Multiple point spaces of finite holomorphic maps

Guillermo Peñafort Sanchis

Supervised by

Juan José Nuño Ballesteros
and Washington Luiz Marar

Facultat de Ciències Matemàtiques
Universitat de València

This dissertation is submitted for the degree of
International Doctor in Mathematics

July 2015

Contents

Resumen

Esta tesis trata sobre espacios múltiples de aplicaciones holomorfas finitas entre variedades complejas. Nuestro enfoque es el de la teoría de singularidades, y las aplicaciones serán consideradas bajo la relación de $\mathcal{A}$-equivalencia, es decir, salvo cambios de coordenadas en partida y llegada. Nos centramos en relacionar propiedades de los espacios de puntos múltiples con propiedades como la $\mathcal{A}$-estabilidad y la $\mathcal{A}$-determinación finita. El trabajo está organizado de la siguiente manera:

El Capítulo 1 contiene los fundamentos básicos necesarios para el resto del trabajo. El único resultado original es el Lema 1.2.6.

En el Capítulo 2 definimos los espacios de puntos múltiples de una aplicación. En la Sección 2.1 demostramos que solo hay una manera de definir estos espacios de forma que satisfagan ciertas propiedades. Si denotamos $D^k(f)$ el espacio de puntos k-múltiples en cuestión, estas condiciones son:

- M1) Si f es estable, entonces $D^k(f)$ es la clausura de los puntos k-multiples estrictos

$$\{(x^{(1)}, \ldots, x^{(k)}) \mid x^{(i)} \neq x^{(j)}, f(x^{(i)}) = f(x^{(j)})\}.$$

- M2) La construcción de $D^k(f)$ se comporta bien por deformaciones.

En [Gaf83] Gaffney introduce un método para calcular estos espacios de puntos múltiples. Este método, aunque teóricamente realizable, es habitualmente impracticable. En la Sección 2.2 introducimos los ideales de Mond, que nos permiten obtener fácilmente los puntos múltiples de gérmenes de corrango 1. La idea es que estos gérmenes cualquier germen $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ se puede llevar a la forma

$$(x, y) \mapsto (x, f_n(x, y), \ldots, f_p(x, y)),$$

con $x \in \mathbb{C}^{n-1}, y \in \mathbb{C}$. Una vez en esta forma, la diferencia entre los puntos que forman un punto k-múltiple de f debe residir en las coordenadas y, así que podemos eliminar las copias innecesarias del resto de coordenadas y trabajar con puntos de la forma

$$(x, y^{(1)}, y^{(2)}, \ldots, y^{(k)}) \in \mathbb{C}^{n-1} \times \mathbb{C}^k.$$

El ideal de puntos dobles de Mond está generado por las diferencias divididas

$$\frac{f_j(x, y^{(2)}) - f_j(x, y^{(1)})}{y^{(2)} - y^{(1)}}.$$

Para obtener el ideal de puntos triples de Mond, añadimos a las diferencias divididas las diferencias divididas iteradas

$$\frac{\frac{f_j(x,y^{(3)})-f_j(x,y^{(1)})}{y^{(3)}-y^{(1)}} - \frac{f_j(x,y^{(2)})-f_j(x,y^{(1)})}{y^{(2)}-y^{(1)}}}{y^{(3)} - y^{(2)}}.$$

En general, el ideal de puntos k-múltiples de Mond se obtiene añadiendo a los generadores del ideal de puntos $(k-1)$-múltiples nuevas diferencias divididas iteradas. En las secciones subsiguientes introducimos otros tipos de espacios de puntos múltiples, definidos en espacios ambientes diferentes de X^k. Sin entrar en detalles, para una aplicacion estable $f\colon X \to Y$ estos espacios son:

- El espacio $D_1^k(f) \subseteq X$ de puntos k-multiples en la partida (Sección 2.3), definido como la clausura de los puntos $x \in X$ tales que $|f^{-1}(f(x))| \geq k$.
- El espacio $M_k(f) \subseteq Y$ de puntos k-múltiples en la llegada (Sección 2.5), que es la clausura de los puntos $y \in Y$ tales que $|f^{-1}(y)| \geq k$.
- El espacio $D^k(f)/S_k$ de puntos k-múltiples cociente (Sección 2.4), que se obtiene al identificar todas las permutaciones de los puntos k-múltiples $(x^{(1)}, \dots, x^{(k)}) \in D^k(f)$.

Por último, en la Sección 2.6 obtendremos un diagrama que relaciona los espacios recién expuestos.

El capítulo 3 está dedicado enteramente a los puntos dobles. Primero introducimos $\mathscr{I}^2(f)$, el haz de ideales de Mond de una aplicación $f\colon X \to Y$ con puntos de cualquier corrango (Sección 3.1). La construcción correspondiente para un germen $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ es la siguiente: Como las funciones $f_j(x) - f_j(x')$ se anulan si $x = x'$, entonces podemos encontrar una matriz α, con entradas en $\mathcal{O}_{2n}$, tal que

$$f(x) - f(x') = \alpha(x, x')(x - x').$$

Dada una tal matriz, el ideal de puntos dobles de Mond es

$$I^2(f) = \langle f_j(x) - f_j(x') \mid 1 \leq j \leq p \rangle + \langle \text{minors } n \times n \text{ of } \alpha \rangle.$$

Como veremos, esta estructura también define los puntos múltiples introducidos en el capítulo 2. La clave es el Teorema 3.1.11: Si los puntos dobles tienen la dimensión adecuada, entonces son espacios de Cohen-Macaulay. En la sección 3.2 demostramos algunas propiedades algebraicas de los puntos dobles, ligadas a la estabilidad y la determinación finita. Por ejemplo, del Corolario 3.2.5 se deduce que, si f es finitamente determinada y $\dim(D^2(f)) > 0$, entonces $D^2(f)$ es reducido. También probamos que el conjunto singular del espacio de puntos dobles de una aplicación estable es

$$\operatorname{Sing} D^2(f) = \{(x,x) \in X \times X \mid \operatorname{corank} f_x \geq 2\}.$$

En la última sección probamos que una estructura alternativa para los puntos múltiples, dada por unos haces de ideales $\mathscr{H}^k(f)$, no satisface las condiciones M1 y M2 del capítulo anterior.

En el Capítulo 4 introducimos otro espacio de puntos dobles, esta vez definido en el blowing-up de $X \times X$ a lo largo de su diagonal. Este espacio, que ya habían sido estudiado por Ronga [Ron72], Kleiman y otros, resulta muy interesante en el caso de corrango ≥ 2. Como el blowing-up es una construcción local, basta que describamos el caso $\mathbb{C}^n \times \mathbb{C}^n$, consistente en los puntos $(x, x', u) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{P}^{n-1}$ tales que para algún $\lambda \in \mathbb{C}$ se tiene

$$(x - x') = \lambda u.$$

Dada una matriz α como en el Capítulo 3, el espacio $B^2(f)$ de puntos dobles de f en el blowing-up, es el subespacio dado por los puntos (x, x', u) del blowing-up que satisfacen

$$\alpha(x, x')u = 0.$$

Existe una proyección propia

$$\pi\colon B^2(f) \to D^2(f),$$

la cual estudiamos en la Sección 4.4. Como veremos, los espacios $B^2(f)$ y $D^2(f)$ son biracionalmente equivalentes bajo condiciones genéricas y, si f es estable, entonces $\pi\colon B^2(f) \to D^2(f)$ es una resolución.

En el Capítulo 5 estudiamos gérmenes de aplicaciones $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$. En las Secciones 5.1 y 5.2 extendemos a corrango 2 algunos resultados ya conocidos en corrango 1. En la primera caracterizamos la determinación finita de un germen $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ en términos de la finitud del número de Milnor de su espacio de puntos doble en la partida, $\mu(D(f))$. En la segunda demostramos las fórmulas de Marar-Mond, que relacionan los números de Milnor de diversos espacios de puntos dobles con el número de puntos triples y cross-caps que se hallan acumulados en el origen. Estas fórmulas son

- $\mu\big(D(f)\big) = \mu\big(D^2(f)\big) + 6T$,
- $\mu\big(D^2(f)\big) = 2\mu\big(D^2(f)/S_2\big) + C - 1$,
- $\mu(D(f)) = 2\mu\big(f(D(f))\big) + C - 2T - 1.$

En la Sección 5.3 estudiamos los ‘double folds’, una familia de gérmenes que contiene multitud de ejemplos interesantes. Los double folds no son más que los gérmenes $f\colon (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$ de la forma

$$(x,y) \mapsto (x^2, y^2, f_3(x,y)).$$

Lo interesante de esta familia es que podemos comprender su geometría en relación con la acción del grupo generado por las reflexiones $(x,y) \mapsto (-x,y)$ y $(x,y) \mapsto (x,-y)$. En la sección 5.4 relacionamos la $\mathcal{A}$-equivalencia de double folds con la clase de la función f_3, respecto a una relación de equivalencia de contacto definida ad hoc.

El Capítulo 6 contiene las conclusiones del trabajo e incluye una lista de problemas abiertos. Por último, los Apéndices A y B contienen resultados sobre espacios complejos y álgebra. Solo una pequeña parte del material en ellos expuesto es original.

Aknowledgements

I want to thank my supervisors, Juan José Nuño Ballesteros and Washington Luiz Marar, my family, my friends and colleagues, for their support and patience.

Introduction

This thesis is about multiple point spaces of finite holomorphic maps between complex manifolds. Our approach comes from singularity theory and we study our maps under $\mathcal{A}$-equivalence, that is, up to changes of coordinates in source and target. We focus on the relation between these spaces and properties such as $\mathcal{A}$-stability and $\mathcal{A}$-finite determinacy. Multiple point spaces are well known for maps without corank ≥ 2 points, hence our contribution belongs mainly to the corank ≥ 2 case.

Given a map $f\colon X \to Y$ between manifolds, we have an intuitive idea of what a k-multiple point of f should be: a tuple $(x^{(1)}, \dots, x^{(k)})$ of different points $x^{(i)} \in X$, such that $f(x^{(1)}) = f(x^{(i)})$, for all i. If we are to define a k-multiple point space, we would like it to satisfy the following natural properties:

a) To be a closed complex subspace of X^k,

b) To behave well under deformations: for any unfolding F of f, the k-multiple point space of F is a deformation of the k-multiple point space of f.

It is clear that the set of multiple points (as defined above) does not satisfy condition a):

Example A. Let $f\colon \mathbb{C}^2 \to \mathbb{C}^3$ be the Cross-Cap, given by

$$(x, y) \mapsto (x, y^2, xy).$$

The pairs of points $(0, t)$ and $(0, -t), t \neq 0$ form a curve of double points that collapse to the diagonal point $((0, 0), (0, 0))$, which is not a double point.

One might try to solve the failure of a) taking the analytic closure of the double points, but then condition b) does not hold:

Example B. Take the family of curves $f_t\colon \mathbb{C} \to \mathbb{C}^2$, $t \in \mathbb{C}$, given by

$$x \mapsto (x^2, x^3 + tx).$$

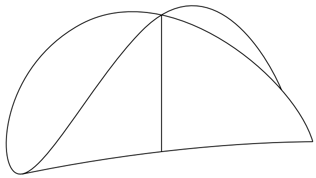

Figure 1: Image of a cross-cap.

Figure 2: Images of f_0 and $f_t, t \neq 0$, respectively.

The curve f_0 is a parameterized cusp $x \mapsto (x^2, x^3)$, which is injective and, thus, has no double points. For $t \neq 0$, f_t has a double point in $(x, x') = (\sqrt{-t}, -\sqrt{-t})$. Therefore, the unfolding $F\colon \mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}^2$, given by

$$(t, x) \mapsto (t, f_t(x)),$$

contains the double points $((t, x), (t', x')) = ((t, \sqrt{-t}), (t, -\sqrt{-t}))$. The closure of these points contains the point $t = t' = 0, x = x' = 0$, but we just said that f_0 has no double points.

As we will see, to avoid the problems in Examples A and B, the multiple point spaces have to include extra points and, sometimes, they have to be to be non-reduced.

Several authors have studied multiple point spaces (Altintas, Gaffney, Houston, Jorge Pérez, Kleiman, Laksov, Lipman, Marar, Mond, Nuño Ballesteros, Pellikaan, Ronga, Ulrich and Wik Atique, just to mention the ones who appear throughout the text). They work in different settings, and thus imposse different conditions on the map they study. Some authors assume Σ-genericity, some assume that the map f is generically one-to-one and some assume that there are no points of corank ≥ 2. Some work locally, studying germs and multigerms. Even more, the authors from the context of algebraic geometry assume that $f\colon X \to Y$ is a finite morphism between algebraic projective schemes over a field of arbitrary characteristic. As a consequence, the relations between all these author's multiple point spaces are sometimes unclear. In this thesis we will show that, in the case of holomorphic maps, there is only one possible definition of multiple point spaces satisfying some natural conditions. The main definitions of multiple point spaces that we consider, namely Gaffney's and Mond's, satisfy them, thus they agree.

The idea behind Gaffney's construction is simple: If we want the multiple point space of f to behave well under deformations, we have to include the multiple points coming from all possible unfoldings of f. The remarkable property that makes this possible is that we do not need to consider all unfoldings of f, but just a stable one. The result will not depend on the chosen stable unfolding, and no new points will come from non stable unfoldings. The truth is that we can only do this locally but, if we ignore the problem for a moment, the procedure to compute the k-multiple point space of $f\colon X \to Y$ looks like this:

1. Take a stable unfolding $F\colon S \times X \to S \times Y$ of f.
2. Let $D^k_S(F) \subseteq (S \times X)^k$ be the closure of the k-multiple points of F (in the sense above).
3. The k-multiple point space of f is the slice
$$D^k(f) = D^k_S(F) \cap (\{0\} \times X)^k.$$

The construction above has two problems. The first is the difficulty of the computations. To put it simply, some maps only admit very complicated stable unfoldings. The second is that it yields little insight on the algebraic structure of the multiple point spaces. These are two problems that Mond's approach does not have. Mond's multiple point spaces, when defined, are given by closed formulas, which can be always computed easily, and they provide good information about the algebraic properties of the multiple points. Hence, the fact that these constructions agree is key to understand multiple point spaces.

Until now, we have only mentioned multiple points consisting on tuples $(x^{(1)}, \dots, x^{(k)}) \in X^k$, but there are further multiple point spaces, defined in ambient spaces other than X^k. Roughly speaking, and for a stable map $f\colon X \to Y$, their definitions are the following:

- The source k-multiple point space $D^k_1(f) \subseteq X$ is the closure of the points $x \in X$, such that $|f^{-1}(f(x))| \geq k$.
- The target k-multiple point space $M_k(f) \subseteq Y$ is the closure of the points $y \in Y$, satisfying $|f^{-1}(y)| \geq k$.
- The quotient k-multiple point space $D^k(f)/S_k$ is obtained by identifying all permutations of k-multiple points $(x^{(1)}, \dots, x^{(k)}) \in D^k(f)$.

For double points we have one more space:

- The blowing-up double point space $B^2(f)$ is what we obtain if, in addition to the points $x, x' \in X$, with $x \neq x'$ and $f(x) = f(x')$, we keep track of the direction from x to x' as well.

Outline of the thesis

Chapter 1 is an expository chapter containing conventions and the background needed for the rest of the work. Except from Lemma 1.2.6, all results are well known and we do not claim originality on them.

In Chapter 2 we give the definitions and show some properties of multiple point spaces. In Section 2.1 we show that there is only one way to define multiple point spaces satisfying some (slightly more demanding than a) and b) above) natural conditions. Roughly speaking, these conditions for multiple point spaces $D^k(f)$ are:

> M1) If f is stable, then $D^k(f)$ is the closure of its strict k-multiple points.
>
> M2) The construction of $D^k(f)$ behaves well under deformations.

As we will see, the satisfactory construction of multiple point spaces corresponds to Gaffney [Gaf83]. In Section 2.2 we introduce Mond's multiple point ideals for corank 1 maps, which Marar and Mond show that agree with Gaffney's construction (Proposition 2.2.2). These ideals give us a nice way to compute the multiple point spaces of corank 1 map germs. They are constructed as follows: First of all, corank 1 map germs can be assumed to be of the form

$$(x, y) \mapsto (x, f_n(x, y), \ldots, f_p(x, y)),$$

with $x \in \mathbb{C}^{n-1}$ and $y \in \mathbb{C}$. For such a germ, the difference between the different points which form a k-multiple point in X^k must relay entirely on their y coordinates, so we can throw away the unnecessary copies of the x coordinates. Thus, we consider points of the form

$$(x, y^{(1)}, y^{(2)}, \ldots, y^{(k)}) \in \mathbb{C}^{n-1} \times \mathbb{C}^k.$$

Mond's double point ideal is generated by the divided differences

$$\frac{f_j(x, y^{(2)}) - f_j(x, y^{(1)})}{y^{(2)} - y^{(1)}}.$$

Mond's triple point ideal is obtained by adding to the previous one the iterated divided differences

$$\frac{\frac{f_j(x,y^{(3)})-f_j(x,y^{(1)})}{y^{(3)}-y^{(1)}} - \frac{f_j(x,y^{(2)})-f_j(x,y^{(1)})}{y^{(2)}-y^{(1)}}}{y^{(3)} - y^{(2)}}.$$

In general, Mond's k-multiple point ideal is obtained by adding to the $(k-1)$-multiple point space some further iterated divided differences. In the following sections we introduce the source multiple point space (Section 2.3), the quotient multiple point space (Section 2.4) and the target multiple point space (Section 2.5). Finally, in Section 2.6 we show a diagram relating these complex spaces.

Chapter 3 is devoted to double points. In Section 3.1 we introduce Mond's double point ideal sheaf $\mathscr{I}^2(f)$ for maps with points of arbitrary corank. The corresponding local construction for germs $f\colon (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$ is as follows: Since the functions $f_j(x)-f_j(x')$ vanish when $x=x'$, we can find a matrix with entries in $\mathcal{O}_{2n}$ satisying

$$f(x)-f(x')=\alpha(x,x')(x-x').$$

Mond's double point ideal is

$$I^2(f)=\langle f_j(x)-f_j(x') \mid 1\leq j\leq p\rangle + \langle \text{minors } n\times n \text{ of } \alpha\rangle.$$

We will show that the space it defines is precisely the double point space in Chapter 2. The key to the proof is Theorem 3.1.11: if the space defined by $\mathscr{I}^2(f)$ has the right dimension, then it is Cohen Macaulay. In Section 3.2 we show algebraic properties of the double point space $D^2(f)$. For instance, from Corollary 3.2.5 it follows that if f is finitely determined and $\dim D^2(f)>0$, then $D^2(f)$ is reduced. As another example, in Proposition 3.2.3 we show that, for any stable map $f\colon X\to Y$, the singular locus of its double point space is

$$\operatorname{Sing} D^2(f)=\{(x,x)\in X\times X \mid \operatorname{corank} f_x\geq 2\}.$$

Finally, in Section 3.3 we introduce an alternative multiple point ideal sheaf $\mathscr{H}^k(f)$ and show that it fails to satisfy the conditions M1 and M2. However, we will also show that under mild conditions the structure yields the right double point space.

In Chapter 4, we introduce the blowing-up double point space $B^2(f)$, which is an interesting tool in the corank ≥ 2 case. This space was defined first by Ronga [Ron72] and it had been studied by Kleiman and others in the context of algebraic geometry. However, our approach is different and we give new proofs of several properties. Locally, $B^2(f)$ is a subspace of the blowing-up of $\mathbb{C}^n\times\mathbb{C}^n$ along its diagonal. This blowing-up can be seen as the space of points $(x,x',u)\in\mathbb{C}^n\times\mathbb{C}^n\times\mathbb{P}^{n-1}$ satisfying

$$(x-x')=\lambda u, \text{ for some } \lambda\in\mathbb{C}.$$

For any matrix α as above, $B^2(f)$ is the subspace of the blowing-up defined by the equations

$$\alpha(x,x')u=0.$$

As we will see in Section 4.4, the space $B^2(f)$ is equiped with a proper projection

$$\pi\colon B^2(f)\to D^2(f).$$

The morphism π, which briefly speaking forgets the direction $u\in\mathbb{P}^{n-1}$, is an isomorphism off the set of diagonal points (x,x) with $\operatorname{corank} f\geq 2$.

Under generic conditions, the spaces $B^2(f)$ and $D^2(f)$ are birrationally equivalent and, if f is stable, then $\pi\colon B^2(f) \to D^2(f)$ is a resolution.

In Chapter 5 we study map germs $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$. In Sections 5.1 and 5.2 we extend to corank 2 some properties already known in corank 1. In the first one, we show that a map germ $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is finitely determined if and only if the Milnor number $\mu(D(f))$ of its source double point space is finite. In the second one, we prove the so called Marar-Mond formulas, relating the Milnor numbers of the different multiple point spaces and the number of cross-caps and triple points collapsed at the origin:

- $\mu\big(D(f)\big) = \mu\big(D^2(f)\big) + 6T$,
- $\mu\big(D^2(f)\big) = 2\mu\big(D^2(f)/S_2\big) + C - 1$,
- $\mu(D(f)) = 2\mu\big(f(D(f))\big) + C - 2T - 1$.

In Section 5.3 we introduce a family of corank 2 map germs, the double folds, of the form

$$(x, y) \mapsto (x^2, y^2, f_3(x, y)).$$

This family contains several interesting examples, such as finitely determined homogeneous maps of corank 2. We study these germs in relation to the reflection group generated by the reflections $(x, y) \mapsto (-x, y)$ and $(x, y) \mapsto (x, -y)$. In Section 5.4, we study the $\mathcal{A}$-equivalence of double folds in terms of the equivalence class of f_3 under a specially adapted contact equivalence relation.

Chapter 6 contains contains a summary of the goals of the thesis and related open problems. Finally, Appendices A and B are about complex spaces and algebra, respectively. Some results are new, but mostly they contain concepts for which our terminology is substantially different from that in the original source, or where there is no standard convention, and some results which are not so well known, or whose proof we have not been able to find in the literature.

Methodology

The research procedure for this thesis has been the usual in the field of mathematics. We started looking for adequate bibliographical resources, both general and specific of the subject of our studies, and have extended these materials as needed for our goals. For the computations we have made use of the software SINGULAR [WGPS15], implementing some algorithms specially adapted to our purposes.

Chapter 1

Preliminaries

This is an expository chapter containing the background needed for the rest of the work. Except from Lemma 1.2.6, all results are well known and we do not claim originality on them. We assume familiarity with the basics of differential topology, complex analytic geometry and commutative algebra. Altough all the spaces considered are constructed over the complex numbers, we will always draw real pictures.

1.1 Previous conventions

Throughout the text, we assume:

1. $f\colon X \to Y$ is a holomorphic map between complex manifolds.

2. The complex manifolds X and Y are assumed equidimensional, and its dimensions are n and p, respectively.

The previous assumptions do not apply to maps and germs denoted by letters other than f. Be aware that some further conventions apply from Section 1.6 onwards.

1.2 Holomorphic maps between complex manifolds

Notation on products

In order to work with products of copies of X, we fix some notation: We write elements in X^k as tuples $w = (x^{(1)}, \dots, x^{(k)})$ of points $x^{(l)} \in X$, each one with local coordinates $x_1^{(l)}, \dots, x_n^{(l)}$. For $k = 2, 3$, we use $x = x^{(1)}, x' = x^{(2)}, x'' = x^{(3)}$ and denote the coordinates by $x_i = x_i^{(1)}, x'_i = x_i^{(2)}$ and

$x_i'' = x_i^{(3)}$. The **small diagonal** of X^k is the subset

$$\Delta(X,k) = \{(x^{(1)},\dots,x^{(k)}) \in X^k \mid x^{(1)} = \cdots = x^{(k)}\}.$$

The **big diagonal** of X^k is the subset

$$D(X,k) = \{(x^{(1)},\dots,x^{(k)}) \in X^k \mid x^{(i)} = x^{(j)} \text{ for some } i \neq j\}.$$

Finally, we write

$$X^{(k)} = X^k \setminus D(X,k).$$

Rank and corank

Definition 1.2.1. A map $f\colon X \to Y$ has **rank** r at a point $x \in X$ (denoted by $\operatorname{rank} f_x = r$) if the differential df_x of f at x has rank r. We also say that f has **corank** k (denoted by $\operatorname{corank} f_x = k$), where $k = n - \operatorname{rank} f_x$. We say that x is a **regular point** of f (or that f is regular at x) if $\operatorname{corank} f_x = 0$. Otherwise, we say that x is a **singular point** of f (or that f is singular at x).

Definition 1.2.2. We say that $f\colon X \to Y$ is a **finite map** if it is closed and finite-to-one (i.e. all the sets $f^{-1}(y), y \in Y$ are finite).

Definition 1.2.3. Given $f\colon X \to Y$, we define

$$\Sigma^k(f) = \{x \in X \mid \operatorname{corank} f_x = k\}$$

and

$$\hat{\Sigma}^k(f) = \bigcup_{i \geq k} \Sigma^i(f).$$

Proposition 1.2.4. *For any map $f\colon X \to Y$*

1. *The subsets $\Sigma^k(f) \subset X$ are locally analytic.*
2. *The subsets $\hat{\Sigma}^k(f) \subset X$ are analytic.*

Proof. Let $d = \min(\dim X, \dim Y)$. Given some local coordinates of X defined at an open subset U, let Z_k be the set of points $x \in U$ where the minors of size $d - k$ of the differential df_x vanish. Then we have $\hat{\Sigma}^k(f) \cap U = Z_k \cap U$ and $\Sigma^k(f) \cap U = (Z_k \setminus Z_{k+1}) \cap U$. We conclude that both spaces are locally analytic. Moreover, X is covered by the coordinate open sets U_i coming from any atlas, and $\hat{\Sigma}^k(f)$ is closed in any of those, so it is closed in X. □

Remark 1.2.5. The closure of $\Sigma^k(f)$ is always contained in the space $\hat{\Sigma}^k(f)$. The equality does not hold in general, as the following map, which we called the Double Cone (Figure 3.1), shows:

$$(x,y) \mapsto (x^2, y^2, xy).$$

An easy computation yields $\Sigma^1(f) = \emptyset$ and $\hat{\Sigma}^1(f) = \Sigma^2(f) = \{0\}$.

Lemma 1.2.6. *Let $f\colon X \to Y$ be a finite-to-one map and let $n = \dim X$. Then*

$$\dim \hat{\Sigma}^k(f) \leq n - k,$$

for all $1 \leq k \leq n$. As a consequence, we have $\dim \Sigma^k(f) \leq n - k$.

Proof. We proceed by induction on k: Assume first $\dim \hat{\Sigma}^1(f) = n$, then there exists a proper analytic space $Z \subsetneq X$, such that f has constant rank $d < n$ at $X \setminus Z$. For any point $x \in X \setminus Z$, by the constant rank theorem, we can perform some local changes of coordinates in source and target to obtain a map of the form $(x_1, \ldots, x_d, 0 \ldots, 0)$, which is not finite. This proves the case $k = 1$.

Assume $\dim \hat{\Sigma}^k(f) \geq n - k + 1$. Then, since $\hat{\Sigma}^k \subseteq \hat{\Sigma}^{k-1}(f)$, for all $k \geq 1$, by induction we have $\dim \hat{\Sigma}^k(f) \leq n - k + 1$, so the dimension of $\hat{\Sigma}^k(f)$ equals $n - k + 1$. Let x be a regular point of $\hat{\Sigma}^k(f)$ where the dimension of $\hat{\Sigma}^k(f)$ is $n - k + 1$. Then, there exists an open neighbohood $U \subseteq X$ of x, such that the restriction $g = f|_{\hat{\Sigma}^k(f) \cap U}$ is a holomorphic map defined on a manifold of dimension $n - k + 1$. Being a restriction of f, the map g is finite-to-one. Since the restriction is done precisely at $\hat{\Sigma}^k(f)$, it is obvious that g has rank $\leq n - k$ at all source points. This means $\dim \hat{\Sigma}^1(g) = n - k + 1$, contradicting the induction hypothesis. □

Example 1.2.7. The previous bound is accurate, since it is exact for the map $\mathbb{C}^n \to \mathbb{C}^p$ given by

$$(x_1, \ldots, x_n) \mapsto (x_1^2, \ldots, x_n^2, 0, \ldots, 0).$$

1.3 Transversality and jet spaces

Definition 1.3.1. Let $f\colon X \to Y$ and let W be a submanifold of Y. We say that f is **transverse to W at a point** $x \in X$ if either

1. $f(x) \notin W$, or
2. $T_{f(x)}Y = T_{f(x)}W + df_x(T_xX)$.

The map f is **transverse** to W (denoted by $f \pitchfork W$) if it is transverse to W at every point $x \in X$.

Definition 1.3.2. We say that a map $f\colon X \to Y$ has **normal crossings** if, for any $k \geq 2$, the restriction of f^k to $X^{(k)}$ is transverse to $\Delta(Y, k)$.

Lemma 1.3.3. *[GG86, Lemma 4.6] Let $F\colon S \times X \to Y$ be a holomorphic map and, for any $s \in S$, let $f_s\colon X \to Y$ be the map given by $f_s(x) = F(s, x)$. If F is transverse to some submanifold W of Y, then the subset*

$$Z = \{s \in S \mid f_s \pitchfork W\}$$

is residual in S.

Proposition 1.3.4. *[GG86, Thm. 4.4] Let $f\colon X \to Y$ be transverse to some submanifold W of Y. Then $f^{-1}(W)$ is a submanifold of X of the same codimension as W.*

Definition 1.3.5. Two maps $f, f'\colon X \to Y$ define the same k-jet at a point $x \in X$ if $f(x) = f'(x)$ and their k-th Taylor expansions –for some local coordinates– $j^k f(x)$ and $j^k f'(x)$ agree. This defines an equivalence relation. The equivalence class of f is called the **k-jet of f at x** and, by abuse of notation, we denote it by $j^k f(x)$. The set of k-jets of maps $f\colon X \to Y$ with $f(x) = y$ is denoted by $J^k(X,Y)_{x,y}$. The **k-jet space** is defined as the disjoint union

$$J^k(X,Y) = \bigsqcup_{(x,y)\in X\times Y} J^k(X,Y)_{x,y}.$$

We have a **source map**

$$\alpha\colon J^k(X,Y) \to X,$$

which maps a jet $\sigma \in J^k(X,Y)_{x,y}$ to x. For each $f\colon X \to Y$ we have its **k-jet extension map**

$$j^k f\colon X \to J^k(X,Y),$$

given by $x \mapsto j^k f(x)$.

