$ and $\mathcal{P}'(c)$ are associated with scalars ζ and ζ^m respectively.

Proof. We first prove that (a) implies (b). Since $(G, N, \varphi)^{(m)} \approx (G, N, \varphi')$ there exist projective representations $\mathcal{Q}$ and $\mathcal{Q}'$ of G associated to φ and φ' having the properties listed in Definition 4.31. Let $\mathcal{P}$ be a projective representation of G associated to φ with the properties described in Lemma 4.34. By Remark 4.15, there exists a map $\mu\colon G \to F^\times$ such that $\mathcal{P}$ is similar to $\mu\mathcal{Q}$. Also μ is constant on N-cosets in G and $\mu(1) = 1$. Let $\mu'\colon G \to F^\times$ be given by $\mu'(g) = \mu(g)^{\omega_m}$ for every $g \in G$. Since μ' is constant on N-cosets in G and $\mu'(1) = 1$, then it is straightforward to check that $\mathcal{P}' = \mu'\mathcal{Q}'$ is a projective representation of G associated to φ'.

In order to verify that $\mathcal{P}$ and $\mathcal{P}'$ satisfy the condition in (b.1) let $g, g' \in G$. According to Definition 4.31, the factor sets β and β' of $\mathcal{Q}$ and $\mathcal{Q}'$ satisfy $\beta(g,g')^{\omega_m} = \beta'(g,g')$. By Remark 4.15, the factor sets α and α' of $\mathcal{P}$ and $\mathcal{P}'$ satisfy

$$\alpha(g,g') = \frac{\mu(g)\mu(g')}{\mu(gg')}\beta(g,g') \text{ and } \alpha'(g,g') = \frac{\mu'(g)\mu'(g')}{\mu'(gg')}\beta'(g,g').$$

This implies that $\alpha(g,g')^{\omega_m} = \alpha'(g,g')$. By the choice of $\mathcal{P}$, we have that $\alpha(g,g')$ is a root of unity in $F^\times$ of order coprime to m, hence $\alpha'(g,g') = \alpha(g,g')^{\omega_m} = \alpha(g,g')^m$ also is. This proves that $\mathcal{P}$ and $\mathcal{P}'$ satisfy the condition (b.1).

In order to verify that $\mathcal{P}$ and $\mathcal{P}'$ satisfy the condition in (b.2) let $c \in \mathbf{C}_G(N)$ and $\zeta \in F^\times$ be the scalar associated to $\mathcal{Q}(c)$. According to Definition 4.31, ζ^{ω_m} is the scalar associated to $\mathcal{Q}'(c)$. By definition $\mathcal{P}(c)$ and $\mathcal{P}'(c)$ are scalar matrices associated with $\zeta\mu(c)$ and $(\zeta\mu(c))^{\omega_m}$ respectively. By the choice of $\mathcal{P}$, we have that $\zeta\mu(c)$ is a root of unity of $F^\times$ of order coprime to m. Hence $(\zeta\mu(c))^{\omega_m} = (\zeta\mu(c))^m \in F^\times$ has also order coprime to m. This proves that $\mathcal{P}$ and $\mathcal{P}'$ satisfy the condition (b.2).

Moreover we see that $\mathcal{P}'$ is a projective representation having the properties mentioned in 4.34.

To prove the converse, just notice that the projective representations $\mathcal{P}$ and $\mathcal{P}'$ in (b) have the properties described in Lemma 4.34 and recall that ω_m acts on roots of unity of $F^\times$ of order coprime to m by raising them to their m-th power. It is then immediate that $\mathcal{P}$ and $\mathcal{P}'$ give $(G, N, \theta)^{(m)} \approx (G, N, \theta')$. □

The following is an easy consequence of Lemma 4.35.

COROLLARY 4.36. *Let (G, N, φ) be a modular character triple. Let m be a positive integer coprime to $|N|$ and let σ_m be defined as in the beginning of this section. If φ is linear, then $(G, N, \varphi)^{(m)} \approx (G, N, \varphi^{\sigma_m})$.*

PROOF. By Remark 4.32, we know that $\varphi^{\sigma_m} = \varphi^m$. Let $\mathcal{P}$ be a projective representation of G associated to φ as in Lemma 4.34. Then, it is straightforward to check that $\mathcal{P}^m$ is a projective representation of G associated to φ^m (as in Lemma 4.34). By Lemma 4.35, we have that $(G, N, \varphi)^{(m)} \approx (G, N, \varphi^{\sigma_m})$. □

We have defined the notion of fake m-th Galois conjugate character triples. We conclude this section by introducing fake m-th Galois actions and verifying their existence on p-solvable groups.

DEFINITION 4.37. Let $N \lhd G$. Let $\mathscr{S} \leqslant \mathrm{IBr}(N)$ be a G-invariant subset. Let m be an integer coprime to $|N|$. We say that there exists a **fake m-th Galois action on $\mathscr{S}$ with respect to** G if there exists a G-equivariant bijection

$$f_m \colon \mathscr{S} \to \mathscr{S}$$

such that

$$(G_\varphi, N, \varphi)^{(m)} \approx (G_\varphi, N, f_m(\varphi)) \text{ for every } \varphi \in \mathscr{S}.$$

Let N be a p-solvable group. Then the Galois group $\mathrm{Gal}(\mathbb{Q}_{|N|}/\mathbb{Q})$ acts on $\mathrm{IBr}(N)$, by Theorem 4.30. In this case, we show that there exists a fake m-th Galois action on $\mathrm{IBr}(N)$ for any integer m coprime to $|N|$ and with respect to any G with $N \lhd G$. We first need to control the values of certain projective (ordinary) representations.

We recall that the theory of projective representations associated to a character triple over $\mathbb{C}$ is very similar to the theory of projective representations associated to character triples over F. In particular, Theorem 8.12 and Lemma 8.27 of [**Nav98**] work for $\mathbb{C}$ instead of F.

Recall we have chosen an ideal $\mathcal{M} < \mathbf{R}$ with $p\mathbf{R} \subseteq \mathcal{M}$ and $F = \mathbf{R}/\mathcal{M}$.

LEMMA 4.38. *Let $N \lhd G$ and $\theta \in \mathrm{Irr}(N)$. Assume that θ is G-invariant and $\theta^0 \in \mathrm{IBr}(N)$. Write $L = \mathbb{Q}_{|G|}$ and $S_L = \{r/s \mid r \in \mathbf{R} \cap L, s \in \mathbf{R} \cap L \backslash \mathcal{M}\}$. There exists a projective representation of G associated to θ with matrix entries in S_L.*

Proof. We notice that by Brauer's Theorem [**Isa76**, Thm. 10.3], L is a splitting field for G. By Problem 2.12 of [**Nav98**], let $\mathfrak{Y}$ be an S_L-representation of N affording θ. We check that there exists a projective representation of G associated to θ extending $\mathfrak{Y}$ with entries in S_L. For every $\overline{g} \in G/N$, the representation $\mathfrak{Y}$ extends to a representation $\mathfrak{Y}_{\overline{g}}$ of $\langle N, g\rangle$ as in Theorem 8.12 of [**Nav98**], where $g \in G$ with $gN = \overline{g}$. Define $\mathcal{D}(g) = \mathfrak{Y}_{\overline{g}}(g)$ for every $g \in G$. Then $\mathcal{D}$ is a projective representation of G extending $\mathfrak{Y}$, by Lemma 8.27 of [**Nav98**]. Hence, it suffices to control the matrix entries of the representations $\mathfrak{Y}_{\overline{g}}$. In other words, we only need to check the case where θ extends to G.

Let $\chi \in \mathrm{Irr}(G)$ be an extension of θ. Let $\mathfrak{X}$ be a representation affording χ with entries in S_L (again such $\mathfrak{X}$ does exist by Problem 2.12 of [**Nav98**]). We have that $\mathfrak{X}_N$ affords θ. Hence, there is some $T \in \mathrm{GL}_n(L)$ such that $\mathfrak{X}_N = T^{-1}\mathfrak{Y}T$. Write $T = (t_{ij})$, where $t_{ij} \in L$. By Lemma 2.5 of [**Nav98**], since all t_{ij} are algebraic over $\mathbb{Q}$, there exists $\beta \in L$ such that all $\beta t_{ij} \in \mathbf{R}$ but not all $\beta t_{ij} \in \mathcal{M}$. Since $\mathfrak{X}_N = (\beta T)^{-1}\mathfrak{Y}(\beta T)$, we may assume that $t_{ij} \in \mathbf{R}$ and $T^* \neq 0$ (replacing T by βT). By assumption, the F-representations $(\mathfrak{X}_N)^*$ and $\mathfrak{Y}^*$ are irreducible. Moreover $T^*(\mathfrak{X}_N)^* = \mathfrak{Y}^*T^*$. By Schur's Lemma [**Isa76**, Lem. 1.5], this implies $T^* \in GL_n(F)$. In particular, we have that $\det(T^*) = \det(T)^* \neq 0$ and so $\det(T) \notin \mathcal{M}$. Thus $T \in \mathrm{GL}_n(S_L)$ and the representation $T\mathfrak{X}T^{-1}$ with entries in S_L extends $\mathfrak{Y}$. ☐

Theorem 4.39. *Let N be a p-solvable group and let m be a positive integer coprime to $|N|$. If $N \lhd G$, then there exists a fake m-th Galois action on* $\mathrm{IBr}(N)$ *with respect to G.*

Proof. By Theorem 4.30, we have that $B_{p'}(N)$ provides a canonical lift of $\mathrm{IBr}(N)$. Moreover, $\mathrm{Aut}(N)$ and $\mathrm{Gal}(\mathbb{Q}_{|N|}/\mathbb{Q})$ act on the set $B_{p'}(N)$ by Theorem 1.25. Thus, the bijection $B_{p'}(N) \to \mathrm{IBr}(N)$ given by $\theta \mapsto \theta^0$ commutes with the action of $\mathrm{Aut}(N)$. Consider σ_m as defined at the beginning of this section, before Definition 4.31. Let $\varphi \in \mathrm{IBr}(N)$ and $\theta \in B_{p'}(N)$ with $\varphi = \theta^0$. Since $\theta^{\sigma_m} \in B_{p'}(N)$, we have that $\varphi^{\sigma_m} = (\theta^{\sigma_m})^0 \in \mathrm{IBr}(N)$. Hence the map $f_m\colon \mathrm{IBr}(N) \to \mathrm{IBr}(N)$ defined by $\varphi \mapsto \varphi^{\sigma_m}$ is an $\mathrm{Aut}(N)$-equivariant bijection.

Let $\varphi \in \mathrm{IBr}(N)$. We want to prove that $(G_\varphi, N, \varphi)^{(m)} \approx (G_\varphi, N, \varphi^{\sigma_m})$. We may assume that $G = G_\varphi$. Let $\theta \in B_{p'}(N)$ be the canonical lift of φ. Then θ is G-invariant and θ^{σ_m} is the canonical lift of φ^{σ_m}. Write $L = \mathbb{Q}_{|G|}$. By Lemma 4.38, there exists a projective representation $\mathcal{D}$ of G associated to θ with matrix entries in $S_L = \{r/s \mid r \in \mathbf{R} \cap L, s \in \mathbf{R} \cap L\backslash\mathcal{M}\}$. In particular, the map $\mathcal{D}^{\sigma_m}$ is a well-defined projective representation of G associated to θ^{σ_m} with matrix entries in S_L. It is straightforward to check that the F-projective representations $\mathcal{P} = \mathcal{D}^*$ and $\mathcal{P}' = (\mathcal{D}^{\sigma_m})^*$ associated to φ and φ^{σ_m} satisfy the required properties of Definition 4.31. ☐

CHAPTER 5

Coprime action and Brauer characters

5.1. Introduction

Let A and G be finite groups. Assume that A acts coprimely on G (recall that this means that A acts by automorphism on G and $(|A|, |G|) = 1$). Then there exists a canonical bijection

$$\pi_{(G,A)} \colon \mathrm{Irr}_A(G) \to \mathrm{Irr}(\mathbf{C}_G(A)),$$

between the set $\mathrm{Irr}_A(G)$ of irreducible characters fixed under the action of A and the irreducible characters of the group $\mathbf{C}_G(A)$ of fixed points under the action of A. This bijection is known as the Glauberman-Isaacs correspondence. (We refer the reader to the discussion preceding Theorem 1.17 and to Theorem 1.17 itself.) The bare fact that these two sets have the same cardinality is highly non-trivial and has already important consequences. For instance, it proves that the actions of A on the irreducible characters of G and on the conjugacy classes of G are permutation isomorphic.

Let us fix a prime p. Recall that $g \in G$ is p-regular if p does not divide $o(g)$. In this chapter we consider the same question on p-Brauer characters and p-regular conjugacy classes.

Conjecture 5.1. *Let A and G be finite groups. Let p be a prime. Suppose that A acts coprimely on G. Then the actions of A on the irreducible Brauer characters of G and on the conjugacy classes of p-regular elements of G are permutation isomorphic.*

Conjecture 5.1 is an open problem posed by G. Navarro in [**Nav94**]. By Corollary 13.10 and Lemma 13.23 of [**Isa76**], it is easy to prove that Conjecture 5.1 is equivalent to the following.

Conjecture 5.2. *Let A and G be finite groups. Let p be a prime. Suppose that A acts coprimely on G. Then the number of A-invariant irreducible Brauer characters of G is equal to the number of irreducible Brauer characters of $\mathbf{C}_G(A)$.*

There is some evidence of the validity of Conjecture 5.2. First, it holds whenever G is p-solvable by work of K. Uno [**Uno83**] (fundamentally based on the fact that there exist canonical lifts of the irreducible Brauer characters). The fact that Conjecture 5.2 holds whenever A is a cyclic group, and hence also Conjecture 5.1, is a well-known consequence of the so-called Brauer's argument on the character table (see Lemma 1.16; the argument is the same in both the ordinary and the modular cases). In particular, this implies that Conjecture 5.1 holds whenever G is a quasi-simple group, using the classification of the finite simple groups (see Theorem 5.17). It has also been proven in [**NST16**] that A fixes a unique irreducible Brauer character of G if and only if $\mathbf{C}_G(A)$ is a p-group, proving Conjecture 5.2 in this case. However, if G is not p-solvable (and therefore A is solvable) no other progress has been made.

Our aim in this chapter is to reduce Conjecture 5.2 to a problem on finite simple groups (in the same spirit as for the McKay conjecture [**IMN07**]). We can prove the following, which is the main result of this chapter.

THEOREM G. *Let G and A be finite groups. Suppose that A acts coprimely on G. Suppose that all finite non-abelian simple groups (of order divisible by p) involved in G satisfy the inductive Brauer Glauberman condition. Then the number of irreducible p-Brauer characters of G fixed by A is the number of irreducible p-Brauer characters of $\mathbf{C}_G(A)$.*

Consequently, the actions of A on the irreducible p-Brauer characters and on the conjugacy classes of p-regular elements of G are permutation isomorphic.

Hence, we prove that if every non-abelian simple group satisfies the *inductive Brauer-Glauberman condition*, then Conjectures 5.1 and 5.2 hold. We do not define this inductive condition right now, but we feel it is worth mentioning that there is a surprising difference between this inductive condition and other inductive conditions coming from global/local conjectures (for instance the inductive McKay condition in [**IMN07**]). This difference is derived from the fact that the Galois group does not act on irreducible Brauer characters together with the fact that Galois action plays an important role in the description of the Glauberman correspondent in a key situation. We will require the existence of fake Galois actions in our inductive condition. See Definition 5.24 for further details.

The content of this chapter is arduous. We think that it is fair to say that our reduction of Conjecture 5.2 is at least as hard as the reduction of the McKay conjecture. Although both reductions bear similarities, there are also differences, as we said. It does not seem possible to conduct both reductions at the same time. Our reduction here is used in the forthcoming paper [**NST16**] in which more evidence for the truth of Conjectures 5.1 and 5.2 is given.

This chapter is structured in the following way: In Section 5.2 we recall some well-known results on character counts above Glauberman-Isaacs correspondents. In Section 5.3 we study a particular case of the Glauberman correspondence. This particular example motivates the definition of fake Galois conjugate modular character triples in Section 4.3. We also explain how to construct centrally isomorphic modular character triples from fake Galois conjugate ones. In Section 5.4 we study coprime actions on direct products of quasi-simple groups. In Section 5.5 we define the inductive Brauer-Glauberman condition on finite simple groups. Then, we assume that a finite non-abelian simple group S satisfies the inductive Brauer-Glauberman condition and we study consequences for the character theory of central extensions of direct products $S \times \cdots \times S$ of S. This section is of a highly technical nature and the results contained in it are key in the inductive process used to prove Theorem G. In Section 5.6, making use of everything proved in preceding sections, we can prove Theorem G. We conclude by studying some natural questions related to Conjecture 5.2 in Section 5.7.

All the results of this chapter are part of an original work of the author together with B. Späth [**SV16**].

5.2. Review on character counts above Glauberman-Isaacs correspondents

The main goal of this section is to count Brauer characters lying above characters of normal p'-subgroups and their Glauberman correspondents.

We shall use the following.

Lemma 5.3. *Assume that a group A acts on a group G. Let $K \lhd G$ be A-invariant. Assume that $(|G : K|, |A|) = 1$ and $\mathbf{C}_{G/K}(A) = G/K$. If $\eta \in \mathrm{IBr}_A(K)$, then every $\chi \in \mathrm{IBr}(G|\eta)$ is A-invariant.*

Proof. The proof follows from the same arguments as the in proof of Lemma 2.5 of [**Wol78a**]. □

Suppose that $K \lhd G$ and η is an irreducible G-invariant character of K. If η is an ordinary character, then $|\mathrm{Irr}(G|\eta)|$ can be determined by a purely group theoretical method due to P. X. Gallagher. There is an analogous result for Brauer characters. If η is a Brauer character we say that $Kg \in (G/K)^0$ (or g) is **η-good** if every extension $\varphi \in \mathrm{IBr}(\langle K, g\rangle)$ of η is U-invariant where $U/K = \mathbf{C}_{G/K}(Kg)$. Notice that η always extends to $\langle K, g\rangle$ since $\langle K, g\rangle/K$ is cyclic (by Theorem 4.10). Also, U acts on $\mathrm{IBr}(\langle K, g\rangle|\eta)$ since $\langle K, g\rangle \lhd U$. It is clear that if $Kg \in (G/K)^0$ is η-good, then every G-conjugate of Kg also is, so we can talk about p-regular η-good classes of G/K.

Theorem 5.4. *Suppose that $K \lhd G$ and that $\eta \in \mathrm{IBr}(K)$ is G-invariant. Then, $|\mathrm{IBr}(G|\eta)|$ is equal to the number of p-regular η-good classes of G/K.*

Proof. See Theorem 6.2 of [**Isa76**]. □

The following is basically a particular case of Theorem 2.12 of [**Wol79**].

Theorem 5.5. *Let A act coprimely on G. Suppose that $K \lhd G$ is A-invariant and $G = KC$, where $C = \mathbf{C}_G(A)$. Then $\eta \in \mathrm{Irr}_A(K)$ extends to G_η iff $\eta' \in \mathrm{Irr}(K \cap C)$ extends to $C_{\eta'}$, where $\eta' \in \mathrm{Irr}(\mathbf{C}_K(A))$ is the Glauberman-Isaacs correspondent of η.*

Proof. Notice that $G_\eta \cap C = C_{\eta'}$ by Lemma 2.5(b) of [**Wol79**]. Hence, we may assume that η and η' are C-invariant.

Suppose that η extends to some $\chi \in \mathrm{Irr}(G)$. By [**Wol78a**, Lem. 2.5] we see that $\chi \in \mathrm{Irr}_A(G)$. Since $[G, A] \leqslant K$, by [**Wol79**, Thm. 2.12]

$$\eta' = \pi_{(K,A)}(\eta) = \pi_{(K,A)}(\chi_K) = (\pi_{(G,A)}(\chi))_{K \cap C}.$$

Hence $\pi_{(G,A)}(\chi)$ is an extension of η'. Analogously we see that η extends to G, if η' extends to C. □

We need similar results for Brauer characters. We state the following easy observation as a lemma for the reader's convenience.

Lemma 5.6. *Let $K \lhd G$, $\theta \in \mathrm{Irr}(K)$ be G-invariant and suppose that K is a p'-group. Then θ extends to an ordinary character of G if and only if θ extends to a Brauer character of G.*

Proof. Let $\chi \in \mathrm{Irr}(G)$ be an extension of θ to G. Then χ^0 is a Brauer character of G extending θ since $K \subseteq G^0$. In particular, $\chi^0 \in \mathrm{IBr}(G)$. Suppose that $\varphi \in \mathrm{IBr}(G)$ extends θ. Let Q/N be a Sylow q-subgroup of G/N for some prime q. If $q \neq p$, then φ_Q is an ordinary character extending θ. If $q = p$, then θ extends to Q by Theorem 1.13. Hence θ extends to G by Theorem 1.15. □

Theorem 5.7. *Suppose that A acts coprimely on G. Let $K \lhd G$ be an A-invariant p'-group. Suppose that $G = KC$, where $C = \mathbf{C}_G(A)$. Let $\eta \in \mathrm{Irr}_A(K)$ be G-invariant and write $\eta' \in \mathrm{Irr}(K \cap C)$ to denote its Glauberman-Isaacs correspondent. Then*

$$|\mathrm{IBr}(G|\eta)| = |\mathrm{IBr}(C|\eta')|.$$

Proof. By Theorem 5.4 it suffices to show that for every $c \in C$, the element cK is η-good if and only if the element $cK \cap C$ is η'-good. By Theorem 4.7 of [**IN96**], it suffices to show that for every U with $K \leqslant U \leqslant G$ and abelian U/K, η extends to U as a Brauer character if and only if η' extends to $U \cap C$ as a Brauer character. We apply Theorem 5.5 and Lemma 5.6 in U. Then we are done. □

Corollary 5.8. *Suppose that A acts coprimely on G. Let $K \lhd G$ be an A-invariant p'-group. Suppose that $G = KC$, where $C = \mathbf{C}_G(A)$. Let $N \lhd G$ be contained in $K \cap C$ and let $\theta \in \mathrm{Irr}(N)$. Then*

$$|\mathrm{IBr}_A(G|\theta)| = |\mathrm{IBr}(C|\theta)|.$$

Proof. Let $\mathcal{B}$ be a set of representatives of the C-orbits of $\mathrm{Irr}_A(K|\theta)$ and $\mathcal{B}' = \{\pi_{(K,A)}(\eta) \mid \eta \in \mathcal{B}\}$. Then $\mathcal{B}' \subseteq \mathrm{Irr}(K \cap C|\theta)$ by [**Wol79**, Lem. 2.4] and $\mathcal{B}'$ is a set of representatives of the C-orbits of $\mathrm{Irr}(K \cap C|\theta)$. By Corollary 5.2 of [**Wol78a**], we deduce that the bijection $\pi_{(K,A)}$ is C-equivariant. Hence $\mathcal{B}'$ is a set of representatives of the C-orbits of $\mathrm{Irr}(K \cap C|\theta)$.

Every element of $\mathrm{IBr}_A(G|\theta)$ lies over a unique element of $\mathcal{B}$ and also every element of $\mathrm{IBr}(C|\theta)$ lies over a unique element of $\mathcal{B}'$. Thus

$$|\mathrm{IBr}_A(G|\theta)| = \sum_{\eta\in\mathcal{B}} |\mathrm{IBr}_A(G|\eta)| \quad \text{and} \quad |\mathrm{IBr}(C|\theta)| = \sum_{\eta\in\mathcal{B}} |\mathrm{IBr}(C|\pi_{(K,A)}(\eta))|.$$

By Lemma 5.3, for every $\eta \in \mathcal{B}$ we have that $|\mathrm{IBr}_A(G|\eta)| = |\mathrm{IBr}(G|\eta)|$. Hence, it suffices to show that $|\mathrm{IBr}(G|\eta)| = |\mathrm{IBr}(C|\pi_{(K,A)}(\eta))|$ for every $\eta \in \mathcal{B}$. By the Clifford correspondence on Brauer characters, we may assume that $\eta \in \mathcal{B}$ is G-invariant. Now, the result follows from Theorem 5.7. □

5.3. More on fake Galois conjugates

We begin this section with the description of the Glauberman correspondence in a very specific situation. This example shows that the Glauberman correspondence is related to a certain Galois action on ordinary characters.

Let G be a group and let $A \leqslant \mathfrak{S}_m$. Recall G^m denotes the external direct product of m copies of G. Write $\widetilde{G} = G^m$. Then A acts on $\widetilde{G}$ by $(g_1, \ldots, g_m)^{a^{-1}} = (g_{a(1)}, \ldots, g_{a(m)})$ for $g_i \in G$ and $a \in A$. If A is transitive, then

$$\mathbf{C}_{\widetilde{G}}(A) = \{(g, \ldots, g) \in \widetilde{G} \mid g \in G\}.$$

The group $\mathbf{C}_{\widetilde{G}}(A)$ is isomorphic to G. An irreducible A-invariant character χ of $\widetilde{G}$ has the form

$$\chi = \theta \times \cdots \times \theta$$

for some $\theta \in \mathrm{Irr}(G)$. If $(|A|, |G|) = 1$, then the Glauberman correspondent of an A-invariant character $\chi = \theta \times \cdots \times \theta$ of $\widetilde{G}$ is some Galois conjugate of θ viewed as a character of $\mathbf{C}_{\widetilde{G}}(A)$. (See Proposition 5.10 below.)

Notation 5.9. Let m be a positive integer. Recall the definition of σ_m from Section 4.3: let π be the set of primes dividing m, then for every positive integer n we define $\sigma_m \in \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ to be the automorphism fixing π-roots of unity and raising to the m-th power π'-roots of unity. For a group G, we denote here by $\widetilde{G}$ the external direct product G^m of m copies of G. Let $\Delta^m\colon G \to \widetilde{G}$ be the injective morphism defined by $g \mapsto (g, \ldots, g)$ for every $g \in G$. Then Δ^m defines natural bijections $\mathrm{Irr}(G) \to \mathrm{Irr}(\Delta^m G)$ and $\mathrm{IBr}(G) \to \mathrm{IBr}(\Delta^m G)$, where $\Delta^m G = \Delta^m(G)$. We also write $\Delta^m\theta = \Delta^m(\theta)$. If from the context m is clear, we omit the superscript m and write Δ, ΔG and $\Delta\theta$ instead of Δ^m, $\Delta^m G$ and $\Delta^m\theta$.

Proposition 5.10. *Let m be a positive integer. Assume that a solvable subgroup $A \leqslant \mathfrak{S}_m$ is transitive. Let G be a finite group with $(|A|, |G|) = 1$. Let $\widetilde{G}$ be the direct product of m copies of G. Let $\theta \in \mathrm{Irr}(G)$ and let $\chi = \theta \times \cdots \times \theta \in \mathrm{Irr}_A(\widetilde{G})$. Then, the Glauberman correspondent of χ is the character $\chi' = \Delta\theta^{\sigma_m}$, where θ^{σ_m} is the image of θ under the Galois automorphism σ_m.*

Proof. This is essentially the content of Exercise 13.11 of [**Isa76**]. To do the case where A is cyclic of prime order, use Exercise 4.7 of [**Isa76**]. The general case, follows by induction on $|A|$. □

It is clear that the Clifford theory over two ordinary irreducible Galois conjugate characters is related.

Proposition 5.11. *Let $N \lhd G$ and $\theta \in \mathrm{Irr}(N)$. Let m be a positive integer coprime to $|N|$. Let $\theta' = \theta^{\sigma_m}$. Then:*

(a) $G_\theta = G_{\theta'}$.

(b) Assume $G = G_\theta$. There exist $\mathcal{P}$ and $\mathcal{P}'$ projective representations of G associated to θ and θ' such that:

(b.1) the factor sets α and α' of $\mathcal{P}$ and $\mathcal{P}'$ satisfy

$$\alpha(g, g')^{\sigma_m} = \alpha'(g, g')$$

for every $g, g' \in G$, and

(b.2) for every $c \in \mathbf{C}_G(N)$ the scalar matrices $\mathcal{P}(c)$ and $\mathcal{P}'(c)$ are associated with ξ and ξ^{σ_m} for some root of unity $\xi \in \mathbb{Q}_{|G|}$.

Proof. By Theorem 4.29 and [**Isa76**, Thm. 10.3], there exists a projective representation $\mathcal{P}$ of G associated to θ whose entries are in $\mathbb{Q}_k$ for some $k \geqslant 1$. Choose $\mathcal{P}' = \mathcal{P}^{\sigma_m}$. The result follows from straightforward calculations. □

It is worth comparing Proposition 5.11 with Definition 4.31. If (G, N, φ) and (G, N, φ') are fake m-th Galois conjugate modular character triples, then the Clifford theory over φ and φ' is related in the same way as the Clifford theory over two σ_m-conjugate ordinary character triples.

We recall below the definition of fake Galois conjugate characters in the alternative version provided by Lemma 4.35.

Definition 5.12. Let (G, N, φ) and (G, N, φ') be modular character triples, and let m be a positive integer coprime to $|N|$. Then we say that φ and φ' are fake m-th Galois conjugate with respect to $N \lhd G$, and we write $(G, N, \varphi)^{(m)} \approx (G, N, \varphi')$, if there exist projective representations $\mathcal{P}$ and $\mathcal{P}'$ of G associated to φ and φ' with factor sets α and α' such that

(i) $\alpha(g, g'), \alpha'(g, g') \in F^\times$ have order coprime to m and

$$\alpha(g, g')^m = \alpha'(g, g')$$

for every $g, g' \in G$, and

(ii) for every $c \in \mathbf{C}_G(N)$, the scalar matrices $\mathcal{P}(c)$ and $\mathcal{P}'(c)$ are associated with elements $\zeta, \zeta' \in F^\times$ of order coprime to m and $\zeta^m = \zeta'$.

The notion of fake Galois conjugate (modular) character triples is important in our later application, since it allows us to construct centrally isomorphic character triples from Galois conjugate character triples (see Theorem 5.14 below).

Notation 5.13. Let m be a positive integer. For groups $H \leqslant G$ we continue writing H^m to denote the external direct product of m copies of H. Recall that $\Delta H \leqslant G^m$. For groups $K, H \leqslant G$ with $K \leqslant \mathbf{N}_G(H)$, we denote by $\check{\Delta}_H K$ the group $H^m \Delta K \leqslant G^m$.

Recall the description of the Brauer characters of a central product of groups in Lemma 4.27.