The k-jet space $J^k(X,Y)$ can be given a manifold structure [GG86, Thm. 2.7], so that α and $j^k f$ are holomorphic.

Definition 1.3.6. Given two manifolds X, Y and an integer $s \geq 2$, we call the **s-fold k-jet space** to the manifold

$$J^k_s(X,Y) = (\alpha \times \cdots \times \alpha)^{-1}(X^{(s)}).$$

Given a map $f\colon X \to Y$, we have a holomorphic map

$$j^k_s(f)\colon X^{(s)} \to J^k_s(X,Y),$$

given by $(x^{(1)},\dots,x^{(s)}) \mapsto (j^k(x^{(1)}),\dots,j^k(x^{(s)}))$. The points of the space $J^k_s(X,Y)$ are called **multijets** and $j^k_s f(x^{(1)},\dots,x^{(s)})$ is called the **multijet** of f at $(x^{(1)},\dots,x^{(s)})$.

Definition 1.3.7. For any k-jet $\sigma \in J^k(X,Y)_{x,y}$, with $k \geq 1$ we define $\operatorname{corank}\sigma = \operatorname{corank} f_x$, for any representative f of σ. We write

$$S_k = \{\sigma \in J^1(X,Y) \mid \operatorname{corank}\sigma = k\}.$$

Obviously, we have $\Sigma^k(f) = (j^1 f)^{-1}(S_k)$.

Proposition 1.3.8. *[GG86, Thm. 5.4] If $n \leq p$, then S_k is a submanifold of $J^1(X,Y)$ of codimension $k(p-n+k)$.*

Definition 1.3.9. We say that a map is Σ^k**-generic** if $j^1 f \pitchfork S_k$. A map is called Σ-generic if it is Σ^k-generic for all $1 \leq k \leq n$.

Remark 1.3.10. Let $f\colon X \to Y$ be a Σ^k-generic map and assume $n \leq p$. By Proposition 1.3.4, the set $\Sigma^k(f)$ is empty or a submanifold of X of dimension $n - k(p - n + k)$.

1.4 Germs

Definition 1.4.1. Given a subset $S \subseteq X$, we say that two subsets $Y_1, Y_2 \subseteq X$ define the same germ along S if there exists an open neighbourhood U of S in X, such that

$$Y_1 \cap U = Y_2 \cap U.$$

This defines an equivalence relation on the set of subsets of X. An equivalence class is called a **germ** (along S) and the germ represented by $Y \subseteq X$ is denoted by (Y, S). When S is a discrete set, germs along S are called germs at S. We omit unnecessary brackets and denote germs of the form $(Y, \{x\})$ by (Y, x).

We say that a germ (Y, S) is contained in a germ (Y', S) (denoted by $(Y, S) \subseteq (Y', S)$) if there exist representatives Z and Z' of (Y, S) and (Y', S) respectively, satisfying $Z \subseteq Z'$. We define $(Y, S) \cap (Y', S) = (Y \cap Y', S)$ and $(Y, S) \cup (Y', S) = (Y \cup Y', S)$. These operations do not depend on the representatives.

Definition 1.4.2. Given a subset $S \subseteq X$ and two maps $f_1\colon U_1 \to Y$ and $f_2\colon U_2 \to Y$, with U_1, U_2 open neighbourhoods of S, we say that f_1 and f_2 define the same germ along S if there exists some open neighbourhood $U \subseteq U_1 \cap U_2$ of S, such that

$$f_1|_U = f_2|_U.$$

Again, this defines an equivalence relation on the set of maps $U \to Y$ defined around S. An equivalence class is called a **map germ** (along S) and the map germ represented by f is denoted by (f, S). A germ of homeomorphism (or germ of biholomorphism, submersion, etc) is a germ which admits a homeomorphism (biholomorphism, submersion, etc.) as a representative. Observe that any map germ (f, S), represented by $f\colon X \to Y$ yields a well defined subset $f(S) \subseteq Y$.

A morphism of germs

$$f\colon (X, S) \to (Y, T)$$

is a map germ f along S such that $f(S) \subseteq T$. Germs and morphisms of germs form a category. The isomorphisms are the germs of biholomorphism $f\colon (X, S) \to (Y, T)$ with $f(S) = T$.

A map germ at a finite set S, satisfying $f(S) = \{y\}$ for some $y \in Y$, is called a **multigerm**. A **monogerm** at x is a map germ of the form $(f, \{x\})$, which we denote by (f, x).

1.5 $\mathcal{A}$-equivalence and stability

Definition 1.5.1. Two multigerms $f\colon (X, S) \to (Y, y)$ and $f'\colon (X', S') \to (Y', y')$ are **$\mathcal{A}$-equivalent** if there exist multigerms of biholomorphisms $\varphi\colon (X, S) \to (X', S')$ and $\psi\colon (Y, y) \to (Y', y')$, such that the following diagram commutes:

$$\begin{tikzcd} (X, S) \arrow[r, "f"] \arrow[d, "\varphi"] & (Y, y) \arrow[d, "\psi"] \\ (X', S') \arrow[r, "f'"] & (Y', y') \end{tikzcd}$$

Observe that a multigerm $f\colon (X, S) \to (Y, y)$, with $S = \{x^{(1)}, \ldots, x^{(s)}\}$ determines (after ordering the elements of S) a unique multijet

$$j_s^k(f) \in \bigoplus_{i=1}^{s} J^k(X, Y)_{x^{(i)}, y}.$$

Since any multigerm $f\colon (X, S') \to (Y, y)$ is $\mathcal{A}$-equivalent to one of the form $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, we shall give our definitions for multigerms of this form.

Definition 1.5.2. A multigerm $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is **k-determined** if any germ satisying $j_s^k f = j_s^k g$ is $\mathcal{A}$-equivalent to f. A multigerm is **finitely determined** if it is k-determined for some k.

Definition 1.5.3. An **unfolding** of a multigerm $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is a multigerm

$$F\colon (\mathbb{C}^r \times \mathbb{C}^n, \{0\} \times S) \to (\mathbb{C}^r \times \mathbb{C}^p, 0),$$

of the form $F(s, x) = (s, f_s(x))$, with $f_0(x) = f(x)$.

Two unfoldings F and F' as above are **$\mathcal{A}$-equivalent** if there exist unfoldings of the multigerms of the identity on $\mathbb{C}^n$ and $\mathbb{C}^p$

$$\Phi\colon (\mathbb{C}^r \times \mathbb{C}^n, \{0\} \times S) \to (\mathbb{C}^r \times \mathbb{C}^n, \{0\} \times S)$$

and

$$\Psi\colon (\mathbb{C}^r \times \mathbb{C}^p, 0) \to (\mathbb{C}^r \times \mathbb{C}^p, 0),$$

such that the following diagram commutes:

$$\begin{tikzcd} (\mathbb{C}^r \times \mathbb{C}^n, \{0\} \times S) \arrow[r, "F"] \arrow[d, "\Phi"] & (\mathbb{C}^r \times \mathbb{C}^p, 0) \arrow[d, "\Psi"] \\ (\mathbb{C}^r \times \mathbb{C}^n, \{0\} \times S) \arrow[r, "F"] & (\mathbb{C}^r \times \mathbb{C}^p, 0) \end{tikzcd}$$

An unfolding F of f is called **trivial** if it is $\mathcal{A}$-equivalent to the constant unfolding $\mathrm{id} \times f$.

A multigerm $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is **stable** if every unfolding of f is trivial.

Example 1.5.4. From Whitney's classification of stable monogerms $\mathbb{C}^n \to \mathbb{C}^{2n-1}$ [Whi44] and Mather criterion for stability of multigerms [Mat69, Prop. 1.6], it follows that the only stable multigerms of maps from complex surfaces to complex 3-manifolds are:

1. Regular points: $(x, y) \mapsto (x, y, 0)$
2. Transverse double points: Given by the branches
$$(x, y) \mapsto (x, y, 0)$$
$$(x, y) \mapsto (x, 0, y)$$
3. Transverse triple points: Given by the branches
$$(x, y) \mapsto (x, y, 0)$$
$$(x, y) \mapsto (x, 0, y)$$
$$(x, y) \mapsto (0, x, y)$$
4. Cross-caps: $(x, y) \mapsto (x, y^2, xy)$

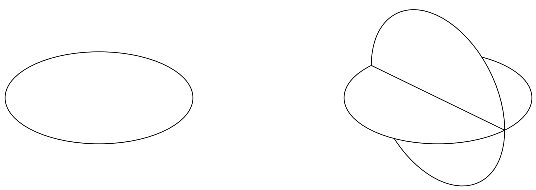

Figure 1.1: Regular point and transverse double point.

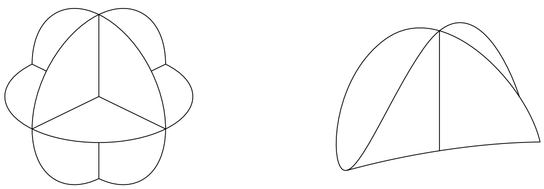

Figure 1.2: Transverse triple point and cross-cap.

Proposition 1.5.5. *Every finite multigerm admits a stable unfolding.*

Proof. If f is finite, then f is $\mathcal{K}$-finite and hence, has finite singularity type in the sense of Mather (see [GWdPL76]). The result follows since any multigerm of finite singularity type admits a stable unfolding. $\square$

Now we define $\mathcal{A}$-stability of maps. In the literature, our definition is sometimes called local stability, to avoid confusion with a different notion, called global stability. Since we do not use the global notion, there is no risk of confusion.

Definition 1.5.6. We say that a finite map $f\colon X \to Y$ is **stable** (or $\mathcal{A}$-stable) if, for any $y \in f(X)$, the multigerm of f at $f^{-1}(y)$ is stable.

Definition 1.5.7. A one-parameter unfolding $F\colon (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C} \times \mathbb{C}^p, 0)$ of the form $(t, x) \mapsto (t, f_t(x))$ is called an **stabilization** of $f = f_0$ if there exist open neighbourhoods $D \subseteq \mathbb{C}$ and $U \subseteq \mathbb{C}^n$ of the origin and a representative of F defined on $D \times U$ such that $f_t\colon U \to \mathbb{C}^p$ is stable for all $t \in D \setminus \{0\}$.

Lemma 1.5.8. *If (n, p) are nice dimensions in the sense of Mather (see [GWdPL76]), then every finitely determined map germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ admits an stabilization.*

Proof. It is well known that every finitely determined map germ admits a versal unfolding. Fixed such unfolding, the bifurcation set (that is, the set of values of the parameters which produce non-stable germs) is analytic. In the nice dimensions the bifurcation set must be proper, and the result follows immediately. $\square$

Proposition 1.5.9. *Any stable map has normal crossings.*

Proof. Fix some k and take any point $y = f(x_0^{(1)}) = \cdots = f(x_0^{(k)})$, for some different points $x^{(l)} \in X^k$. Let $U_1, \ldots, U_k$ be pairwise disjoint open coordinate neighbouhoods of $x_0^{(1)}, \ldots, x_0^{(k)}$. Let $A_1, \ldots, A_k$ be the $n \times p$ matrices containing the coordinates of $\mathbb{C}^N = \mathbb{C}^{np} \times \overset{k}{\ldots} \times \mathbb{C}^{np}$ and let $A = (A_1, \ldots, A_k)$. Now define the map $\phi\colon U_1 \times \cdots \times U_k \times \mathbb{C}^N \to \mathbb{C}^{kp}$ given by

$$(x, A) \mapsto (f(x^{(1)}) + A_1 x^{(1)}, \ldots, f(x^{(k)}) + A_k x^{(k)}).$$

This map is clearly a submersion. Therefore, there exists a curve $\gamma\colon D \to \mathbb{C}^N$, defined on an open neighbouhood $D \subseteq \mathbb{C}$ of the origin, such that, for all $t \in D \setminus \{0\}$, the map $\phi_t\colon U_1 \times \ldots U_k \to \mathbb{C}^{kp}$ given by

$$\phi_t(x) = \phi(x, \gamma(t))$$

is transverse to $\Delta(Y, k)$.

Let $F\colon D \times U \to D \times \mathbb{C}^p$ be the unfolding of f of the form $F(t, x) = (t, f_t(x))$, where f_t is given at any U_i by

$$f_t|_{U_i}(x) = f(x) + A_i x.$$

It is obvious that $f_t \times \cdots \times f_t = \phi_t$ on $U_1 \times \cdots \times U_k$. Therefore, the above mentioned transversality of ϕ_t means that $(f_t)^k = \phi_t$ is transverse to $\Delta(Y, k)$ on $U_1 \times \cdots \times U_k$. The stability of f implies that the family f_t is locally trivial. Then, since the transversality of f^k to $\Delta(Y, k)$ is preserved under $\mathcal{A}$-equivalence, we have that f^k is transverse to $\Delta(Y, k)$ at $(x_0^{(1)}, \ldots, x_0^{(k)})$. □

Proposition 1.5.10. *[GG86, Prop. 3.12] For $p = 2n$, a finite map $X \to Y$ is stable if and only if it is an immersion with normal crossings.*

A similar argument to that of Proposition 1.5.9 shows the following well known result:

Proposition 1.5.11. *Any finite stable map $f\colon X \to Y$ is Σ-generic. In particular $\Sigma^k(f)$ is empty or a manifold of dimension $n - k(p - n + k + 1)$.*

Theorem 1.5.12 (Mather-Gaffney criterion of finite determinacy [Wal81])**.** *A finite germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is finitely determined if and only if it admits a representative $f\colon U \to V$, such that f is stable on $U \setminus \{0\}$.*

Definition 1.5.13. We say that two maps $f\colon X \to Y$ and $f'\colon X' \to Y'$ are $\mathcal{A}$-equivalent if there exist two biholomorphisms $\varphi\colon X \to X'$ and $\psi\colon Y \to Y'$, making the following diagram commutative:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"] & Y \arrow[d, "\psi"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

Definition 1.5.14. Given a map $f\colon X \to Y$, we call an **unfolding of** f over a pointed manifold (S, s_0) to any map $F\colon \mathcal{X} \to \mathcal{Y}$, endowed with two embeddings $i\colon X \to \mathcal{X}, j\colon Y \to \mathcal{Y}$ and two submersions $\alpha\colon \mathcal{X} \to S, \beta\colon \mathcal{Y} \to S$, satisfying:

1. The following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[d, "i"] & & Y \arrow[d, "j"] \\ \mathcal{X} \arrow[rr, "F"] \arrow[dr, "\alpha"'] & & \mathcal{Y} \arrow[dl, "\beta"] \\ & S & \end{tikzcd}$$

2. Let $X_s = \alpha^{-1}(s)$ and $Y_s = \beta^{-1}(s)$, for any $s \in S$. Then i and j map X and Y isomorphically to X_{s_0} and Y_{s_0}.

For any $s \in S$, we write $f_s = F|_{X_s} \colon X_s \to Y_s$. We will call a **local unfolding** (at a subset $A \subseteq X$) to any unfolding F of the restriction of f to some open neighbourhood $U \subseteq X$ of A.

Remark 1.5.15. Some comment are due:

1. If F is an unfolding of f, then f and f_{s_0} are $\mathcal{A}$-equivalent. Indeed, every $\mathcal{A}$-equivalence can be seen as an unfolding where S consists just on one point.

2. If F is an unfolding of f over (S, s_0) and $\mathscr{F}$ is an unfolding of F over (T, t_0), then $\mathscr{F}$ is also an unfolding of f over $(S \times T, (s_0, t_0))$.

3. Every unfolding admits the following local form: take local coordinates so that the maps i, j, α, β are given by $i(x) = (x, 0)$, $j(y) = (y, 0)$, $\alpha(s, x) = s$ and $\beta(s, y) = s$. The map F is written in such coordinates as $F(s, x) = (s, f_s(x))$, with $f_0 = f$. Hence, the definition of unfolding can be seen as a global coordinate-free version of the local Definition 1.5.3.

Definition 1.5.16. With the notations above, we say that $F \colon \mathcal{X} \to \mathcal{Y}$ is a $\mathcal{A}$-trivial unfolding if there exists an open neighbourhood $S' \subseteq S$ of s_0, an open neighbourhood $U \subset \mathbb{C}^n$ of 0 and two biholomorphisms $\Phi \colon \mathcal{X}' \to X \times U$ and $\Psi \colon \mathcal{Y}' \to Y \times U$, where $\mathcal{X}' = \alpha^{-1}(S)$ and $\mathcal{Y}' = \beta^{-1}(S)$, satisfying:

1. The diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "F'"] \arrow[d, "\Phi"] & \mathcal{Y}' \arrow[d, "\Psi"] \\ X \times U \arrow[r, "f \times \mathrm{id}"] & Y \times U \end{tikzcd}$$

   commutes, where F' is the corresponding restriction of F.

2. The biholomorphisms Φ and Ψ map respectively X_{s_0} and Y_{s_0} to $X \times \{0\}$ and $Y \times \{0\}$.

1.6 Further conventions

In what follows, in addition to the conventions in Section 1.1, the maps $f \colon X \to Y$ and the multigerms $f \colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ are assumed to be finite. In particular, this implies $n \leq p$.

Chapter 2

Multiple points

The aim of this chapter is to stablish the definitions and some properties of multiple points of finite holomorphic mappings between manifolds, without any extra assumptions.

In the first section we show that there is only one way to define multiple point spaces satisfying some natural conditions (the conditions M1 and M2 in Definition 2.1.4, slightly more demanding than a) and b) in the introduction). As we will see, the satisfactory construction of multiple point spaces corresponds to Gaffney [Gaf83]. Section 2.2 is devoted to Mond's multiple point ideals, which give us an easier way to compute the multiple point spaces of corank 1 map germs. (Proposition 2.2.2). In Section 2.3 we introduce the source multiple point space and show how to compute it in some cases. In Section 2.4 we study the quotient of the multiple point space by the action of permutation groups. We give an effective way to compute the quotient and we explain the difficulties which arise in corank 2. In Section 2.5 we introduce the target multiple point space. We show an improvement of a method by Mond to compute some presentation matrices, which is more efficient for computational purposes. Finally, in Section 2.6 we show a diagram relating the above complex spaces.

2.1 The multiple point space

Definition 2.1.1. Given a map $f\colon X \to Y$, we say that $(x^{(1)}, \dots, x^{(k)}) \in X^k$ is a **strict k-multiple point of** f if $f(x^{(i)}) = f(x^{(j)})$ and $x^{(i)} \neq x^{(j)}$, for all $i \neq j$. We denote by $D^k_S(f)$ the analytic closure of the set of strict k-multiple points of f, that is,

$$D^k_S(f) = \overline{(f^k)^{-1}(\Delta(Y,k)) \setminus D(X,k)}.$$

We regard $D^k_S(f)$ as a complex space with the reduced structure. If we denote by $\mathscr{I}_{\Delta(X,k)}$ and $\mathscr{I}_{D(X,k)}$ the ideal sheaves in X^k of the small and big diagonal respectively, then the ideal sheaf of $D^k(f)$ is

$$\sqrt{\mathscr{P}(f,k) : \mathscr{I}_{D(X,k)}{}^{\infty}},$$

where $\mathscr{P}(f,k) = (f^k)^* \mathscr{I}_{\Delta(Y,k)}$ and $A : B^\infty$ stands for the saturation of A with respect to B.

We have local versions of the above definitions as well: For any finite multigerm $f\colon (X,A) \to (Y,y)$, we take a representative $\hat{f}$ of f, defined at a small enough neighbourhood U of A. Then, we define $D^k_S(f)$ as the multigerm in A^k of $D^k_S(\hat{f})$ (see Section A.2 for the definition of germs of complex spaces). We fix some particular notation for the case of monogerms: we denote by $\Delta(n,k)$ and $D(n,k)$ the germs of $\Delta(\mathbb{C}^n,k)$ and $D(\mathbb{C}^n,k)$ at 0. The ideals in $\mathscr{O}_{kn}$ defining these two space germs are

$$I_{\Delta(n,k)} = \sum_{l=2}^{k} \langle x_i^{(1)} - x_i^{(l)} \mid i = 1,\dots,n\rangle,$$

$$I_{D(n,k)} = \bigcap_{1\leq l<m\leq k} \langle x_i^{(l)} - x_i^{(m)} \mid i = 1,\dots,n\rangle.$$

As in the global case, we have $D^k_S(f) = \overline{(f^k)^{-1}(\Delta(p,k)) \setminus D(n,k)}$ and its defining ideal in $\mathscr{O}_{kn}$ is $\sqrt{P(f,k) : I_{D(n,k)}{}^{\infty}}$, with $P(f,k) = (f^k)^* I_{\Delta(p,k)}$.

Example 2.1.2. Take, as in Example B, the family of curves $f_t\colon \mathbb{C} \to \mathbb{C}^2$ given by

$$x \mapsto (x^2, x^3 + tx).$$

The map f_0 is just a usual cusp and, since it is a injective map, the space $D^k_S(f_0)$ is empty. For $t \neq 0$, a straightforward computation shows that $D^k_S(f_t)$ is the zero set of $\langle x^2 + t, x + x'\rangle$.

Take the ideal $I = \langle x^2 + t, x + x'\rangle$ in $\mathscr{O}_3$ (variables x, x' and t) and let I_{t_0} be the ideal in $\mathscr{O}_2$ (variables x and x') obtained by the substitution $t = t_0$ in I. We observe

1. the ideal I_0 does not define $D^k_S(f_0)$, and

2. the ring $\mathscr{O}_2/I_0$ is not reduced.

The previous example shows that the closure of the set of strict multiple points does not define a satisfactory multiple point space. As we will show soon, a satisfactory definition of multiple point spaces has to include more points (not just the ones in the closure of the strict points) and has to sometimes yield non-reduced spaces. Before getting into the details, we need the following lemma:

Lemma 2.1.3. *For any multigerm $f\colon (\mathbb{C}^n, A) \to (\mathbb{C}^p, 0)$.*

1. *If $F = (s, f_s)$ and $F' = (s, f'_s)$ are $\mathcal{A}$-equivalent unfoldings of f, then*
$$D^k_S(F) \cap \{s = 0\} = D^k_S(F') \cap \{s = 0\}.$$

2. *If f is stable, then for any unfolding $F = (s, f_s)$ of f,*
$$D^k_S(F) \cap \{s = 0\} = D^k_S(f).$$

3. *If $F = (s, f_s)$ and $F' = (t, f'_t)$ are stable unfoldings of f, then*
$$D^k_S(F) \cap \{s = 0\} = D^k_S(F') \cap \{t = 0\}.$$

4. *Let $F = (s, f_s)$ and $F'(t, f'_t)$ be stable unfoldings of two germs f, f', respectively. If $f' = \psi \circ f \circ \phi^{-1}$, where ϕ, ψ are biholomorphisms, then*
$$\phi^k(D^k_S(F) \cap \{s = 0\}) = D^k_S(F') \cap \{t = 0\}.$$

Proof. 1) By hypothesis, we have $F' = \Psi \circ F \circ \Phi^{-1}$, where Φ, Ψ are unfoldings of the identity in $\mathbb{C}^n, \mathbb{C}^p$ respectively. On one hand, $\Phi(D^k_S(F)) = D^k_S(F')$. On the other hand, we write $\Phi = (s, \phi_s)$ with $\phi_0 = \mathrm{id}$, so
$$D^k_S(F) \cap \{s = 0\} = \Phi(D^k_S(F)) \cap \{s = 0\} = D^k_S(F') \cap \{s = 0\}.$$

2) Since f is a stable map and F is an unfolding of f, then F is $\mathcal{A}$-equivalent to the constant unfolding $(\mathrm{id} \times f)$. From (1), it follows
$$D^k_S(F) \cap \{s = 0\} = D^k_S(\mathrm{id} \times f) \cap \{s = 0\} = D^k_S(f).$$

3) Let $\mathcal{F}$ be the germ given by $\mathcal{F}(s, t, x) = (s, t, f_s(x) + f'_t(x) - f(x))$. This is an unfolding of both F and F'. Since F and F' are stable, (2) implies
$$D^k_S(F) \cap \{s = 0\} = D^k_S(\mathcal{F}) \cap \{s = t = 0\} = D^k_S(F') \cap \{t = 0\}.$$

4) Let G be the unfolding of f' given by $G(s, x) = (s, \psi \circ f_s \circ \phi^{-1})$. We have that $\Psi \circ F \circ \Phi^{-1} = G$, where $\Phi = \mathrm{id} \times \phi$ and $\Psi = \mathrm{id} \times \psi$, hence G is also stable. From (3) we obtain
$$D^k_S(G) \cap \{s = 0\} = D^k_S(F') \cap \{t = 0\}.$$

On the other hand, $\Phi^k(D^k_S(F)) = D^k_S(G)$ and thus,
$$\phi^k(D^k_S(F) \cap \{s = 0\}) = \Phi^k(D^k_S(F)) \cap \{s = 0\} = D^k_S(G) \cap \{s = 0\}.$$

□

To be precise about our scheme-theoretic requirements for multiple point spaces, we need to introduce some definitions. We call a **multiple point space structure** to any rule, denoted by D^k, which associates to any finite map $f\colon X \to Y$ a closed complex subspace $D^k(f)$ of X^k. The space $D^k(f)$ is called the **k-multiple point space of** f. Since we ask $D^k(f)$ to be a closed subspace of X^k, the structure sheaf $\mathcal{O}_{D^k(f)}$ is defined by some coherent ideal sheaf $\mathscr{I}^k(f)$ in $\mathcal{O}_{X^k}$.

Definition 2.1.4. For any multiple point space structure D^k, we define two conditions:

M1. If f is a stable map, then $D^k(f) = D^k_S(f)$.

M2. For any local unfolding F of f at an open neighbourhood $U \subseteq X$ (Definition 1.5.14), the map i^k sends $D^k(f) \cap U^k$ isomorphically to $D^k(F) \cap (U_{s_0})^k$.

Condition M1 may be thought as D^k being a simple choice for the multiple point structure. Condition M2 means, first, that D^k behaves well under deformations and, second, that $D^k(f)$ is determined by the multilocal behaviour of f at every collection of points. Now we show that this two conditions determine D^k uniquely.

Proposition-Definition 2.1.5. There exists a unique multiple point structure D^k satisfying M1 and M2. For any map $f\colon X \to Y$, we call $D^k(f)$ the **k-multiple point space** of f. For any point $w \in X^k$, the space $D^k(f)$ is given locally around w by

$$D^k(f) = (i^k)^{-1}(D^k_S(F) \cap (X_{s_0})^k),$$

where F is any local stable unfolding of f at w.

Proof. First of all, we set $D^k(f) \cap (X^k \setminus (f^k)^{-1}(\Delta(Y,k))) = \emptyset$. Now let $x^{(1)}, \dots, x^{(k)} \in (f^k)^{-1}(\Delta(Y,k))$ and take the finite multigerm $f\colon (X,A) \to (Y,y)$, where $A = \{x^{(1)}, \dots, x^{(k)}\}$ and $f(x^{(i)}) = y$. Taking local coordinates in X and Y, we have biholomorphisms ϕ and ψ such that $f = \psi \circ f' \circ \phi^{-1}$, for some $f'\colon (\mathbb{C}^n, A') \to (\mathbb{C}^p, 0)$. We define $D^k(f)$ in a neighbourhood of $(x^{(1)}, \dots, x^{(k)})$ as

$$D^k(f) = \phi^k(D^k_S(F') \cap \{s = 0\}),$$

where F' is any stable unfolding of f'. By Lemma 2.1.3, $D^k(f)$ is well defined and does not depend on the choice of ϕ, ψ and F'. Therefore, these spaces can be glued together to get a complex space defined globally. We will denote it by $D^k(f)$ and its defining ideal sheaf by $\mathscr{I}^k(f)$. It follows from the definition that $D^k(f)$ is given by

$$D^k(f) = (i^k)^{-1}(D^k_S(F) \cap (X_{s_0})^k),$$

where F is any local stable unfolding of f.

If f is stable, then we can take $F = f$ and hence $D^k(f) = D^k_S(f)$, so condition M1 is satisfied. Condition M2 is obvious as well. Taking local coordinates, it suffices to prove the claim for a multigerm $f\colon (\mathbb{C}^n, A) \to (\mathbb{C}^p, 0)$. Given any unfolding $F = (t, f_t)$ of f, we take $\mathcal{F}(s, t, f_{s,t})$ a stable unfolding of F. Then,

$$\begin{aligned} D^k(f) &= D^k_S(\mathcal{F}) \cap \{s = t = 0\} = (D^k_S(\mathcal{F}) \cap \{s = 0\}) \cap \{t = 0\} \\ &= D^k(F) \cap \{t = 0\}. \end{aligned}$$

Finally, we show the unicity. Let $\hat{D}^k$ be another k-multiple point structure satisfying M1 and M2. For any map f and for any local stable unfolding F of f, we have locally the following equalities:

$$D^k(f) = (i^k)^{-1}(D^k_S(F) \cap (X_{s_0})^k) = (i^k)^{-1}(\hat{D}^k(F) \cap (X_{s_0})^k) = \hat{D}^k(f).$$

$\square$

Example 2.1.6. We are going to compute the double point space of the cusp $f\colon (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ given by

$$x \mapsto (x^2, x^3).$$

Take the family of curves of Example 2.1.2 as an unfolding, that is, take the map germ $F\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, given by

$$(t, x) \mapsto (t, x^2, x^3 + tx).$$

Let ϕ and ψ be the local changes of coordinates $(t, x) \mapsto (t - x^2, x)$ and $(X, Y, Z) \mapsto (X - Y^2, Y, Z)$. The map $\phi \circ F \circ \psi$ is a cross-cap, and therefore F is stable. The strict double point space $D^2_S(F)$ is defined by the ideal

$$\sqrt{P(F, 2) : I_{D(n,2)}{}^{\infty}} = \langle t - t', x + x', t + x^2 \rangle.$$

The substitution $t = 0$ yields the double point space

$$D^2(f) = V(x + x', x^2).$$

Example 2.1.7. Let $A_\mu\colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ be the germ given by $x \mapsto x^{\mu+1}$. To compute its multiple point spaces, we need to take a stable unfolding F of A_μ. One can check (see [Gib79] for details) that the map germ $F\colon (\mathbb{C}^{\mu-1} \times \mathbb{C}, 0) \to (\mathbb{C}^{\mu-1} \times \mathbb{C}, 0)$, given by

$$(u_1, \ldots, u_{\mu-1}, x) \mapsto (u_1, \ldots, u_{\mu-1}, x^{\mu+1} + u_{\mu-1}x^{\mu-1} + \cdots + u_1 x),$$

is a minimal (in the sense of the number of parameters u_i) stable unfolding of A_μ.