Theorem 5.14. *Let (G, N, φ) and (G, N, φ') be modular character triples. Let $Z = \mathbf{Z}(N)$. Write $\widetilde{Z} = Z^m$, $\widetilde{N} = N^m$, $\check{G} = \check{\Delta}_Z G$ and $\check{N} = \check{\Delta}_Z N$. Let $\nu \in \mathrm{IBr}(\varphi_Z)$, $\widetilde{\varphi} = \varphi \times \cdots \times \varphi \in \mathrm{IBr}(\widetilde{N})$, $\widetilde{\nu} = \nu \times \cdots \times \nu \in \mathrm{IBr}(\widetilde{Z})$ and $\check{\varphi} = \Delta\varphi' \cdot \widetilde{\nu} \in \mathrm{IBr}(\check{N})$. Then the following are equivalent:*

(a) $(G, N, \varphi)^{(m)} \approx (G, N, \varphi')$,
(b) $(\widetilde{N}(\check{G} \rtimes \mathfrak{S}_m), \widetilde{N}, \widetilde{\varphi}) \succ_{Br,c} (\check{G} \rtimes \mathfrak{S}_m, \check{N}, \check{\varphi})$.

Proof. Notice that

$$\widetilde{N}(\check{G} \rtimes \mathfrak{S}_m) = (\widetilde{N}\check{G}) \rtimes \mathfrak{S}_m.$$

Also $\widetilde{N} \cap (\check{G} \rtimes \mathfrak{S}_m) = \check{N}$ is the central product of $\widetilde{Z}$ and ΔN. Hence, the characters in $\mathrm{IBr}(\check{N})$ are dot products of characters in $\mathrm{IBr}(\widetilde{Z})$ and characters in $\mathrm{IBr}(\Delta N)$ that lie over the same $\lambda \in \mathrm{IBr}(\Delta Z)$ (see Theorem 4.27). Note that $\check{\varphi} = \Delta\varphi' \cdot \widetilde{\nu}$ is well-defined and lies in $\mathrm{IBr}(\check{N})$ since $\mathrm{IBr}(\widetilde{\nu}_{\Delta Z}) = \mathrm{IBr}((\Delta\varphi')_{\Delta Z})$

Now, we prove that (a) implies (b). Let $\mathcal{Q}$ and $\mathcal{Q}'$ be projective representations of G giving

$$(G, N, \varphi)^{(m)} \approx (G, N, \varphi')$$

as in Definition 5.12.

We construct projective representations $\mathcal{P}_0$ and $\mathcal{P}'_0$ of $\widetilde{N}\check{G}$ and $\check{G}$ associated to $\widetilde{\varphi}$ and $\check{\varphi}$ respectively. Note that $\widetilde{N}\check{G} = \check{\Delta}_N G$. It is straightforward to show that the map $\mathcal{P}_0 \colon \widetilde{N}\check{G} \to \mathrm{GL}_{\varphi(1)^m}(F)$ given by

$$\mathcal{P}_0((n_1, \ldots, n_m)\Delta g) = \mathcal{Q}(n_1 g) \otimes \cdots \otimes \mathcal{Q}(n_m g)$$

for every $(n_1, \ldots, n_m) \in \widetilde{N}$ and $g \in G$, defines a projective representation associated to $\widetilde{\varphi}$. The factor set α_0 of $\mathcal{P}_0$ satisfies

$$\alpha_0(\widetilde{n}\Delta g, \widetilde{n}'\Delta g') = \beta(g, g')^m \text{ for every } \widetilde{n}, \widetilde{n}' \in \widetilde{N} \text{ and } g, g' \in G,$$

where β denotes the factor set of $\mathcal{Q}$. Let τ be an F-representation of $\widetilde{Z}$ affording $\widetilde{\nu}$. The map $\mathcal{P}'_0 \colon \check{G} \to \mathrm{GL}_{\check{\varphi}(1)}(F)$ given by

$$\mathcal{P}'_0(\widetilde{z}\Delta g) = \tau(\widetilde{z})\mathcal{Q}'(g) \text{ for every } \widetilde{z} \in \widetilde{Z} \text{ and } g \in G$$

is a projective representation associated to $\check{\varphi}$. The factor set α'_0 of $\mathcal{P}'_0$ satisfies

$$\alpha'_0(\widetilde{z}\Delta g, \widetilde{z}'\Delta g') = \beta'(g, g') = \beta(g, g')^m = \alpha_0(\widetilde{z}\Delta g, \widetilde{z}'\Delta g')$$

for every $g, g' \in G$ and $z, z' \in \widetilde{Z}$, where β' denotes the factor set of $\mathcal{Q}'$. (Recall that $\beta'(g, g') = \beta(g, g')^m$ since $\mathcal{Q}$ and $\mathcal{Q}'$ satisfy Definition 5.12(b.1)).

In the next step we extend $\mathcal{P}_0$ to a projective representations $\mathcal{P}$ of $(\widetilde{N}\breve{G}) \rtimes \mathfrak{S}_m$. Note that $\mathfrak{S}_m$ has a natural action on the tensor space $\bigotimes F^{\varphi(1)}$ by permuting the tensors. This induces a representation $\mathcal{R}\colon \mathfrak{S}_m \to \mathrm{GL}_{\varphi(1)^m}(F)$. The map $\mathcal{P}\colon \breve{\Delta}_N G \rtimes \mathfrak{S}_m \to \mathrm{GL}_{\varphi(1)^m}(F)$ given by

$$\mathcal{P}(x\sigma) = \mathcal{P}_0(x)\mathcal{R}(\sigma) \text{ for every } x \in \widetilde{N}\breve{G} \text{ and } \sigma \in \mathfrak{S}_m$$

is a projective representation of $\widetilde{N}\breve{G} \rtimes \mathfrak{S}_m = \widetilde{N}(\breve{G} \rtimes \mathfrak{S}_m)$. Note that $\mathcal{P}_{N\wr\mathfrak{S}_m}$ is a representation, as defined in [**Hup98**, Thm. 25.6]. It is easy to check, using the definition of $\mathcal{R}$, that the factor set α of $\mathcal{P}$ satisfies

$$\alpha(\widetilde{n}\Delta g\sigma, \widetilde{n}'\Delta g'\sigma') = \alpha_0(\widetilde{n}\Delta g, \widetilde{n}'\Delta g')$$

for every $g, g' \in G$, $\widetilde{n}, \widetilde{n}' \in \widetilde{N}$ and $\sigma, \sigma' \in \mathfrak{S}_m$.

In the next step we extend $\mathcal{P}_0'$ to a projective representation $\mathcal{P}'$ of $\breve{G}\rtimes\mathfrak{S}_m$. Note that $[\Delta G, \mathfrak{S}_m] = 1$. Hence $\widetilde{\nu}$ and τ are $\mathfrak{S}_m$-invariant and the map $\mathcal{P}'\colon \breve{G} \rtimes \mathfrak{S}_m \to \mathrm{GL}_{\varphi'(1)}(F)$ defined by $\mathcal{P}'(g\widetilde{z}\sigma) = \mathcal{P}_0'(g)\tau(\widetilde{z})$ for every $g \in \breve{G}$, $\widetilde{z} \in \widetilde{Z}$ and $\sigma \in \mathfrak{S}_m$ is a projective representation whose factor set α' satisfies

$$\alpha'(g\sigma, g'\sigma') = \alpha_0'(g, g')$$

for every $g, g' \in \breve{G}$ and $\sigma, \sigma' \in \mathfrak{S}_m$. In particular, we see that the projective representations $\mathcal{P}$ and $\mathcal{P}'$ that we have constructed satisfy the property described in Definition 4.19(ii.1).

In the last step we compare $\mathcal{P}(x)$ and $\mathcal{P}'(x)$ for $x \in \mathbf{C}_{(\widetilde{N}\breve{G})\rtimes\mathfrak{S}_m}(\widetilde{N})$. We have that

$$\mathbf{C}_{(\widetilde{N}\breve{G})\rtimes\mathfrak{S}_m}(\widetilde{N}) = \breve{\Delta}_Z\mathbf{C}_G(N) \leqslant \breve{G} \rtimes \mathfrak{S}_m.$$

Then $x = \widetilde{z}\Delta c$ for some $\widetilde{z} \in \widetilde{Z}$ and $c \in \mathbf{C}_G(N)$. Recall that by Definition 5.12(b.2), $\mathcal{Q}(c)$ and $\mathcal{Q}'(c)$ are scalar matrices associated with some ζ and ζ^m. Thus $\mathcal{P}(\widetilde{z}\Delta c)$ and $\mathcal{P}'(\widetilde{z}\Delta c)$ are scalar matrices associated with $\tau(\widetilde{z})\zeta^m$, by the definition of $\mathcal{P}$ and $\mathcal{P}'$. This implies that $\mathcal{P}$ and $\mathcal{P}'$ satisfy the property in Definition 4.19(ii.2).

This proves (a) implies (b), since we have already seen that the group theory conditions of Definition 4.19 are satisfied.

We only sketch the steps to prove that (b) implies (a). We start by choosing a projective representation $\mathcal{Q}$ of G associated to φ as in Lemma 4.34. Then one can construct a projective representation $\mathcal{P}$ of $(\widetilde{N}\breve{G}) \rtimes \mathfrak{S}_m$ associated to $\widetilde{\varphi}$ as in the first part of the proof. Let $\mathcal{P}'$ be the projective representation of $\breve{G} \rtimes \mathfrak{S}_m$ associated to $\breve{\varphi}$ given by Lemma 4.21(a). Then $\mathcal{P}'|_{\Delta G}$ defines via the natural isomorphism $\Delta G \to G$ a projective representation $\mathcal{Q}'$ of G associated to φ', because $\mathcal{P}'|_{\Delta N}$ affords $\Delta\varphi'$. It is easy to check that $\mathcal{Q}$ and $\mathcal{Q}'$ give $(G, N, \varphi)^{(m)} \approx (G, N, \varphi')$ using Lemma 4.35. □

Corollary 5.15. *Assume the notation and the situation of Theorem 5.14. Let $H \lhd G$ with $H \leqslant \mathbf{C}_G(N)$ and write $\breve{G}_1 = \breve{\Delta}_{ZH}G$. Then*

$$(\widetilde{N}(\breve{G}_1 \rtimes \mathfrak{S}_m), \widetilde{N}, \widetilde{\varphi}) \succ_{Br,c} (\breve{G}_1 \rtimes \mathfrak{S}_m, \breve{N}, \breve{\varphi}).$$

Proof. Note that $\widetilde{N}\check{G}_1 = \check{\Delta}_{NH}G$ and $\check{G}_1 = \check{\Delta}_{HZ}G$ are well-defined. The group $\check{\Delta}_{NH}G \rtimes \mathfrak{S}_m$ induces on $\widetilde{N}$ the same automorphisms as $\check{\Delta}_N G \rtimes \mathfrak{S}_m$. Hence, the statement follows from Theorem 5.14 after applying Theorem 4.26. ☐

Also as a consequence of Theorem 5.14, we obtain the analogue of Theorem 4.26 for fake Galois conjugate modular character triples: $(G, N, \varphi)^{(m)} \approx (G, N, \varphi')$ is a property that only depends on the characters φ and φ' as well as the automorphisms induced by G on N.

Corollary 5.16. *Let (G, N, φ) and (G, N, φ') be modular character triples. Suppose that for a positive integer m coprime to $|N|$,*

$$(G, N, \varphi)^{(m)} \approx (G, N, \varphi').$$

Let G_1 be a group such that $N \lhd G_1$ and $G_1/\mathbf{C}_{G_1}(N)$ is equal to $G/\mathbf{C}_G(N)$ as a subgroup of $\mathrm{Aut}(N)$. *Then*

$$(G_1, N, \varphi)^{(m)} \approx (G_1, N, \varphi').$$

Proof. This follows from combining Theorem 5.14 and Theorem 4.26. ☐

5.4. Coprime action on simple groups and their direct products

In this section we study the situation where a group A acts coprimely on the direct product of isomorphic non-abelian simple groups (as well as on the direct product of their universal covering groups). If a group A acts on a set Λ, then we write A_{Λ_0} to denote the stabilizer of $\Lambda_0 \subseteq \Lambda$ in A, so that

$$A_{\Lambda_0} = \{a \in A \mid \Lambda_0^a = \Lambda_0\}.$$

We begin by studying coprime actions on finite simple non-abelian groups. Let S be a non-abelian simple group and let X be its universal covering group (unique up to isomorphism). We identify $\mathrm{Aut}(S)$ and $\mathrm{Aut}(X)$ as in [**Asc00**, Ex. 6, Chapt. 11].

Theorem 5.17. *Let S be a simple non-abelian group. Let B act on S faithfully with $(|S|, |B|) = 1$. Then B is cyclic, $\mathbf{N}_{\mathrm{Aut}(S)}(B) = \mathbf{C}_{\mathrm{Aut}(S)}(B)$ and*

$$\mathbf{Z}(\mathbf{C}_X(B)) = \mathbf{C}_X(B) \cap \mathbf{Z}(X), \tag{5.1}$$

where X is the universal covering group of S and B acts on X via the canonical identification $\mathrm{Aut}(S) = \mathrm{Aut}(X)$.

Proof. We identify B with the corresponding subgroup of $\mathrm{Aut}(S)$. We may assume $B \neq 1$, otherwise the result is trivial. According to the classification of finite simple groups the group S has to be a simple group of Lie type and B is $\mathrm{Aut}(S)$-conjugate to some group of field automorphisms of S, see for example Section 2 of [**MNS15**]. In particular, B is cyclic. The structure of $\mathrm{Aut}(S)$ is described in Theorem 2.5.12 of [**GLS03**]. Straightforward computations with $\mathrm{Aut}(S)$ prove that $\mathbf{N}_{\mathrm{Aut}(S)}(B) = \mathbf{C}_{\mathrm{Aut}(S)}(B)$.

Let X be the universal covering group of S. In order to prove that $\mathbf{Z}(\mathbf{C}_X(B)) = \mathbf{C}_X(B) \cap \mathbf{Z}(X)$ we may assume that $B \neq 1$. By the first paragraph of this proof S is a simple group of Lie type. Arguing as in the beginning of Section 2 of [**MNS15**] we see that the Schur multiplier of S is generic or $S = {}^2\mathrm{B}_2(8)$ and B is a cyclic group of order 3.

Let us consider first the case where $S = {}^2\mathrm{B}_2(8)$ and that B is a cyclic group of order 3. Then $X = 2^2.{}^2\mathrm{B}_2(8)$ and $\mathbf{C}_X(B) = {}^2\mathrm{B}_2(2)$. The group ${}^2\mathrm{B}_2(2)$ is a Frobenius group of order $5 \cdot 4$ and has trivial centre. It follows that $\mathbf{Z}(X) \cap \mathbf{C}_X(B)$ is also trivial.

Hence we can assume that $X = \mathbf{X}^F$, for some simply-connected simple algebraic group $\mathbf{X}$ and some Steinberg endomorphism $F\colon \mathbf{X} \to \mathbf{X}$. We can assume that B is generated by some automorphism that is induced by some Steinberg endomorphism $F_0\colon \mathbf{X} \to \mathbf{X}$. Without loss of generality we can assume that some power of F_0 coincides with F. From Theorem 24.15 of [**MT11**] we deduce that

$$\begin{aligned}\mathbf{Z}(\mathbf{C}_X(B)) &= \mathbf{Z}(\mathbf{X}^{F_0}) = \mathbf{Z}(\mathbf{X})^{F_0}(\mathbf{Z}(\mathbf{X})^F)^{F_0}\\ &= (\mathbf{Z}(\mathbf{X}^F))^{F_0} = \mathbf{Z}(X) \cap \mathbf{C}_X(B).\end{aligned}$$

This is exactly Equation 5.1 □

Notation 5.18. Let S be a non-abelian finite simple group. Let X be the universal covering group of S. Let r be a positive integer. Recall from previous sections that X^r denotes the external direct product of r copies of X. We write $\widetilde{X} = X^r$. Note that $\widetilde{X}$ is also the internal direct product of $X_1, \dots, X_r$, where the group X_i is defined by

$$X_i = 1 \times \cdots \times X \times \cdots \times 1$$

with an X at the i-th position, for each $i \in \{1, \dots, r\}$.