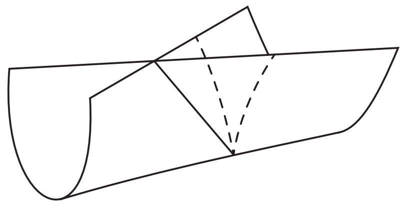

Figure 2.1: Image of the cross-cap, as an unfolding of the cusp.

For any k, the ideal $P(f,k)$ is generated by the germs $u_i^{(j)} - u_i^{(1)}$ and $\sum_i u_i^{(j)}(x^{(j)})^i - \sum_i u_i^{(1)}(x^{(1)})^i$, for $j = 2, \dots, k$,. The first generators allow us to eliminate the unnecessary copies $u^{(j)}$ to produce an isomorphic space, embeeded in $\mathbb{C}^{\mu-1} \times \mathbb{C}^k$, given by

$$\sum_i u_i((x^{(j)})^i - (x^{(1)})^i).$$

Now the ideal defining the k-multiple point space is obtained by, first, computing the saturation of this ideal with respect to the ideal $I_{D(1,k)}$, and then radical of the resulting ideal. A much easier way to obtain to this ideal is given by Proposition 2.2.2

As the previous example shows, the previous construction of multiple point space forces us to study maps between manifolds of arbitrarily big dimension. For double points in any corank and for k-multiple points of corank 1 map germs, this problem is adressed by Mond's ideals (Section 2.2 and Theorem 3.1.12).

Proposition 2.1.8. *Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a rank r map germ.*

1. *If f is of the form $(s,x) \mapsto (s, f_s(x)), s \in \mathbb{C}^r, x \in \mathbb{C}^{n-r}$, then the projection $P\colon \mathbb{C}^{nk} \to \mathbb{C}^r \times \mathbb{C}^{k(n-r)}$ which forgets the variables $s^{(2)}, \dots, s^{(k)}$ induces an isomorphism*

$$D^k_S(f) \cong \overline{\{(s,w) \in \mathbb{C}^r \times \mathbb{C}^{k(n-r)} \mid w \text{ is a strict multiple point of } f_s\}}.$$

2. *$D^k(f)$ embeds into $\mathbb{C}^r \times \mathbb{C}^{k(n-r)}$.*

Proof. 1) Let $Z = \{(s,w) \in \mathbb{C}^r \times \mathbb{C}^{kn} \mid w \text{ is a strict multiple point of } f_s\}$ and observe that 1) is a set theoretical question, since both $D^k_S(f)$ and $\overline{Z}$ are reduced. It is obvious that P restricts to a bijection

$$\{\text{strict } k\text{-multiple points of } f\} \to Z.$$

Therefore, $P(D^k_S(f)) \subseteq \overline{Z}$. Let $\gamma\colon D \to \overline{Z}$ be a curve defined in a neighboohood of the origin of $\mathbb{C}$, satisfying $\gamma(D \setminus \{0\}) \subseteq Z$. Let γ_i be the coordinate functions of γ and let

$$\sigma = (\gamma_1, \dots, \gamma_r) \text{ and } \omega = (\gamma_{r+1}, \dots, \gamma_{r+k(n-r)}).$$

Let $\gamma'\colon D \to X^k$ be defined by

$$\gamma'(t) = (\sigma(t), \dots, \sigma(t), \omega(t)),$$

with σ repeated k times. It is obvious that $\gamma'(t)$ is a strict k-multiple point of f, for all $t \in D \setminus \{0\}$. Therefore, we have $\gamma'(D) \subseteq D^k_S(f)$. Since $P \circ \gamma' = \gamma$, we obtain $\overline{Z} \subseteq P(D^k_S(f))$, as desired.

To show 2) take $F\colon (\mathbb{C}^l \times \mathbb{C}^n, 0) \to (\mathbb{C}^l \times \mathbb{C}^p)$ a stable unfolding of f of the form $F(t, s, x) = (t, s, f_{t,s}(x))$. By 1), we can eliminate the variables $t^{(2)}, \dots, t^{(k)}$ and $s^{(2)}, \dots, ^{(k)}$ to obtain an isomorphic image of $D^k_S(F)$ into $\mathbb{C}^{l+r} \times \mathbb{C}^{k(n-r)}$. Now, since $D^k(f) = D^k_S(F) \cap \{t^{(i)} = 0 \mid 1 \leq i \leq k\}$, the claim follows immediately. $\square$

Lemma 2.1.9. *If $f\colon X \to Y$ is a stable map, then the set of strict k-multiple points of f is empty or a manifold of dimension $kn - (k-1)p$. In particular, $D^k(f)$ is empty or a reduced unmixed space of dimension $kn - (k-1)p$ (see Definition A for unmixedness).*

Proof. The statement follows directly from Proposition 1.5.9 and Proposition 1.3.4. $\square$

Proposition 2.1.10. *For any $f\colon X \to Y$, any integer $k \geq 2$ and any point $w \in D^k(f)$, all non embedded irreducible components of the germ $(D^k(f), w)$ have dimension $\geq kn - (k-1)p$. In particular, the k-multiple point space $D^k(f)$ is empty or has dimension greater than or equal to $kn - (k-1)p$.*

Proof. Let $F(s, x) = (s, f_s(x)), s \in \mathbb{C}^r$ be a local stable unfolding of f, so that $D^k(f) = D^k(F) \cap \{s = 0\}$. By Lemma 2.1.9, $D^k(F)$ is empty or has dimension $k(r+n)-(k-1)(r+p) = kn-(k-1)p+r$. Let $w \in D^k(f)$, then $(0, w) \in D^k(F)$. By Lemma 2.1.9, all the irreducible components of $D^k(F)$ have dimension exactly $kn-(k-1)p+r$. Obviously, $\{s = 0\}$ is a manifold of dimension r. Thus, at every point, it consists of just one component of dimension r. Now the result follows directly from Proposition A.0.7. $\square$

We finish this section with an example of a corank 2 map germ (extracted from [MNB08]), to which we will come back several times in order to illustrate different constructions.

Example 2.1.11. Let $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be given by

$$(x, y) \mapsto (x^2, y^2, x^3 + y^3 + xy).$$

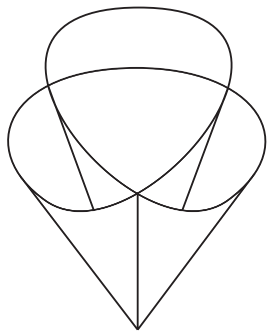

Figure 2.2: Image of the map germ in Example 2.1.11.

Computing the double point space of f as explained in Proposition-Definition 2.1.5 can be quite difficult. However, as we will see in Theorem 3.1.12, there is a straightforward way to calculate double point spaces (the details for this particular map are given in Example 3.1.2). The resulting ideal $I^2(f)$ is generated in $\mathcal{O}_{2n}$ by

$$\begin{aligned} g_1 &= (x+x')(x-x'), & g_4 &= (y+y')(2x^2+2xx'+2x'^2+y+y'),\\ g_2 &= (y+y')(y-y'), & g_5 &= (x+x')(x+x'+2y^2+2yy'+2y'^2),\\ g_3 &= (x+x')(y+y'), & g_6 &= (x-x')(2x^2+2xx'+2x'^2+y+y')+\\ & & &+(y-y')(x+x'+2y^2+2yy'+2y'^2). \end{aligned}$$

The reason why we factorize the generators this way will become clear in Example 2.4.4.

2.2 Multiple points of corank 1 monogerms

In [Mon87] Mond introduces some ideals $I^k(f)$ for corank 1 map germs which define the multiple point spaces. These ideals are obtained directly from the original map, with no need to take any unfolding. Moreover, in [MM89] Marar and Mond show that stability and finite determinacy of corank 1 map germs $f\colon(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$, $n<p$, can be characterized by the geometry of the multiple point spaces.

Let $f\colon(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$ be a corank 1 map germ with $n\leq p$. Up to $\mathcal{A}$-equivalence, f can be written in the form

$$(x,y)\mapsto(x,f_n(x,y),\dots,f_p(x,y)),$$

with $x\in\mathbb{C}^{n-1}$ and $y\in\mathbb{C}$. We can think of $f(x,y)$ as a $(n-1)$-parameter family of functions of one variable $f_x(y)=(f_n(x,y),\dots,f_p(x,y))$. Embedding $D^2(f)$ in $\mathbb{C}^{n-1}\times\mathbb{C}^2$ (see Proposition 2.1.8), a point (x,y,y') is a double point if and only if the coefficients of the Newton interpolating polynomial of degree 1 for the points $(y, f_{j,x}(y)), (y', f_{j,x}(y'))$ are equal to 0, for all $n \leq j \leq p$. These coefficients, the generators of **Mond's double point ideal** $I^2(f)$, are the divided differences

$$f_{j,x}[y, y'] = \frac{f_j(x, y) - f_j(x, y')}{y - y'}.$$

Similarly, the triple points are $(x, y, y', y'') \in \mathbb{C}^{n-1} \times \mathbb{C}^3$ such that every coefficient of the Newton interpolating polynomial of degree 2 for the points $(y, f_{j,x}(y)), (y', f_{j,x}(y')), (y'', f_{j,x}(y''))$ are equal to 0 for every $n \leq j \leq p$. These coefficients are the divided differences $f_{j,x}[y, y']$ and the iterated divided differences

$$f_{j,x}[y, y', y''] = \frac{f_{j,x}[y, y'] - f_{j,x}[y, y'']}{y' - y''}.$$

Hence, **Mond's triple point ideal** is

$$I^3(f) = \langle f_{j,x}[y, y'], f_{j,x}[y, y', y''] \mid n \leq j \leq p \rangle.$$

Higher order k-tuple ideals $I^k(f)$ are defined analogously.

Remark 2.2.1. As observed by Marar and Mond [MM89], the coefficients of the Lagrange interpolation polynomial provide another set of generators for the ideal defining the k-tuple points, with the advantage that they are invariant under the action of the symmetric group S_k.

Proposition 2.2.2. *[MM89, Prop. 2.16] For any corank 1 map germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, the ideal $I^k(f)$ defines $D^k(f)$.*

Theorem 2.2.3. *[MM89, Thm. 2.14] Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a finite corank 1 map, with $n < p$. Then*

1. *f is stable if and only if $D^k(f)$ is empty or smooth of dimension $p - k(p - n)$, for every $k \geq 2$.*
2. *f is finitely determined if and only if $D^k(f)$ is empty or an ICIS of dimension $p - k(p - n)$, for all k satisfying $p - k(p - n) \geq 0$, and $D^k(f) \subseteq \{0\}$ for $p - k(p - n) < 0$.*

We will show in Proposition 3.2.3 that the double point locus of a stable map may contain singularities, provided that the map has some corank ≥ 2 points. Thus, we can not expect a criterion so simple for the corank ≥ 2 case.

The previous theorem motivates the following definition, due to Kevin Houston [Hou94]:

Definition 2.2.4. For any $f\colon X \to Y$ the k-multiple point space $D^k(f)$ is **dimensionally correct** if it is empty or has dimension $kn - (k - 1)p$.

The following is still, to our knowledge, an open question: Is it true that the multiple point spaces are Cohen Macaulay if they are dimensionally correct? See the Open Problem 3 for a more detailed explanation.

2.3 Source multiple points

Definition 2.3.1. Let $f\colon U \to V$ and let $p\colon D^k(f) \to U$ be the restriction to $D^k(f)$ of the projection $U^k \to U$ on the first coordinate. The **source k-multiple point space** is the complex space defined by the 0-Fitting ideal sheaf (Definition A.3.1) of the push-forward module $p_*\mathcal{O}_{D^k(f)}$, that is:

$$D_1^k(f) = V(\mathcal{F}_0(p_*\mathcal{O}_{D^k(f)})).$$

In the case $k = 2$ we write $D(f) = D_1^2(f)$. From Proposition B.1.2, we obtain the set-theoretical equality

$$D_1^k(f) = p(D^k(f)).$$

The definition goes analogously for multigerms $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, that is, $D_1^k(f) = V(F_0(p_*\mathcal{O}_{D^k(f)}))$. For any map $f\colon U \to V$ and any point $z \in U$, we have $D(f)_z = D(f_z)$.

Remark 2.3.2. Computing the ideal $F_0(p_*\mathcal{O}_{D^k(f)})$ which defines the source double points of a map germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ can be quite involved. However, for double points in the case $p = n + 1$, if $D^2(f)$ is dimensionally correct, then $p\colon D^2(f) \to \mathbb{C}^n$ is a map from a Cohen Macaulay space of dimension $n - 1$ to $\mathbb{C}^n$ (Lemma 3.1.10). Then we can proceed as explained in Section 2.5.

Example 2.3.3. Let $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, as in Example 2.1.11, be given by

$$(x, y) \mapsto (x^2, y^2, x^3 + y^3 + xy).$$

The projection $p\colon D^2(f) \to (\mathbb{C}^2, 0)$, is just $(x, y, x', y') \mapsto (x, y)$. The ideal $F_0(p_*\mathcal{O}_{D^2(f)})$ is generated by the determinant of the following presentation matrix of $p_*\mathcal{O}_{D^2(f)}$, obtained by means of the SINGULAR libary mentioned in Remark 2.5.8 :

$$\begin{pmatrix} y & -1 & 0 & 0 & 0 & 0 \\ 0 & y & 0 & 0 & -1 & 0 \\ 2xy + x^3 & 0 & y - x^2 & x - x^2y & -x^2 & -2 \\ 0 & x^2 & 0 & y + x^2 & 1 & 0 \\ 0 & x^2y & 0 & x^2y & y & 1 \\ x^2 & -x^3 & x + x^2y & -xy & -x + x^2y & 2y + x^2 \end{pmatrix}$$

Thus, the source double point space of f is the zero set of

$$F_0(p_*\mathcal{O}_{D^2(f)}) = \langle (x^3 + y^3)(x + y^2)(y + x^2) \rangle.$$

In Corollary 3.2.9 we will show that for generic maps germs $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$, the space $D(f)$ is a reduced hypersurface. This will provide a useful criterion of finite determinacy for corank 2 map germs $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ (Corollary 5.1.3): f is finitely determined if and only if $D(f)$ is a reduced curve.

Remark 2.3.4. There is another way to define multiple point spaces in the source, which consists in taking the preimage by f of the target multiple point space $M_k(f)$ (Section 2.5). This is the approach used by Kleiman [Kle81] and others (notation caution: Kleiman denotes our $M_k(f)$ by N_k, and denotes by M_k the space $f^{-1}(M_k(f))$). We make use of this space in Section 5.3. The relation between both structures is, to our knowledge, still unclear (see Open Problem 6).

2.4 Quotient multiple points

Let S_k be the permutation group of k points, acting on X^k coordinatewise.

Lemma 2.4.1. *For any map $f\colon X \to Y$, the ideal sheaf $\mathscr{I}^k(f)$ is S_k-invariant.*

Proof. Let $F\colon S \times X \to Y$ be an unfolding of f of the form $F(s,x) = (s, f_s(x))$, then $h \in \mathscr{I}^k(F)$ if and only if h vanishes on all points

$$(s^{(1)}, x^{(1)}, \dots, s^{(k)}, x^{(k)}) \in (S \times X)^{(k)}$$

satisfying $F(s^{(1)}, x^{(1)}) = F(s^{(i)}, x^{(i)})$, for $i = 2, \dots, k$. It is obvious that $\mathscr{I}^k(F)$ is S_k-invariant. The claim follows, since the ideal $\mathscr{I}^k(f)$ is obtained by adding to $\mathscr{I}^k(F)$ the ideal $\langle s^{(1)}, \dots, s^{(k)}\rangle$, which is also S_k-invariant. $\square$

Definition 2.4.2. For any map $f\colon U \to V$, we define its **quotient multiple point space** as the quotient space $D^k(f)/S_k$ of the complex space $D^k(f)$ by the action of the permutation group S_k (see [Fis76, 1.26]). The definition extends to germs taking representatives.

The underlying space of $D^k(f)/S_k$ is the quotient, as a topological space, of $D^k(f)$ by S_k. The structure sheaf $\mathcal{O}_{D^k(F)/S_k}$ is given locally at a class $[w] \in D^k(f)/S_k$ by the invariant subalgebra:

$$\mathcal{O}_{D^k(F)/S_k,[w]} = \Big(\bigoplus_{w' \in [w]} \mathcal{O}_{D^k(F),w'}\Big)^{S_k},$$

Observe that since $\mathcal{O}_{X/S_2,[w]}$ is a subalgebra of $\bigoplus_{w' \in [w]} \mathcal{O}_{D^k(F),w'}$, then X/S_2 is reduced at $[w]$ if X is reduced at all $w' \in [w]$.

How to compute $D^k(f)/S_k$

Let $D^k(f) = V(I^k(f))$ be the k-multiple point space of a monogerm $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. To embed $D^2(f)/S_k$ in some $\mathbb{C}^m$, we have to express the invariant algebra

$$\mathcal{O}_{D^k(f)/S_k,0} = \mathcal{O}_{kn}^{S_k}/(I^k(f)^{S_k}).$$

as an analytic algebra $\mathcal{O}_m/J$. This can be done as follows:

1. Find a system of invariants $\alpha_i, 1 \leq i \leq kn$, and some funtions $\beta_j, 1 \leq j \leq l$, such that

$$\mathcal{O}_{nk} = \bigoplus \beta_j \alpha^* \mathcal{O}_{kn},$$

where $\alpha\colon \mathbb{C}^{kn} \to \mathbb{C}^{kn}$ is the map with coordinate functions $\alpha_1, \ldots, \alpha_{kn}$ (see Theorem B.2.3). Let $m = kn + l$ and define

$$\psi\colon \mathbb{C}^{kn} \to \mathbb{C}^m$$

as the map with coordinate functions α_i and β_j. It is immediate that $\psi^*\colon \mathcal{O}_m \to \mathcal{O}_{kn}^{S_k}$ is an epimorphism.

2. Obtain the ideal $(I^k(f))^{S_k} = I^k(f) \cap \mathcal{O}_{kn}^{S_k}$. If $I^k(f)$ is generated by $h_1, \ldots, h_r$ then, from Lemma B.2.8 it follows that $(I^k(f))^{S_k}$ is generated in $\mathcal{O}_{kn}^{S_k}$ by the elements

$$(\beta_j h_i)^{\#}, \quad j = 1, \ldots, l, \quad i = 1, \ldots, r.$$

3. Now $\mathcal{O}_{D^k(f)/S_k} \cong \mathcal{O}_m/J$, with $J = (\psi^*)^{-1}((I^k(f))^{S_k})$.

In the following examples we compute the embedding of $D^2(f)/S_2$ for two given monogerms, one of corank 1 and the other of corank 2.

Example 2.4.3. Let $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be the corank 1 map germ given by

$$(x, y) \mapsto (x, xy + y^3, xy^2 + cy^4), c \in \mathbb{C}.$$

As we saw in Section 2.2, $D^2(f)$ embeds in $\mathbb{C} \times \mathbb{C}^2$ (variables x, y, y') and the ideal $I^2(f)$ is generated by

$$g_1 = x + y^2 + yy' + y'^2, \text{ and}$$

$$g_2 = x(y + y') + c(y^3 + y^2y' + yy'^2 + y'^3).$$

Observe that after the embedding in $\mathbb{C} \times \mathbb{C}^2$, the permutation group S_2 acts only in the variables y, y'. Since g_1 and g_2 are both S_2-invariant, from Lemma B.2.8 we get that $I^2(f)^{S_2}$ is precisely the ideal generated by g_1, g_2 in $\mathcal{O}_3^{S_2}$. Let $\psi\colon (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ be given by

$$(x, y, y') \mapsto (x, y + y', (y - y')^2).$$

From Example B.3.4 it follows that $\psi^*\colon \mathcal{O}_3 \to \mathcal{O}_3^{S_2}$ is an isomorphism. Therefore $J = \psi^*(I^2(f)^{S_2})$ is just the ideal generated by the expressions of g_1 and g_2 in the variables $x, \sigma_1 = (y + y')$ and $\sigma_2 = (y - y')^2$. That is, $D^2(f)/S_2 \cong V(J)$, where

$$J = \langle 4x + 3\sigma_1^2 + \sigma_2, 2x\sigma_1 + c(\sigma_1^3 + \sigma_1\sigma_2)\rangle.$$

Example 2.4.4. Let $f\colon (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$, as in Example 2.1.11, be given by
$$(x,y) \mapsto (x^2, y^2, x^3+y^3+xy).$$
$I^2(f)$ is generated by the germs $g_1,\dots,g_6$ given in Example 2.1.11. The germs g_3, g_4 and g_5 are symmetric whereas g_1, g_2 and g_6 are antisymmetric. Thus, from Example B.3.9 it follows that $I^2(f)^{S_2}$ is generated by g_1, g_2, g_6 and $(x-x')g_j, (y-y')g_j, j = 3,4,5$. Let $\psi\colon \mathbb{C}^4 \to \mathbb{C}^5$ be given by
$$(x,y,x',y') \mapsto (x+x', (x-x')^2, y+y', (y-y')^2, (x-x')(y-y')).$$

From Lemma B.3.2 and Lemma B.3.3, it follows that morphism
$$\psi^*\colon \mathcal{O}_5 \to \mathcal{O}_4^{S_2}$$
is an epimorphism with
$$\ker\psi^* = \langle r_{12}^2 - r_{11}r_{22}\rangle,$$
where s_1, s_2, r_{11}, r_{12} and r_{22} are the variables in $\mathcal{O}_5$. To compute $J = \psi^*(I^2(f)^{S_2})$ we have to add $\ker\psi^*$ to the ideal generated by the expressions of the generators of $I^2(f)^{S_2}$ in the variables $s_1 = x+x'$, $s_2 = y+y'$, $r_{11} = (x-x')^2$,, $r_{12} = (x-x')(y-y')$ $r_{22} = (y-y')^2$ and . Therefore, $D^2(f)/S_2 \cong V(J)$, where J is the ideal generated by

$$\begin{aligned}
h_1 &= s_1 r_{11}, & h_6 &= 2s_1^2 + s_1 r_{22},\\
h_2 &= s_1 s_2, & h_7 &= 2s_2^2 + s_2 r_{11},\\
h_3 &= s_1 r_{12}, & h_8 &= r_{11}^2 + r_{22}r_{12} + 2r_{11}s_2,\\
h_4 &= s_2 r_{22}, & h_9 &= r_{22}^2 + r_{11}r_{12} + 2s_1 r_{22},\\
h_5 &= s_2 r_{12}, & h_{10} &= r_{12}^2 - r_{11}r_{22}.
\end{aligned}$$

The explicit method to compute $D^k(f)/S_k$ given above yields the next result:

Proposition 2.4.5. *Let $f\colon (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$ be a corank 1 map germ. If $\dim D^k(f) = kn-(k-1)p$, then $D^k(f)/S_k$ is a complete intersection.*

Proof. By Proposition 2.1.8, $D^k(f)$ embeds in $\mathbb{C}^{n-1}\times\mathbb{C}^k$ and the action S_k induces by permutation of the coordinates in $\mathbb{C}^k$. The group S_k acting on $\mathbb{C}^k$ by permutation of coordinates is a reflection group (i.e. generated by reflections). Therefore, Shephard-Todd's Theorem B.2.5 implies that $\mathbb{C}^k/S_k$ is a manifold of dimension k. From Remark 2.2.1, it follows that the ideal defining $D^k(f)$ in $\mathbb{C}^{n-1}\times\mathbb{C}^k$ can be generated by exactly $(k-1)(p-n+1)$ S_k-invariant germs g_i in $\mathcal{O}_{n-1+k}$. From Lemma B.2.8 it follows that $D^k(f)/S_k$ is isomorphic to the subspace of $\mathbb{C}^{n-1}\times\mathbb{C}^k/S_k$

given by the vanishing of the expressions of g_i in the symmetric variables. We have $(k-1)(p-n+1)$ generators in regular ring of dimension $n-1+k$, and the claim follows. □

Remark 2.4.6. In the corank ≥ 2 case the situation is more complicated. We can not embeed $D^k(f)/S_k$ in $\mathbb{C}^{n-1}\times\mathbb{C}^k$, where the group acts as a reflection group. Therefore, the ambient space $\mathbb{C}^{kn}/S_k$, where we embeed $D^k(f)/S_k$, is not smooth anymore (it is, however, a Cohen Macaulay space of dimension kn, by Corollary B.2.4). Moreover, we can not expect the generators of $I^k(f)$ to be S_k-invariant. Thus, when we apply Lemma B.2.8, the number of generators of $(I^k(f))^{S_k}$ grows substantially (see Example 2.4.4, were we obtained 9 generators of $(I^k(f))^{S_k}$ from a collection of 6 generators of $I^k(f)$).

2.5 Target multiple points

The definition of the target multiple point space we use is due to Mond and Pellikaan [MP89] and also Kleiman [Kle81]. This definition, unlike the source and quotient multiple point spaces we introduced before, is independent of the multiple point structure $D^k(f)$. As a matter of fact, the relations between these two approaches are not well known (see Open Problem 6). The idea is simple: Given a map $f\colon X\to Y$, a k-multiple point in the target is a point $y\in Y$ which has at least k preimages (counting with multiplicity). As Proposition A.3.2 shows, the adequate tool for this definition are the Fitting ideal sheaves of the pushforward module $f_*\mathcal{O}_X$.

Definition 2.5.1. The **k-multiple point space of $f\colon X\to Y$ in the target** is the complex space

$$M_k(f)=V(\mathcal{F}_{k-1}(f_*\mathcal{O}_X)).$$

In the case $k=2$, we write $f(D(f))=M_2(f)$. For a multigerm $f\colon(\mathbb{C}^n,S)\to(\mathbb{C}^p,0)$, the k-multiple point space in the target is $M_k(f)=V(F_{k-1}(f_*\mathcal{O}_{\mathbb{C}^n,S}))$. Of course, this space agrees with the germ at 0 of the k-multiple point space in the target of any representative defined in a small enough neighborhood of S.

Observe that the terminology $f(D(f))$ is just a notation, we are not exactly taking the scheme-theoretical image of the complex space $D(f)$ by the finite morphism f. However, we will see that the underlying set (or set germ) of $f(D(f))$ is precisely the image of $D(f)$ by f (Remark 2.6.1).

Proposition 2.5.2. *[MP89, Thm. 3.4] If $p=n+1$ and f is a generically one-to-one, then $f(D(f))$ is determinantal (in particular, it is Cohen Macaulay).*

Algorithms to compute presentation matrices

Here we discuss how to compute the presentation matrix of the module $f_*\mathcal{O}_X$ for a finite map $f\colon X \to \mathbb{C}^{n+1}$, when X is a Cohen Macaulay space of dimension n. First we explain an algorithm, due to Mond and Pellikaan [MP89], to compute such a presentation matrix. It turns out that the algorithm has some issues when restricted to work with polynomial data. We introduce a slight improvement of the algorithm, obtained in collaboration with M. E. Hernandes and A. J. Miranda, which behaves better for computational purposes.

Mond-Pellikaan's algorithm consists briefly on this: Let $f_1, \dots, f_{n+1}$ be the coordinate functions of f. After a generic linear change of coordinates in the target, we may assume that the map

$$\tilde{f} = (f_1, \dots f_n)\colon X \to (\mathbb{C}^n, 0)$$

is finite. If $g_1, \dots g_h$ are generators of $\tilde{f}_*\mathcal{O}_X$, then they are generators of $f_*\mathcal{O}_X$ as well. Therefore, we obtain an epimorphism $\psi\colon \mathcal{O}^r_{n+1} \to \mathcal{O}_X$ mapping the canonical vector e_i to the generator g_i. For any $1 \leq i \leq h$, there exist germs $a_{ij} \in \mathcal{O}_n$, $1 \leq j \leq h$ satisfying

$$f_{n+1} g_i = \sum_{j=1}^{h} \tilde{f}^* a_{ij} g_j.$$

If we denote by $X_1, \dots, X_{n+1}$ the variables in $\mathbb{C}^{n+1}$ and δ_{ij} is the Kronecker's delta function, then the matrix $M(f)$ with entries

$$a_{ij}(X_1, \dots, X_n) - \delta_{ij} X_{n+1}$$

is a presentation matrix for $f_*\mathcal{O}_X$.

Example 2.5.3. Let $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, as in Example 2.1.11, be given by

$$(x, y) \mapsto (x^2, y^2, x^3 + y^3 + xy).$$

We have $\tilde{f}(x, y) = (x^2, y^2)$ and the pushforward module $\tilde{f}_*\mathcal{O}_2$ is generated by $g_1 = 1, g_1 = x, g_3 = y, g_4 = xy$. If we denote by X, Y, Z the target coordinates, then the presentation matrix of $f_*\mathcal{O}_2$ obtained by Mond-Pellikaan's algorithm is

$$\begin{pmatrix} -Z & X^2 & Y^2 & XY \\ X & -Z & Y & Y^2 \\ Y & X & -Z & X^2 \\ 1 & Y & X & -Z \end{pmatrix}.$$

We obtain the following Fitting ideals:

1. $F_0(f_*\mathcal{O}_2) = \langle X^2Y^2 - 2XYZ^2 + Z^4 - 2X^4Y - 2XY^4 - 8X^2Y^2Z - 2X^3Z^2 - 2Y^3Z^2 + X^6 - 2X^3Y^3 + Y^6 \rangle$, which defines the image of f.

2. $F_1(f_*\mathcal{O}_2) = \langle Y^2+XZ, Z+XY, -Y+X^2\rangle \cap \langle X+Y-Z, Y^2-YZ+Z^2\rangle \cap \langle Y+Z, X+Z\rangle \cap \langle -X+Y^2, Z+XY, X^2+YZ\rangle$, which defines the target double point space of f.

3. $F_2(f_*\mathcal{O}_2) = \langle X, Y, Z\rangle$, which defines the target triple point space.

The rest of this section is a collaboration with M. E. Hernandes, A. J. Miranda [HMPS15]. We call a Mond-Pellikaan matrix (M-P matrix for short) the matrix $M(f)$ with entries $a_{ij}(X_1,\dots,X_n) - \delta_{ij}X_{n+1}$ of Mond-Pellikaan's algorithm. It is obvious that, if $\tilde{f}$ is finite, then a M-P matrix with entries in $\mathcal{O}_{n+1}$ can be obtained. However, to make an computer implementation of Mond-Pellikaan's algorithm, we have to face the fact that most commutative algebra software (for example SINGULAR [WGPS15]) only deal with polynomial data. The problem, as the following example shows, is that certain maps do not admit a P-M matrix with polynomial entries (even if they do admit very easy polynomial presentations!).

Example 2.5.4. Let be $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ the map germ given by

$$(x, y) \mapsto (xy, x + y + x^2, x).$$

We have $\tilde{f}(x, y) = (xy, x + y + x^2)$. A minimal system of generators of $\tilde{f}_*\mathcal{O}_2$ is $g_1 = 1, g_2 = y$. If $\tilde{f}_*\mathcal{O}_2$ admits a polynomial M-P matrix, then there exist polynomials $a_1, a_2 \in \mathbb{C}[X_1, X_2]$, satisfying

$$x = \tilde{f}_*a_1 \cdot g_1 + \tilde{f}_*a_2 \cdot g_2 = a_1(xy, x+y+x^2)\cdot 1 + a_2(xy, x+y+x^2)\cdot y. \quad (2.1)$$

Since $\{\overline{1}, \overline{x}, \overline{y}\}$ is a basis for the $\mathbb{C}$-vector space $\mathbb{C}[x, y]/\tilde{f}^*\mathfrak{m}$, then x does not belong to the submodule of $f_*\mathbb{C}[x, y]$ generated by g_1 and g_2. We conclude that the equation above cannot be satisfied by polynomial elements a_1 and a_2, and thus there is no M-P type presentation for $f_*\mathcal{O}_2$.

Although the previous example does not admit a polynomial M-P matrix, one can check (or just wait to Theorem 2.5.6) that the following polynomial matrix is, with respect to the generators 1 and y, a presentation of $f_*\mathcal{O}_2$:

$$\lambda = \begin{pmatrix} Y\cdot(1+Y) - X_2 & 1 \\ -X_1 & Y \end{pmatrix}.$$

Thus, we must consider a different, wider than M-P, class of matrices for our presentations.

Definition 2.5.5. Let $f\colon \mathcal{X} \to (\mathbb{C}^{n+1}, 0)$ be map germ, with $\mathcal{X} = V(I)$ a germ of n-dimensional Cohen Macaulay space and let $g_1, \dots, g_h$ be a minimal set of generators of $\tilde{f}_*\mathcal{O}_\mathcal{X}$. For any $h \times h$ matrix λ with entries in $\mathcal{O}_{n+1}$, we define the following conditions:

C1. $\displaystyle\sum_{i=1}^{h} f^*\lambda_{ij}\cdot g_i \equiv 0 \mod I,\ j = 1, \dots, h.$

C2. $\det(\lambda(0, X_{n+1}) - \lambda(0,0)) = X_{n+1}^h \cdot u(X_{n+1}),\ u(0) \neq 0.$

Observe that **C1** and **C2** hold for all M-P matrices.

Theorem 2.5.6. *In the conditions above, every matrix λ satisfying* **C1** *and* **C2** *is a presentation matrix of $f_*\mathcal{O}_{n+1}$.*

Proof. We follow the steps in the proof of Mond-Pellikaan algorithm.

Consider $\lambda : \mathcal{O}_{n+1}^h \to \mathcal{O}_{n+1}^h$ the homomorphism associated to the matrix Λ and $\psi : \mathcal{O}_{n+1}^h \to \mathcal{O}_{\mathcal{X}}$ such that $\psi(e_i) = g_i$ where $\{e_i;\ i = 1, \dots, h\}$ is the canonical basis for $\mathcal{O}_{n+1}^h$. It is immediate that **C1** implies $\operatorname{Im}\lambda \subseteq \operatorname{Ker}\psi$. Since ψ is surjective, it suffices to show $\operatorname{Coker}\lambda \cong \mathcal{O}_{\mathcal{X}}$ to prove the desired exactness.

We embed $\mathcal{X} \subset \mathcal{X} \times (\mathbb{C}, 0)$ and define the map germ

$$\begin{array}{rccc} F: & \mathcal{X} \times (\mathbb{C},0) & \longrightarrow & (\mathbb{C}^{n+1},0) \\ & (x,t) & \longmapsto & (f_1, \dots, f_{n+1}+t). \end{array}$$

Since $g_1, \dots, g_h$ generate the module $\tilde{f}_*\mathcal{O}_n$, it is immediate that they generate $F_*\mathcal{O}_{n+1}$ as well. For $j = 1, \dots, h$, we have

$$\begin{array}{rcl} \sum_{i=1}^h F^*\Lambda_{ij} \cdot g_i & = & \sum_{i=1}^h (\Lambda_{ij}(f_1, \dots, f_{n+1}+t)) \cdot g_i \\ & = & \sum_{i=1}^h (\Lambda_{ij}(f_1, \dots, f_{n+1})) \cdot g_i + \\ & & + \sum_{i=1}^h (\sum_{k=1}^\infty \frac{1}{k!} \frac{\partial^k \Lambda_{ij}}{\partial Y^k}(f_1, \dots, f_{n+1}) t^k) \cdot g_i \\ & = & t \sum_{i=1}^h (\sum_{k=1}^\infty \frac{1}{k!} \frac{\partial^k \Lambda_{ij}}{\partial Y^k}(f_1, \dots, f_{n+1}) t^{k-1}) \cdot g_i. \end{array}$$

Let R be the $h \times h$ matrix with entries

$$R_{ij} := \sum_{k=1}^{\infty} \frac{1}{k!} (f^* \frac{\partial^k \Lambda_{ij}}{\partial Y^k}) t^{k-1}.$$

We claim that R is invertible in $\mathcal{O}_{\mathcal{X}\times(\mathbb{C},0)}$. This is equivalent to say that $\det(R)$ does not vanish at the origin of $\mathcal{X} \times (\mathbb{C}, 0)$. By construction of R, and since f is a finite map taking the origin of $\mathcal{X}$ to the origin of $\mathbb{C}^{n+1}$, this is equivalent to say that $\det(\frac{\partial \Lambda_{ij}}{\partial Y}(0,0)) \neq 0$. The claim follows, since condition **C2** tells us

$$\det(\frac{\partial \Lambda_{ij}}{\partial Y}(0,0)) = \det(\lim_{Y \to 0} \frac{\Lambda_{ij}(0,Y) - \Lambda_{ij}(0,0)}{Y}) = u(0) \neq 0.$$

Since $\mathcal{O}_{\mathcal{X}\times(\mathbb{C},0)}$ is Cohen Macaulay, it is generated freely by $g_1, \dots, g_h$ via F. Therefore, we can define a map

$$\varphi\colon \mathcal{O}_{\mathcal{X}\times(\mathbb{C},0)} \to \mathcal{O}_{\mathcal{X}\times(\mathbb{C},0)},$$

by extending $g_j \mapsto \sum_{i=1}^h F^*\Lambda_{ij} \cdot g_i$ linearly. By construction, we have the matrix equality

$$\begin{bmatrix} \varphi(g_1) \\ \vdots \\ \varphi(g_h) \end{bmatrix} = t \begin{bmatrix} R_{11} & \cdots & R_{1h} \\ \vdots & \ddots & \vdots \\ R_{h1} & \cdots & R_{hh} \end{bmatrix} \cdot \begin{bmatrix} g_1 \\ \vdots \\ g_h \end{bmatrix}.$$

Since R is invertible and $g_1, \dots, g_h$ generate $\mathcal{O}_{\mathcal{X}\times(\mathbb{C},0)}$ as a ring, we obtain $\mathrm{Im}(\varphi) = \langle t \rangle \subset \mathcal{O}_{\mathcal{X}\times(\mathbb{C},0)}$. Since $\{g_1, \dots, g_h\}$ is a free $\mathcal{O}_{n+1}$-basis, we have an isomorphism $\eta : \mathcal{O}_{n+1}^h \to \mathcal{O}_{\mathcal{X}\times(\mathbb{C},0)}$ given by

$$(a_1, \dots, a_h) \mapsto \sum_{i=1}^{h} F^* a_i g_i$$

The following diagram comutes:

$$\begin{tikzcd}
\mathcal{O}_{n+1}^h \arrow[r, "\lambda"] \arrow[d, leftrightarrow, "\eta"] & \mathcal{O}_{n+1}^h \arrow[r, "\psi"] \arrow[d, leftrightarrow, "\eta"] & \operatorname{Coker}\lambda \arrow[r] & 0 \\
\mathcal{O}_{\mathcal{X}\times(\mathbb{C},0)} \arrow[r, "\varphi"] & \mathcal{O}_{\mathcal{X}\times(\mathbb{C},0)} \arrow[r] & \operatorname{Coker}\varphi \arrow[r] & 0.
\end{tikzcd}$$

Thus, we obtain $\operatorname{Coker}\lambda \cong \operatorname{Coker}\varphi = \frac{\mathcal{O}_{\mathcal{X}\times(\mathbb{C},0)}}{im(\varphi)} = \frac{\mathcal{O}_{\mathcal{X}\times(\mathbb{C},0)}}{\langle t \rangle} \cong \mathcal{O}_{(\mathcal{X},0)}$, as desired. $\square$

The following easy proposition shows that, in a computationally reasonable setting, matrices satisfying **C1** and **C2** exist. We assume that f is a polynomial map taking the origin to the origin, restricted to a variety $\mathcal{X} = V(I)$ which contains the origin. Assume that I is generated by polynomials, and let $\mathbb{C}[\mathcal{X}]$ be the affine coordinate ring of $\mathcal{X}$ and $\mathfrak{m}$ its maximal ideal at the origin. We denote by $\mathbb{C}[\mathcal{X}]_\mathfrak{m}$ the localization of $\mathbb{C}[\mathcal{X}]$ at $\mathfrak{m}$.

Proposition 2.5.7. *With the previous notations, if $g_1, \dots, g_h$ are rational functions generating $\tilde{f}_*(\mathbb{C}[\mathcal{X}]_\mathfrak{m})$, then f admits a matrix with polynomial entries satisfying* **C1** *and* **C2**.

Proof. The procedure is very similar to the construction of a M-P matrix: Denote by $X_1, \dots, X_{n+1}$ the variables in $\mathbb{C}^{n+1}$. By hypothesis, there exist $a_{ij} \in \mathbb{C}[X_1, \dots, X_n]$ and $b_{ij} \in \mathbb{C}[X_1, \dots, X_n] \setminus \mathfrak{m}$, $1 \le j \le h$, satisfying

$$f_{n+1} g_i = \sum_{j=1}^{h} \tilde{f}^* (\frac{a_{ij}}{b_{ij}}) g_j.$$

Now let $B_i = \prod_{j=1}^{h} b_{ij} \in \mathbb{C}[X_1, \dots, X_n] \setminus \mathfrak{m}$ and let $M(f)$ be the matrix with entries

$$a_{ij}(X_1, \dots, X_n) - \delta_{ij} B_i X_{n+1}.$$

The matrix $M(f)$ satisfies **C1** and **C2** immediately. $\square$

Remark 2.5.8. At the webpage of A. J. Miranda [Mir] one can find a Singular library to compute presentation matrices based on these results.

2.6 A diagram of multiple points

For any $f\colon X \to Y$, there is a commutative diagram

$$\begin{array}{ccc} D^k(f) & \longrightarrow & D^k(f)/S_k \\ \big\downarrow & & \big\downarrow \\ D_1^k(f) & \longrightarrow & M_k(f) \end{array}$$

with arrows as follows:

The top arrow $D^k(f) \to D^k(f)/S_k$ is the quotient map. It is always surjective and it is generically $k!$-to-one whenever the k-multiple points of f are dense in $D^k(f)$.

The left arrow $p\colon D^k(f) \to D_1^k(f)$ is the restriction of the projection on the first coordinate (see Definition 2.3.1). It is always surjective and it is generically $(k-1)!$-to-one whenever the strict k-multiple points of f are dense in $D^k(f)$.

The bottom arrow $f\colon D_1^k(f) \to M_k(f)$ is the corresponding restriction of f. It is generically k-to-one whenever the strict k-multiple points of f are dense in $D^k(f)$. Observe that the fact that the target of this restriction is $M_k(f)$ needs a proof: Assume first that f is stable. Therefore the space $D_1^k(f)$ is the closure of the projection of all strict k-multiple points of f. For every k-multiple strict point $x \in X$, we have $\dim_{\mathbb{C}} \mathcal{O}_{X,x}/f^*\mathfrak{m}_{f(x)} > 0$. Since there are at least k different points in $f^{-1}(f(x))$, the claim follows by Proposition A.3.2. If f is not stable, then we take, around $f^{-1}(f(x))$, a local stable unfolding of the form $F(s,x) = (s, f_s(x))$. The claim holds for F and, since all these spaces behave well under deformations, the result follows.

The right arrow $D^k(f)/S_k \to f(D_1^k(f))$ is the map that the S_k-compatible map $f \circ p$ induces on the quotient. It is generically one-to-one whenever the strict k-multiple points of f are dense in $D^k(f)$.

Remark 2.6.1. If $k = 2$, then the bottom and right arrows are also surjective. This follows immediately from the first item of Theorem 3.2.1 and Proposition A.3.2, just counting multiplicities: At singular points $x \in X$, we have multiplicity $\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,x}/f^*\mathfrak{m}_{f(x)} \geq 2$. At strict double points, there are at least two different preimages of $f(x)$, and every one of them has multiplicity ≥ 1. For $k \geq 2$, the situation is more complicated (see Open Problem 5).

Chapter 3

Double points

In the previous chapter we have shown that there is a unique multiple point structure satisfying some reasonable conditions. We have also shown how to compute these structure by taking a stable unfolding and slicing its strict k-multiple point space. However, since stable unfoldings may need a big number of parameters (see Example 2.1.7), this is not a practical approach for the computation of k-multiple points, and has to be considered just as an existence and unicity property. This chapter is devoted to double points, which we do know how to compute in an effective way. In [Mon87], Mond gives an explicit, easily computable set of generators of a double point ideal for any map germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. This structure can be obtained without unfolding the map f.

In the first section of this chapter we show: a) How Mond's local construction glues to a global scheme for holomorphic maps. b) The double point structure obtained satisfies conditions M1 and M2, and thus it agrees with the double point structure of Section 2. This extends previous results about the relation of Mond's structure and the standard one (see Proposition 2.2.2 and also [Alt11, Section 2.1.2]).

In the second section we use the insight provided by the explicit structure to obtain properties of the double point space of a map.

In the third section we consider an alternative multiple point structure. We give criteria for the two structures to agree and, finally, show that the new structure does not satisfy condition M2.

3.1 The double point ideal sheaf

The following definition is due to Mond [Mon87]:

Proposition-Definition 3.1.1. For any map germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, the germs $f_j(x) - f_j(x'), 1 \leq j \leq p$ vanish on the diagonal $\Delta(n, 2)$. Therefore, they are contained in the ideal generated by $x_i - x'_i, 1 \leq i \leq n$. In other words, for all $j \leq p$, there exist some function germs $\alpha_{ij} \in \mathcal{O}_{2n}$, satisfying

$$f_j(x) - f_j(x') = \sum_{i=1}^{n} \alpha_{ji}(x, x')(x_i - x'_i).$$

This can be expressed as the matrix equality

$$f(x) - f(x') = \alpha(x - x'),$$

where α represents the $p \times n$ matrix (α_{ji}) and $x - x'$ and $f(x) - f(x')$ are taken as column vectors of sizes n and p respectively. **Mond's double point ideal** $I^2(f)$ is the sum

$$I^2(f) = P(f, 2) + \langle n \times n \text{ minors of } \alpha \rangle.$$

The matrix α may not be unique, but $I^2(f)$ does not depend on the choice of α [Mon87, Prop 3.1].

Example 3.1.2. Let $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, as in Example 2.1.11, be given by

$$(x, y) \mapsto (x^2, y^2, x^3 + y^3 + xy).$$

A solution for the equation $f(x) - f(x') = \alpha(x - x')$ is

$$\begin{pmatrix} x^2 - x'^2 \\ y^2 - y'^2 \\ x^3 + y^3 + xy - x'^3 - y'^3 - x'y' \end{pmatrix} =$$

$$= \begin{pmatrix} x + x' & 0 \\ 0 & y + y' \\ x^2 + xx' + x'^2 + y & y^2 + yy' + y'^2 + x' \end{pmatrix} \begin{pmatrix} x - x' \\ y - y' \end{pmatrix}.$$

The ideal generated by the 2×2 minors of α and the germs $f_j(x) - f_j(x'), i = 1, 2, 3$ is precisely the ideal $I^2(f)$ in Example 2.1.11.

Remark 3.1.3. There are multiple ways to obtain a matrix α as above. Besides the ones given here, in Section B.3 we give further expressions for α, with the advantage of providing the expression of α in terms of some given symmetric functions (see Definition B.3.1 for details).

a) Let $w^i = (x_1, \dots, x_{n-i}, x'_{n-i+1}, \dots, x'_n), i = 0, \dots, n$. We can add and subtract terms to obtain $f(x) - f(x') = f(w^0) - f(w^1) + f(w^1) - \cdots - f(w^{n-1}) + f(w^{n-1}) - f(w^n)$. Since $x_i - x'_i$ divides $f(w^{i-1}) - f(w^i)$ we can take the holomorphic functions

$$\alpha_{ji}(x, x') = \frac{f_j(w^{i-1}) - f_j(w^i)}{x_i - x'_i}.$$

b) From the sequence of equalities

$$\begin{aligned} f_j(x) - f_j(x') &= \int_0^1 \frac{d}{dt}\{f_j(tx + (1-t)x')\}dt \\ &= \int_0^1 \sum_{i=1}^n \frac{\partial f_j}{\partial x_i}(tx + (1-t)x')(x_i - x_i')dt \\ &= \sum_{i=1}^n \left(\int_0^1 \frac{\partial f_j}{\partial x_i}(tx + (1-t)x')dt\right)(x_i - x_i'), \end{aligned}$$

follows that we can take

$$\alpha_{ji}(x, x') = \int_0^1 \frac{\partial f_j}{\partial x_i}(tx + (1-t)x')dt.$$

Lemma 3.1.4. *Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a map germ and let α be any matrix satisfying $f(x) - f(x') = \alpha(x, x')(x - x')$, then $\alpha(x, x) = df_x$.*

Proof. Let e_i be the i-th vector of the canonical basis of $\mathbb{C}^n$. Then

$$\alpha_{ji}(x, x) = \lim_{\lambda \to 0} \alpha_{ji}(x, x + \lambda e_i) = \lim_{\lambda \to 0} \frac{f_j(x) - f_j(x + \lambda e_i)}{-\lambda} = \frac{\partial f_j}{\partial x_i}.$$

$\square$

Now we show that $I^2(f)$ behaves well under $\mathcal{A}$-equivalence.

Lemma 3.1.5. *Let f and g be $\mathcal{A}$-equivalent map germs with $f = \psi \circ g \circ \varphi$. Then, $(\varphi \times \varphi)^*(I^2(f)) = I^2(g)$.*

Proof. We proceed in two steps. First we show that if $f = \psi \circ g$, then $I^2(f)$ equals $I^2(g)$. On one hand we have

$$P(f, 2) = (f \times f)^* I_{\Delta(p,2)} = (g \times g)^*(\psi \times \psi)^* I_{\Delta(p,2)} = (g \times g)^* I_{\Delta(p,2)} = P(g, 2).$$

On the other hand, take a matrix α satisfying

$$g(x) - g(x') = \alpha(x, x')(x - x')$$

and a matrix M satisfying

$$\psi(y) - \psi(y') = M(y, y')(y - y').$$

We obtain

$$\begin{aligned} f(x) - f(x') &= \psi \circ g(x) - \psi \circ g(x') = M(g(x), g(x'))(g(x) - g(x')) \\ &= M(g(x), g(x'))\alpha(x, x')(x - x') = \beta(x, x')(x - x'), \end{aligned}$$

where $\beta = (M \circ (g \times g)) \cdot \alpha$. The matrix $M \circ (g \times g)$ is locally invertible, since $M(0, 0)$ is just the differential of ψ at the origin. It follows that the $n \times n$ minors of α and β generate the same ideal.

Second, assume $f = g \circ \varphi$, then obviously $(\varphi \times \varphi)^*(P(f,2)) = P(g,2)$. Let Q be a matrix satisfying

$$\varphi(x) - \varphi(x') = Q(x,x')(x - x').$$

We obtain

$$\begin{aligned} f(x) - f(x') &= \alpha(\varphi(x), \varphi(x'))(\varphi(x) - \varphi(x')) \\ &= \alpha(\varphi(x), \varphi(x'))(Q(x,x')(x - x')) = \beta(x,x')(x - x'), \end{aligned}$$

where $\beta = (\alpha \circ (\varphi \times \varphi)) \cdot Q$. Since $Q(0,0)$ is the differential of φ at the origin, the matrix Q is locally invertible. Therefore, the ideal generated by the minors of β equals the ideal generated by the minors of $\alpha \circ (\varphi \times \varphi)$, which is the image under $(\varphi \times \varphi)^*$ of the ideal generated by the minors of α, as desired. □

Given a map $f \colon X \to Y$ and a point $x \in X$, we take local coordinates so that the germ f_x of f at x is $f_x = \psi \circ f' \circ \varphi^{-1}$, for some biholomorphisms φ, ψ and some map germ $f' \colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. We define the ideal $I^2(f_x)$ in $\mathcal{O}_{X \times X,(x,x)}$ as

$$I^2(f_x) = (\varphi \times \varphi)^*(I^2(f')).$$

Lemma 3.1.5 ensures that this definition does not depend on the choice of φ and ψ. The following lemma allows us to extend the local definition of $I^2(f)$ to a global ideal sheaf of double points.

Lemma 3.1.6. *Given a map $f \colon X \to Y$ and a point $x \in X$, denote by f_x the germ of f at x. There exists an open neighborhood U of (x,x), and some representatives of the generators of $I^2(f)$, defined in U, such that the ideal sheaf $\mathscr{I}^2(f)$ on U defined by such representatives satisfies:*

1. *$\mathscr{I}^2(f)_{(x',x')} = I^2(f_{x'})$, for any $(x',x') \in \Delta(U,2)$.*
2. *$\mathscr{I}^2(f)_{(x',x'')} = \mathscr{P}(f,2)_{(x',x'')}$, for any $(x',x'') \in U \setminus \Delta(U,2)$.*

Proof. By Lemma 3.1.5 we can assume $X = \mathbb{C}^n$ and $Y = \mathbb{C}^p$, taking local coordinates. To show (1), we just need to shrink U so that we have representatives of the germs $f_j(x) - f_j(x')$ and of the entries of the matrix α defined on all U. Therefore, the germs of these representatives at (x',x') produce the corresponding germs and the corresponding matrix around (x',x').

To show (2) we need to show that the ideal generated by the germs at (x',x'') of the $n \times n$ minors of α is contained in $\mathscr{P}(f,2)_{(x',x'')}$. Let A be the submatrix of α obtained by picking the rows $j_1, \dots, j_n$ of α. Let b be the vector with entries $f_{j_1}(x) - f_{j_1}(x'), \dots, f_{j_n}(x) - f_{j_n}(x')$. Since (x',x'') is not a diagonal point, there exists $i \leq n$ such that $x'_i \neq x''_i$. Let A' be the matrix obtained by substitution of the i-th column of A by b. By Cramer's Rule we obtain $|A| = |A'|/(x'_i - x''_i) \in \mathscr{P}(f,2)_{(x',x'')}$ □

The second item of the previous lemma is equivalent to [Alt11, Lemma 2.1.17]

Definition 3.1.7. The **sheaf of double points** $\mathscr{I}^2(f)$ of a map $f\colon X \to Y$ is defined as the glueing of the following local structures: Off the diagonal, $\mathscr{I}^2(f)$ is just the restriction of the sheaf $\mathscr{P}(f,k)$ to $X^2 \setminus D(X,2)$. If (x,x) is a diagonal point, then at a neighbourhood of (x,x) the sheaf is locally given by the double point ideal $I^2(f)$ of the germ of f centered at x.

To glue these local structures, we need to check that, if we compute the structure locally around some point, then the stalk of this local structure at any other close enough point agrees with the structure computed at this other point. This is precisely Lemma 3.1.6

Lemma 3.1.8. *Set theoretically, $V(\mathscr{I}^2(f))$ is the union of the strict double points of f and the pairs (x,x) such that f is singular at x.*

Proof. Let $(x,x'), x \neq x'$ a non diagonal point in X^2. Locally, $\mathscr{I}^2(f)$ equals $\mathscr{P}(f,2)$, which vanishes if and only if (x,x') is a strict double point of f. Let (x,x) be a diagonal point in $X \times X$, then $\mathscr{P}(f,2)$ vanishes trivially at (x,x). Moreover, if α is the matrix in the definition of $I^2(f_x)$, by Lemma 3.1.4 $\alpha(x,x)$ equals the differential matrix of f at x. Therefore, the $n \times n$ minors of α vanish at x if and only if f is singular at x. $\square$

The following lemma can be obtained easily from results about Cohen-Macaulay modules contained in [Mat89] (details in [BA01, Lemma 2.5.1]).

Lemma 3.1.9. *Let $\phi\colon (\mathbb{C}^m,0) \to (\mathbb{C}^r,0)$ be any map germ. Let I be an ideal in $\mathcal{O}_r$ and let $J = \phi^*(I)$. If $\mathcal{O}_r/I$ is Cohen-Macaulay and* $\operatorname{codim} V(I) = \operatorname{codim} V(J)$*, then $\mathcal{O}_m/J$ is Cohen-Macaulay.*

With a trivial modification of the proof of [Alt11, Prop. 2.1.11], we obtain

Lemma 3.1.10. *Let $f\colon (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$ with $n \leq p$. Then*

1. *If $\mathcal{O}_{2n}/I^2(f) \neq 0$ then $\dim \mathcal{O}_{2n}/I^2(f) \geq 2n-p$.*
2. *If $\dim \mathcal{O}_{2n}/I^2(f) = 2n-p$, then $\mathcal{O}_{2n}/I^2(f)$ is Cohen Macaulay.*

Proof. We identify the space of $n \times p$ matrices $A = (a_{ji})$ and $n \times 1$ vectors $(d_1,\dots,d_n)^T$ with $\mathbb{C}^{np} \times \mathbb{C}^n$. Let I be the ideal in $\mathcal{O}_{np+n}$ generated by the entries of Ad and the $n \times n$ minors of A. Write $D = V(I) \subseteq \mathbb{C}^{np} \times \mathbb{C}^n$. It turns out that D is a Buchsbaum-Eisenbud variety of complexes (more precisely $D = W(n-1,1)$, with $n_0 = p, n_1 = n$ and $n_2 = 1$, in the notation of [dCS81]). By [dCS81, Thm 2.7, Lemma 2.3] D is a Cohen Macaulay subspace of $\mathbb{C}^{np} \times \mathbb{C}^n$ of codimension p.

Now given a matrix α satisfying $f(x)-f(x')=\alpha(x-x')$, we take the map $\phi\colon \mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}^{np}\times\mathbb{C}^n$ given by

$$(x,x')\mapsto(\alpha(x,x'),(x_1-x_1',\dots,x_n-x_n')).$$

We obtain $I^2(f)=\phi^*(I)$, and the result follows directly from Lemma 3.1.9 $\square$

We can patch the local results above to obtain the following

Theorem 3.1.11. *For any map $f\colon X\to Y$, if $V(\mathscr{I}^2(f))$ has dimension $2n-p$, then it is a Cohen Macaulay complex space.*

Proof. This is a local question at $X\times X$. Let $Z=V(\mathscr{I}^2(f))$. At strict double points (x,x') of f, the stalk $\mathscr{I}^2(f)_{(x,x')}$ agrees with $\mathscr{P}(f,2)_{(x,x')}$, which is generated locally by the p function germs $f_j(x)-f_j(x'), 1\leq j\leq p$ (where f_j is the composition of f with the j-th coordinate function of Y around $f(x)$). Thus, Z is locally a complete intersection. Let $(x,x)\in X\times X$ be a diagonal point and denote by f_x the germ of f at x. Then, $\mathscr{I}^2(f)_{(x,x)}=I^2(f_x)$ and the result follows directly from Lemma 3.1.10. $\square$

Theorem 3.1.12. *$\mathscr{I}^2(f)$ defines the double point space $D^2(f)$.*

Proof. By Proposition 2.1.5, we only need to show that $\mathscr{I}^2$ satisfies conditions M1 and M2. To show M1, let f be a stable map and denote by Z the zero set of $\mathscr{I}^2(f)$. By Theorem 3.1.11, Z is a Cohen Macaulay space of dimension $2n-p$. Now we claim that Z is smooth out of the set $C=\{(x,x)\mid x\in\hat{\Sigma}^2(f)\}$. By Lemma 3.1.8, Z consists of strict double points of f and diagonal points (x,x), such that f is singular at x.

If (x,x) is a diagonal point with $x\in\Sigma^1(f)$, then the stalk $\mathscr{I}^2(f)_{(x,x)}$ is the double point ideal $I^2(f_x)$ of the corank 1 map germ f_x defined by f at x and the claim follows by Proposition 2.2.2. If (x,x') is a strict double point of f, then $\mathscr{I}^2(f)$ agrees with $\mathscr{P}(f,2)$ locally at (x,x') by Lemma 3.1.6. The claim follows since $\mathscr{P}(f,2)=(f\times f)^*\mathscr{I}_{\Delta(Y,2)}$ and, for every stable map f, the restriction of $f\times f$ to $X\times X\setminus\Delta(X,2)$ is transverse to $\Delta(Y,2)$ (Proposition 1.5.9).

By Proposition 1.5.11, the dimension of C is less than or equal to $n-2(p-n+3)<2n-p$. Hence, Z is a generically smooth Cohen Macaulay space and, thus, reduced. This reduces M1 to show that Z is, set theoretically, the closure of the strict double points of f. In other words, it suffices to show that there are no irreducible components of Z consisting of points (x,x) with f singular at x. Assume that there is such a component. Then, since Z is Cohen Macaulay (and hence equidimensional), the dimension of this component is $2n-p$. Therefore, the dimension of the set of singular points of f is at least $2n-p$, which contradicts Proposition 1.5.11.