Recall we identify $\mathrm{Aut}(S)$ and $\mathrm{Aut}(X)$. Also $\mathrm{Aut}(X) \wr \mathfrak{S}_r$ acts on $\widetilde{X}$ via

$$(x_1, \dots, x_r)^{(\alpha_1, \dots, \alpha_r)\sigma} = ((x_{\sigma^{-1}(1)})^{\alpha_1}, \dots, (x_{\sigma^{-1}(r)})^{\alpha_r}),$$

for every $x_i \in X$, $\alpha_i \in \mathrm{Aut}(X)$ and $\sigma \in \mathfrak{S}_r$. It is easy to show that this defines an isomorphism between $\mathrm{Aut}(X) \wr \mathfrak{S}_r$ and $\mathrm{Aut}(\widetilde{X})$. Hence we can identify $\mathrm{Aut}(S) \wr \mathfrak{S}_r$ and $\mathrm{Aut}(\widetilde{X})$.

Finally, for each $i = 1, \dots, r$, the natural isomorphism $\mathrm{pr}_i\colon X_i \to X$ induces the epimorphism

$$\overline{\mathrm{pr}}_i\colon \mathrm{Aut}(\widetilde{X})_{X_i} \to \mathrm{Aut}(X) \text{ given by } \alpha \mapsto \mathrm{pr}_i^{-1} \circ \alpha|_{X_i} \circ \mathrm{pr}_i,$$

where $\circ$ denotes the usual composition of maps.

Suppose $A \leqslant \mathrm{Aut}(\widetilde{X})$ with $(|X|, |A|) = 1$. Let $\widetilde{\Gamma} \leqslant \mathrm{Aut}(\widetilde{X})$ be such that $\widetilde{\Gamma} \leqslant \mathbf{C}_{\mathrm{Aut}(\widetilde{X})}(A)$. Suppose further that $\widetilde{\Gamma}A$ acts transitively on $\{X_1, \dots, X_r\}$.

Our objective is to control the structure of the groups A and $\widetilde{\Gamma}A$. We need two preliminary results.

Lemma 5.19. *Assume the notation in Notation 5.18. Write $B_i = \overline{\mathrm{pr}}_i(A_{X_i})$ for each $i \in \{1, \ldots, r\}$.*

(a) The groups B_i are cyclic.
(b) If $\widetilde{\Gamma}A$ acts transitively on $\{X_1, \ldots, X_r\}$, then B_i and B_1 are $\mathrm{Aut}(X)$-conjugate and the A-orbits on the set $\{X_1, \ldots, X_r\}$ all have the same length.

Proof. Notice that B_i acts on X with $(|X|, |B_i|) = 1$. By Theorem 5.17, we have that B_i is cyclic. This proves part (a).

Now, since $\widetilde{\Gamma}A$ acts transitively on $\{X_1, \ldots, X_r\}$, there is some $\alpha_i \in \widetilde{\Gamma}A$ such that $X_1^{\alpha_i} = X_i$. It is easy to check that $B_i = \overline{\mathrm{pr}}(A_{X_i}) = \overline{\mathrm{pr}}((A_{X_1})^{\alpha_i}) = B_1^{\beta_i}$, where $\beta_i \in \mathrm{Aut}(X)$ is given by $\beta_i \circ \mathrm{pr}_i = \mathrm{pr}_1 \circ \alpha_i^{-1}|_{X_i}$.

Furthermore $\widetilde{\Gamma}$ permutes transitively the A-orbits on $\{X_1, \ldots, X_r\}$. Hence these A-orbits all have the same length. This concludes the proof of part (b). □

According to the classification of finite simple groups Schreier's conjecture holds, i.e., the outer automorphisms group $\mathrm{Out}(S)$ of every simple non-abelian group S is solvable. In particular, if X is the universal covering group of the non-abelian simple group S and π is the set of primes dividing $|X|$, then $\mathrm{Aut}(X)$ is π-separable, and hence there are Hall π'-subgroups in $\mathrm{Aut}(X)$.

Using this fact one can determine a *convenient* group containing A.

Proposition 5.20. *Assume the notation in Notation 5.18. Write $B_i = \overline{\mathrm{pr}}_i(A_{X_i})$ for each $i \in \{1, \ldots, r\}$. Suppose that $\widetilde{\Gamma}A$ acts transitively on $\{X_1, \ldots, X_r\}$. Let π be the set of prime divisors of $|X|$. Let H be a Hall π'-subgroup of $\mathrm{Aut}(X)$. Then A is $\mathrm{Aut}(X)^r$-conjugate to a subgroup of $H \wr \mathfrak{S}_r$. Also, $B_i = B_1$ for all $i = 1, \ldots, r$.*

Proof. By the discussion preceding the statement of this proposition $\mathrm{Aut}(X)$ is π-separable. Hence $\mathrm{Aut}(X)^r A$ is also π-separable. Let $K = \mathrm{Aut}(X)^r A \cap \mathfrak{S}_r$. Notice that K is a π'-subgroup of $\mathfrak{S}_r$. Moreover, $H^r \rtimes \mathfrak{S}_r$ is a Hall π'-subgroup of $\mathrm{Aut}(X)^r A$. Since A is a π'-subgroup of $\mathrm{Aut}(X)^r A$, there exist $a \in A$ and $\alpha \in \mathrm{Aut}(X)^r$, such that

$$A^{a\alpha} = A^{\alpha} \leqslant H^r \rtimes K \leqslant H \wr \mathfrak{S}_r.$$

For the latter part we may assume $A \leqslant H \wr \mathfrak{S}_r$. Now, $B_1, B_i \leqslant H$ are $\mathrm{Aut}(X)$-conjugate by Lemma 5.19(b). In particular $|B_i| = |B_1|$. Since H is cyclic, by Theorem 5.17, this implies $B_i = B_1$. □

The next two propositions describe the structure of A and $\widetilde{\Gamma}A$ in the case where A acts transitively on $\{X_1, \ldots, X_r\}$.

Proposition 5.21. *Assume the notation in Notation 5.18. Suppose that* $A \leqslant H \wr \mathfrak{S}_r$. *Write* $B = \overline{\mathrm{pr}}(A_{X_1})$. *Then* A *is* H^r*-conjugate to a subgroup of* $B \wr \mathfrak{S}_r$.

Proof. It is enough to prove the statement in the case where A acts transitively on $\{X_1, \cdots, X_r\}$ by working on A-orbits.

If $A \leqslant H \wr \mathfrak{S}_r$ acts transitively on $\{X_1, \ldots, X_r\}$, then for every $i \in \{1, \ldots, r\}$, there exist $a_i \in A$ such that

$$X_1^{a_i} = X_i,$$

and $h_i \in H$ such that for every $x \in X$

$$(x, 1, \ldots, 1)^{a_i} = (1, \ldots, x^{h_i}, \ldots, 1) \in X_i.$$

Let $h = (1_H, h_2^{-1}, \ldots, h_r^{-1}) \in H^r$. We claim that $A^h \leqslant B \wr \mathfrak{S}_r$. First notice that $\overline{\mathrm{pr}}_1((A^h)_{X_1}) = \overline{\mathrm{pr}}_1((A^h_{X_1})) = B$. Now, write $\widehat{a}_i = h^{-1}a_ih \in A^h \leqslant H \wr \mathfrak{S}_r$ for each $i \in \{1, \ldots, r\}$. Then

$$(x, 1, \ldots, 1)^{\widehat{a}_i} = (1, \ldots, x, \ldots, 1) \in X_i$$

for every $x \in X$.

Let $y \in A^h$. Then $y = (y_1, \ldots, y_r)\rho \in H \wr \mathfrak{S}_r$. Let $i \in \{1, \ldots, r\}$. Write $j = \rho^{-1}(i)$, so that

$$X_i^y = X_j.$$

Since

$$(x, 1 \ldots, 1)^{\widehat{a}_i y \widehat{a}_j^{-1}} = (x^{y_i}, 1 \ldots, 1) \in X_1,$$

for every $x \in X$, we have that $\widehat{a}_i y \widehat{a}_j^{-1} \in (A^h)_{X_1}$. Consequently $y_j \in B$. This argument applies for every $i \in \{1, \ldots, r\}$, hence the claim follows. □

Recall the notation from the previous section: For $H, K \leqslant G$, we denote by H^m the direct product of m copies of H and we write $\Delta K \leqslant G^m$ for the diagonally embedded group K. We denote by $\check{\Delta}_H K \leqslant G^m$ the product of H^m and ΔK, whenever $K \leqslant \mathbf{N}_G(H)$. If we want to emphasize that $\check{\Delta}_H K = H^m \Delta K$ is constructed in G^m we write $\check{\Delta}_H^m K$.

Proposition 5.22. *Assume the notation in Proposition 5.20. Let* $B = B_1$ *and let* $\Gamma = \mathbf{C}_{\mathrm{Aut}(X)}(B)$. *Suppose that* $A \leqslant B \wr \mathfrak{S}_r$ *and that* A *acts transitively on* $\{X_1, \ldots, X_r\}$. *Then*

$$\widetilde{\Gamma} A \leqslant (\check{\Delta}_B \Gamma) \rtimes \mathfrak{S}_r.$$

Proof. Let $c \in \widetilde{\Gamma}$. Then $c = (c_1, \ldots, c_r)\rho$ with $c_i \in \mathrm{Aut}(X)$ and $\rho \in \mathfrak{S}_r$. Let $a \in A_{X_1}$. Then $a = (b_1, \ldots, b_r)\sigma$ with $b_i \in B$ and $\sigma \in \mathfrak{S}_r$. The equation $ac = ca$ implies that

$$b_1 = c_1^{-1} b_{\rho^{-1}(1)} c_1 \in B.$$

This holds for every $a \in A_{X_1}$. Hence $c_1 \in \mathbf{N}_{\mathrm{Aut}(X)}(B)$. By Theorem 5.17, we have that $\mathbf{N}_{\mathrm{Aut}(X)}(B) = \mathbf{C}_{\mathrm{Aut}(X)}(B) = \Gamma$. Proceeding like this for elements $a \in A_{X_i}$, we conclude $c_i \in \Gamma$ for every $i \in \{1, \ldots, r\}$. Hence $\widetilde{\Gamma} \leqslant \Gamma \wr \mathfrak{S}_r$.

Now, for $a \in A$ with $a = (b_1, \ldots, b_r)\sigma \in B \wr \mathfrak{S}_r$ and $c \in \widetilde{\Gamma}$ with $c = (c_1, \ldots, c_r)\rho \in \Gamma \wr \mathfrak{S}_r$ the equation $ac = ca$ implies that

$$b_i c_{\sigma^{-1}(i)} = c_i b_{\rho^{-1}(i)}.$$

Hence $c_{\sigma^{-1}(1)} c_i^{-1} \in B$ for every $i \in \{1, \ldots, r\}$. Since A acts transitively on $\{X_1, \ldots, X_r\}$, we have that

$$c_j c_1^{-1} \in B,$$

for every $j \in \{1, \ldots, r\}$. This proves that $(c_1, \ldots, c_r) \in B^r \Delta\Gamma = \check{\Delta}_B \Gamma$. This proves that $\widetilde{\Gamma} \leqslant (\check{\Delta}_B \Gamma) \rtimes \mathfrak{S}_r$. □

We finally consider the case where $\widetilde{\Gamma} A$ acts transitively on $\{X_1, \ldots, X_r\}$.

Proposition 5.23. *Assume the notation in Proposition 5.20. Let $B = B_1$ and let $\Gamma = \mathbf{C}_{\mathrm{Aut}(X)}(B)$. Suppose that $A \leqslant B \wr \mathfrak{S}_r$ and that $\widetilde{\Gamma} A$ acts transitively on $\{X_1, \ldots, X_r\}$. Let m be the length of an A-orbit in $\{X_1, \ldots, X_r\}$. Then for some $\tau \in \mathfrak{S}_r$ we have that*

$$A^\tau \leqslant (B \wr \mathfrak{S}_m)^{r/m} \quad \textit{and} \quad (\widetilde{\Gamma} A)^\tau \leqslant ((\check{\Delta}_B^m \Gamma) \rtimes \mathfrak{S}_m) \wr \mathfrak{S}_{\frac{r}{m}}.$$

Proof. By Proposition 5.22 the statement holds when $m = r$.

Let $d = r/m$. We may assume that the A-orbits on $\{X_1, \ldots, X_r\}$ are exactly $\{X_1, \ldots, X_m\}, \ldots, \{X_{(d-1)m+1}, \ldots, X_{dm}\}$ after conjugating $\widetilde{\Gamma} A$ by some $\tau \in \mathfrak{S}_r$. (Notice that A^τ and $\widetilde{\Gamma}^\tau$ satisfy the same hypotheses as A and $\widetilde{\Gamma}$.) This proves the first statement.

Let $a = (b_1, \ldots, b_r)\sigma \in A^\tau \leqslant (B \wr \mathfrak{S}_m)^d$ and $c = (c_1, \ldots, c_r)\rho \in \widetilde{\Gamma}^\tau$. Note that $\sigma \in (\mathfrak{S}_m)^d$, hence we can write $\sigma = \sigma_1 \cdots \sigma_d$ where $\sigma_l \in \mathfrak{S}_m$ permutes the set $\{(l-1)m + 1, \ldots, lm\}$. The equation $ac = ca$ implies that

$$b_i c_{\sigma^{-1}(i)} = c_i b_{\rho^{-1}(i)} \quad \text{and} \quad \sigma^\rho = \sigma.$$

Notice that $\sigma^\rho = \sigma$ implies that for every $l \in \{1, \ldots, d\}$, we have that $\sigma_l^\rho = \sigma_k$ for a unique $k \in \{1, \ldots, d\}$. Proceeding as in the first paragraph of the proof of Proposition 5.22 we can prove that $c_i \in \Gamma$. Also, arguing as in the second paragraph of the proof of Proposition 5.22 we see that

$$c_{\sigma^{-1}(i)} c_i^{-1} \in B$$

for every $i \in \{1, \ldots, r\}$. We can proceed like this for every $a \in A$. Since A is transitive on each $\{(l-1)m + 1, \ldots, lm\}$ we conclude that

$$c_j c_l^{-1} \in B$$

for every $j \in \{(l-1)m + 1, \ldots, lm\}$ and for every $l \in \{1, \ldots, d\}$. Hence $(c_1, \ldots, c_r) \in (B^m \Delta\Gamma)^d = (\check{\Delta}_B^m \Gamma)^d$.

Finally, since $\sigma^\rho = \sigma$ for every σ coming from an element $a \in A$, we conclude that ρ permutes the set $\{\{(l-1)m + 1, \ldots, lm\} \mid l = 1, \ldots, d\}$ and also permutes the elements of the set $\{(l-1)m + 1, \ldots, lm\}$ for each $l = 1, \ldots, d$. Hence $\rho \in \mathfrak{S}_m \wr \mathfrak{S}_d$. We conclude that $c = (c_1, \ldots, c_r)\rho \in ((\check{\Delta}_B^m \Gamma) \rtimes \mathfrak{S}_m) \wr \mathfrak{S}_d$. □

5.5. The inductive Brauer-Glauberman condition

We begin this section by defining the inductive Brauer-Glauberman condition (see Definition 5.24 below). After that, we study consequences of the validity of the inductive Brauer-Glauberman condition for a simple non-abelian group S. The main result of this section is Theorem 5.27.