To show M2, first notice that, by Lemma 3.1.5, we can take local coordinates and assume $X = \mathbb{C}^n$, $Y = \mathbb{C}^p$ and that the unfolding is given by $F(s,x) = (s, f_s(x))$, $s \in \mathbb{C}^r$, with $f_0 = f$. Therefore it suffices to show, for any point (x,x'), the equality

$$\mathscr{I}^2(f)_{(x,x')} + \langle s_i, s_i' \mid 1 \le i \le r \rangle = \mathscr{I}^2(F)_{(0,x,0,x')} + \langle s_i, s_i' \mid 1 \le i \le r \rangle,$$

where both stalks are seen as ideals in $\mathcal{O}_{\mathbb{C}^{2(r+n)},(0,x,0,x')}$.

If $x \neq x'$, then

$$\mathscr{I}^2(F)_{(0,x,0,x')} = \mathscr{P}(F,2)_{(0,x,0,x')} = \langle s_i - s_i' \rangle + \langle (f_s)_j(x) - (f_s)_j(x') \rangle,$$
$$\mathscr{I}^2(f)_{(x,x')} = \mathscr{P}(f,2)_{(x,x')} = \langle f_j(x) - f_j(x') \rangle,$$

so these two ideals agree modulo $\langle s_i, s_i' \rangle$.

If $x = x'$, $\mathscr{I}^2(F)_{(0,x,0,x')}$ is given by the sum of $\mathscr{P}(F,2)_{(0,x,0,x')}$ and the ideal generated by the $n+r$-minors of some matrix A, satisfying

$$F(s,x) - F(s',x') = A(s,x,s',x')(s - s', x - x').$$

The local form of the unfolding F forces A to be of the form

$$A(s,x,s',x') = \left(\begin{array}{c|c} I_r & 0 \\ \hline * & \alpha_{s,s'} \end{array} \right).$$

Taking $s, s' = 0$ we see that the submatrix $\alpha_{s,s'}$ satisfies

$$f(x) - f(x') = \alpha_{0,0}(x,x')(x - x').$$

Therefore $\mathscr{I}^2(f)_{(x,x)}$ is the sum of $\mathscr{P}(f,2)_{(x,x)}$ and the ideal generated by the n-minors of $\alpha_{(0,0)}$, which are exactly the $n+r$-minors of $A(0,x,0,x')$. Again the equality modulo $\langle s_i, s_i' \rangle$ is immediate. $\square$

The first two items of the followin theorem can also be found in [Lak77] and [Ron72].

3.2 Properties of the double point space

Putting together Theorem 3.1.12, Theorem 3.1.11, Lemma 3.1.8 and Proposition 2.1.10, we obtain the following

Theorem 3.2.1. *For any $f\colon X \to Y$*

1. *Set theoretically, $D^2(f)$ is the union of the strict double points of f and the pairs (x,x) such that f is singular at x.*

2. *$D^2(f)$ has dimension $\ge 2n - p$ at every point. In particular, $D^2(f)$ is empty or $\dim(D^2(f)) \ge 2n - p$*

3. *If $\dim D^2(f) = 2n - p$, then $D^2(f)$ is Cohen Macaulay.*

Let $(X, \mathcal{O}_{X,x})$ be a germ of complex space and let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{X,x}$. We define the embedding dimension of X as

$$\operatorname{edim} X = \dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}^2.$$

It is well known that a germ X is regular if and only if $\dim X = \operatorname{edim} X$.

Lemma 3.2.2. *Let* $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, *with* $\operatorname{corank} f = k \geq 2$, *then*

$$\operatorname{edim} D^2(f) = n + k.$$

Proof. We may assume that f is of the form

$$(x, y) \mapsto (x, f_{n-k+1}(x, y), \ldots, f_p(x, y)),$$

with $x = x_1, \ldots, x_{n-k}, y = y_1, \ldots, y_k$ and $f_j \in \mathfrak{m}^2$, where $\mathfrak{m}$ stands for the maximal ideal of $\mathcal{O}_{2n}$. Then, the ideal $P(f, 2) + \mathfrak{m}^2$ is generated by $n - k$ linearly independent elements in $\mathfrak{m}/\mathfrak{m}^2$. Now let α be a matrix satisfying $f(x) - f(x') = \alpha(x - x')$. The rows corresponding to the coordinate functions $f_j, n - k + 1 \leq j \leq p$ have all entries in $\mathfrak{m}$. Since there are only $n - k$ remaining rows, it follows that all the $n \times n$ minors of α are contained in $\mathfrak{m}^k \subseteq \mathfrak{m}^2$. We obtain $\operatorname{edim} D^2(f) = 2n - (n - k) = n + k$. □

Proposition 3.2.3. *Let* $f\colon X \to Y$ *be stable. Set theoretically, the singular locus of* $D^2(f)$ *is*

$$\{(x, x) \in X^2 \mid x \in \hat{\Sigma}^2(f)\}.$$

Proof. Let $C = \{(x, x) \in X^2 \mid x \in \hat{\Sigma}^2(f)\}$. In the proof of Theorem 3.1.12 we have shown that $V(\mathscr{I}^2(f))$ is smooth out of C. Since $D^2(f) = V(\mathscr{I}^2(f))$, the singular locus of $D^2(f)$ is contained in C. Now let $(x_0, x_0) \in C$ and let $k = \operatorname{corank} f_{x_0} \geq 2$. By definition, $\mathscr{I}^2(f)_{(x_0,x_0)} = I^2(f_{x_0})$, where f_{x_0} stands for the germ of f at x_0. From 2.1.9 we obtain $\dim D^2(f) = 2n - p$, and from Lemma 3.2.2, since $2n - p \leq n < n + k$, the statement follows. □

Corollary 3.2.4. *If f is stable, then* $p\colon D^2(f) \to D(f)$ *is a normalization.*

Proof. By Proposition 2.1.10, $D^2(f)$ is dimensionally correct. Thus, by Theorem 3.2.1, it is a Cohen Macaulay space. By Proposition 1.5.11 and Proposition 3.2.3, the singular locus of $D^2(f)$ is empty or has dimension $n - 2(p - n + 2)$. Hence $\dim D^2(f) - \dim Z \geq p - n + 4 \geq 4$. Thus $D^2(f)$ is a normal complex space by Serre's criterion [Mat80, Thm. 39]. Since f is stable, the strict double points are dense in $D^2(f)$ and p is genenerically one-to-one (see Section 2.6). □

Corollary 3.2.5. *Let* $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ *be finitely determined, then*

1. *If $2n-p\geq 2$, then $D^2(f)$ is a normalization of $D(f)$.*

2. *If $2n-p\geq 1$, then $D^2(f)$ is reduced.*

Proof. By Mather-Gaffney Criterion 1.5.12, there exists a representative of f defined on a open neighbourhood of the origin U (denoted also by f), such that $f^{-1}(0)=\{0\}$ and such that the restriction $f|_{U\setminus\{0\}}$ is stable. Then, we have $D^2(f)=D^2(f|_{U\setminus\{0\}})\cup\{(0,0)\}$ and $D^2(f|_{U\setminus\{0\}})$ is normal by Corollary 3.2.4. Moreover, $D^2(f)$ has dimension $2n-p$ and is Cohen-Macaulay by Theorem 3.1.11. By Serre's criterion [Mat80, Thm. 39], if $\dim D^2(f)\geq 1$ then it is reduced, and if $\dim D^2(f)\geq 2$ then it is also normal. □

The first item of the previous corollary can be found for $n=3,p=4$ in [Alt11, Prop. 4.3.1]. The two following examples justify the conditions in the previous corollary:

Example 3.2.6. Let f be any non-stable finitely determined map germ $(\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$. We claim that $D^2(f)$ is not normal. By Mather-Gaffney Criterion 1.5.12, f is stable out of the origin and thus, from Example 1.5.4, it follows $\dim D^2(f)=1$. Now it suffices to show $\operatorname{Sing} D^2(f)=\{0\}$. Assume first $\operatorname{corank} f=1$, then the claim follows from Theorem 2.2.3, taking into account that f is not stable. If $\operatorname{corank} f\geq 2$, then the claim follows by Lemma 3.2.2.

Example 3.2.7. Let $f\colon(\mathbb{C}^2,0)\to(\mathbb{C}^4,0)$ be the map given by

$$(x,y)\mapsto(x^2,y^2,x^3+xy,y^3+xy)$$

In Example 4.2.4 we will justify that f is finitely determined and $\dim D^2(f)=0$. But $\operatorname{edim} D^2(f)=4$, by Lemma 3.2.2. It follows that $D^2(f)$ is singular and, since it has dimension equal to 0, it must be non-reduced.

The case $p=n+1$

Proposition 3.2.8. *Let $f\colon X\to Y$ with $p=n+1$. If f is generically one-to-one, then $D^2(f)$ is empty or a Cohen-Macaulay space of dimension $n-1$.*

Proof. If f is finite and generically one-to-one, then the dimension of the set of strict double points of f is $\leq n-1$. By Lemma 1.2.6, the dimension of the space of pairs (x,x) with f singular at x is $\leq n-1$. The claim follows immediately from Theorem 3.2.1. □

Corollary 3.2.9. *Let $f\colon(\mathbb{C}^n,0)\to(\mathbb{C}^{n+1},0)$ be generically one-to-one. Then $D(f)$ is reduced if and only if $D^2(f)$ is reduced and the projection $p_1\colon D^2(f)\to(\mathbb{C}^n,0)$ is generically one-to-one.*

Proof. If f is generically one-to-one, then $D^2(f)$ is Cohen-Macaulay by the previous proposition and the statement follows by Lemma A.3.3. □

Lemma 3.2.10. *Let F be an r-parametric unfolding of a finitely determined map germ $f\colon (\mathbb{C}^n,0) \to (\mathbb{C}^{n+1},0)$. Then, the projections of $D(F)$, $D^2(F)$, $D^2(F)/S_2$ and $F(D(F))$ to the parameter space $\mathbb{C}^r$ are flat deformations of $D(f)$, $D^2(f)$, $D^2(f)/S_2$ and $f(D(f))$, respectively.*

Proof. If $\pi\colon (X,0) \to (\mathbb{C},0)$ is a deformation of $(X_0,0)$, with $(X,0)$ Cohen-Macaulay, $\dim(X,0) = d + r$ and $\dim(X_0,0) = d$, then the deformation is flat [Mat89, Thm. 23.1]). This applies to $D(F), D^2(F)$ and $F(D(F))$, since they are Cohen-Macaulay (see Proposition 2.5.2) of dimension $n - 1 + r$ and $D(f), D^2(f)$ and $f(D(f))$ have dimension $n-1$. For $D^2(F)/S_2$, just observe that $\mathcal{O}_{D^2(F)/S_2}$ is a subalgebra of $\mathcal{O}_{D^2(F)}$. If there is no element in $\mathcal{O}_{D^2(F)}$ killing $t_1,\dots,t_r$, then there is no such an element in $\mathcal{O}_{D^2(F)/S_2}$ neither. □

3.3 Another multiple point structure

The present section is devoted to the study of a different approach to the computation of multiple points. This alternative structure, was introduced for double points by Mond in [Mon87], where he shows that it agrees with the usual double point structure $I^2(f)$, provided that f has corank 1. We will give some criteria for the equality of both structures of double points and show one example where they disagree. Therefore, we conclude that the new structure does not satisfy the properties M1 and M2.

Recall that the ideal sheaf $\mathscr{P}(f,k)$ defines the locus of points

$$(x^{(1)},\dots,x^{(k)}) \in X^k,$$

such that $f(x^{(1)}) = f(x^{(l)})$ for all $l \leq k$. It is clear that the zeros of $\mathscr{P}(f,k)$ may contain contain some points which are not what we defined as k-multiple points. Indeed, for any $(x^{(1)},\dots,x^{(k-1)})$ belonging to the zeros of $\mathscr{P}(f,k-1)$, the point $(x^{(1)},\dots,x^{(k-1)},x^{(k-1)})$ belongs to $\mathscr{P}(f,k)$ without imposing further conditions. For instance, the zero set of $\mathscr{P}(f,k)$ contains always the small diagonal $\Delta(X,k)$. It seems a good idea to erase, taking multiplicities into account, the trivial copies of the diagonal which appear in the zeros of $\mathscr{P}(f,k)$. Locally, given two subspaces $A = V(I)$ and $B = V(J)$ of $(\mathbb{C}^n,0)$, to erase B from A corresponds to take the zeros of the ideal $I : J = \{h \in \mathcal{O}_n \mid hJ \subseteq I\}$. This local definition extends to the corresponding operation between sheaves which, furthermore, preserves coherence.

Definition 3.3.1. For any map $f\colon X \to Y$, we define

$$\mathscr{H}^k(f) = \mathscr{P}(f,k) : \mathscr{I}_{D(X,k)}.$$

We denote by $\tilde{D}^k(f)$ the complex space defined by $\mathscr{H}^k(f)$. If f is a map germ, then we define $H^k(f) = P(f,k) : I_{D(X,k)}$ and $\tilde{D}^k(f) = V(H^k(f))$.

Lemma 3.3.2. *[Mon87] For any map germ f of corank k:*

1. $I^2(f) \subseteq H^2(f)$,
2. $H^2(f)^k \subseteq I^2(f)$.

In particular, if $\operatorname{corank} f = 1$, *then* $I^2(f) = H^2(f)$.

Corollary 3.3.3. *For any $f\colon X \to Y$, the spaces $D^2(f)$ and $\tilde{D}^2(f)$ agree set-theoretically. At the level of schemes, the ideal sheaf $\mathscr{I}^2(f)$ is a subsheaf of $\mathscr{H}^2(f)$ and they both agree out of the space*

$$\{(x,x) \in \Delta(X,2) \mid x \in \hat{\Sigma}^2(f)\}.$$

Proof. Out of the diagonal, both $\mathscr{H}^2(f)$ and $\mathscr{I}^2(f)$ agree with $\mathscr{P}(f,2)$. Let $(x,x) \in D(X,2)$ be a diagonal point and let f_x be the germ of f at x. We have the equalities $\mathscr{I}^2(f)_{(x,x)} = I^2(f_x)$ and $\mathscr{H}^2(f)_{(x,x)} = H^2(f_x)$. The result follows directly from the previous lemma. □

Lemma 3.3.4. *Let I, J be ideals in a noetherian ring R and let be $I = \bigcap_{i=1}^s \mathfrak{q}_i$ be a minimal primary decomposition, so that the associated primes of R/I are $Ass(R/I) = \{\sqrt{\mathfrak{q}_1}, \dots, \sqrt{\mathfrak{q}_s}\}$. Then, the associated primes of $R/(I:J)$ satisfy:*

$$Ass(R/(I:J)) \subseteq \{\sqrt{q_i} \mid J \not\subseteq \mathfrak{q}_i\}.$$

Proof. We claim that, if $\mathfrak{q}$ is a primary ideal, then $\mathfrak{q} : J = R$ if $J \subseteq \mathfrak{q}$ and $\sqrt{\mathfrak{q} : J} = \sqrt{\mathfrak{q}}$ otherwise. From this we obtain the (possibly non minimal) primary decomposition $I : J = \bigcap_{J \not\subseteq \mathfrak{q}_i} \mathfrak{q}_i : J$. The result follows, since this decomposition can be refined to obtain a minimal one, which will determine the associated primes of $I : J$. Now we show the claim. If $J \subseteq \mathfrak{q}$, then $\mathfrak{q} : J = R$ obviously. On one hand, $\sqrt{\mathfrak{q}} \subseteq \sqrt{\mathfrak{q} : J}$ by monotony of the radical and colon ideal operators. Now assume that there exists an element $a \in J \setminus \mathfrak{q}$. Let $b \in \sqrt{\mathfrak{q} : J}$, then $b^n a \in \mathfrak{q}$. Since $\mathfrak{q}$ is a primary ideal and $a \notin \mathfrak{q}$, it follows that $b^n \in \sqrt{\mathfrak{q}}$ and, thus, $b \in \sqrt{\mathfrak{q}}$, as desired. □

Example 3.3.5. The inclusion is strict in general. Let $\mathfrak{q}_1 = \langle x^2, y^2 \rangle$ and $\mathfrak{q}_2 = \langle x^2, xy^2, y^3, z \rangle$ be primary ideals and $J = \langle x, y \rangle$. The associated primes of $I = \mathfrak{q}_1 \cap \mathfrak{q}_2$ are $J = \sqrt{\mathfrak{q}_1}$ and $\langle x, y, z \rangle = \sqrt{\mathfrak{q}_2}$, since $\mathfrak{q}_1 \not\subset \mathfrak{q}_2$ and $\mathfrak{q}_2 \not\subset \mathfrak{q}_1$. Moreover, we have $J \not\subset \mathfrak{q}_2$. However, the only associated prime of $I : J$ is J.

Theorem 3.3.6. *If $f\colon X \to Y$ satisfies*

1. $\dim D^2(f) = 2n - p$,
2. $\dim \hat{\Sigma}^2(f) < 2n - p$,

then $\mathscr{I}^2(f) = \mathscr{H}^2(f)$.

Proof. By Corollary 3.3.3, the stalks of $\mathscr{I}^2(f)$ and $\mathscr{H}^2(f)$ agree out of the space $\{(x,x) \in U \times U \mid x \in \hat{\Sigma}^2(f)\}$, which is a space of dimension $< 2n - p$.

Again by Corollary 3.3.3, we have $\mathscr{I}^2(f) \subseteq \mathscr{H}^2(f)$. Therefore, the space Z where $\mathscr{I}^2(f)$ and $\mathscr{H}^2(f)$ disagree is

$$Z = \{(x,x') \in U{\times}U \mid \mathscr{I}^2(f)_{(x,x')} \subsetneq \mathscr{H}^2(f)_{(x,x')}\} = \operatorname{supp}(\mathscr{H}^2(f)/\mathscr{I}^2(f)).$$

Since the sheaf $\mathscr{H}^2(f)/\mathscr{I}^2(f)$ is coherent , we have

$$Z = V(\mathscr{A}nn(\mathscr{H}^2(f)/\mathscr{I}^2(f))).$$

The stalks of this sheaf are

$$\begin{aligned}\mathscr{A}nn(\mathscr{H}^2(f)/\mathscr{I}^2(f))_{(x,x')} &= \operatorname{Ann}(\mathscr{H}^2(f)_{(x,x')}/\mathscr{I}^2(f)_{(x,x')}) \\ &= \mathscr{I}^2(f)_{(x,x')} : \mathscr{H}^2(f)_{(x,x')}.\end{aligned}$$

Since the zero set of a sheaf depends only of its stalks, we have the equality $Z = V(\mathscr{I}^2(f) : \mathscr{H}^2(f))$. We already know that Z is contained in the diagonal. By hypothesis, $D^2(f)$ has dimension $2n - p$ and thus, by Theorem 3.1.11 and Theorem 3.1.12, $D^2(f)$ is Cohen-Macaulay, and hence equidimensional. The germ of Z at any point $(x,x) \in \Delta(X,2)$ is $V(\mathscr{I}^2(f)_{(x,x)} : \mathscr{H}^2(f)_{(x,x)}) = V(I^2(f_x) : H^2(f_x))$ and, by Lemma 3.3.4, all the associated primes of $I^2(f_x) : H^2(f_x)$ are associated primes of $I^2(f_x)$. It follows that Z is equidimensional of dimension $2n-p$ or empty. Since we have shown before that Z is a subspace of a space of dimension $< 2n - p$, we conclude that Z is empty. $\square$

Corollary 3.3.7. *Assume* $\dim X = \dim Y$. *Then* $\mathscr{I}^2(f) = \mathscr{H}^2(f)$ *for any* $f\colon X \to Y$.

Proof. If $D^2(f)$ is empty, then the result is trivial. Otherwise, let $n = \dim X$. Since f is finite, then $\dim D^2(f) = n$ and, by Lemma 1.2.6, we have $\dim \hat{\Sigma}^2(f) \leq n - 2$. $\square$

Corollary 3.3.8. *Assume* $\dim Y = \dim X + 1$. *If* $f\colon X \to Y$ *is generically one-to-one, then* $I^2(f) = H^2(f)$.

Proof. As in the previous corollary, if $D^2(f)$ is empty, then the result is trivial. Otherwise, let $n = \dim X$. If the map f is finite and generically one-to-one, then $\dim D^2(f) = n-1$. By Lemma 1.2.6 we have $\dim \hat{\Sigma}^2(f) \leq n - 2$. $\square$

Corollary 3.3.9. *Any stable map* f *satisfies* $\mathscr{I}^2(f) = \mathscr{H}^2(f)$.

Proof. If $D^2(f) = \emptyset$, then the statement follows trivially from Lemma 3.3.2. If $D^2(f) \neq \emptyset$, then the stability of f implies $\dim D^2(f) = 2n - p$ (Lemma 2.1.9) and we have $\dim \hat{\Sigma}^2(f) = n - 2(p - n + 3) < 2n - p$ (Proposition 1.5.11). □

Corollary 3.3.10. *If $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is a finitely determined germ and $p < 2n$, then $I^2(f) = H^2(f)$ and both ideals are reduced.*

Proof. By the Mather-Gaffney criterion 1.5.12, we can find a representative f which is stable out of $\{0\}$. Then, $D^2(f)$ is reduced and, by the previous corollary, $\mathscr{I}^2(f)$ and $\mathscr{H}^2(f)$ agree out of $\{0\}$. Moreover, $\mathscr{O}_{2n}/I^2(f)$ is Cohen Macaulay and, thus, equidimensional of dimension $2n - p > 0$. As we saw in the proof of Theorem 3.3.6, these sheaves can only differ on some zeros of associated primes of $\mathscr{I}^2(f)$, which are spaces of dimension > 0. Therefore $\mathscr{I}^2(f) = \mathscr{H}^2(f)$ □

Example 3.3.11. Let be $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ the 'Double Cone', given by

$$(x, y) \mapsto (x^2, y^2, xy).$$

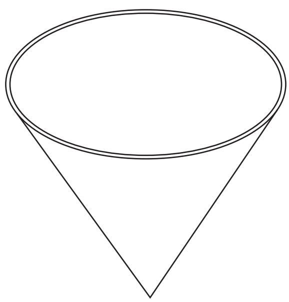

Figure 3.1: Image of the Double Cone

A straightforward computation with SINGULAR yields $I^2(f) = A \cap B_1$ and $H^2(f) = A \cap B_2$, where

$$A = \langle x + x', y + y' \rangle,$$

$$B_1 = \langle x^2, xx', xy, x'^2, x'y', y^2, yy', y'^2, xy' + x'y \rangle,$$

$$B_2 = \langle x^2, xy, y^2, x', y' \rangle.$$

The ideal A defines a reduced plane, while $V(B_i) = \{0\}$. Therefore, $\dim D^2(f) = 2$ and Theorem 3.3.6 doesn't apply here. Indeed, $D^2(f)$ has an embedded component, namely $V(B_i)$, and thus it is is not a Cohen-Macaulay space.

Now we show that in this situation $H^2(f)$ may not behave well under deformations: Take the unfolding $F\colon (\mathbb{C}^3,0)\to(\mathbb{C}^4,0)$ given by

$$(t,x,y)\mapsto (t,f_t(x,y)),\ \ f_t(x,y)=(x^2,y^2,xy+t(y^3+x^3)).$$

For sufficiently small $t\neq 0$, f_t is $\mathcal{A}$-equivalent to the map in Example 2.1.11. Since f_t is generically one-to-one, for all $t\neq 0$, we conclude that the map F is generically one-to-one. From Corollary 3.3.8 it follows $I^2(F)=H^2(F)$ and, since $I^2(f)$ behaves well under deformations, we have $H^2(F)+\langle t\rangle = I^2(f)\neq H^2(f)$.

Chapter 4

Blowing-up double points

The properties of the double space of maps change, depending on the existence of points of corank ≥ 2. For instance: The double point space $D^2(f)$ of a stable map $f\colon X \to Y$ is smooth if and only if f has no corank ≥ 2 points (Proposition 3.2.3). If the double point space of a corank 1 map is dimensionally correct (Definition 2.2.4), then it is locally a complete intersection (just count the number of generators provided in Section 2.2). On the other hand, it is easy to find corank 2 map germs (for instance Example 2.1.11) whose double point space is dimensionally correct, but not a complete intersection.

In this chapter we introduce a different double point space that satisfies, for any corank, some of the nice properties of double points spaces of corank 1 maps. This space was introduced first by Ronga [Ron72] and Laksov [Lak77]. In the context of enumerative geometry, the corresponding space has been studied by Kleiman, Lipman and Ulrich [Kle81, KLU92, KLU96] for finite morphisms $f\colon X \to Y$ between projective algebraic schemes over a field of any characteristic. Our goal is to give a different, more explicit approach to the construction of the mentioned space (only in the holomorphic case). We obtain some new results and transparent proofs of some of the ones that the authors above had obtained before.

4.1 The space $B^2(f)$

Throughout this chapter, $B(X)$ represents the blowing-up of $X \times X$ along the diagonal $\Delta(X, 2)$ and $\pi\colon B(\mathbb{C}^n) \to X \times X$ its associated projection. The notation we use and necessary background can be found in Section A.1).

Definition 4.1.1. Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. We define

$$K_f = \{0\} \times \{0\} \times \mathbb{P}(\ker df_0) \subseteq B(\mathbb{C}^n).$$

As in Proposition-Definition 3.1.1, let α be a $p \times n$ matrix with entries in $\mathcal{O}_{2n}$, satisfying

$$f(x') - f(x) = \alpha(x, x')(x - x').$$

We denote by $K^2(f)$ the ideal sheaf in $\mathcal{O}_{B(X),K_f}$ generated by the germs at K_f of the functions

$$\sum_{i=1}^{n} \alpha_{ji}(x, x')u_i, \quad 1 \leq j \leq p.$$

We define $B^2(f)$ as the germ of complex space defined by $K^2(f)$, and call it the **blowing-up double point space of** f.

Remark 4.1.2. Some comments are due:

1. The space $B^2(f)$ does not depend on the choice of α (this will follow from Lemma 4.1.8, simply because there is no α in the statement).

2. Since $\alpha(0,0) = df_0$ (see Lemma 3.1.4), the subspace K_f is exactly the intersection of $B^2(f)$ with the fiber $\pi^{-1}(0,0)$.

3. In general $B^2(f)$ is a germ of complex space along a subset (see Section A.2). More precisely, $B^2(f)$ is a germ along the subset $K_f \cong \mathbb{P}^{k-1}$, where $k = \operatorname{corank} f$. Therefore, $B^2(f)$ is a usual germ of complex space at a point if and only if $\operatorname{corank} f = 1$.

Remark 4.1.3. The blowing-up double points behave well under the following operation: Let f and g be maps $X \to Y$. Let $(f,g)\colon X \to Y \times Y$ be the map given by $x \mapsto (f(x), g(x))$. It is immediate that the blowing-up double point spaces satisfy:

$$B^2((f,g)) = B^2(f) \cap B^2(g).$$

In other words, we have $K^2(F) = K^2(f) + K^2(g)$.

Observe that the property just mentioned does not hold for usual double point spaces $D^2(f)$. Although the remark is trivial, it has nice consequences. For instance, it makes blowing-up double points of map germs of the form $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, with $p < n$, into useful mathematical objects, as the following examples show:

Example 4.1.4. Let $p^n_r\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^r, 0), r < n$, be the projection $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_r)$. Then, there is an isomorphism

$$B^2(p^n_r) \cong \mathbb{C}^r \times B(\mathbb{C}^{n-r}),$$

given by the identification of $\mathbb{C}^r \times B(\mathbb{C}^{n-r})$ with the space of points $(x, x', [u]) \in B(\mathbb{C}^n)$, satifying $x'_1 = x_1, \dots, x'_r = x_r$ and $u_1 = \cdots = u_r = 0$.

Example 4.1.5. Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a corank 1 map germ of the form

$$(x_1, \dots, x_{n-1}, y) \mapsto (x_1, \dots, x_{n-1}, f_n(x, y), \dots, f_p(x, y)).$$

Then $B^2(f)$ is isomorphic to $D^2(f)$.

We have $B^2(f) = B^2(p^n_{n-1}) \cap B^2(f_n, \dots, f_p)$. Thus $K^2(f)$ is given by $K^2(f_n, \dots, f_p)$ at the subspace $B^2(p^n_{n-1})$ of $B(\mathbb{C}^n)$ given by $x'_1 = x_1, \dots, x'_{n-1} = x_{n-1}$ and $u = (0 : \cdots : 0 : 1)$. Modulo $K^2(p^n_{n-1})$, the functions $f_j(x'_1, \dots, x'_{n-1}, y') - f_j(x_1, \dots, x_{n-1}, y)$ are equivalent to $f_j(x_1, \dots, x_{n-1}, y') - f_j(x_1, \dots, x_{n-1}, y)$. Therefore, the function germs $\sum_{i=1}^n \alpha_{ji}(x, x')u_i$, $n \leq j \leq p$ are precisely the divided differences. The claim follows eliminating the variables $x'_1, \dots, x'_{n-1}$ and u.

Remark 4.1.6. Let $\varphi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a germ of biholomorphism. Let P be a matrix, with entries in $\mathcal{O}_{2n}$, satisfying

$$\varphi(x) - \varphi(x') = P(x, x')(x - x').$$

Then, for any vector subspace $K \leq \mathbb{C}^n$, φ induces a germ of biholomorphism $\hat{\varphi}\colon B_1 \to B_2$, given by

$$(x, x', [u]) \mapsto (\varphi(x), \varphi(x'), [P(x, x')u]),$$

where B_1 is the germ of $B(\mathbb{C}^n)$ at $\{0\} \times \{0\} \times K$ and B_2 is the germ of $B(\mathbb{C}^n)$ at $\{0\} \times \{0\} \times d\varphi_0(K)$

Proposition 4.1.7. *If $f, g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ are $\mathcal{A}$-equivalent map germs with $f = \psi \circ g \circ \varphi$, then the spaces $B^2(f)$ and $B^2(g)$ are isomorphic via the map germ $\hat{\varphi}$ above, induced by φ at $K = \ker df_0$.*

Proof. We divide the proof in two steps. First, assume $f = \psi \circ g$. Let $g(x) - g(x') = \alpha(x, x')(x - x')$. With the notations above, we obtain $f(x) - f(x') = \psi(g(x)) - \psi(g(x')) = P(g(x), g(x'))(g(x) - g(x')) = P(g(x), g(x'))(\alpha(x, x'))(x - x')$. Taking

$$\beta(x, x') = P(g(x), g(x'))(\alpha(x, x'))$$

we have $f(x) - f(x') = \beta(x, x')(x - x')$. The result follows in this case since $\alpha(x, x')u = 0$ and $\beta(x, x')u = 0$ are equivalent systems of equations.

Now, we can assume $f = g \circ \varphi$. Let $g(x) - g(x') = \alpha(x, x')(x - x')$, for some matrix α. With the notations above, we have $f(x) - f(x') = \alpha(\varphi(x), \varphi(x'))P(x, x')(x - x')$. Taking the matrix

$$\beta(x, x') = \alpha(\varphi(x), \varphi(x'))P(x, x')$$

we obtain $f(x) - f(x') = \beta(x, x')(x - x')$. Therefore $B^2(f)$ is generated by the germs at K of the functions $\sum_{i=1}^n \beta_{ji}(x, x')u_i$, for $j = 1, \dots, p$. These germs are precisely the images under the pullback $(\hat{\varphi})^*$ of the germs of functions $\sum_{i=1}^n \alpha_{ji}(x, x')u_i$, which define $B^2(g)$, and the claim follows. $\square$

Lemma 4.1.8. *With the notations in Example A.1.3, the chart ϕ_l yields an isomorphism between $B^2(f) \cap U_l$ and the zero set of the ideal in $\mathcal{O}_{A,\phi_l(K)}$, generated by the germs*

$$h_j^l(x, \lambda, a) = \frac{f_j(x + \lambda \widetilde{a}) - f_j(x)}{\lambda}, \quad 1 \le j \le p.$$

Proof. This is just

$$\phi_l^*(\frac{f_j(x + \lambda \widetilde{a}) - f_j(x)}{\lambda}) = \phi_l^*(\frac{\sum_{i=1}^n \alpha_{ij}(x, x + \lambda \widetilde{a}) \lambda \widetilde{a}_i}{\lambda}) = \sum_{i=1}^n \alpha_{ij}(x, x') u_i.$$

□

Example 4.1.9. Let $f \colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be the Folded Handkerchief, given by

$$(x, y) \mapsto (x^2, y^2).$$

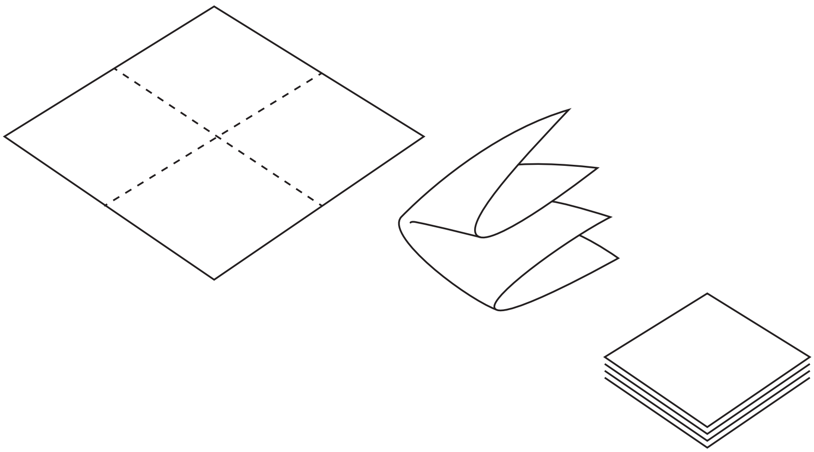

Figure 4.1: Image of the Folded Handkerchief.

Following the definition of $B^2(f)$, the first thing to do is to solve the equation $f(x) - f(x') = \alpha(x, x')(x - x')$. In this case, it is just

$$\begin{pmatrix} x^2 - x'^2 \\ y^2 - y'^2 \end{pmatrix} = \begin{pmatrix} x + x' & 0 \\ 0 & y + y \end{pmatrix} \begin{pmatrix} x - x' \\ y - y' \end{pmatrix}.$$

Thus, $B^2(f)$ is the germ along $\{0\} \times \{0\} \times \mathbb{P}^1$ of the space of points $((x, y), (x', y'), (u_1 : u_2)) \in \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{P}^1$, satisfying the equation

$$(x - x')u_2 = (y - y')u_1,$$

which defines $B(\mathbb{C}^2)$, and the equations

$$(x+x')u_1 = 0, \quad (y+y')u_2 = 0,$$

which determine the ideal $K^2(f)$.

Now we compute $B^2(f)$ piecewise, by means of Lemma 4.1.8:

The chart ϕ_1 maps $B^2(f) \cap U_1$ to the space of points $(x, y, \lambda, a) \in A$, where the functions h_1^1 and h_2^1 vanish. Taking into account that for this chart we have $\widetilde{a} = (1, a)$, the functions h_1^1, h_2^1 yield the equations

$$\lambda + 2x = 0, \quad a(2y + \lambda a) = 0.$$

This space consists on two irreducible components $\{\lambda = -2x, a = 0\}$ and $\{\lambda = -2x, y = xa\}$. We can take these components back to U_1 as the space of points $((x, y), (x', y'), P))$ with $(x', y') = (x, y) + \lambda(1, a)$ and $P = (1 : a)$. The corresponding components are the subsets

$$Z_1 = \{((x, y), (-x, y), (1 : 0)) \mid x, y \in \mathbb{C}\}$$

and

$$Z_2 = \{((x, y), (-x, -y), (1 : a)) \mid ax = y, \; x, y \in \mathbb{C}\}$$

of $B(\mathbb{C}^2)$, both with reduced structure. Observe that Z_2 is not a projective, but a quasi-projective variety. This is because the points

$$((x, y), (-x, -y), (1 : a)),$$

with $ax = y$ and $x \neq 0$, accumulate at points

$$((0, y), (0, -y), (0 : 1)),$$

which simply do not belong to U_1. This is not in contradiction with $B^2(f)$ being a projective space. As we will see shortly, these boundary points belong to $B^2(f) \cap U_2$.

Now we deal with $B^2(f) \cap U_2$. We are only interested in $(B^2(f) \cap U_2) \setminus U_1$, since we have computed the rest of $B^2(f)$ already by means of ϕ_1. Therefore, the only point $\widetilde{a} = (a, 1)$ we have to consider is $(0, 1)$. After some computations analogous the previous ones, we obtain the component

$$Z_3 = \{((x, y), (x, -y), (0 : 1)) \mid x, y \in \mathbb{C}\},$$

also with reduced structure. It is obvious that Z_3 contains the points $((x, 0), (-x, 0), (0 : 1)) \in B(\mathbb{C}^n)$ in the boundary of Z_2, making $B^2(f) = Z_1 \cup Z_2 \cup Z_3$ into a projective space.

There are three different kinds of points in $B^2(f)$ (Figure 4.2), namely:

1. Non diagonal points: Their image by π is not contained in $\Delta(X, 2)$. They consist of

- points $((x, y), (-x, y), (1 : 0))$, with $x \neq 0$. These belong to Z_1, and also to Z_2 if $y = 0$,
- points $(x, y), (x, -y), (0 : 1)$, with $y \neq 0$. These belong to Z_3, and also to the boundary of Z_2 if $x = 0$,
- points $(x, y), (-x, -y), (0 : 1)$, with $x, y \neq 0$. These belong to $Z_2 \setminus (Z_1 \cup Z_3)$.

2. Diagonal corank 1 points: Their image by π belongs to $\Delta(\Sigma^1(f), 2)$ They consist of

   - Diagonal points $((0, y), (0, y), (1 : 0))$, with $y \neq 0$. These points belong to Z_1,
   - Diagonal points $((x, 0), (x, 0), (0 : 1))$, with $x \neq 0$. These points belong to Z_3.

3. Diagonal corank 2 points: Their image by π belongs to $\Delta(\Sigma^2(f), 2)$. They are of the form $((0, 0), (0, 0), P)$, with $P \in \mathbb{P}^1$. They belong to Z_2, and also to Z_1 and Z_3 in the respective cases of $P = (1 : 0)$ and $P = (0 : 1)$. The space $B^2(f)$ is a germ along the set of these points.

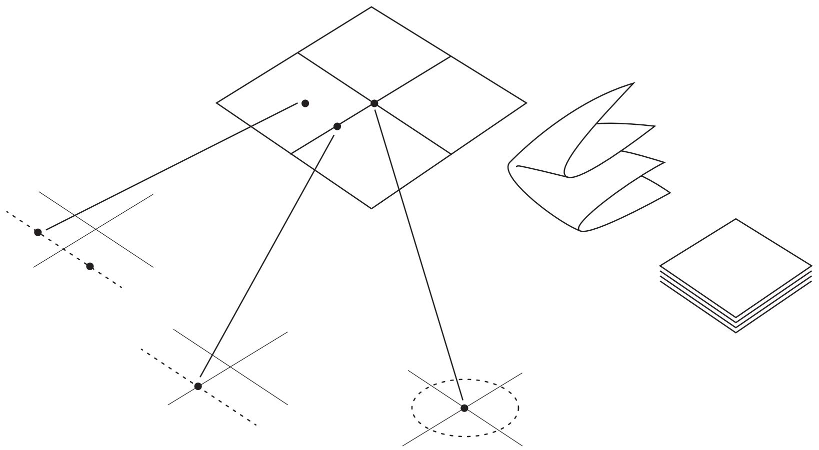

Figure 4.2: Blowing-up double points of a Folded Handkerchief.

Lemma 4.1.10. *Let $F(t, x) = (t, f_t(x))$ be an r-parametric unfolding of $f = f_0$, then:*

1. $B^2(F) \cap U_l = \emptyset$, for $l = 1, \ldots, r$.

2. *For any $l \leq n$, the space $B^2(F) \cap U_{l+r}$ is isomorphic to the zero set of the germs along $\{0\} \times \phi_l(K_{f_0})$ of the following functions, defined in $\mathbb{C}^r \times A$:*

$$\frac{(f_t)_j(x+\lambda\widetilde{a}) - (f_t)_j(x)}{\lambda}, \quad 1 \leq j \leq p,$$

where $(f_t)_j(x)$ is the j-th coordinate of $f_t(x)$.

Proof. To show the first item, let $1 \leq l \leq r$. The space $B^2(F) \cap U_l$ is isomorphic to its image via ϕ_l. We make use of Lemma 4.1.8 to compute it. We have to consider points of type (t, x, λ, a), with $a = a_1, \dots, a_{r+n-1}$ and $\widetilde{a} = (a_1, \dots, 1, \dots, a_{n-1})$, where we have added a coordinate with value 1 at the l-th position. The equations defining the space are $h^l_j = 0$, for $j = 1, \dots, r+p$. In particular, since $F_l(t, x) = t_l$, we have $h^l_l(t, x, \lambda, a) = (t_l + \lambda - t_l)/\lambda = 1$. This forces $B^2(F) \cap U_l = \emptyset$.

Now we show the second item. If we assume $l \geq r$ and proceed as before, we will obtain functions $h^l_j(t, x, \lambda, a) = t_j - a_j$, for $j = 1, \dots, r$. The claim follows eliminating the variables a_j, $1 \leq j \leq r$, and labeling the remainig variables $a_{r+1}, \dots, a_{r+n-1}$ as new variables $a_1, \dots, a_{n-1}$. $\square$

Lemma 4.1.11. *GIven $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, let $\mathscr{F}$ be the ideal sheaf defined by some representatives of the generators of $\mathscr{K}^2(f)$ at some open neighbourhood U of K_f. Let f be a representative of f defined in U. Then (possibly shrinking U) the following hold for every point $w \in U$:*

1. *If $w = (x, x, [u]) \in \pi^{-1}(\Delta(\mathbb{C}^n, 2))$, then the stalk $\mathscr{F}_w$ equals the germ at w of $\mathscr{K}^2(f_x)$, where f_x is the germ of f at x.*

2. *If $w \in B(X) \setminus \pi^{-1}(\Delta(\mathbb{C}^n, 2))$, then $\mathscr{F}_w \cong \mathscr{P}(f, 2)_{\pi(w)}$, via the homomorphism π^*_w induced by π on the stalks.*

Proof. 1) follows shrinking U so that we can find representatives of the germs $\sum_{i=1}^n \alpha_{ji}(x, x')u_i$ defined on all U. To show 2) we use Lemma 4.1.8. Let $w = (x, x', u) \in B(C^n)$and take a chart ϕ_l making $\phi_l(x, x', u) = (x_0, \widetilde{a}_0, \lambda_0)$. The ideal $\mathscr{F}_w$ is generated by the germs $h^l_j(x, \lambda, a)$. Since $w \notin \Delta(\mathbb{C}^n, 2)$, we have $\lambda_0 \neq 0$, so the generators h^l_j can be changed by $f_j(x + \lambda\widetilde{a}) - f_j(x)$. These germs are precisely the image by the pullback of $\phi^{-1} \circ \pi$ of the generators $f_j(x') - f_j(x)$ of $\mathscr{P}(f, 2)_{\pi(w)}$. $\square$

Definition 4.1.12. Given $f\colon X \to Y$, the **blowing-up double point ideal sheaf of** f is the ideal sheaf $\mathscr{K}^2(f)$ of $\mathcal{O}_{B(\mathbb{C}^n)}$, given locally by:

1. $\mathscr{K}^2(f) = \pi^*(\mathscr{P}(f, 2))$ on the open subset $B(X) \setminus \pi^{-1}(\Delta(X, 2))$.

2. For any point $w = (x, x, u) \in \pi^{-1}(\Delta(X, 2))$, the stalk $\mathscr{K}^2(f)_w$ is the germ at w of $K^2(f_x)$, where f_x is the germ of f at x.

We define $B^2(f)$ as the zero set of $\mathscr{K}^2(f)$, and we call it the **blowing-up double point space of** f.

Proposition 4.1.13.

1. *If f and g are $\mathcal{A}$-equivalent, then the spaces $B^2(f)$ and $B^2(g)$ are isomorphic.*

2. *The space $B^2(f)$ is locally isomorphic to $D^2(f)$, out of the subspace*

$$\pi^{-1}(\{(x,x) \in \Delta(X,2) \mid x \in \hat{\Sigma}^2(f)\}).$$

3. *$B^2(f)$ behaves well under deformations: If $F(t,x) = (t, f_t(x))$ is an unfolding of f, then*

$$B^2(f) = B^2(F) \cap \pi^{-1}(\{t = 0\}).$$

Proof. Off the fiber $\pi^{-1}(\Delta(X,2)$ of the diagonal, the spaces $B^2(f)$ and $B^2(g)$ are isomorphic to $D^2(f)$ and $D^2(g)$, so 1), 2) and 3) hold. On $\pi^{-1}(\Delta(X,2)$ 1) follows directly from Proposition 4.1.7. To show 2), it remains to show the isomorphism on diagonal points of corank 1. Around this points, the map can be taken to the form

$$(x_1, \ldots, x_{n-1}, f_n(x,y), \ldots, f_p(x,y)),$$

and the result follows from Lemma 4.1.10. To be precise, the first assertion of the lemma forces $\widetilde{a} = (0, \ldots, 0, 1)$ and thus the generators given on the second item are exactly the divided differences (see Section 2.2). On diagonal points, 3) follows directly from Lemma 4.1.10. $\square$

The following proposition can also be found in [Ron72] and [Kle81]:

Proposition 4.1.14. *If $f\colon X \to Y$ satisfies $\dim B^2(f) = 2n - p$, then $B^2(f)$ is locally a complete intersection.*

Proof. This is a local question at points $w \in B^2(f)$. If $w \notin \pi^{-1}(\Delta(X,2))$, then $K^2(f)_w$ is isomorphic to the stalk $\mathscr{P}(f,2)_{\pi(w)}$, which is generated by the germs $f_j(x) - f_j(x') \in \mathcal{O}_{2n},\ 1 \leq j \leq p$ in $\mathcal{O}_{2n}$, where f_j are local coordinates of the function f at $f(x)$. If $w = (x,x,u) \in \pi^{-1}(\Delta(X,2))$, then $\mathscr{K}^2(f)_w$ is locally generated by the germs at w of $\sum_{i=1}^n \alpha_{ij}(x,x')u_i, 1 \leq j \leq p$. In both cases, $\mathscr{K}^2(f)_w$ is generated locally by p function germs in a smooth space of dimension $2n$. $\square$

4.2 $B^2(f)$ and stability and finite determinacy

The next result follows from a result due to Ronga [Ron72, Corollary 2.3], because any stable map is Σ-generic and has normal crossings.

Theorem 4.2.1. *If $f\colon X \to Y$ is a stable map, then $B^2(f)$ is empty or a complex manifold of dimension $2n - p$.*

Proof. We proceed locally and in two cases. First, assume that $w \in B^2(f)$ is not contained in the diagonal fibre $\pi^{-1}(\Delta(X,2)$. Then, locally at w, $B^2(f)$ is isomorphic to the zero set of $\mathscr{P}(f,2)$, and then the result follows directly from Proposition 1.3.4 and the fact that every stable map has normal crossings (Proposition 1.5.9).

Now let $w = (x,x,u) \in \pi^{-1}(\Delta(X,2))$. By definition, the stalk $\mathscr{K}^2(f)_w$ equals the germ at w of $K^2(f_x)$. Therefore, it suffices to show that $K^2(f)$ is a germ of smooth space, for every stable map germ $f\colon (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$. Let f be such a stable germ and define, for every $p \times n$ matrix $M \in L(n,p)$, the map germ $f_M\colon (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$, given by

$$x \mapsto f(x) + Mx.$$

Identifying $L(n,p)$ with $\mathbb{C}^{np}$ we define an unfolding $F\colon (L(n,p)\times\mathbb{C}^n,0) \to (L(n,p)\times\mathbb{C}^p,0)$, given by $(M,x) \mapsto (M, f_M(x))$. Let $w = (0,0,u) \in K_f$. By Proposition 4.1.7, we can perform a change of coordinates and assume $u = (1 : 0 : \cdots : 0)$. From Lemma 4.1.10 and item (3) in Proposition 4.1.13 it follows that the space $B^2(f_M)$, at the neighbourhood U_1 of w, is isomorphic to the zeros of the functions

$$h_{j,M}(x,\lambda,a) = \frac{f_M(x+\lambda\widetilde{a}) - f_M(x)}{\lambda} = h_j^1(x,\lambda,a) + M\widetilde{a},$$

for some representative F and every matrix M close enough to the zero matrix. Therefore, for any fixed matrix M close enough to 0, the space $B^2(f)$ is regular of dimension $2n-p$ around w if the map

$$h_M = (h_{1,M},\ldots,h_{p,M})\colon A \to \mathbb{C}^p$$

is transverse to 0 at all points of the form $(0,0,a) \in A$. Let $H\colon L(n,p)\times A \to \mathbb{C}^p$ be the map given by

$$H(M,x,\lambda,a) = h_M(x,\lambda,a).$$

If we label the coordinates in $L(n,p)$ as $m_{ji}, 1 \le i \le n, 1 \le j \le p$, then it is immediate that the partial derivative $\frac{\partial H_j}{\partial m_{ji}}|_{(M,x,\lambda,a)}$ equals the i-th coordinate of $\widetilde{a}$. In particular, since the first coordinate of $\widetilde{a}$ is 1, we have $\frac{\partial H_j}{\partial m_{j1}}|_{(M,x,\lambda,a)} = 1$. Thus, the $p \times n(p+2)$ matrix $dH_{(M,x,\lambda,a)}$ contains the identity matrix of size $p \times p$ as a submatrix. It follows that H is a submersion at all points $(M,x,\lambda,a) \in L(n,p)\times A$. By Lemma 1.3.3, the map $(h_{1,M},\ldots,h_{p,M})\colon A \to \mathbb{C}^p$ is transverse to 0, for all M out of a proper subspace $Z \subset L(n,p)$. Thus, we can produce a family of maps $f_t = f_{M_t}$, with $M_t \in L(n,p)\setminus Z$ and $M_0 = 0$, so that $B^2(f_t)$ is smooth of dimension $2n-p$, for all $t \neq 0$. Since $f = f_0$ is stable, the unfolding must be trivial. Therefore, from the first item of Proposition 4.1.13, it follows that $B^2(f)$ is smooth of dimension $2n-p$. $\square$

Corollary 4.2.2. *Let* $f\colon (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$ *be a finitely determined map germ and* $\operatorname{corank} f = k \ge 1$*, then:*

1. *$B^2(f)$ is smooth of dimension $2n-p$ out of K_f.*

2. *If $2n-p-k-1 \geq 0$, then the space $B^2(f)$ is normal.*

3. *$B^2(f)$ is locally a complete intersection of dimension $2n-p$ if and only if $2n-p-k+1 \geq 0$.*

Proof.
1) Follows directly from Theorem 1.5.12 and Theorem 4.2.1.
2) Since $B^2(f)$ is not empty, we have $\dim B^2(f) \geq 2n-p$. By 1), the singular locus of is contained in $K_f \cong \mathbb{P}^{k-1}$, and the result follows from Serre's criterion [Mat80, Thm. 39].
3) Since $K^2(f)$ is generated by $2n-p$ elements, $B^2(f)$ is not a locally complete intersection of dimension $2n-p$ if and only if it contains an irreducible component of dimension $> 2n-p$. By 1) this component must be $K_f \cong \mathbb{P}^{k-1}$. □

Corollary 4.2.3. *A map germ $f\colon (\mathbb{C}^n,0) \to (\mathbb{C}^{2n},0)$ is finitely determined if and only if $B^2(f)$ is contained in $\pi^{-1}(0,0)$.*

Proof. Assume f is finitely determined. By Proposition 1.5.10, a small representative of f has to be an immersion with normal crossings. Thus, we have $B^2(f) \setminus K_f$ equals $D^2(f) \setminus \{0\}$ which consists on isolated points. Shrinking our reprensentative, we obtain $B^2(f) = K_f \subseteq \pi^{-1}(0,0)$. If we assume $B^2(f) \subseteq \pi^{-1}(0,0)$, then there exists a representative of f which an injective immersion out of the origin. Thus, again by Proposition 1.5.10, this representative is stable off the origin. □

Example 4.2.4. Let $f\colon (\mathbb{C}^2,0) \to (\mathbb{C}^4,0)$ be the map germ given by

$$(x,y) \mapsto (x^2, y^2, x^3+xy, y^3+xy).$$

Due to the symmetric role played by the variables x and y at this example, it suffices to compute $B^2(f) \cap U_1$. According to Lemma 4.1.8, this space is isomorphic to the zeros of the ideal H generated by the germs $h^1_i(x,y,\lambda,a)$:

- $h^1_1 = 2x + \lambda$,
- $h^1_2 = a(2y + \lambda a)$,
- $h^1_3 = 3x^2 + 3x\lambda + \lambda^2 + xa + y + \lambda a$,
- $h^1_4 = 3y^2a + 3y\lambda a^2 + \lambda^2 a^3 + xa + y + \lambda a$.

A straightforward computation shows $H = \langle x^3, y + x(x-a), \lambda + 2x, ax^2 \rangle$. We conclude that $B^2(f)$ equals $\{0\} \times \{0\} \times \mathbb{P}^1$, set theoretically. Therefore, f is a finitely determined map germ. Observe that $\dim B^2(f) = 1$, while $\dim D^2(f) = 0$. This justifies the need of condition $2n-p-k+1$ in the third item 3 of Corollary 4.2.2.

Corollary 4.2.5. *A map germ $f\colon(\mathbb{C}^n,0)\to(\mathbb{C}^{2n-1},0)$, $n\geq 2$ is finitely determined if and only if $D^3(f)\subseteq\{0\}$ and $B^2(f)$ is a reduced curve out of $\pi^{-1}(0,0)$.*

Proof. Assume f is finitely determined. By Mather-Gaffney criterion 1.5.12, the map is stable off the origin. By Proposition 1.5.11 f has corank 1 at all these stable points, and thus $B^2(f)\setminus\pi^{-1}(0,0)$ is isomorphic to $D^2(f)\setminus\{0,0\}$, by Proposition 4.1.13. Therefore, by Theorem 2.2.3, $B^2(f)$ is a smooth curve out of $\pi^{-1}(0,0)$. Off the origin, $D^3(f)$ is empty or a manifold of dimension $2-n$, by Lemma 2.1.9. Thus, we have $D^3(f)\subseteq\{0\}$.

Assume that $B^2(f)$ is a reduced curve out of $\pi^{-1}(0,0)$. For any point x of corank k, we have a fiber $\pi^{-1}(x,x)\cong\mathbb{P}^{k-1}$. Hence, points $x\in\hat{\Sigma}^2(f)$ must be isolated. Thus, we can find a representative of f defined at a neighbouhood U not containing corank two points and such that $B^2(f)$ is a smooth curve at $U\setminus\pi^{-1}(0,0)$. Since $B^2(f)$ is isomorphic to $D^2(f)$ off the origin and $D^3(f)\subseteq\{0\}$, from Theorem 2.2.3 it follows that f is stable at $U\setminus\{0\}$. Now the result follows from Mather-Gaffney criterion 1.5.12. □

An equivalent criterion, in terms of $D^2(f)$ and $D^3(f)$, can be found in [NBJP09, Proposition 3].

Example 4.2.6. Let $F=(t,f_t)\colon(\mathbb{C}^3,0)\to(\mathbb{C}^5,0)$ be the unfolding of f given by

$$(t,x,y)\mapsto(t,x^2,y^2,x^3+xy+ty,y^3+xy+tx).$$

By Lemma 4.1.10, we consider $B^2(F)$ as a deformation $B^2(f_t)$ of the space $B^2(f)$, with parameter t. As for f, the roles of the variables x and y are symmetric for F, and thus we only need to compute $B^2(f_t)\cap U_1$. The ideal H defining the family $B^2(f_t)\cap U_1$ is generated by the germs h^1_j, $1\leq j\leq 4$, given by

$$h^1_j(t,x,y,\lambda,a)=\frac{(f_t)_j(x+\lambda,y+\lambda a)-(f_t)_j(x,y)}{\lambda}.$$

A primary decomposition is given by $H=\bigcap_{i=0}^4 H_i\cap J$, with

- $J=\langle t,x^2,y-xa,\lambda+2x,x^2a\rangle$,
- $H_0=\langle x^2-t,y+t,\lambda+2x,a\rangle$,
- $H_i=\langle x^2-\tau_it,y-\tau_ix,\lambda+2x,a-\tau_i\rangle$, where $\tau_1,\dots,\tau_4$ are the different fourth roots of the unity in $\mathbb{C}$.

If we write $Z_i=V(H_i)$, it turns out that Z_0 is a non-reduced curve contained in $\pi^{-1}(0)$, whereas $Z_1,\dots,Z_4$ are reduced curves. After checking that F has just one triple point, we conclude that F is a finitely determined map germ.

4.3 Strict blowing-up double points

Now we show that $B^2(f)$ can be obtained in an analogous way to the definition of $D^2(f)$ in Proposition-Definition 2.1.5. That is, we can compute $B^2(f)$ by slicing the closure of the 'strict double points' of an unfolding F of f.

Definition 4.3.1. Given $f\colon X \to Y$, the **strict blowing-up double point space** $B^2_S(f)$ of f is the closure of the preimage by π of the set of strict double points

$$\{(x,x') \in X \times X \mid x \neq x', f(x) = f(x')\}.$$

The following result is equivalent to [Ron72, Corollary 2.4]

Proposition 4.3.2. *If f is a stable map, then $B^2_S(f)$ equals $B^2(f)$.*

Proof. Observe that this is a set theoretical question, since $B^2_S(f)$ is reduced by construction and $B^2(f)$ is reduced by Theorem 4.2.1. $B^2_S(f) \subseteq B^2(f)$ is obvious. Assume $B^2_S(f) \neq B^2(f)$. Then $B^2(f)$ contains an irreducible component Z contained in the closure of the union of the fibers $\pi^{-1}(x,x)$, with f singular at x. On one hand, the fiber $\pi^{-1}(x,x)$ of a point $x \in \Sigma^k(f)$ is a projective space of dimension $k-1$. On the other hand, if f is stable, then $\Sigma^k(f)$ is a complex manifold of dimension $n-k(p-n+k)$. Thus, the dimension of Z cannot exceed the maximum of the numbers $k-1+n-k(p-n+k), 1 \leq k \leq n$. However, we have the inequality $2n-p-((k-1)+n-k(p-n+k)) \geq k-p-((k-1)-kp) = 1-p+kp \geq 1$. It follows $\dim Z < \dim B^2(f)$, which is in contradiction with Theorem 4.2.1. □

As a consequence of the previous property, we obtain necessary conditions for finite determinacy in terms of $B^2_S(f)$. For instance, from Corollary 4.2.3 and Corollary 4.2.5 we get the following

Corollary 4.3.3.

1. *If a map germ $f\colon (\mathbb{C}^n,0) \to (\mathbb{C}^{2n},0)$ is finitely determined then $B^2_S(f) = \emptyset$.*

2. *If a map germ $f\colon (\mathbb{C}^n,0) \to (\mathbb{C}^{2n-1},0)$ is finitely determined then $B^2_S(f)$ is (the germ along K_f of) a reduced curve.*

Proof. Assume f is finitely determined, then from Proposition 4.3.2 and Mather-Gaffney criterion 1.5.12 follows the equality

$$B^2_S(f) \setminus \pi^{-1}(0,0) = B^2(f) \setminus \pi^{-1}(0,0).$$

Since $(0,0)$ can not be a strict double point, we have

$$B^2_S(f) = \overline{B^2(f) \setminus \pi^{-1}(\Delta(X,2))}.$$

Now 1) and 2) follow immediately from Corollary 4.2.3 and Corollary 4.2.5, respectively. □

Corollary 4.3.4. *Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a finitely determined map germ of corank k. If $2n - p - k \geq 0$, then $B^2_S(f) = B^2(f)$ as schemes.*

Proof. If $2n - p - k \geq 0$, then $B^2(f)$ is locally a complete intersection of dimension $2n - p$ by Corollary 4.2.2. Since $\dim K_f = k - 1 < 2n - p$, we conclude that there is no component of $B^2(f)$ contained in $\pi^{-1}(0, 0)$. Now, from Proposition 4.3.2 and Mather-Gaffney criterion 1.5.12, we obtain $B^2_S(f) \setminus \pi^{-1}(0, 0) = B^2(f) \setminus \pi^{-1}(0, 0)$ and the result follows. □

4.4 The morphism $\pi\colon B^2(f) \to D^2(f)$

For any complex manifold X, we let π^* be the morphism $\mathcal{O}_{X\times X} \to \mathcal{O}_{B(\mathbb{C}^n)}$ given by $h \mapsto h \circ \pi$.