DEFINITION 5.24. Let S be a non-abelian simple group and let X be the universal covering group of S. We say that S satisfies the **inductive Brauer-Glauberman condition** for the prime p if for every $B \leqslant \mathrm{Aut}(X)$ with $(|X|, |B|) = 1$ the following conditions are satisfied:

(i) For $Z = \mathbf{Z}(X)$, $\Gamma = \mathbf{C}_{\mathrm{Aut}(X)}(B)$, $C_0 = \mathbf{C}_X(B)$ and $C = C_0 Z$, there exists a Γ-equivariant bijection

$$\Omega_B \colon \mathrm{IBr}_B(X) \to \mathrm{IBr}_B(C),$$

such that for every $\theta \in \mathrm{IBr}_B(X)$

$$(X \rtimes \Gamma_\theta, X, \theta) \succ_{Br,c} (C \rtimes \Gamma_\theta, C, \Omega_B(\theta)).$$

(ii) For every positive integer m with $(|X|, m) = 1$, there exists a fake m-th Galois action on $\mathrm{IBr}(C_0)$ with respect to $C_0 \rtimes \Gamma$.

We can simplify a bit the verification of Definition 5.24 for all non-abelian simple groups. First we only need to consider B up to $\mathrm{Aut}(X)$-conjugation. Moreover condition (ii) is true for every B if (ii) is true for $B = 1$. Part of these simplifications is possible thanks to the fact that fake Galois actions do exist in p-solvable groups.

REMARK 5.25. Let S be a non-abelian simple group and X the universal covering group of S.

(a) The group S satisfies the inductive Brauer-Glauberman condition, if conditions (i) and (ii) in Definition 5.24 hold for some complete set of representatives of classes of $\mathrm{Aut}(X)$-conjugate subgroups B of $\mathrm{Aut}(X)$ with $(|X|, |B|) = 1$.

(b) Let $B \leqslant \mathrm{Aut}(X)$ with $(|X|, |B|) = 1$ and $|B| \neq 1$ and assume that $\mathbf{C}_X(B)$ is quasi-simple. Then condition (ii) in Definition 5.24 holds for X and B, if for every integer m with $(|X|, m) = 1$ there exists a fake m-th Galois action on $\mathrm{IBr}(X_1)$ with respect to $X_1 \rtimes \mathrm{Aut}(X_1)$, where X_1 is the universal covering group of the unique non-abelian composition factor of $\mathbf{C}_X(B)$.

(c) Let $B \leqslant \mathrm{Aut}(X)$ with $(|X|, |B|) = 1$ and $|B| \neq 1$ and assume that $\mathbf{C}_X(B)$ is not quasi-simple. Then condition (ii) in Definition 5.24 holds for X and B.

PROOF. For the proof of (a) we suppose that (i) and (ii) from Definition 5.24 are satisfied for the universal covering group X of some simple non-abelian group and for some $B \leqslant \mathrm{Aut}(X)$ with $(|X|, |B|) = 1$. Assume the notation Definition 5.24(i) with respect to B. Let $\alpha \in \mathrm{Aut}(X)$. Define $\Omega_{B^\alpha}(\chi^\alpha) = \Omega_B(\chi)^\alpha$ for every $\chi \in \mathrm{IBr}_B(X)$. Then Ω_{B^α} is a Γ^α-equivariant bijection from $\mathrm{IBr}_{B^\alpha}(X)$ onto $\mathrm{IBr}_{B^\alpha}(C^\alpha)$. By Lemma 4.22 we have that condition (i) in Definition 5.24 is satisfied for Ω_{B^α}. Let m be an integer with $(|X|, m) = 1$. Let $f_m\colon \mathrm{IBr}(C_0) \to \mathrm{IBr}(C_0)$ give the fake m-th Galois action on $\mathrm{IBr}(C_0)$ with respect to $C_0 \rtimes \Gamma$, as in Definition 4.37. Define $f'_m(\varphi^\alpha) = f_m(\varphi)^\alpha$ for every $\varphi \in \mathrm{IBr}(C_0)$. It is easy to prove an analogue of Lemma 4.22 for m-th Galois conjugate modular character triples via Lemma 4.35. This implies that f'_m gives a fake m-th Galois action on $\mathrm{IBr}(C_0^\alpha)$ with respect to $C_0^\alpha \rtimes \Gamma^\alpha$.

Now we prove parts (b) and (c). Let X be the universal covering group of some non-abelian simple group S. Let $B \leqslant \mathrm{Aut}(X)$ with $(|B|, |X|) = 1$. If $B \neq 1$, then the group S is a simple group of Lie type and B is $\mathrm{Aut}(X)$-conjugate to some subgroup of the field automorphisms of X, see for example Section 2 of [**MNS15**]. By part (a) we may assume that B consists of field automorphisms. By Theorem 2.2.7 of [**GLS03**], the group $\mathbf{C}_X(B)$ is either quasi-simple, solvable or $\mathbf{C}_X(B) \in \{\mathrm{B}_2(2), \mathrm{G}_2(2), {}^2\mathrm{F}_4(2), {}^2\mathrm{G}_2(3)\}$.

Suppose that $\mathbf{C}_X(B)$ is not quasi-simple. Write $C_0 = \mathbf{C}_X(B)$. If $C_0 \in \{\mathrm{B}_2(2), \mathrm{G}_2(2), {}^2\mathrm{F}_4(2), {}^2\mathrm{G}_2(3)\}$, then $\mathrm{Out}(C_0)$ is cyclic and $\mathbf{Z}(C_0)$ is trivial. Hence, it is easy to show that the identity yields a fake m-th Galois action on $\mathrm{IBr}(C_0)$ with respect to $C_0 \rtimes \mathrm{Aut}(C_0)$, for every positive integer m with $(|X|, m) = 1$. If C_0 is a solvable group, then Theorem 4.39 guarantees that for every m with $(|X|, m) = 1$, there exists a fake m-th Galois action on $\mathrm{IBr}(C_0)$ with respect to any G in which C_0 is normal. This proves part (c).

In all other cases $\mathbf{C}_X(B)$ is quasi-simple and there exists some non-abelian simple group S_1 such that $\mathbf{C}_X(B)$ is a central quotient of the universal covering group X_1 of S_1. By assumption for every m with $(|X_1|, m) = 1$, there exists a fake m-th Galois action on $\mathrm{IBr}(X_1)$ with respect to $X_1 \rtimes \mathrm{Aut}(X_1)$. Since $\mathbf{C}_{\mathrm{Aut}(X)}(B) \leqslant \mathrm{Aut}(X_1)$ this gives the required fake m-th Galois action on $\mathrm{IBr}(\mathbf{C}_X(B))$ according to an analogue of Lemma 4.23 for fake Galois conjugate modular character triples. We use that if m is such that $(|\mathbf{C}_X(B)|, m) = 1$, then also $(|X_1|, m) = 1$ by [**Asc00**, 33.12]. This proves (b). □

Notation 5.26. Let S be a non-abelian simple group and X its universal covering. We write $Z = \mathbf{Z}(X)$. If $B \leqslant \mathrm{Aut}(X)$, then we write $C_0 = \mathbf{C}_X(B)$, $C = C_0 Z$ and $\Gamma = \mathbf{C}_{\mathrm{Aut}(X)}(B)$. Recall Notation 5.13, and for a positive integer r, write $\widetilde{X} = X^r$, $\widetilde{Z} = Z^r$, $\widetilde{C} = C^r$. Let $\Delta\colon X \to \widetilde{X}$ be the map defined as in Notation 5.9. We also write $\check{C} = \check{\Delta}_Z C_0$. For each $i \in \{1, \dots, r\}$ let X_i and $\overline{\mathrm{pr}}_i$ be defined as in Notation 5.18.

Our aim in this section is to prove the following result and study some consequences of it.

Theorem 5.27. *Let S be a non-abelian simple group satisfying the inductive Brauer-Glauberman condition for the prime p. Let X be the universal covering group of S, r a positive integer and $A \leqslant \mathrm{Aut}(\widetilde{X})$ with $(|A|, |X|) = 1$.*

Write $\widetilde{\Gamma} = \mathbf{C}_{\mathrm{Aut}(\widetilde{X})}(A)$. *Suppose that* $\widetilde{\Gamma}A$ *acts transitively on the factors* $\{X_1, \ldots, X_r\}$ *of* $\widetilde{X}$. *Then there exists a* $\widetilde{\Gamma}$*-equivariant bijection*

$$\widetilde{\Omega}_{\widetilde{X},A} \colon \mathrm{IBr}_A(\widetilde{X}) \to \mathrm{IBr}_A(\mathbf{C}_{\widetilde{X}}(A)\widetilde{Z}),$$

such that for every $\chi \in \mathrm{IBr}_A(\widetilde{X})$ *and* $\chi' = \widetilde{\Omega}_{\widetilde{X},A}(\chi)$

$$(\widetilde{X} \rtimes \widetilde{\Gamma}_\chi, \widetilde{X}, \chi) \succ_{Br,c} (\mathbf{C}_{\widetilde{X}}(A)\widetilde{Z} \rtimes \widetilde{\Gamma}_\chi, \mathbf{C}_{\widetilde{X}}(A)\widetilde{Z}, \chi'). \tag{5.2}$$

Remark 5.28. The above result illustrates why we need to require the existence of fake Galois actions in Definition 5.24. Let X be the universal covering group of a simple non-abelian group S. Let r be a positive integer. Let $A \leqslant \mathfrak{S}_r$ act on $\widetilde{X}$ by transitively permuting $\{X_1, \ldots, X_r\}$. Then $\mathbf{C}_{\widetilde{X}}(A) = \Delta X \leqslant \widetilde{X}$. Let $\chi = \varphi \times \cdots \times \varphi \in \mathrm{IBr}_A(\widetilde{X})$. In particular Theorem 5.27 requires the existence of $\chi' \in \mathrm{IBr}(\Delta X)$ such that $(\widetilde{X}, \widetilde{X}, \chi) \succ_{Br,c} (\Delta X, \Delta X, \chi')$. If we write $\chi' = \Delta\varphi'$ for $\varphi' \in \mathrm{IBr}(X)$, then φ' must lie above λ^m, where $\lambda \in \mathrm{IBr}(\varphi_{\mathbf{Z}(X)})$. If one analyzes a little more this example (namely the factor sets condition in this example), then it is easy to see that φ' needs to be a fake r-th Galois conjugate of φ.

We continue identifying $\mathrm{Aut}(\widetilde{X}) = \mathrm{Aut}(X) \wr \mathfrak{S}_r$ as in Notation 5.18. For $A \leqslant \mathrm{Aut}(\widetilde{X})$, write $B = \overline{\mathrm{pr}}_1(A_{X_1}) \leqslant \mathrm{Aut}(X)$.

We prove Theorem 5.27 in a series of steps. We first prove a particular case and after that we use this particular case to prove the general statement.

Now, we concentrate on proving Theorem 5.27 in the case where A acts transitively on $\{X_1, \ldots, X_r\}$ and $A \leqslant B \wr \mathfrak{S}_r$. Among other things, we want to define a bijection

$$\widetilde{\Omega}_{\widetilde{X},A} \colon \mathrm{IBr}_A(\widetilde{X}) \to \mathrm{IBr}_A(\mathbf{C}_{\widetilde{X}}(A)\widetilde{Z})$$

such that corresponding characters give centrally isomorphic character triples. We first determine the group $\mathbf{C}_{\widetilde{X}}(A)\widetilde{Z}$ and some character sets with which we will work.

Recall Notation 5.26. If $B \leqslant \mathrm{Aut}(X)$, then $C_0 = \mathbf{C}_{\mathrm{Aut}(X)}(B)$, $C = C_0 Z$ and $\check{C} = \Delta_Z C_0$.

Lemma 5.29. *If* A *acts transitively on* $\{X_1, \ldots, X_r\}$ *and* $A \leqslant B \wr \mathfrak{S}_r$, *then*

(a) $\mathbf{C}_{\widetilde{X}}(A) = \Delta C_0 = \mathbf{C}_{\widetilde{X}}(B \wr \mathfrak{S}_r)$. *In particular,* $\mathbf{C}_{\widetilde{X}}(A)\widetilde{Z} = \check{C}$,
(b) $\mathrm{IBr}_A(\widetilde{X}) = \{\theta \times \cdots \times \theta \mid \theta \in \mathrm{IBr}_B(X)\}$,
(c) $\mathrm{IBr}_A(\widetilde{Z}) = \{\nu \times \cdots \times \nu \mid \nu \in \mathrm{IBr}_B(Z)\}$,
(d) $\mathrm{IBr}_B(C) = \{\varphi \cdot \nu \mid \varphi \in \mathrm{IBr}(C_0)$ *and* $\nu \in \mathrm{IBr}_B(Z)\}$,
(e) $\mathrm{IBr}_A(\widetilde{C}) = \{(\varphi \times \cdots \times \varphi) \cdot (\nu \times \cdots \times \nu) \mid \varphi \in \mathrm{IBr}(C_0)$ *and* $\nu \in \mathrm{IBr}_B(Z)\}$,
(f) $\mathrm{IBr}_A(\check{C}) = \{(\Delta\varphi) \cdot \mu \mid \varphi \in \mathrm{IBr}(C_0)$ *and* $\mu \in \mathrm{IBr}_A(\widetilde{Z})\}$.

Proof. The equalities $\mathbf{C}_{\widetilde{X}}(A) = \Delta C_0 = \mathbf{C}_{\widetilde{X}}(B \wr \mathfrak{S}_r)$ easily follow from Lemma 2.2 of [**IN96**]. Then $\check{C} = \check{\Delta}_Z C_0 = \mathbf{C}_{\widetilde{X}}(A)\widetilde{Z}$, as wanted. The rest easily follow from the definitions (use Lemma 4.27 for parts (d), (e) and (f)). □

In the following proposition, we introduce a key bijection $\widetilde{f}_r \colon \mathrm{IBr}_A(\widetilde{C}) \to \mathrm{IBr}_A(\check{C})$ that will help us to define the map $\widetilde{\Omega}_{\widetilde{X},A}$.

Proposition 5.30. *In the situation of Theorem 5.27, suppose that A acts transitively on $\{X_1, \ldots, X_r\}$ and $A \leqslant B \wr \mathfrak{S}_r$. Let $\check{\Gamma} = \check{\Delta}_B \Gamma$ and $\check{Y} = \check{C} \rtimes (\check{\Gamma} \rtimes \mathfrak{S}_r)$. Then there exists a $\Delta\Gamma$-equivariant bijection*

$$\widetilde{f}_r \colon \mathrm{IBr}_A(\widetilde{C}) \to \mathrm{IBr}_A(\check{C})$$

such that for every $\widetilde{\psi} \in \mathrm{IBr}_A(\widetilde{C})$ and $\check{\psi} = \widetilde{f}_r(\widetilde{\psi})$ we have

$$(\widetilde{C}\check{Y}_{\widetilde{\psi}}, \widetilde{C}, \widetilde{\psi}) \succ_{Br,c} (\check{Y}_{\widetilde{\psi}}, \check{C}, \check{\psi}).$$

Proof. Write $Z_0 = \mathbf{Z}(C_0)$, $\widetilde{Z}_0 = Z_0^r$ and $\widetilde{C}_0 = C_0^r$. Note that the assumption that A acts transitively on $\{X_1, \ldots, X_r\}$ with $(|X|, |A|) = 1$ implies $(r, |X|) = 1$. Since S satisfies the inductive Brauer-Glauberman condition, there exists a fake Galois r-th action on $\mathrm{IBr}(C_0)$ with respect to $C_0 \rtimes \Gamma$. Let f_r give a fake r-th Galois action as in Definition 4.37.