Lemma 4.4.1. *For any map $f\colon X \to Y$, the morphism π^* takes $\mathscr{I}^2(f)$ into $\mathscr{K}^2(f)$*

Proof. Since π is an isomorphism out of $\pi^{-1}(\Delta(X, 2))$, it suffices to show the equivalent claim for a germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. Moreover, we can check this locally using the atlas $\{\phi_i\colon U_i \to A \mid 1 \leq i \leq n\}$. Thus, we need to check, for any $i \leq n$, the inclusion $(\phi_i^{-1} \circ \pi)^*(I^2(f)) \subseteq (\phi_i^{-1})^*(K^2(f))$. Recall that the ideal $(\phi^{-1})^*(K^2(f))$ is generated by the germs $h^i_j(x, \lambda, a)$ given in Lemma 4.1.8, while $I^2(f)$ is generated by the germs $f_j(x) - f_j(x')$ and the $n \times n$ minors of any matrix α satisfying $f(x) - f(x') = \alpha(x, x')(x - x')$. On one hand, we have $(\phi_i^{-1} \circ \pi)^*(f_j(x) - f_j(x')) = f_j(x) - f_j(x + \lambda\widetilde{a}) = \lambda h^i_j$, which is clearly contained in $(\phi^{-1})^*(K^2(f))$. On the other hand, let A be the image by $(\varphi_i^{-1} \circ \pi)^*$ of an $n \times n$ submatrix obtained by choosing some rows $j_1, \ldots, j_n$ of α. By Cramer's rule, it follows

$$|A|\lambda(\widetilde{a})_i = |A'_i|,$$

where A'_i stands for the matrix obtained by substitution of the i-th column of A by the elements $f_{j_i}(x) - f_{j_i}(x + \lambda\widetilde{a}), i = 1, \ldots, n$. Since the i-th coordinate of $\widetilde{a}$ equals 1, we obtain $|A| = |A'_i|/\lambda$. Expanding the determinant $|A'_i|$ by the i-th column, we obtain

$$|A| = \frac{1}{\lambda}\sum_{i=1}^{n}(f_{j_i}(x) - f_{j_i}(x + \lambda\widetilde{a}))|A''_{j_i}| = \sum_{i=1}^{n} h^i_j |A''_{j_i}|,$$

for the corresponding submatrices A''_{j_i} of A'_i. Hence $|A| \in (\phi^{-1})^*(K^2(f))$. □

From the previous lemma and Lemma 3.1.8, it follows that the restriction of $\pi\colon B(X) \to X \times X$ yields a proper and surjective morphism of complex spaces $B^2(f) \to D^2(f)$, which, by abuse of notation, we denote also by

$$\pi\colon B^2(f) \to D^2(f).$$

The following property is very similar to [Lak77, Proposition 25]

Proposition 4.4.2. *For any map $f\colon X \to Y$, if the space*

$$\{(x,x) \in \Delta(X,2) \mid x \in \hat{\Sigma}^2(f)\}$$

does not contain an irreducible component of $D^2(f)$, then the complex spaces $B^2(f)$ and $D^2(f)$ are birationally equivalent.

Proof. The statement follows directly from Proposition 4.1.13. □

Corollary 4.4.3. *If $f\colon X \to Y$ satisfies:*

1. $\dim D^2(f) = 2n - p$,
2. $\dim \hat{\Sigma}^2(f) < 2n - p$,

then $B^2(f)$ and $D^2(f)$ are birationally equivalent.

Proof. If $\dim D^2(f) = 2n - p$ then, by Theorem 3.2.1, $D^2(f)$ is Cohen Macaulay and, hence, ummixed. Therefore $\hat{\Sigma}^2(f)$ can not be an irreducible component of $D^2(f)$, since its dimension is $< 2n - p$. □

Corollary 4.4.4. *If $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is a finitely determined map germ and $p < 2n$, then $B^2(f)$ and $D^2(f)$ are birationally equivalent.*

Proof. If f is finitely determined then, from Mather-Gaffney criterion 1.5.12 and Proposition 1.5.11, it follows $\dim(\hat{\Sigma}^2(f) \setminus \{0\}) \leq n - k(p - n + k + 1)$, and therefore $\dim \hat{\Sigma}^2(f) < 2n - p$. The claim follows since, by Theorem 3.2.1, all the irreducible components of $D^2(f)$ have dimension $\geq 2n - p$, □

Corollary 4.4.5. *If f is a stable map, then $\pi\colon B^2(f) \to D^2(f)$ is a resolution of $D^2(f)$.*

Proof. If f is stable then, by Lemma 2.1.9 and Proposition 1.5.11, it satisfies the hipothesis of Corollary 4.4.3. Hence, π is a birational equivalence and the claim follows, since $B^2(f)$ is smooth by Theorem 4.2.1. □

Chapter 5

Map germs $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$

Stability and finite determinacy of corank 1 map germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ are characterized by some algebraic properties of its multiple point spaces (Theorem 2.2.3). To achieve such a result, we need to have a good understanding of the algebraic structure of these spaces. For instance, we know a explicit set of generators of their defining ideals (Section 2.2), and it is immediate that the k-multiple point space of a map of corank 1 is a complete intersection, if it is dimensionally correct. In the corank ≥ 2 case, we must face the fact that in this setting we do not know much about the algebraic properties of the spaces $D^k(f)$. However, as we will see in Section 5.1, stability and finite determinacy of map germs $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ can be determined by looking just at double points, which we do know well even in corank two (see Chapter 3). In Section 5.2 we extend to corank 2 some nice formulas relating the Milnor number of double point spaces to the number of crosscaps and triple points collapsed at the origin in a finitely determined map $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$. In Section 5.3 we introduce the Double Fold family of map germs, which provides interesting examples. Finally, in Section 5.4 we relate the $\mathcal{A}$-equivalence of double folds (indeed, of a much greater class of map germs) to a new equivalence relation defined ad hoc. The results contained in Sections 5.1 and 5.2 have been published in [MNBPS12], the ones in Sections 5.3 and 5.4 have been published in [PS14].

5.1 Mond number and finite determinacy

Theorem 5.1.1. *Let $f\colon X^2 \to Y^3$. Then f is stable if and only if $D^2(f)$ is a smooth curve and the projection $p\colon D^2(f) \to U$ is an immersion with normal crossings.*

Proof. Assume f is stable, then f has corank 1 (Example 1.5.4). By Theorem 2.2.3, $D^2(f)$ is a smooth curve, $D^3(f)$ is smooth of dimension 0 and $D^4(f)=\emptyset$. From [Alt11, Proposition 2.4.6] we have $D^k(f)=D^{k-1}(p)$ as schemes, for $k=3$ and $k=4$. Since p has corank 1 at any point, this implies that $p\colon D^2(f)\to U$ is stable. From Proposition 1.5.10, we have that p is an immersion with normal crossings.

Conversely, assume that $D^2(f)$ is a smooth curve and the projection $p\colon D^2(f)\to U$ is an immersion with normal crossings. Then triple points are isolated. First we show that f is finitely determined at any point. By Theorem 2.2.3, it suffices to show that f has corank ≤ 1 at all points. Since $D^2(f)$ is a regular curve, we have $\operatorname{edim} D^2(f)=\dim D^2(f)=1$, and thus the claim follows from Lemma 3.2.2. Now, since f is finitely determined and has corank 1, again by [Alt11, Proposition 2.4.6], we have $D^k(f)=D^{k-1}(p)$ as schemes for $k=3$ and $k=4$. Since p is an immersion with normal crossings, Example 1.5.10 implies that $D^3(f)=D^2(p)$ is smooth. Now the result follows from Theorem 2.2.3. □

Theorem 5.1.2. *A germ $f\colon(\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$ is finitely determined if and only if $D^2(f)$ is a germ of reduced curve and the projection $p\colon D^2(f)\to(\mathbb{C}^2,0)$ is generically one-to-one.*

Proof. If f is finitely determined, by Mather-Gaffney's criterion 1.5.12, there is a finite representative of $f\colon U\to V$, such that $f^{-1}(0)=\{0\}$ and $\hat{f}=f|_{\hat{U}}\colon\hat{U}\to\hat{V}$ is stable, with $\hat{U}=U\setminus\{0\}$ and $\hat{V}=V\setminus\{0\}$. By Theorem 5.1.1, $D^2(\hat{f})$ is a smooth curve and $\hat{p}\colon D^2\big(\hat{f}\big)\to\hat{U}$ is an immersion with normal crossings and hence generically one-to-one. Since $D^2(f)\setminus\{0\}=D^2(\hat{f})$ is a smooth curve, the germ $D^2(f)$ has dimension 1 an is generically reduced. From Proposition 3.2.8 it follows that $D^2(f)$ is Cohen Macaulay and, therefore, unmixed. Since $D^2(f)$ is generically reduced and unmixed, we conclude that $D^2(f)$ is reduced.

Conversely, assume that $D^2(f)$ is a germ of reduced curve and the projection $p\colon D^2(f)\to(\mathbb{C}^2,0)$ is generically one-to-one. We can choose a finite representative $f\colon U\to V$, such that $f^{-1}(0)=\{0\}$, $D^2(f)\setminus\{0\}$ is a smooth curve and p is an embedding on $D^2(f)\setminus\{0\}$. As above, we write $\hat{f}=f|_{\hat{U}}\colon\hat{U}\to\hat{V}$, with $\hat{U}=U\setminus\{0\}$ and $\hat{V}=V\setminus\{0\}$. Then, $D^2(\hat{f})=D^2(f)\setminus\{0\}$ and hence, $\hat{f}$ is stable by Theorem 5.1.1. Finally, from Mather-Gaffney's criterion 1.5.12 follows that the map germ f is finitely determined. □

Corollary 5.1.3. *A germ $f\colon(\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$ is finitely determined if and only if its **Mond number** $\mu(D(f))$ is finite.*

Proof. The statement follows immediately from Theorem 5.1.2 and Corollary 3.2.9. □

5.2 Geometric invariants

Let $f\colon (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$ be a finitely determined map germ. By Mather-Gaffney's criterion 1.5.12, there is a representative $f\colon U \subset \mathbb{C}^2 \to V \subset \mathbb{C}^3$ such that $f^{-1}(0) = \{0\}$ and f is stable on $U \setminus \{0\}$. By shrinking U if necessary, we can assume that there are no cross-caps nor triple points in U. Then, since we are in the nice dimensions, we can take a stabilization (see definition 1.5.7) of f, $F\colon D \times U \to \mathbb{C}^4$, $F(s,z) = (s, f_s(z))$, with D a neighbourhood of 0 in $\mathbb{C}$. We define

- $C = \#$ cross-caps of f_s,
- $T = \#$ triple points of f_s,

for $s \neq 0$. These are analytic invariants of f which can be computed as follows [Mon87, MP89]:

$$C = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{Jf}, \quad T = \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{F_2(f_*\mathcal{O}_2)},$$

where Jf is the ramification ideal of f, generated by the minors of df, and $F_2(f_*\mathcal{O}_2)$ is the defining ideal of the space of target triple point space discussed in Section 2.5. These formulae also imply the independence of the invariants C and T with regards to the choosen stabilization and to the (small enough) parameter s.

Example 5.2.1. Let $f\colon (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$, as in Example 2.1.11, be given by

$$(x,y) \mapsto (x^2, y^2, x^3 + y^3 + xy).$$

The computation of the ramification ideal J_f is straightforward and, together with the calculations in Example 2.5.3, we obtain:

$$C = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle x^2, xy, y^2\rangle} = 3, \quad T = \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{\langle X, Y, Z\rangle} = 1.$$

We claim that these are the lowest possible values of C and T for corank 2 germs. For C, the claim is straightforward. If f has corank 2, then all entries of the differential of f are in the maximal ideal $\mathfrak{m}$, and therefore its 2×2 minors are in $\mathfrak{m}^2$. For triple points, the statement follows from Proposition A.3.2. If f has corank 2, then $f^*\mathfrak{m} \subseteq \mathfrak{m}^2$, and thus $\dim_{\mathbb{C}}(\mathcal{O}_2/f^*\mathfrak{m}) > 2$. Therefore, the triple point space of f is non empty, and $T \geq 1$.

The reader can find in Figure 5.1 a real stabilization of this map germ which exhibits the three cross-caps and the triple point.

We are going to show formulas relating C, T and the Milnor numbers of the double point curves. Observe that the double point curves may be non complete intersection. We use the definition of Buchweitz and Greuel

[BG80] of Milnor number of a germ of reduced curve. Given a reduced space curve $(X_0,0)\subset(\mathbb{C}^n,0)$ its Milnor number is denoted by $\mu(X_0,0)$. All we need to know for our purposes are the following properties:

1. Milnor formula: $\mu(X_0,0)=2\delta-r+1$, where δ stands for the delta invariant and r is the number of branches of X_0.

2. If $\pi\colon(X,0)\to(\mathbb{C},0)$ is a flat deformation of $(X_0,0)$, then there is a representative $\pi\colon X\subset D\times U\to D$ such that, for any $t\in D$,

$$\mu(X_0,0)-\mu(X_t)=1-\chi(X_t),$$

   where X_t is the fibre $X_t=\pi^{-1}(t)$, $\mu(X_t)=\sum_{x\in X_t}\mu(X_t,x)$ is the global Milnor number and $\chi(X_t)$ the Euler characteristic.

The following construction will allow us to apply properties (1) and (2) to our double point curves: Let $F(s,z)=(s,f_s(z))$ be a stabilization of a finitely determined map germ $f_0\colon(\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$. Let X be any of the double point curves $D(f_s),D^2(f_s),D^2(f_s)/S_2,f_s(D(f_s))$, with $s\neq0$. Let X_s^1 be the complex space obtained by removing all the points in X_s that are related to cross-caps or triple points (that is, cross-caps in the source, or pairs (z,z) in the lifting, with z a cross-cap, etc.). Then X^1 is a smooth complex curve and we have the following relations among the Euler characteristics of the spaces:

$$\begin{aligned}
&\chi\big(D(f_s)\big)=\chi\big(D(f_s)^1\big)+C+3T,\\
&\chi\big(D^2(f_s)\big)=\chi\big(D^2(f_s)^1\big)+C+6T,\\
&\chi(D^2(f_s)/S_2)=\chi\big((D^2(f_s)/S_2)^1\big)+C+3T,\\
&\chi\big(f_s(D(f_s))\big)=\chi\big(f_s(D(f_s))^1\big)+C+T.
\end{aligned}$$

Take the diagram

$$\begin{array}{ccc}
D^2(f_s)^1 & \longrightarrow & (D^2(f_s)/S_2)^1\\
\big\downarrow & & \big\downarrow\\
D(f_s)^1 & \longrightarrow & f_s(D(f_s))^1,
\end{array}$$

obtained by restriction of the diagram in Section 2.6 corresponding to f_s (see also Remark 2.6.1). Obviously, the vertical arrows are homeomorphisms and the horizontal ones are unramified double covers, hence:

$$\chi\big(D^2(f_s)^1\big)=2\chi\big((D^2(f_s)/S_2)^1\big),\qquad \chi\big(D(f_s)^1\big)=2\chi\big(f_s(D(f_s))^1\big),$$
$$\chi\big(D^2(f_s)^1\big)=\chi\big(D(f_s)^1\big).$$

Finally, we can easily compute the global Milnor number $\mu(X_s)$ in each case. Since the only singularities of $D(f_s)$ are Morse points (3 for each triple point), then $\mu\big(D(f_s)\big) = 3T$. Analogously, $f_s(D(f_s))$ has just ordinary triple points, so $\mu\big(f_s(D(f_s))\big) = 2T$, and $\mu\big(D^2(f_s)\big) = \mu\big(D^2(f_s)/S_2\big) = 0$ because they are smooth.

Now we are ready to obtain formulae relating these invariants. See [MM89] for corank 1 analogue.

Theorem 5.2.2. *If $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is finitely determined, then*

$$\begin{aligned}
&\mu\big(D(f)\big) = \mu\big(D^2(f)\big) + 6T,\\
&\mu\big(D^2(f)\big) = 2\mu\big(D^2(f)/S_2\big) + C - 1,\\
&\mu(D(f)) = 2\mu\big(f(D(f))\big) + C - 2T - 1.
\end{aligned}$$

Proof. The proofs of the three equations are analogous. For the first one, from the system

$$\begin{cases}
\chi\big(D^2(f_t)\big) = \chi\big(D^2(f_t)^1\big) + C + 3T\\
\chi\big(\tilde{D}^2(f_t)\big) = \chi\big(\tilde{D}^2(f_t)^1\big) + C + 6T\\
\chi\big(D^2(f_t)^1\big) = \chi\big(\tilde{D}^2(f_t)^1\big)
\end{cases}$$

we obtain $\chi\big(\tilde{D}^2(f_t)\big) - 2\chi\big(\tilde{D}^2(f_t)/S_2\big) = -C$. Taking into account $\mu\big(D^2(f_t)\big) = 3T$ and $\mu\big(\tilde{D}^2(f_t)\big) = 0$, from the Milnor formula $\mu(X_0, 0) = 2\delta - r + 1$ we obtain the system

$$\begin{cases}
\mu\big(D^2(f)\big) = 3T + 1 - \chi\big(D^2(f_t)\big)\\
\mu\big(\tilde{D}^2(f)\big) = 1 - \chi\big(\tilde{D}^2(f_t)\big)\\
\chi\big(\tilde{D}^2(f_t)\big) - 2\chi\big(\tilde{D}^2(f_t)/S_2\big) = -C
\end{cases}$$

Putting everything together, we get $\mu\big(D^2(f)\big) = \mu\big(D^2(f)\big) + 6T$.

For the second equation, the system is

$$\begin{cases}
\chi\big(\tilde{D}^2(f_t)\big) = \chi\big(\tilde{D}^2(f_t)^1\big) + C + 6T\\
\chi\big(\tilde{D}^2(f_t)/S_2\big) = \chi\Big(\big(\tilde{D}^2(f_t)/S_2\big)^1\Big) + C + 3T\\
\chi\big(\tilde{D}^2(f_t)^1\big) = 2\cdot\chi\Big(\big(\tilde{D}^2(f_t)/S_2\big)^1\Big)
\end{cases}$$

and we obtain $\chi\big(\tilde{D}^2(f_t)\big) - 2\chi\big(\tilde{D}^2(f_t)/S_2\big) = -C$. From $\mu\big(\tilde{D}^2(f_t)\big) = \mu\big(\tilde{D}^2(f_t)/S_2\big) = 0$ we get

$$\begin{cases}
\mu\big(\tilde{D}^2(f)\big) = 1 - \chi\big(\tilde{D}^2(f_t)\big)\\
\mu\big(\tilde{D}^2(f)/S_2\big) = 1 - \chi\big(\tilde{D}^2(f_t)/S_2\big)\\
\chi\big(\tilde{D}^2(f_t)\big) - 2\chi\big(\tilde{D}^2(f_t)/S_2\big) = -C
\end{cases}$$

which implies $\mu\big(\tilde{D}^2(f)\big) = 2\mu\big(\tilde{D}^2(f)/S_2\big) + C - 1$.

For the last equation, we take the system

$$\begin{cases} \chi\big(D(f_s)\big) = \chi\big(D(f_s)^1\big) + C + 3T \\ \chi\big(f_s(D(f_s))\big) = \chi\big(f_s(D(f_s))^1\big) + C + T \\ \chi\big(D(f_s)^1\big) = 2\chi\big(f_s(D(f_s))^1\big) \end{cases}$$

and obtain $\chi\big(D^2(f_s)\big) - 2\chi\big(f_s(D(f_s))\big) = -C+T$. From $\mu\big(D^2(f_s)\big) = 3T$ and $\mu\big(f_s(D(f_s))\big) = 2T$ we get

$$\begin{cases} \mu\big(D(f)\big) = 3T + 1 - \chi\big(D(f_s)\big) \\ \mu\big(f(D(f))\big) = 2T + 1 - \chi\big(f_s(D(f_s))\big) \\ \chi\big(D(f_s)\big) - 2\chi\big(f_s(D(f_s))\big) = -C + T \end{cases}$$

From this it follows $\mu\big(D(f)\big) = 2\mu\big(f(D(f))\big) + C - 2T - 1$.

□

Example 5.2.3. Let $f\colon(\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$, as in Example 2.1.11, be given by

$$(x,y)\mapsto(x^2,y^2,x^3+y^3+xy).$$

In Example 2.3.3, we computed the double point curve

$$D(f) = V((x+y^2)(x^2+y)(x^3+y^3)),$$

which has Milnor number $\mu(D(f)) = 16$.

Now we compute the Milnor number of $D^2(f)$ (its defining ideal was computed in Example 2.1.11). To do this, we compute the δ invariant and use the formula above to obtain the Milnor number. We use the same notation as in [BG80]. If $C_1 = V(I_1)$ and $C_2 = V(I_2)$ are two curves (non necessarily irreducible) intersecting only at $x_0\in\mathbb{C}^n$, then we write

$$C_1\cdot C_2 = \dim_{\mathbb{C}}\mathcal{O}_{\mathbb{C}^n,x_0}/(I_1+I_2).$$

As a particular case of a lemma due to Hironaka [BG80, Lemma 1.2.2], we have

$$\delta(C_1\cup C_2) = \delta(C_1) + \delta(C_2) + C_1\cdot C_2.$$

The curve $D^2(f)$ has 5 smooth branches, namely 2 parabolas $P_1 = V(I_1)$, $P_2 = V(I_2)$, and 3 coplanar lines $L = L_1\cup L_2\cup L_3 = V(I_3)$, where

$$I_1 = \langle x - x', y + y', x + y^2\rangle,$$

$$I_2 = \langle x + x', y - y', x^2 + y\rangle,$$

$$I_3 = \langle x + x', y + y', x^3 + y^3\rangle.$$

Its obvious that $\delta(P_1 \cup P_2) = 1$ and $\delta(L) = 3$ and we also have

$$(P_1 \cup P_2) \cdot L = \dim_{\mathbb{C}} \frac{\mathcal{O}_4}{I_1 \cap I_2 + I_3} = 3.$$

By the lemma mentioned above, we have $\delta(D^2(f)) = 1 + 3 + 3 = 7$, and hence $\mu(D^2(f)) = 2\delta - r + 1 = 10$.

Analogously, $D^2(f)/S_2$ is composed by the five lines (the ideal is computed in Example 2.4.4)

$$\hat{L}_0 \cup \hat{L}_1 = V(J_1), \quad \hat{L}_2 = V(J_2), \quad \hat{L}_3 = V(J_3), \quad \hat{L}_4 = V(J_4),$$

where,

$$J_1 = \langle s_1, s_2, r_{11} + r_{22} - r_{12}, r_{22}^2 - r_{22}r_{12} + r_{12}^2 \rangle,$$

$$J_2 = \langle s_1, s_2, r_{11} + r_{12}, r_{22} + r_{12} \rangle,$$

$$J_3 = \langle s_1, 2s_2 + r_{11}, r_{22}, r_{12} \rangle,$$

$$J_4 = \langle s_2, 2s_1 + r_{22}, r_{11}, r_{12} \rangle.$$

We have $\delta(\hat{L}_0 \cup \hat{L}_1) = 1$ and by Hironaka's lemma:

$$\delta(\hat{L}_0 \cup \hat{L}_1 \cup \hat{L}_2) = \delta(\hat{L}_0 \cup \hat{L}_1) + \delta(\hat{L}_2) + (\hat{L}_0 \cup \hat{L}_1 \cdot \hat{L}_2) = 1 + 0 + 1.$$

$$\delta(\hat{L}_0 \cup \hat{L}_1 \cup \hat{L}_2 \cup \hat{L}_3) = 2 + \delta(\hat{L}_3) + (\hat{L}_0 \cup \hat{L}_1 \cup \hat{L}_2 \cdot \hat{L}_3) = 2 + 0 + 1.$$

$$\delta(D^2(f)/S_2) = 3 + \delta(\hat{L}_4) + (\hat{L}_0 \cup \hat{L}_1 \cup \hat{L}_2 \cup \hat{L}_3 \cdot \hat{L}_4) = 3 + 0 + 1 = 4.$$

Then $\mu(D^2(f)/S_2) = 2\delta - r + 1 = 4$. Finally, we use the computation of $F_1(f_*\mathcal{O}_2)$ in Example 2.5.3 to obtain $\mu(f(D(f))) = 8$ in the same way. Since $C = 3$ and $T = 1$, we can easily check that the formulas in Theorem 5.2.2 hold.

5.3 Double folds

Here we study the geometry of a particular family of singular map germs $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ called **double folds**. As we will see, these family provides interesting germs, such as finitely determined homogeneous corank 2 germs. Our family is created by analogy to David Mond's **fold maps** (see [Mon85]), which we explain next:

A map germ $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is a fold map if its first two coordinate functions form a **Whitney fold** $T\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, given by

$$(x, y) \mapsto (x, y^2).$$

The image of a fold map $f(x, y) = (x, y^2, f_3)$ looks like the graph of the function f_3 'folded' along the OX axis. The third coordinate function of a fold map can be any but, under $\mathcal{A}$-equivalence, we can assume that it is of the form yp, where $p = T^*P$ for some germ $P \in \mathcal{O}_2$. Hence, the normal form of a fold map is

$$f(x,y) = (x, y^2, yp).$$

A remarkable thing about fold maps is that they are related to the action of the group $G = \{1, i\}$, generated by the reflection $i(x,y) = (x,-y)$. For instance, it is immediate that all double points of a fold map are of the form $(z, i(z))$, for some $z = (x,y) \in \mathbb{C}^2$.

To produce a family of corank 2 maps related to a reflection group, we are going to 'fold' twice, once through OX and once through OY axis. Let $\alpha\colon (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$, as in Example 4.1.9, be the Folded Hankerchief, given by

$$(x,y) \mapsto (x^2, y^2).$$

Take the reflections $i_1(x,y) = (-x,y)$ and $i_2(x,y) = (x,-y)$ and the rotation $i_3(x,y) = (-x,-y)$. We write G for the group $\{1, i_1, i_2, i_3\}$. The orbit of any $z \in \mathbb{C}^2$ is $Gz = \alpha^{-1}(\alpha(z))$ and z is a singular point of α if and only if z belongs to $Fix(i_1) \cup Fix(i_2) = OX \cup OY$. Now, related to the group G, we have the following family of maps:

Definition 5.3.1. A map germ $f\colon (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$ is a **double fold** (**DF** for short) if it is of the form

$$(x,y) \mapsto (x^2, y^2, f_3(x,y)).$$

The function germ $f_3 \in \mathcal{O}_2$ can be written in the form

$$f_3(x,y) = P_0(x^2,y^2) + xP_1(x^2,y^2) + yP_2(x^2,y^2) + xyP_3(x^2,y^2),$$

for some $P_i \in \mathcal{O}_2$. Under $\mathcal{A}$-equivalence, we can eliminate P_0. Then we obtain a **double fold in normal form**

$$f(x,y) = (x^2, y^2, xp_1 + yp_2 + xyp_3),$$

with $p_i = \alpha^* P_i$, for some $P_i \in \mathcal{O}_2$. We call **special double folds** (**SDF** for short) the double folds in normal form satisfying $p_3 = 0$.

Remark 5.3.2. Fold and double fold families are not mutually exclusive. The cross-cap is usually parameterized as a fold in normal form

$$(x,y) \mapsto (x, y^2, xy),$$

but it can also be regarded (Lemma 5.3.9) as double fold

$$(x,y) \mapsto (x^2, y^2, x+y).$$

Target multiple points of double folds

Given a double fold $f(x,y) = (x^2, y^2, xp_1+yp_2+xyp_3)$, we use the method explained in Section 2.5 to find a presentation matrix $M(f)$ of $f_*\mathcal{O}_2$: Take $g_1 = 1, g_2 = x, g_3 = y, g_4 = xy$ as generators of $\alpha_*\mathcal{O}_2$. For $i = 1$, we have

$$f_3 g_1 = xp_1 + yp_2 + xyp_3 = 0 \cdot g_1 + \alpha^* P_1 g_2 + \alpha^* P_2 g_3 + \alpha^* P_3 g_4.$$

Therefore, the elements of the first column of the matrix are $-Z, P_1, P_2, P_3$. After computing $f_3 g_i$ for $i = 2, 3, 4$, we get the matrix

$$M(f) = \begin{pmatrix} -Z & XP_1 & YP_2 & XYP_3 \\ P_1 & -Z & YP_3 & YP_2 \\ P_2 & XP_3 & -Z & XP_1 \\ P_3 & P_2 & P_1 & -Z \end{pmatrix},$$

where P_i represents $P_i(X,Y)$.

Observe that, since $M(f)$ has size 4×4, f has no points with multiplicity greater than 4. For special double folds, the space of quadruple points in the image is given by the ideal $F_3(f) = \langle P_1(X,Y), P_2(X,Y), Z\rangle$ and $F_2(f) = (F_3(f))^2$. Hence, triple points of special double folds appear concentrated at quadruple points.

Source multiple points of double folds

We denote by $N_2(f)$ the germ of complex space defined by the pull back $f^*(F_1(f_*\mathcal{O}_2))$. For a double fold in normal form, we have

$$N_2(f) = V\big((p_1 + yp_3)(p_2 + xp_3)(xp_1 + yp_2)\big).$$

Observe that, even though $D(f)$ and $N_2(f)$ agree set-theoretically (Remark 2.6.1), it is not clear that they agree as schemes (see Open Problem 6, and observe that, indeed, both structures agree for Example 2.3.3). We factorize $f^*F_1(f_*\mathcal{O}_2)$ as the product of the ideals

$$I_1 = \langle p_1 + yp_3\rangle,$$

$$I_2 = \langle p_2 + xp_3\rangle,$$

$$I_3 = \langle xp_1 + yp_2\rangle.$$

The analogous alternative source triple point space $V(f^*(F_2(f_*\mathcal{O}_2))$ is given by the two minors of the presentation matrix above. Set theoretically, it decomposes as the union of the zero sets of the ideals

$$I_{1,2} = \langle p_1 + yp_3, p_2 + xp_3\rangle,$$

$$I_{1,3} = \langle p_1 + yp_3, p_2 - xp_3\rangle,$$

$$I_{2,3} = \langle p_2 + xp_3, p_1 - yp_3\rangle.$$

Finally, quadruple points (again with the structure induced by the target) are defined by

$$I_{1,2,3} = \langle p_1, p_2, p_3 \rangle.$$

Observe that, in the special double fold case, all triple points appear collapsed into quadruple points: If p_3 equals zero, then the radical of $I_{1,2}I_{1,3}I_{2,3}$ is $\langle p_1, p_2 \rangle$, which is the ideal defining the quadruple point locus.