Let $\widetilde{\psi} \in \mathrm{IBr}_A(\widetilde{C})$. Recall $\widetilde{C} = \widetilde{C}_0\widetilde{Z}$ is the central product of $\widetilde{C}_0$ and $\widetilde{Z}$. By Theorem 5.17, $\widetilde{C}_0 \cap \widetilde{Z} = \widetilde{Z}_0$. By Lemma 5.29(f), we have that $\widetilde{\psi} = \widetilde{\varphi} \cdot \widetilde{\nu}$ for some $\widetilde{\varphi} = \varphi \times \cdots \times \varphi \in \mathrm{IBr}_A(\widetilde{C}_0)$ and $\widetilde{\nu} = \nu \times \cdots \times \nu \in \mathrm{IBr}_A(\widetilde{Z})$ where φ and ν lie over the same $\lambda \in \mathrm{IBr}(Z_0)$. We define

$$\widetilde{f}_r(\widetilde{\psi}) = \Delta(f_r(\varphi)) \cdot \widetilde{\nu}.$$

By Lemma 5.29, we have that $\widetilde{f}_r$ is a bijection. Note that $\widetilde{f}_r$ is $\Delta\Gamma$-equivariant as f_r is Γ-equivariant. In particular, for $\check{\psi} = \widetilde{f}_r(\widetilde{\psi})$ we have that $\check{Y}_{\widetilde{\psi}} = \check{Y}_{\check{\psi}}$.

Let $\varphi' = f_r(\varphi)$ and write $Y_0 = C_0 \rtimes \Gamma_\varphi$. Since f_r yields a fake r-th Galois action on $\mathrm{IBr}(C_0)$ with respect to Y_0, we have that

$$(Y_0, C_0, \varphi)^{(r)} \approx (Y_0, C_0, \varphi'). \tag{5.3}$$

Let $\check{C}_0 = \check{\Delta}_{Z_0} C_0$. Write $\widetilde{\lambda} = \lambda \times \cdots \times \lambda \in \mathrm{IBr}(\widetilde{Z}_0)$, where $\mathrm{IBr}(\varphi_{Z_0}) = \{\lambda\} = \mathrm{IBr}(\nu_{Z_0})$, and $\check{\varphi} = \Delta\varphi' \cdot \widetilde{\lambda} \in \mathrm{IBr}(\check{C}_0)$. Write also $\check{Y}_0 = \check{\Delta}_{Z_0 B} Y_0$. Since $Z_0 B \leqslant \mathbf{C}_{Y_0}(C_0)$, by Corollary 5.15 we have that Equation (5.3) yields

$$(\widetilde{C}_0(\check{Y}_0 \rtimes \mathfrak{S}_r), \widetilde{C}_0, \widetilde{\varphi}) \succ_{Br,c} (\check{Y}_0 \rtimes \mathfrak{S}_r, \check{C}_0, \check{\varphi}). \tag{5.4}$$

We have that $\check{C} = \widetilde{Z}\check{C}_0$. Then Equation (5.4) together with Lemma 4.28 imply

$$(\widetilde{C}\check{Y}_{\widetilde{\psi}}, \widetilde{C}, \widetilde{\psi}) \succ_{Br,c} (\check{Y}_{\check{\psi}}, \check{C}, \check{\psi}).$$ □

Proposition 5.31. *If A acts transitively on $\{X_1, \ldots, X_r\}$ and $A \leqslant B \wr \mathfrak{S}_r$, then Theorem 5.27 holds.*

Proof. Let Ω_B be the map given by Definition 5.24(i). For $\theta \in \mathrm{IBr}_B(X)$, we write $\theta' = \Omega_B(\theta)$. We define a map $\widetilde{\Omega} = \widetilde{\Omega}_{\widetilde{X},A}$ by

$$\begin{aligned}\widetilde{\Omega}\colon \mathrm{IBr}_A(\widetilde{X}) &\to \mathrm{IBr}_B(\check{C})\\ \theta \times \cdots \times \theta &\mapsto \widetilde{f}_r(\theta' \times \cdots \times \theta')\end{aligned}$$

where $\widetilde{f}_r$ is given by Proposition 5.30. By Lemma 5.29, this map is well-defined. In fact, since $\widetilde{f}_r$ is bijective, then $\widetilde{\Omega}$ is also bijective (see Lemma 5.29(f)).

Recall $\widetilde{\Gamma} = \mathbf{C}_{\mathrm{Aut}(\widetilde{X})}(A)$. Write $\Upsilon = \check{\Delta}_B\Gamma \rtimes \mathfrak{S}_r = \Delta\Gamma(B \wr \mathfrak{S}_r)$. By Proposition 5.22

$$\widetilde{\Gamma} \leqslant \widetilde{\Gamma}A \leqslant \Upsilon.$$

In order to prove that $\widetilde{\Omega}$ is $\widetilde{\Gamma}$-equivariant we show that $\widetilde{\Omega}$ is actually Υ-equivariant. In view of the description of $\mathrm{IBr}_A(\widetilde{X})$ and $\mathrm{IBr}_A(\widetilde{C})$ given in Lemma 5.29, it follows that $B \wr \mathfrak{S}_r$ acts trivially on these sets. Hence our bijection $\widetilde{\Omega}$ is $B \wr \mathfrak{S}_r$-equivariant. By definition, Ω_B is Γ-equivariant and by Proposition 5.30, $\widetilde{f}_r$ is $\Delta\Gamma$-equivariant. Hence $\widetilde{\Omega}$ is also $\Delta\Gamma$-equivariant (so Υ-equivariant).

It remains to prove that for every $\chi \in \mathrm{IBr}_A(\widetilde{X})$ we have that

$$(\widetilde{X} \rtimes \widetilde{\Gamma}_\chi, \widetilde{X}, \chi) \succ_{Br,c} (\widetilde{C} \rtimes \widetilde{\Gamma}_\chi, \widetilde{C}, \widetilde{\Omega}(\chi)).$$

Let $\chi \in \mathrm{IBr}_A(\widetilde{X})$ and $\theta \in \mathrm{IBr}_B(X)$ with $\chi = \theta \times \cdots \times \theta$. By Definition 5.24(i), we have that θ and $\theta' = \Omega_B(\theta)$ satisfy

$$(X \rtimes \Gamma_\theta, X, \theta) \succ_{Br,c} (C \rtimes \Gamma_\theta, C, \theta').$$

Let $\widetilde{\psi} = \theta' \times \cdots \times \theta' \in \mathrm{IBr}(\widetilde{C})$. The equation above together with Corollary 4.25 imply that

$$(\widetilde{X} \rtimes (\Gamma_\theta \wr \mathfrak{S}_r), \widetilde{X}, \chi) \succ_{Br,c} (\widetilde{C} \rtimes (\Gamma_\theta \wr \mathfrak{S}_r), \widetilde{C}, \widetilde{\psi}).$$

Using $\Upsilon \leqslant \Gamma \wr \mathfrak{S}_r$ and $\Upsilon_\chi \leqslant \Gamma_\theta \wr \mathfrak{S}_r$ we deduce that

$$(\widetilde{X} \rtimes \Upsilon_\chi, \widetilde{X}, \chi) \succ_{Br,c} (\widetilde{C} \rtimes \Upsilon_\chi, \widetilde{C}, \widetilde{\psi}). \tag{5.5}$$

Let $\check{Y} = \check{C} \rtimes \Upsilon$ and $\check{\psi} = \widetilde{f}_r(\widetilde{\psi})$. Then $\check{\psi} = \widetilde{\Omega}(\chi)$. By Proposition 5.30 we know that

$$(\widetilde{C}\check{Y}_{\widetilde{\psi}}, \widetilde{C}, \widetilde{\psi}) \succ_{Br,c} (\check{Y}_{\widetilde{\psi}}, \check{C}, \check{\psi}).$$

Because of $\widetilde{C} \rtimes \Upsilon_\chi = \widetilde{C}\check{Y}_{\widetilde{\psi}}$ and $\check{C} \rtimes \Upsilon_{\check{\psi}} = \check{Y}_{\check{\psi}}$, the equation above is exactly

$$(\widetilde{C} \rtimes \Upsilon_\chi, \widetilde{C}, \widetilde{\psi}) \succ_{Br,c} (\check{C} \rtimes \Upsilon_{\widetilde{\psi}}, \check{C}, \check{\psi}). \tag{5.6}$$

Since $\succ_{Br,c}$ is transitive, Equations (5.5) and (5.6) imply that

$$(\widetilde{X} \rtimes \Upsilon_\chi, \widetilde{X}, \chi) \succ_{Br,c} (\breve{C} \rtimes \Upsilon_{\breve{\psi}}, \breve{C}, \breve{\psi}).$$

Since $\widetilde{\Gamma}_\chi \leqslant \Upsilon_\chi = \Upsilon_{\breve{\psi}}$ and $\breve{\psi} = \widetilde{\Omega}(\chi)$ this proves the statement. □

Remark 5.32. Assume the situation described in Proposition 5.31 as well as the notation of Theorem 5.27. The proof of Proposition 5.31 actually shows that the conclusions of Theorem 5.27 also hold with $\Upsilon = \breve{\Delta}_B\Gamma \rtimes \mathfrak{S}_r \supseteq \widetilde{\Gamma}$ in place of $\widetilde{\Gamma}$.

Our next step is to prove Theorem 5.27 in the case where A no longer acts transitively on $\{X_1, \ldots, X_r\}$ but the structure of A is somehow controlled, namely $A \leqslant (B \wr \mathfrak{S}_r)^{r/m}$ for some divisor m of r.

Proposition 5.33. *Let m be the length of some A-orbit on $\{X_1, \ldots, X_r\}$. If $\widetilde{\Gamma}A$ acts transitively on $\{X_1, \ldots, X_r\}$ and $A \leqslant (B \wr \mathfrak{S}_m)^{r/m}$, then Theorem 5.27 holds.*

Proof. Let $\Lambda = \{X_1, \ldots, X_r\}$ and $\Lambda_1, \ldots, \Lambda_d$ be the A-orbits on Λ. Notice that $\widetilde{\Gamma}$ permutes the A-orbits transitively so that $d = r/m$. The assumption $A \leqslant (B \wr \mathfrak{S}_m)^{\frac{r}{m}}$ implies that $\Lambda_j = \{X_{(j-1)m+1}, \ldots, X_{jm}\}$ for every $1 \leqslant j \leqslant d$. By Proposition 5.23 we have that

$$A\mathbf{C}_{\mathrm{Aut}(\widetilde{X})}(A) = A\widetilde{\Gamma} \leqslant \Upsilon,$$

where $\Upsilon = (\breve{\Delta}_B^m\Gamma \rtimes \mathfrak{S}_m) \wr \mathfrak{S}_d$ (because in this case we can take $\tau = 1$ in Proposition 5.23).

For every $j \in \{1, \ldots, d\}$, let

$$X_{\Lambda_j} = \prod_{Y \in \Lambda_j} Y \quad \text{and} \quad Z_{\Lambda_j} = \prod_{Y \in \Lambda_j} \mathbf{Z}(Y).$$

Clearly A acts on X_{Λ_j} with $(|A|, |X_{\Lambda_j}|) = 1$. Let Υ_j be the projection of $\mathrm{Stab}_\Upsilon(X_{\Lambda_j})$ into $\mathrm{Aut}(X_{\Lambda_j})$. Then, it is easy to show that Υ_j is isomorphic to $\breve{\Delta}_B^m\Gamma \rtimes \mathfrak{S}_m$. By Lemma 5.31 (and using Remark 5.32), there is a Υ_1-equivariant bijection

$$\widetilde{\Omega}_{\Lambda_1,A} \colon \mathrm{IBr}_A(X_{\Lambda_1}) \to \mathrm{IBr}_A(\breve{C}_{\Lambda_1}),$$

where $\breve{C}_{\Lambda_j} = \mathbf{C}_{X_{\Lambda_j}}(A)Z_{\Lambda_1}$. Furthermore for every $\chi_1 \in \mathrm{IBr}_A(X_{\Lambda_1})$ and $\breve{\chi}_1 = \widetilde{\Omega}_{\Lambda_1,A}(\chi_1)$ we have

$$(X_{\Lambda_1} \rtimes (\Upsilon_1)_{\chi_1}, X_{\Lambda_1}, \chi_1) \succ_{Br,c} (\breve{C}_{\Lambda_1} \rtimes (\Upsilon_1)_{\chi_1}, \breve{C}_{\Lambda_1}, \breve{\chi}_1). \tag{5.7}$$

For $j \in \{2, \ldots, d\}$, we define Υ_j-equivariant bijections

$$\widetilde{\Omega}_{\Lambda_j,A} \colon \mathrm{IBr}_A(X_{\Lambda_j}) \to \mathrm{IBr}_A(\breve{C}_{\Lambda_j})$$

from $\widetilde{\Omega}_{X_{\Lambda_1},A}$ via the permutation action of $\mathfrak{S}_d$ on $\{X_{\Lambda_1},\dots,X_{\Lambda_d}\}$. For every $\chi_j \in \mathrm{IBr}_A(X_{\Lambda_j})$ and $\check{\chi}_j = \widetilde{\Omega}_{\Lambda_j,A}(\chi_j)$ we have

$$(X_{\Lambda_j} \rtimes (\Upsilon_j)_{\chi_j}, X_{\Lambda_j}, \chi_j) \succ_{Br,c} (\check{C}_{\Lambda_j} \rtimes (\Upsilon_j)_{\chi_j}, \check{C}_{\Lambda_j}, \check{\chi}_j) \tag{5.8}$$

by a transfer of Equation (5.7) via Lemma 4.22.

Note that $\widetilde{X} = X_{\Lambda_1} \times \cdots \times X_{\Lambda_d}$ is an internal direct product and A stabilizes each X_{Λ_j}. Hence $\mathrm{IBr}_A(\widetilde{X})$ is in natural correspondence with $\mathrm{IBr}_A(X_{\Lambda_1}) \times \cdots \times \mathrm{IBr}_A(X_{\Lambda_d})$. Analogously $\mathbf{C}_{\widetilde{X}}(A)\widetilde{Z} = \check{C} = \check{C}_{\Lambda_1} \times \cdots \times \check{C}_{\Lambda_d}$ is an internal direct product, so that $\mathrm{IBr}_A(\mathbf{C}_{\widetilde{X}}(A)) = \mathrm{IBr}_A(\check{C}_{\Lambda_1}) \times \cdots \times \mathrm{IBr}_A(\check{C}_{\Lambda_d})$.

Define $\widetilde{\Omega}_{\widetilde{X},A} \colon \mathrm{IBr}_A(\widetilde{X}) \to \mathrm{IBr}(\check{C})$ by

$$\chi_1 \times \cdots \times \chi_d \mapsto \widetilde{\Omega}_{\Lambda_1,A}(\chi_1) \times \cdots \times \widetilde{\Omega}_{\Lambda_d,A}(\chi_d).$$

We see that $\widetilde{\Omega}_{\widetilde{X},A}$ is a well-defined $(\Upsilon_1 \times \cdots \times \Upsilon_d)$-equivariant bijection. By definition $\widetilde{\Omega}_{\widetilde{X},A}$ is $\mathfrak{S}_d$-equivariant, where $\mathfrak{S}_d$ is identified with the subgroup of Υ acting on the groups X_{Λ_i} by permutation. Hence $\widetilde{\Omega}_{\widetilde{X},A}$ is Υ-equivariant.

It is easy to prove that every character in $\mathrm{IBr}_A(\widetilde{X})$ is $(\Upsilon_1 \times \cdots \times \Upsilon_d)$-conjugate to some $\chi = \chi_1 \times \cdots \times \chi_d$ where either χ_i and χ_j are $\mathfrak{S}_d$-conjugate or χ_i and χ_j are not Υ-conjugate (again $\mathfrak{S}_d$ is identified with the subgroup of Υ acting on the X_{Λ_j} groups by permutation). In particular, the stabilizer Υ_χ of χ in Υ satisfies

$$\Upsilon_\chi = ((\Upsilon_1)_{\chi_1} \times \cdots (\Upsilon_d)_{\chi_d}) \rtimes (\mathfrak{S}_d)_\chi.$$

The Equation (5.8) for each $j \in \{1,\dots,d\}$ together with Corollary 4.25 imply that the character χ satisfies

$$(\widetilde{X} \rtimes \Upsilon_\chi, \widetilde{X}, \chi) \succ_{Br,c} (\check{C} \rtimes \Upsilon_\chi, \check{C}, \chi'),$$

where $\chi' = \widetilde{\Omega}(\chi) = \check{\chi}_1 \times \cdots \times \check{\chi}_d$. Of course, since $\widetilde{\Gamma} \leqslant \Upsilon$, we deduce

$$(\widetilde{X} \rtimes \widetilde{\Gamma}_\chi, \widetilde{X}, \chi) \succ_{Br,c} (\check{C} \rtimes \widetilde{\Gamma}_\chi, \check{C}, \chi').$$

By Lemma 4.22 this finishes the proof. □

Proof of Theorem 5.27 . By Proposition 5.20 there exists some $\alpha \in \mathrm{Aut}(X)^r$ such that $A^\alpha \leqslant B \wr \mathfrak{S}_r$, where B is the projection of A_{X_1} on $\mathrm{Aut}(X)$. Let m be the length of an A-orbit on $\{X_1,\dots,X_r\}$. Since $\widetilde{\Gamma}$ acts transitively on the A-orbits, $d = r/m$ is the number of A-orbits. Let $\tau \in \mathfrak{S}_d$ be as given in Proposition 5.22. Then $A^{\alpha\tau} \leqslant (B \wr \mathfrak{S}_m)^d$. Let $\widetilde{\Omega}_{\widetilde{X},A^{\alpha\tau}}$ be the $\widetilde{\Gamma}^{\alpha\tau}$-equivariant bijection given by Proposition 5.33. Define $\widetilde{\Omega}_{\widetilde{X},A}$ by $\chi \mapsto \widetilde{\Omega}_{\widetilde{X},A^{\alpha\tau}}(\chi^{\alpha\tau})$ for every $\chi \in \mathrm{IBr}_A(\widetilde{X})$. It is easy to check that $\widetilde{\Omega}_{\widetilde{X},A}$ is a $\widetilde{\Gamma}$-equivariant bijection. Use Lemma 4.22 to check the central character triple isomorphism condition with respect to $\widetilde{\Omega}_{\widetilde{X},A}$. □

The importance of Theorem 5.27 for us is illustrated in the following two results, which are consequences of it.

THEOREM 5.34. *Suppose that A acts coprimely on a finite group G. Let $K \lhd G$ be an A-invariant perfect subgroup. Suppose that $G = K\mathbf{C}_G(A)$. Write $C = \mathbf{C}_G(A)$ and $M = K \cap C$. Suppose further that A acts trivially on $N = \mathbf{Z}(G) \leqslant K$, $K/N = S_1 \times \cdots \times S_r \cong S^r$, where S is a non-abelian simple group, and CA permutes transitively $\{S_1, \ldots, S_r\}$. If S satisfies the inductive Brauer-Glauberman condition, then there exists a C-equivariant bijection*

$$\Omega' \colon \mathrm{IBr}_A(K) \to \mathrm{IBr}(M),$$

such that

$$(G_\chi, K, \chi) \succ_{Br,c} (C_\chi, M, \chi')$$

for every $\chi \in \mathrm{IBr}_A(K)$ and $\chi' = \Omega'(\chi)$.

PROOF. Since $G = K\mathbf{C}_G(A)$, then $\mathbf{C}_A(K) = \mathbf{C}_A(G)$ and we may assume that A acts faithfully on K, i.e., $A \leqslant \mathrm{Aut}(K)$. Let $\widetilde{S} = S^r$. Let X be the universal covering group of S. Then, $\widetilde{X} = X^r$ is the universal covering group of $\widetilde{S}$. Write $\widetilde{Z} = \mathbf{Z}(\widetilde{X})$. Since K is a covering of $\widetilde{S}$, there exists an epimorphism $\epsilon \colon \widetilde{X} \to K$ with $L = \ker(\epsilon) \leqslant \widetilde{Z}$. In fact, $\widetilde{X}$ is the universal covering of K.

The map ϵ induces an isomorphism $\mathrm{Aut}(\widetilde{X})_L \to \mathrm{Aut}(K)$. Hence, the groups A and $\overline{C} = C\mathbf{C}_G(K)/\mathbf{C}_G(K)$ can be seen as groups of automorphisms of $\widetilde{X}$. In fact, under this identification, the group $A\overline{C} \leqslant A\mathbf{C}_{\mathrm{Aut}(\widetilde{X})}(A)$ acts transitively on the factors of $\widetilde{X}$. Since $(|A|, |K|) = 1$, we have $(|A|, |\widetilde{X}|) = 1$ by [**Asc00**, 33.12]. Write $\widetilde{\Gamma} = \mathbf{C}_{\mathrm{Aut}(\widetilde{X})}(A)$ and $\check{C} = \mathbf{C}_{\widetilde{X}}(A)\widetilde{Z}$. By Theorem 5.27, there exists a $\widetilde{\Gamma}$-equivariant bijection

$$\widetilde{\Omega} = \widetilde{\Omega}_{\widetilde{X},A} \colon \mathrm{IBr}_A(\widetilde{X}) \to \mathrm{IBr}_A(\check{C})$$

such that

$$(\widetilde{X} \rtimes \widetilde{\Gamma}_\chi, \widetilde{X}, \chi) \succ_{Br,c} (\check{C} \rtimes \widetilde{\Gamma}_\chi, \check{C}, \widetilde{\Omega}(\chi))$$

for every $\chi \in \mathrm{IBr}_A(\widetilde{X})$. Since $\overline{C} \leqslant \widetilde{\Gamma}$, we have that $\widetilde{\Omega}$ is $\overline{C}$-equivariant and

$$(\widetilde{X} \rtimes \overline{C}_\chi, \widetilde{X}, \chi) \succ_{Br,c} (\check{C} \rtimes \overline{C}_\chi, \check{C}, \chi') \tag{5.9}$$

for every $\chi \in \mathrm{IBr}_A(\widetilde{X})$ and $\chi' = \widetilde{\Omega}(\chi)$. By Definition 4.19(ii), we have that for every $\chi \in \mathrm{IBr}_A(\widetilde{X})$, the characters χ and $\chi' = \widetilde{\Omega}(\chi)$ lie over the same character $\lambda \in \widetilde{Z}$. We deduce easily that

$$\widetilde{\Omega}(\mathrm{IBr}_A(\widetilde{X} \mid 1_L)) = \mathrm{IBr}_A(\check{C} \mid 1_L).$$

Note that $\epsilon(\check{C}) = \epsilon(\mathbf{C}_{\widetilde{X}}(A)\widetilde{Z}) = \mathbf{C}_K(A) = M$ and $\mathrm{IBr}_A(M) = \mathrm{IBr}(M)$. Hence $\widetilde{\Omega}$ defines, via ϵ, a C-equivariant bijection

$$\Omega' \colon \mathrm{IBr}_A(K) \to \mathrm{IBr}(M).$$

Notice that $\mathbf{C}_{\widetilde{X}\rtimes\overline{C}}(\widetilde{X}) = \widetilde{Z}$, $\epsilon(\widetilde{Z}/L) = N = \mathbf{Z}(K) = \mathbf{C}_{K\rtimes\overline{C}}(K)$ and $L \leqslant \ker(\chi) \cap \ker(\chi')$ for every $\chi \in \mathrm{IBr}_A(\widetilde{X}/L)$ and $\chi' = \widetilde{\Omega}(\chi) \in \mathrm{IBr}_A(\check{C}/L)$. By Lemma 4.23, Equation (5.9) implies that

$$(K \rtimes \overline{C}_\chi, K, \chi) \succ_{Br,c} (M \rtimes \overline{C}_\chi, M, \chi')$$

for every $\chi \in \mathrm{IBr}_A(K)$ and $\chi' = \Omega'(\chi)$. Finally, a direct application of Theorem 4.26 yields

$$(G_\chi, K, \chi) \succ_{Br,c} (C_\chi, M, \chi'). \qquad \square$$

Corollary 5.35. *Suppose A that acts coprimely on G. Let $K \lhd G$ be A-invariant. Suppose that $G = K\mathbf{C}_G(A)$. Write $C = \mathbf{C}_G(A)$ and $M = K \cap C$. Suppose further that A acts trivially on $N = \mathbf{Z}(G) \leqslant K$, $K/N = S_1 \times \cdots \times S_r \cong S^r$, where S is a non-abelian simple group, and CA permutes transitively $\{S_1, \ldots, S_r\}$. If S satisfies the inductive Brauer-Glauberman condition for the prime p, then there exists a bijection*

$$\Omega' \colon \mathrm{IBr}_A(K) \to \mathrm{IBr}(M),$$

such that

$$(G_\chi, K, \chi) \succ_{Br,c} (C_\chi, M, \chi')$$

for every $\chi \in \mathrm{IBr}_A(K)$ and $\chi' = \Omega'(\chi)$.

Proof. Let $K_1 = [K, K]$. Since K/N is a direct product of simple non-abelian groups, it follows that K_1 is perfect and $K = K_1 N$. Let $N_1 = N \cap K_1$. Notice that $\mathbf{C}_G(K_1) = \mathbf{C}_G(K)$. Also K is the central product of K_1 and N. Write $M_1 = M \cap K_1$.

Let $\Omega'_1 \colon \mathrm{IBr}_A(K_1) \to \mathrm{IBr}(M_1)$ be the bijection given by Theorem 5.34. Every $\chi \in \mathrm{IBr}_A(K)$ has the form $\chi_1 \cdot \mu$, where $\chi_1 \in \mathrm{IBr}_A(K_1)$, $\mu \in \mathrm{IBr}(N)$ and both characters lie over the same Brauer character of N_1. Define

$$\Omega' \colon \mathrm{IBr}_A(K) \to \mathrm{IBr}(M)$$
$$\chi_1 \cdot \mu \mapsto \Omega'_1(\chi_1) \cdot \mu.$$

It is clear that Ω' is a C-equivariant bijection. Let $\chi = \chi_1 \cdot \mu \in \mathrm{IBr}_A(K)$. By Theorem 5.34 we have $(G_{\chi_1}, K_1, \chi_1) \succ_{Br,c} (C_{\chi_1}, M_1, \Omega'_1(\chi_1))$. A direct application of Lemma 4.28 implies

$$(G_\chi, K, \chi) \succ_{Br,c} (C_\chi, M, \Omega'(\chi)). \qquad \square$$

5.6. A reduction theorem

We are finally ready to prove Theorem G. Since we need to use a strong inductive argument, we first prove a relative to normal subgroups version of Theorem G below.

Theorem 5.36. *Let A act coprimely on G. Let $N \lhd G$ be stabilized by A and write $C = \mathbf{C}_G(A)$. Let $\theta \in \mathrm{IBr}_A(N)$. Suppose that the non-abelian* *simple groups involved in G/N satisfy the inductive Brauer-Glauberman condition. Then*

$$|\mathrm{IBr}_A(G|\theta)| = |\mathrm{IBr}(CN|\theta)|.$$

PROOF. We proceed by induction on $|G:N|$.

Step 1. We may assume θ is G-invariant.
Let $T = G_\theta$. Then $CN \cap T = (CN)_\theta$. By the Clifford correspondence for Brauer characters (see Theorem 4.7) we have that

$$|\mathrm{IBr}_A(G|\theta)| = |\mathrm{IBr}_A(T|\theta)| \text{ and } |\mathrm{IBr}(CN|\theta)| = |\mathrm{IBr}(CN \cap T|\theta)|.$$

If $T < G$, then by induction hypothesis with respect to $|T:N| < |G:N|$, we have that

$$|\mathrm{IBr}_A(T|\theta)| = |\mathrm{IBr}(\mathbf{C}_T(A)N|\theta)| = |\mathrm{IBr}(CN \cap T|\theta)|.$$

Step 2. We may assume that $N \leqslant \mathbf{Z}(GA)$ is a p'-group.
By Theorem 8.28 of [**Nav98**], there exists a strong isomorphism of modular character triples

$$(\sigma, \tau)\colon (GA, N, \theta) \to (\Gamma, M, \varphi)$$

such that $M \leqslant \mathbf{Z}(\Gamma)$ is a p'-group. Whenever $N \leqslant H \leqslant GA$, we write H^τ to denote the subgroup of Γ such that $\tau(H/N) = H^\tau/M$. Then $A \cong \tau(AN/N) = (AN)^\tau/M$, so that $(AN)^\tau/M$ acts on G^τ/M as A acts on G/N and $(AN)^\tau/M$ acts trivially on M. Therefore

$$(CN)^\tau/M = \tau(CN/N) = \mathbf{C}_{G^\tau/M}((AN)^\tau/M) = \mathbf{C}_{G^\tau}((AN)^\tau)/M.$$

By Theorem 4.12(a), θ extends to AN, and hence φ extends to $(AN)^\tau$. Recall that φ is a linear character since $M \leqslant \mathbf{Z}(\Gamma)$. Let π be the set of primes dividing $|A|$. Write $\varphi = \varphi_\pi\varphi_{\pi'}$. Recall that φ_π and $\varphi_{\pi'}$, the π-part and π'-part of φ, are powers of φ. In particular, φ_π extends to $(AN)^\tau$. If $q \notin \pi$, then by Theorem 4.12(b) we have that φ_π extends to Q for every $Q/M \in \mathrm{Syl}_q(\Gamma/M)$. Thus φ_π extends to Γ by Theorem 4.11. By parts (*d*) and (*b*) of Lemma 4.17, the modular character triple (Γ, M, φ) is strongly isomorphic to $(\Gamma, M, \varphi_{\pi'})$ and we may assume that $\varphi_{\pi'}$ is faithful. Write $\varphi' = \varphi_{\pi'}$. We have that $|M| = o(\varphi')$ is a π'-number. Hence M has a complement B in $(AN)^\tau$ by Schur-Zassenhaus' theorem [**Isa08**, Thm. 3.5]. Thus B acts coprimely on G^τ and $(CN)^\tau/M = \mathbf{C}_{G^\tau}(B)/M$. Since (GA, N, θ) is strongly isomorphic to (Γ, M, φ') we have that

$$|\mathrm{IBr}_A(G|\theta)| = |\mathrm{IBr}_B(G^\tau|\varphi')| \text{ and } |\mathrm{IBr}(CN|\theta)| = |\mathrm{IBr}(\mathbf{C}_{G^\tau}(B)|\varphi')|.$$

Moreover, since $G/N \cong G^\tau/M$, then the simple groups involved in G^τ/M satisfy the inductive Brauer-Glauberman condition for the prime p. Hence, the claim follows if it follows for G^τ, M, B and φ.

Step 3. We may assume $G = KC$ for every A-invariant K with $N < K \lhd G$.

Let $N < K \lhd G$ be A-invariant. We have that C acts on $\mathrm{IBr}_A(K|\theta)$. Let $\mathcal{B}$ be a complete set of representatives of C-orbits on $\mathrm{IBr}_A(K|\theta)$. Let $K \leqslant H \leqslant G$ and $\psi \in \mathrm{IBr}_A(H|\theta)$. By Theorem 4.6, we have that H/K acts transitively on $\mathrm{IBr}(\psi_K)$. Also A acts on $\mathrm{IBr}(\psi_K)$. Since $(|A|, |H/K|) = 1$, by Glauberman's Lemma [**Isa76**, Lem. 13.8] and [**Isa76**, Cor. 13.9] there is some A-invariant character in $\mathrm{IBr}(\psi_K)$ and any two of them are C-conjugate. This proves that every $\psi \in \mathrm{IBr}_A(H|\theta)$ lies over a unique element of $\mathcal{B}$. By the previous argument for $H = G$ and $H = CK$ we have that

$$|\mathrm{IBr}_A(G|\theta)| = \sum_{\eta \in \mathcal{B}} |\mathrm{IBr}_A(G|\eta)| \quad \text{and} \quad |\mathrm{IBr}_A(CK|\theta)| = \sum_{\eta \in \mathcal{B}} |\mathrm{IBr}_A(CK|\eta)|.$$

By the inductive hypothesis $|\mathrm{IBr}_A(G|\eta)| = |\mathrm{IBr}(CK|\eta)|$ for every $\eta \in \mathcal{B}$. Since A acts coprimely on CK/K and $\mathbf{C}_{CK/K}(A) = CK/K$, we have that $\mathrm{IBr}_A(CK|\eta) = \mathrm{IBr}(CK|\eta)$ by Lemma 5.3. Hence $|\mathrm{IBr}_A(G|\theta)| = |\mathrm{IBr}_A(CK|\theta)|$. If $CK < G$, then by induction $|\mathrm{IBr}_A(CK|\theta)| = |\mathrm{IBr}(C|\theta)|$, and the claim follows.

Step 4. We may assume $\mathbf{O}_p(G) = 1$.
Write $O = \mathbf{O}_p(G)$. If $O > 1$, then $|G/O : NO/O| < |G : N|$. By Lemma 4.4, $O \leqslant \ker(\varphi)$ for every $\varphi \in \mathrm{IBr}(G)$. To prove the claim use that $\mathbf{C}_{G/O}(A) = CO/O$ by coprime action and the inductive hypothesis.

Step 5. Every chief factor K/N of GA with $K \leqslant G$ is a direct product of isomorphic non-abelian simple groups and $N = \mathbf{Z}(G)$.
Let K/N be a chief factor of GA with $K \leqslant G$. We may assume that $G = KC$ by Step *3* and that K/N is not a p-group by Step *4*. If K/N is a p'-group, then $|\mathrm{IBr}_A(G|\theta)| = |\mathrm{IBr}(C|\theta)|$ by Corollary 5.8. Hence we can assume that GA has no abelian chief factor of the form K/N with $K \leqslant G$. In particular $N = \mathbf{Z}(G)$.

Final Step. Let K/N be a chief factor of GA with $K \leqslant G$. By Step *4* we may assume $G = KC$. By Step *5* we have that $K/N \cong S_1 \times \cdots \times S_r$, where the S_i are simple non-abelian groups. Notice that CA permutes transitively the groups S_i in K/N and hence they are all isomorphic. Since $S = S_1$ is involved in G/N, then S satisfies the inductive Brauer-Glauberman condition. Write $M = C \cap K$. By Corollary 5.35 there is a C-equivariant bijection

$$\Omega' \colon \mathrm{IBr}_A(K) \to \mathrm{IBr}(M),$$

such that $(G_\eta, K, \eta) \succ_{Br,c} (C_\eta, M, \Omega'(\eta))$ for every $\eta \in \mathrm{IBr}_A(K)$. We write $\eta' = \Omega'(\eta)$ for every $\eta \in \mathrm{IBr}_A(K)$. Since $N \leqslant \mathbf{C}_G(K)$, then Ω' actually yields a bijection $\mathrm{IBr}_A(K|\theta) \to \mathrm{IBr}(M|\theta)$. Let $\mathcal{B}$ be a set of representatives of C-orbits on $\mathrm{IBr}_A(K|\theta)$. Every element of $\mathrm{IBr}_A(G|\theta)$ lies over a unique element of $\mathcal{B}$ as in Step *3*. Since Ω' is C-equivariant, we have that the set $\{\eta' \mid \eta \in \mathcal{B}\}$ is a complete set of representatives of C-orbits on $\mathrm{IBr}(M|\theta)$.

Hence

$$|\mathrm{IBr}_A(G|\theta)| = \sum_{\eta \in \mathcal{B}} |\mathrm{IBr}_A(G|\eta)| \quad \text{and} \quad |\mathrm{IBr}(C|\theta)| = \sum_{\eta \in \mathcal{B}} |\mathrm{IBr}(C|\eta')|.$$

For every $\eta \in \mathcal{B}$, we have $\mathrm{IBr}_A(G|\eta) = \mathrm{IBr}(G|\eta)$ by Lemma 5.3 and

$$(G_\eta, K, \eta) \succ_{Br,c} (C_\eta, M, \eta')$$

by Corollary 5.35. In particular, $|\mathrm{IBr}(G_\eta|\eta)| = |\mathrm{IBr}(C_\eta|\eta')|$ for every $\eta \in \mathcal{B}$. The result follows then by using Theorem 4.7. □

Corollary 5.37. *Let A be a group that acts coprimely on a group G. Suppose that every simple non-abelian group involved in G satisfies the inductive Brauer-Glauberman condition with respect to the prime p. Then the actions of A on the irreducible Brauer characters of G and on the p-regular classes of G are permutation isomorphic*

Proof. For every $B \leqslant A$, Theorem 5.36 with $N = 1$ and B playing the role of A guarantees that $|\mathrm{IBr}_B(G)| = |\mathrm{IBr}(\mathbf{C}_G(B))|$. The map $K \mapsto K \cap \mathbf{C}_G(B)$ is a well-defined bijection between the set of B-invariant p-regular classes of G and the set of p-regular conjugacy classes of $\mathbf{C}_G(B)$. Hence the number of B-invariant irreducible Brauer characters of G equals the number of B-invariant p-regular conjugacy classes of G. By Lemma 13.23 of [**Isa76**], this proves the statement. □

5.7. Some examples

In this final section, we try to answer some questions naturally related to our topic. Recall that if A acts coprimely on G and p is a prime, then we are studying if

$$|\mathrm{IBr}_A(G)| = |\mathrm{IBr}(C)|,$$

where $C = \mathbf{C}_G(A)$ (with respect to p-Brauer characters). If G is p-solvable, then we have already mentioned that K. Uno [**Uno83**] proved the above equality. In fact, he established a canonical bijection

$$^* : \mathrm{IBr}_A(G) \to \mathrm{IBr}(\mathbf{C}_G(A))$$

which behaves exactly as the Glauberman correspondence. In particular, if A is a q-group for some prime q and $\varphi \in \mathrm{IBr}_A(G)$, then

$$\varphi_C = e\varphi^* + q\Delta,$$

where q does not divide e, and Δ is zero or a Brauer character of C. It is natural to ask if the equation above holds without restrictions on the structure of G. Unfortunately, the next example shows that the answer is negative.

We let G be the simple group ${}^2\mathrm{B}_2(2)$ of order $2^6 \cdot 5 \cdot 7 \cdot 13$. It is well known that G has an automorphism σ of order 3. Call $A = \langle \sigma \rangle$. Then

$$C = \mathbf{C}_G(A) = C_5 : C_4$$

is a Frobenius group of order $2^2{\cdot}5$. Let us set $p = 13$. The irreducible characters of G can be written as $\{\chi_1, \alpha_{14}, \beta_{14}, \alpha_{35}, \beta_{35}, \gamma_{35}, \chi_{64}, \alpha_{65}, \beta_{65}, \gamma_{65}, \chi_{91}\}$, where the subscript in each case denotes the degree of the character. Furthermore, every irreducible character lifts an ordinary p-Brauer character, except the character of degree 64. We can write

$$\mathrm{IBr}(G) = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6, \varphi_7, \varphi_8\},$$

where $\varphi_1 = (\chi_1)^0$, $\varphi_2 = (\alpha_{14})^0$, $\varphi_3 = (\beta_{14})^0$, $\varphi_4 = (\alpha_{35})^0 = (\beta_{35})^0 = (\gamma_{35})^0$, $\varphi_5 = (\alpha_{65})^0$, $\varphi_6 = (\beta_{65})^0$, $\varphi_7 = (\gamma_{65})^0$, $\varphi_8 = (\chi_{91})^0$. Also

$$(\chi_{64})^0 = \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4.$$

We have checked that

$$\mathrm{IBr}_A(G) = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_8\}.$$

The centralizer C is a p'-group, and $\mathrm{Irr}(C) = \{1, \lambda, \epsilon, \bar{\epsilon}, \delta\}$, where ϵ is a linear character of order 4, $\lambda = \epsilon^2$ and $\delta(1) = 4$. We have computed the restrictions of the A-invariant Brauer characters to C.

$$\begin{aligned}
(\varphi_1)_C &= 1_{C^0}\\
(\varphi_2)_C &= 2\epsilon + 3\delta\\
(\varphi_3)_C &= 2\bar{\epsilon} + 3\delta\\
(\varphi_4)_C &= 2\cdot 1_{C^0} + 3\lambda + \epsilon + \bar{\epsilon} + 7\delta\\
(\varphi_8)_C &= 3\cdot 1_{C^0} + 4\lambda + 6\epsilon + 6\bar{\epsilon} + 18\delta\,.
\end{aligned}$$

Hence, we see that all the restrictions have a unique irreducible constituent with multiplicity not divisible by 3, except for φ_4 whose restriction has 4 irreducible constituents with multiplicity not divisible by 3. This fact is not surprising in this case where C is a p'-group, since every $\varphi \in \mathrm{IBr}_A(G)$ different from φ_4 lifts to a unique $\chi \in \mathrm{Irr}(G)$, that is, therefore, A-invariant. Hence $\varphi_C = \chi_C$ has the desired decomposition by Glauberman's correspondence. Also, we see that φ_4 does not lift to an A-invariant ordinary irreducible character. This raises a natural question on the Glauberman-Isaacs correspondence: if $\chi \in \mathrm{Irr}_A(G)$ lifts $\varphi \in \mathrm{IBr}(G)$, is it true that the Glauberman-Isaacs correspondent $\chi^* \in \mathrm{Irr}(\mathbf{C}_G(A))$ lifts an irreducible Brauer character of $\mathbf{C}_G(A)$? This is the case for p-solvable groups, as shown in [**SG94**].

We come back to our example. If we were asked to establish a natural bijection between $\mathrm{IBr}_A(G)$ and $\mathrm{IBr}(C)$, we would give

$$\begin{aligned}
\varphi_1 &\mapsto 1_{C^0}\\
\varphi_2 &\mapsto \epsilon\\
\varphi_3 &\mapsto \bar{\epsilon}\\
\varphi_4 &\mapsto \delta\\
\varphi_8 &\mapsto \lambda\,.
\end{aligned}$$

As a curiosity, we remark that the fields of values between character correspondents under this bijection are the same.

$$\mathbb{Q}(\varphi_2) = \mathbb{Q}(i) = \mathbb{Q}(\epsilon)$$
$$\mathbb{Q}(\varphi_3) = \mathbb{Q}(i) = \mathbb{Q}(\bar{\epsilon})$$
$$\mathbb{Q}(\varphi_4) = \mathbb{Q} = \mathbb{Q}(\delta)$$
$$\mathbb{Q}(\varphi_8) = \mathbb{Q} = \mathbb{Q}(\lambda)\,.$$

It is not totally reasonable to expect that something like this is going to happen in general, since as we know by now $\mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$ does not act on $\mathrm{IBr}(G)$. (Moreover, for every $\varphi \in \mathrm{IBr}_A(G)$, we see that there exists some $g \in C^0$ such that $o(g) = f_\varphi$. In the ordinary case, since the Glauberman-Isaacs correspondence preserves fields of values, Feit's conjecture implies that if $\chi \in \mathrm{Irr}_A(G)$, then there exists some $c \in \mathbf{C}_G(A)$ such that $o(c) = f_\chi$.)

We come back for a moment to ordinary character theory. Let q be a prime. It is well-known that the proof of the McKay conjecture in the q-solvable case heavily depends on the Glauberman correspondence. In fact, in groups with a normal q-complement, the McKay conjecture is essentially the Glauberman count. Namely, if $G = K \rtimes Q$, where $Q \in \mathrm{Syl}_q(G)$, then $N = \mathbf{N}_G(Q) = \mathbf{C}_K(Q) \times Q$, by elementary group theory. Therefore

$$|\mathrm{Irr}_{q'}(N)| = |\mathrm{Irr}(\mathbf{C}_K(Q))||Q/Q'|.$$

Also, every $\chi \in \mathrm{Irr}_{q'}(G)$ restricts to $\theta = \chi_K \in \mathrm{Irr}_Q(K)$ (by Theorem 1.10). Conversely, every $\theta \in \mathrm{Irr}_K(Q)$ extends canonically to $\hat{\theta} \in \mathrm{Irr}_{q'}(G)$ (by Theorem 1.13). Hence, by Gallagher's theory (see Theorem 1.12, we have that

$$\mathrm{Irr}_{q'}(G) = \{\beta\hat{\theta} \mid \text{ for } \theta \in \mathrm{Irr}_Q(K) \text{ and } \beta \in \mathrm{Irr}(Q/Q')\},$$

where we identify the characters of Q/Q' with the characters of G/KQ') it follows that

$$|\mathrm{Irr}_{q'}(G)| = |Q : Q'||\mathrm{Irr}_K(Q)|.$$

Therefore, the McKay conjecture in this case reduces to the Glauberman count $|\mathrm{Irr}_Q(K)| = |\mathrm{Irr}(\mathbf{C}_K(Q))|$.

Now, we slightly change the notation that we have used throughout this chapter. Recall that we have been studying whether or not the equality

$$|\mathrm{IBr}_Q(K)| = |\mathrm{IBr}(\mathbf{C}_K(Q))|$$

holds whenever a group Q acts coprimely on K (with respect to p-Brauer characters, where p is any prime). Suppose that Q is a q-group and let $G = K \rtimes Q$. In view of the previous discussion it makes sense to ask if

$$|\mathrm{IBr}_{q'}(G)| = |\mathrm{IBr}_{q'}(\mathbf{N}_G(Q))|.$$

(Notice that this is a version of McKay for (p)-Brauer characters in the simplest case in which the group has a normal q-complement.) The answer is negative, as pointed out to us by P. H. Tiep. If $K = \mathrm{SL}_3(32)$, $q = 5$, $Q = C_5$, $G = K \rtimes Q$ and $p = 31$, then

$$|\mathrm{IBr}_{q'}(G)| \neq |\mathrm{IBr}_{q'}(\mathbf{N}_G(Q))|.$$

Since Q is cyclic, in this case, we do have that $|\mathrm{IBr}_Q(K)| = |\mathrm{IBr}(\mathbf{C}_K(Q))|$. The problem is that degrees of irreducible Brauer characters do not divide the order of the group. In some sense, this example explains why the McKay and the coprime counting conjectures, need separate reductions. Although closely related, there does not seem a simultaneous generalization of both of them.

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