Definition 5.3.3. Given a double fold $f = (\alpha, xp_1 + yp_2 + xyp_3)$, we decompose the source double points as the union of $D_i(f), 1 \leq i \leq 3$, with

$$D_i(f) = V(I_i)$$

and the triple points as the union of $D_{i,j}(f), 1 \leq i < j \leq 3$, with

$$D_{i,j}(f) = V(I_{i,j}).$$

Finally, the quadruple points are

$$D_{1,2,3}(f) = V(I_{1,2,3}).$$

Remark 5.3.4. It is immediate that:

1. A point w belongs to $D_l(f)$ if and only if $i_l(w)$ does so. Moreover $f(w) = f(i_l(w))$.
2. A point w belongs to $D_{l,k}(f)$ if and only if $i_l(w)$ and $i_k(w)$ do so. Moreover $f(w) = f(i_l(w)) = f(i_k(w))$.
3. A point w belongs to $D_{1,2,3}(f)$ if and only if $i_1(w), i_2(w)$ and $i_3(w)$ do so. Moreover $f(w) = f(i_1(w)) = f(i_2(w)) = f(i_3(w))$.

Example 5.3.5. Take the family

$$(x, y) \mapsto (x^2, y^2, \lambda_1 x + \lambda_2 y + \lambda_3 xy), \lambda_i \in \mathbb{C}.$$

Assume $\lambda_3 \neq 0$, then its double points are the following (see Figure 5.1):

- $D_1(f) = V(\lambda_1 + y\lambda_3)$ is the i_1-invariant line $y = -\lambda_1/\lambda_3$.
- $D_2(f)$ is the i_2-invariant line $x = -\lambda_2/\lambda_3$,
- $D_3(f)$ is the i_3-invariant line $\lambda_2 y + \lambda_1 x = 0$, if $\lambda_1 \neq 0$ or $\lambda_2 \neq 0$. Otherwise, we have $D_3(f) = \mathbb{C}^2$ and the map is not generically one-to-one (see the Double Cone in Example 3.3.11).

We find the triple points where these lines meet:

- $D_{1,2}(f) = \{(-\lambda_2/\lambda_3, -\lambda_1/\lambda_3)\}$,
- $D_{1,3}(f) = \{(\lambda_2/\lambda_3, -\lambda_1/\lambda_3)\}$,

- $D_{2,3}(f) = \{(-\lambda_2/\lambda_3, \lambda_1/\lambda_3)\}$.

In the case $\lambda_3 = 0$, we have a special double fold. Thus, its triple points should appear collapsed at quadruple points, with equations $p_1 = p_2 = 0$. Since we have $p_1 = \lambda_1$ and $p_2 = \lambda_2$ for this example, the appearance of quadruple point forces $\lambda_1 = \lambda_2 = 0$ and hence, the map is the Folded Hankerchief, seen as a map germ $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$.

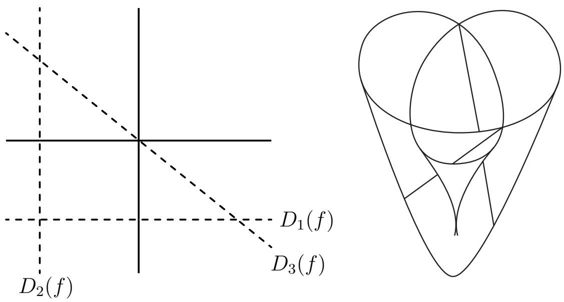

Figure 5.1: Image and double points of a double fold (see Example 5.3.5).

Double fold stability

In this section we study the singularity types which are characteristic of double fold map germs. By a singularity type we mean an $\mathcal{A}$-equivalence class of multigerms $f\colon (\mathbb{C}^2, S) \to (\mathbb{C}^3, y)$. We know that the stable types $\mathbb{C}^2 \to \mathbb{C}^3$ are transverse double points, transverse triple points and cross-caps. Our goal is to make a version of the concept of stability adapted specifically for double folds. The idea is that some types, despite not being stable, are preserved by deformations within the double fold world. We call them DF-stable types and these deformations DF-deformations. This concept can be adapted to the special double fold case and we shall use the notation (S)DF to refer respectively to both, the double fold and the special double fold case.

Definition 5.3.6. A **(S)DF-deformation** of f_0 is a germ $F\colon (\mathbb{C}^2 \times \mathbb{C}, 0) \to (\mathbb{C}^3, 0)$ of the form $F(x,t) = f_t(x)$, such that the germ $f_t\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is a (special) double fold for all t. A (S)DF-unfolding is a map germ $F\colon (\mathbb{C}^2 \times \mathbb{C}, 0) \to (\mathbb{C}^3 \times \mathbb{C}, 0)$ of the form $F(x,t) = (t, f_t(x))$, such that $f_t(x)$ is a (S)DF-deformation.

Definition 5.3.7. A multigerm f is **(S)DF-stable** if any (S)DF-unfolding F of f is trivial. A (special) double fold $f\colon U \to \mathbb{C}^3$ is (S)DF-stable if all its multigerms at $f^{-1}(f(w)), w \in U$ are (S)DF-stable.

Remark 5.3.8. Every stable type is (S)DF-stable.

A priori, it might seem difficult to identify all possible (S)DF-stable maps, but a better understanding of the map α will help us to do so. The map α is the invariant map associated to the Coxeter group G (see [Hum90] for Coxeter group theory). For any Coxeter Group there is a Coxeter complex, in this case $\mathcal{C} = \{\mathbb{C}^2 \setminus (OX \cup OY), OX \setminus \{0\}, OY \setminus \{0\}, \{0\}\}$. The Coxeter complex stratifies the space in a way such that the behavior of the group, and thus that of α, changes whenever we go from a facet to another. Consequently, much information about a double fold is contained in the way its multiple point spaces meet the Coxeter complex. The following proposition is an example of this.

Lemma 5.3.9. *The germ of a double fold $f(x, y) = (x^2, y^2, xp_1 + yp_2 + xyp_3)$ centered at a point $w \in \mathbb{C}^2$ is a cross-cap if and only if one of the three conditions is verified:*

i) $w \in OX \setminus \{0\}$ *and the restricted function* $(p_2 + xp_3)|_{OX}$ *has a simple zero at* w*.*

ii) $w \in OY \setminus \{0\}$ *and the restricted function* $(p_1 + yp_3)|_{OY}$ *has a simple zero at* w*.*

iii) $w = 0$ *and* $p_1(w) \neq 0 \neq p_2(w)$*.*

Proof. From Theorem 5.1.1, a singular monogerm of map from $\mathbb{C}^2$ to $\mathbb{C}^3$ is a cross-cap if and only if its source double point space is smooth. Since cross-caps are singular monogerms, they lie on $OX \cup OY$. Assume first that $w \in OX \setminus \{0\}$. Looking at the 2×2 minors of the differential of f at w it follows that f is singular at w if and only if $p_2 + xp_3$ vanishes at w. Now the source double point space of the germ of f at w is $D_2(f)$, given by the zeros of $p_2 + xp_3$ (notice that, by Remark 5.3.4, the branches of double points $D_1(f)$ and $D_3(f)$ at $OX \setminus \{0\}$ produce multigerms, not monogerms). Therefore, the double point space of the germ of f at w is smooth if and only if the Milnor number of the germ of function $p_2 + yp_3$ at w equals 0. This happens if and only if at least one of the partial derivatives $\frac{\partial p_2 + xp_3}{\partial x}$ and $\frac{\partial p_2 + xp_3}{\partial y}$ does not vanish at w. Since p_2 and p_3 are functions of x^2 and y^2, we deduce that $\frac{\partial p_2 + xp_3}{\partial y}$ vanishes at OX. Hence, f has a cross-cap at $w \in OX \setminus \{0\}$ if and only $p_2 + xp_3$ vanishes at w and $\frac{\partial p_2 + xp_3}{\partial x}$ does not, that is, if and only if the restriction $(p_2 + xp_3)|_{OX}$ has a simple zero at w. The case $w \in OY \setminus \{0\}$ is analogous. Assume now $w = 0$. The source double point of f is the germ of complex space given by the zeros of $(p_1 + xp_3)(p_2 + p_3)(xp_1 + yp_2)$. The non vanishing of p_1 and p_2 at 0 is a necessary and sufficient condition for this germ of complex space to be smooth. $\square$

Points where the source double point space meets the facets of the Coxeter complex in a generic way are called (S)DF-generic. We shall determine the different possible (S)DF-generic singularities and then show that they are exactly the (S)DF-stable singularities. Let us state the (S)DF-genericity conditions rigorously:

Definition 5.3.10. Let $f = (\alpha, xp_1 + yp_2 + xyp_3): U \to \mathbb{C}^3$ be a double fold. We say that a point $w \in \mathbb{C}^2$, that belongs to a facet $C \in \mathcal{C}$, is a **DF-generic point** if:

1) $(p_1 + yp_3)|_C$, $(p_2 + xp_3)|_C$ and $(xp_1 + yp_2)|_C$ are transverse to $\{0\}$ at w, with the exception $(xp_1 + yp_2)|_{\{0\}}$ (notice that no double fold in canonical form could verify this transversality condition).

2) $(p_1 + yp_3, p_2 + xp_3)|_C$, $(p_1 + yp_3, p_2 - xp_3)|_C$ and $(p_2 + xp_3, p_1 - yp_3)|_C$ are transverse to $\{(0, 0)\}$ at w.

3) w is not a quadruple point of f.

A double fold $f: U \to \mathbb{C}^3$ is **DF-generic** if all points $w \in U$ are DF-generic

Conditions 1) and 2) adapt to the special double fold case just taking $p_3 = 0$ but, since quadruple points are more likely to appear at special double folds (they are the zeros of just two equations in $\mathbb{C}^2$), the SDF genericity conditions don't include condition 3).

Definition 5.3.11. Let $f = (\alpha, xp_1 + yp_2): U \to \mathbb{C}^3$ be a special double fold, we say that a point $w \in \mathbb{C}^2$, that belongs to a facet $C \in \mathcal{C}$, is a **SDF-generic point** if:

1) $p_1|_C$, $p_2|_C$ and $(xp_1 + yp_2)|_C$ are transverse to $\{0\}$ at w, with the exception $(xp_1 + yp_2)|_{\{0\}}$.

2) $(p_1, p_2)|_C$ is transverse to $\{(0, 0)\}$ at w.

A special double fold $f: U \to \mathbb{C}^3$ is **SDF-generic** if all points $w \in U$ are SDF-generic

Remark 5.3.12. It is immediate from its defining ideals that every point belonging to $D_1(f) \cap OX$ or to $D_2(f) \cap OY$ must belong to $D_3(f)$ too. It is also immediate that $D_3(f)$ always crosses the facet $\{0\}$. Apart from these exceptions, which are inherent to the double fold family, the genericity conditions imply the following more geometric assertion: Given a regular stratification of $D(f)$, the strata have their expected dimension (double points have dimension 1 and triple (quadruple) points have dimension 0) and are transverse to the strata of the Coxeter complex $\mathcal{C}$.

Now we introduce our new candidates to be (S)DF-generic multigerms.

Definition 5.3.13. We call a **standard self tangency** the multigerm formed by two smooth branches with Morse contact. We call a **standard quadruple point** the multigerm formed by four smooth branches such that every three of them meet transversally. These singularities are depicted in Figure 5.2.

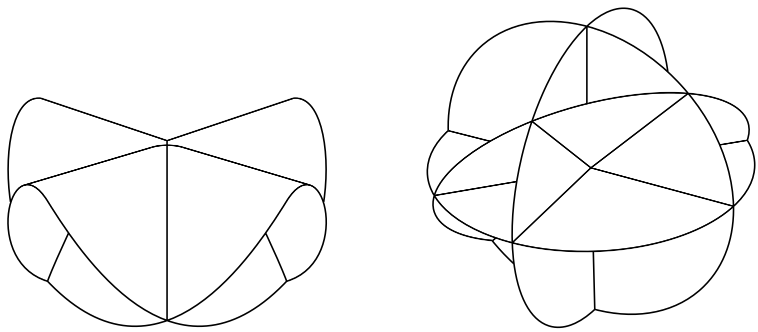

Figure 5.2: A standard self tangency and a standard quadruple point.

Proposition 5.3.14. *All standard self tangencies are $\mathcal{A}$-equivalent. All standard quadruple points are $\mathcal{A}$-equivalent.*

Proof. In [WA00] it is shown that the $\mathcal{A}$-class of a bigerm with smooth branches is determined by the contact type of its branches. Since there is only one contact class of Morse type, all standard self tangencies are equivalent. Let f be a multigerm of standard quadruple point. Any three of its branches form a triple point and there is only one $\mathcal{A}$-class of triple points. Therefore, there exists a change of coordinates that takes f to a multigerm whose branches send (x, y) respectively to $(x, y, 0), (x, 0, y), (0, x, y)$ and $g(x, y)$ for some regular monogerm g with

$$\operatorname{Im} g = \{U_1X + U_2Y + U_3Z = 0\},$$

$U_i \in \mathcal{O}_3$. The plane tangent to $\operatorname{Im} g$ is determined by the equation

$$t_1X + t_2Y + t_3Z = 0,$$

with $t_i = U_i(0, 0)$. If we assume $t_1 = 0$, then the intersection of the tangent plane with the branches $\{Y = 0\}$ and $\{Z = 0\}$ is the line $\{Y = Z = 0\}$. This contradicts the transversality of these three branches. We deduce $t_1 \neq 0$ and, analogously, $t_2 \neq 0 \neq t_3$. The change

$$(X, Y, Z) \mapsto (U_1X, U_2Y, U_3Z)$$

defines a germ of diffeomorphism that takes our multigerm to the one with image $\{XYZ(X + Y + Z) = 0\}$. Now the four branches of our multigerm are given by

- $(x, y) \mapsto (u_1x, u_2y, 0)$,
- $(x, y) \mapsto (u_1x, 0, u_3y)$,
- $(x, y) \mapsto (0, u_2x, u_3y)$, and

- $(x,y) \mapsto (a_1x+b_1y, a_2x+b_2y, -(a_1+b_1)x-(a_2+b_2)y)$,

where $u_i = U_i \circ f$, and a_1, a_2, b_1, b_2 are some function germs in $\mathcal{O}_2$. We take germs of diffeomorphisms at the source, at the four different points where our multigerm is centered. The first three diffeomorphisms send (x,y) respectively to $(x/u_1, y/u_2)$, $(x/u_1, y/u_3)$ and $(x/u_2, y/u_3)$. The fourth diffeomorphism is the inverse of the germ $(x,y) \mapsto (a_1x+b_1y, a_2x+b_2y)$. These four source coordinate changes take the multigerm to one multigerm defined by four branches sending (x,y) respectively to $(x,y,0), (x,0,y), (0,x,y)$ and $(x,y,-x-y)$. Hence, all germs of standard quadruple point are equivalent. □

Lemma 5.3.15. *The (S)DF-generic points are regular points, transverse double points, cross-caps, standard self tangencies and triple points (resp. standard quadruple points).*

Proof. Given a (special) double fold f and a point $w = (x_0, y_0) \in \mathbb{C}^2$ satisfying the (S)DF-genericity conditions, we shall determine the type of singularity of the multigerm of f at $f^{-1}(f(w))$. First of all, notice that singular points lie in $OX \cup OY$ and the genericity condition 2) implies that all triple points belong to the facet $\mathbb{C}^2 \setminus (OX \cup OY)$. Hence, from genericity condition 1), together with Lemma 5.3.9, it follows that all points where f is singular are cross-caps.

Now assume that f is regular at w and the point w belongs to $D_l(f)$, $1 \le l \le 3$. Take the vector fields along f defined by the cross product

$$\eta = \frac{\partial f}{\partial x} \times \frac{\partial f}{\partial y}$$

and

$$\eta_l = \eta \circ i_l, \quad l = 1,2,3.$$

The branches of the multigerm of f at w and $i_l w$ are transverse unless $\eta \times \eta_l$ or, equivalently,

$$\xi_l = (\eta - \eta_l) \times (\eta + \eta_l)$$

vanish at w. We study the different cases a), b) and c), where w belongs to $D_1(f), D_2(f)$ and $D_3(f)$ respectively.

Case a) Let w belong to $D_1(f)$, then we have:

$$\xi_1|_w = 4x_0y_0\Big(4x_0\frac{\partial(xp_1+xyp_3)}{\partial y}|_w,\ 4y_0\frac{\partial(xp_1+xyp_3)}{\partial x}|_w, \\ ,\ (\frac{\partial(xp_1+xyp_3)}{\partial y}|_w\frac{\partial yp_2}{\partial x}|_w - \frac{\partial(xp_1+xyp_3)}{\partial x}|_w\frac{\partial yp_2}{\partial y}|_w)\Big).$$

Assume first $w \notin OX \cup OY$, then $\xi_1|_w$ vanishes if and only if

$$\frac{\partial p_1 + yp_3}{\partial x}|_w = \frac{\partial p_1 + yp_3}{\partial y}|_w = 0,$$

that is, if and only if p_1+yp_3 is not transverse to $\{0\}$ at w. This is in contradiction with the first genericity condition. Now assume $w\in OX\cup OY$ and notice $w\notin OY$, because it would be a singular point. Thus, we have $w\in OX\setminus\{0\}$. We claim that the bigerm of f at $(\pm x_0,0)$ forms a standard self tangency at $(X_0,0,0)$, where $X_0=x_0^2$. The genericity conditions imply that P_1 has a simple zero at $(X_0,0)$ and P_2 does not vanish at $(X_0,0)$. Let the germ of $f\colon\mathbb{C}^2\to\mathbb{C}^3$ at x_0 parameterize one of the branches and let $\phi\colon\mathbb{C}^3\to\mathbb{C}$ be the germ at $(X_0,0,0)$ which defines the other branch implicitly. Then, following Montaldi [Mon86], the contact between the branches is given by the $\mathcal{K}$-class of the composition $\phi\circ f$. The branches are given by

$$(Z^2\pm\sqrt{X}P_1)^2-YP_2^2\pm 2Y\sqrt{X}P_2P_3-XYP_3^2=0.$$

After choosing the preimage $(x_0,0)$ and composing, we get the function $4x(p_1+yp_3)(xp_1+yp_2)$, which is of Morse type in $(x_0,0)$. Therefore, the multigerm of f at $(\pm x_0,0)$ is a standard self tangency.

Case b) is symmetric interchanging indices 1 and 2, and OX and OY.

Case c) If $w\in D_3(f)$, then we can assume $w\in D_3(f)\setminus(OX\cup OY)$ because otherwise $w\in D_1(f)\cup D_2(f)$, by Remark 5.3.12. We have

$$\begin{aligned}\xi_3|_w = 4x_0y_0\Big(&4x_0\frac{\partial(xp_1+yp_2)}{\partial y}|_w,\ -4y_0\frac{\partial(xp_1+yp_2)}{\partial x}|_w,\\ &,\ (\frac{\partial(xp_1+yp_2)}{\partial y}|_w\frac{\partial xyp_3}{\partial x}|_w-\frac{\partial(xp_1+yp_2)}{\partial x}|_w\frac{\partial xyp_3}{\partial y}|_w)\Big),\end{aligned}$$

which vanishes if and only if

$$\frac{\partial xp_1+yp_2}{\partial x}|_w=\frac{\partial xp_1+yp_2}{\partial y}|_w=0,$$

that is, if and only if xp_1+yp_2 is not transverse to $\{0\}$ at w.

As we have seen before, all triple points (and therefore all quadruple points) belong to the facet $\mathbb{C}^2\setminus(OX\cap OY)$, where the second genericity condition implies that the branches are transverse. Therefore, all triple points are transverse (respectively, all quadruple points are standard quadruple points). □

Lemma 5.3.16. *Every (special) double fold admits a (S)DF-deformation f_t defined in a neighborhood $U\times V$ of $(0,0)\in\mathbb{C}^2\times\mathbb{C}$ such that, for every $t\in V$, f_t is (S)DF-generic.*

Proof. Let $f=(\alpha,xp_1+yp_2+xyp_3)$ be a representative defined at some neighborhood U of the origin. we consider DF-deformations of the form $f_{a,b,c}=(\alpha,x(p_1+a)+y(p_2+b)+xy(p_3+c))$. Let Z be the analytic space consisting on points $(a,b,c)\in\mathbb{C}^3$ such that, for some point w in U, the map $f_{a,b,c}$ does not satisfy all genericity conditions. We claim that Z

is a proper subspace of $\mathbb{C}^3$. Take the first function, $p_1 + yp_3$, of the first condition and any facet of the Coxeter complex $C \in \mathcal{C}$. We consider the map $\psi\colon C \times \mathbb{C}^3 \to \mathbb{C}$, given by $\psi(w, a, b, c) = p_1(w) + a + y(p_3(w) + c)$. This is clearly a submersion. Therefore, Lemma 1.3.3 tells us that, for almost every $(a, b, c) \in \mathbb{C}^3$, the map $f_{a,b,c}$ is transverse to 0. We can proceed analogously for all the maps given by the DF-genericity conditions to finally show that, for almost every $(a, b, c) \in \mathbb{C}^3$, all the genericity conditions hold at every point in U. Thus, Z is a proper subspace. Hence, we can find some particular $(a, b, c) \in \mathbb{C}^3$ and some neighborhood V of 0, such that $t(a, b, c) \notin \mathbb{C}^3$, for all $t \in V$. If we take the DF-deformation $f_t(x, y) = (x^2, y^2, x(p_1 + ta) + y(p_2 + tb) + xy(p_3 + tc)$ defined at $U \times V$ then, for any $t \in V$, the map f_t has only DF-generic points at U. The special double fold case is analogous. □

Theorem 5.3.17. *(S)DF-stable and (S)DF-generic points are the same. As a consequence:*
The DF-stable singularities are

- *Transverse double points, cross-caps and triple points.*
- *Standard self tangencies.*

The SDF-stable singularities are

- *Transverse double points and cross-caps.*
- *Standard self tangencies.*
- *Standard quadruple points.*

Proof. By Lemma 5.3.16, the DF-stable singularities must be DF-generic. Now take a DF-generic point w of a double fold f. If w is a transverse double point, a cross-cap or a triple point, then it is stable and, hence, DF-stable. Suppose w is a standard self tangency and Let $F = (f_t, t)$ be a DF-unfolding of f. Assume $w \in D_1(f)$. Then, as we have seen in the proof of Lemma 5.3.15, the point belongs to $OX \setminus \{0\}$, $(p_1 + yp_3)|_{OX}$ has a simple zero at w and the functions $p_2 + xp_3$ and $xp_1 + yp_2$ don't vanish at w. Therefore, there exist a neighborhood $U \times V$ of $(w, 0)$ and a curve of points $w_t \in U \cap OX \setminus \{0\}$, with $t \in V$ and $w_0 = w$, such that $(p_1 + yp_3)|_{OX}$ has a simple zero and the functions $p_2 + xp_3$ and $xp_1 + yp_2$ don't vanish at w_t. All this points are also standard self tangencies and, since they are all $\mathcal{A}$-equivalent by Proposition 5.3.14, they are DF-stable. The proof holds in the special case and is analogous for standard quadruple points. □

Counting (S)DF-stable points

A usual way to study germs is to count the number of stable 0-dimensional points of each type which appear in a stabilization of the original germ.

One can show that these numbers can be obtained as the dimension (as $\mathbb{C}$-vector space) of certain local algebras related to the different stable 0-dimensional types. We adapt these techniques specifically to (S)DF-deformations and to (S)DF-stable points.

Definition 5.3.18. We call **(S)DF-stabilization** any (S)DF-deformation F such that there exists a neighborhood $U \times V$ of $(0,0) \in \mathbb{C}^2 \times \mathbb{C}$ such that, for every $t \in V$, f_t is (S)DF-stable.

Remark 5.3.19. By Lemma 5.3.16 and Theorem 5.3.17, every (special) double fold admits a (S)DF-stabilization.

Definition 5.3.20. For any (special) double fold f we define:

- $ST_i(f) = \frac{1}{2}\dim_{\mathbb{C}} \mathcal{O}_1/j_i^* I_i(f)$, for $i = 1, 2$,
- $C_i(f) = \dim_{\mathbb{C}} \mathcal{O}_1/j_k^* I_i(f)$, for $(i,k) = (1,2),(2,1)$,
- $T(f) = \dim_{\mathbb{C}} \mathcal{O}_2/I_{1,2}(f)$ (in the special double fold case: $QD(f) = \frac{1}{4}\dim_{\mathbb{C}} \mathcal{O}_2/\langle p_1, p_2\rangle$),

where j_1 and j_2 are the inclusions of OX and OY into $\mathbb{C}^2$ respectively.

Remark 5.3.21. We avoid indices for triple points in different branches because the complex spaces $D_{i,j}(f)$ are all isomorphic, since $\mathcal{O}_2/I_{1,2}(f) \cong \mathcal{O}_2/I_{1,3}(f) \cong \mathcal{O}_2/I_{2,3}(f)$ via the isomorphisms induced by i_1 and i_2.

Proposition 5.3.22. *Let $ST_i(f), C_i(f)$ and $T(f)$ (respectively $QD(f)$) be finite. Let f_s be a (S)DF-stabilization of f. Then, for a small enough $s \neq 0$, the following equalities hold:*

- $ST_i(f) = \#$ *standard self tangencies* $f(D_i(f_s))$,
- $C_i(f) = \#$ *cross-caps in* $D_i(f_s) \setminus \{0\}$,
- $T(f) = \#$ *triple points of* f_s *(in the special double fold case:* $QD(f) = \#$ *standard quadruple points of* f_s*).*

Proof. Take the zero set of the different ideals which appear in Definition 5.3.20. If $ST_i(f), C_i(f)$ and $T(f)$ (respectively $QD(f)$) are finite, then the spaces are 0-dimensional. In this case, the codimension of any of these spaces equals the number of generators of its defining ideal. Hence, the spaces are complete intersection and the Principle of Conservation of Number (see [dJP00, Theorem 6.4.7]) applies to them. We only need to check that, if the multigerm of f_s at $f_s^{-1}(f_s(w))$ is (S)DF-generic, then the numbers are 1 if it is the considered singularity, and 0 otherwise. □

Example 5.3.23. Take the family of special double folds

$$(x,y) \mapsto (x^2, y^2, x(a_1x^2 + b_1y^2 - c_1) + y(a_2x^2 + b_2y^2 - c_2)).$$

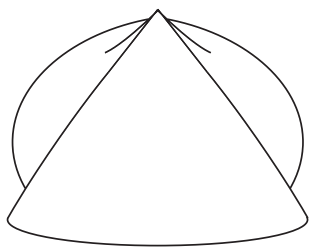

Figure 5.3: A non SDF-stable special double fold (see Example 5.3.23).

The double points $D_1(f)$ and $D_2(f)$ are given by $a_1x^2 + b_1y^2 = c_1$ and $a_2x^2 + b_2y^2 = c_2$.

For the germ

$$(x, y) \mapsto (x^2, y^2, x(x^2 + 2y^2) + y(2x^2 + y^2))$$

(Figure 5.3), we can easily compute

$$ST_1 = 1/2 \dim_{\mathbb{C}}(\mathcal{O}_1/\langle x^2\rangle) = 1$$

and similarly $ST_2 = 1$ and

$$C_1 = C_2 = 2.$$

We also have

$$QD = 1/4 \dim_{\mathbb{C}}(\mathcal{O}_2/\langle 2x^2 + y^2, 2y^2 + x^2\rangle) = 1.$$

Now take the 2-parameter (S)DF-deformation

$$(x, y) \mapsto (x^2, y^2, x(x^2 + 2y^2 - t_1) + y(2x^2 + y^2 - t_2)),$$

where $t = (t_1, t_2) \in \mathbb{C}^2$. We see that, for almost every fixed t with $t_1 \neq 0 \neq t_2$, f_t is a SDF-stable map where we can find (Figure 5.4) a standard self tangency and two cross-caps along $D_1(f_t) \setminus \{0\}$ and the same on $D_2(f_t) \setminus \{0\}$. We also see the cross-cap at $f_t(0)$ and a standard quadruple point. For these good values of t we can also see that, apart from the restrictions on $D_i(f) \cap D_3(f)$ and $D_3(f) \cap \{0\}$ (see Remark 5.3.12), the regular stratification of $D(f_t)$ is transverse to every facet of the Coxeter complex.

Example 5.3.24. If we take the Double Cone (Example 5.3.5), given by

$$(x, y) \mapsto (x^2, y^2, xy).$$

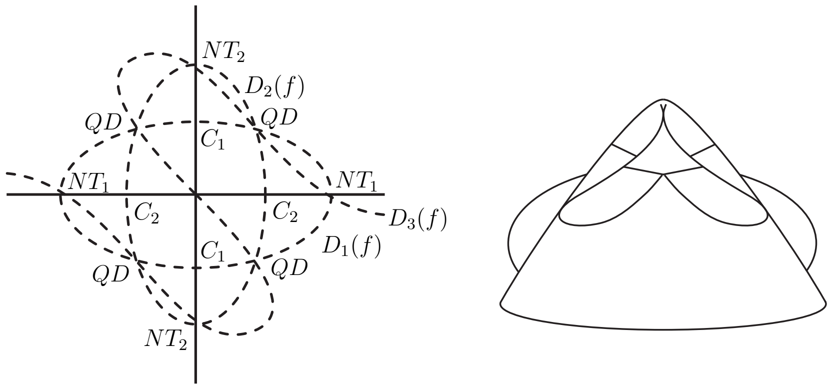

Figure 5.4: A SDF-stable deformation of the surface shown in figure 5.3.

We see easily that

$$ST_i = 0,$$

$$C_i = \dim_{\mathbb{C}} \mathcal{O}_1/\mathfrak{m}_1 = 1, \quad i = 1,2$$

and

$$T = \dim_{\mathbb{C}} \mathcal{O}_2/\mathfrak{m}_2 = 1.$$

Indeed we can take the stabilization DF-stabilization

$$(x,y) \mapsto (x^2, y^2, tx + ty + xy),$$

which has, for any $t \neq 0$, three cross-caps (one in $D_1(f) \setminus \{0\}$, one in $D_2(f) \setminus \{0\}$ and the other at 0) and one triple point (as in Figure 5.1).

Remark 5.3.25. Let $ST(f)$, $C(f)$, $T(f)$ (and respectively $QD(f)$ in the special case) be the number of standard self tangencies, cross-caps, triple points (and standard quadruple points) respectively that appear taking a (S)DF-stabilization of f. It is known that $C(f)$ and $T(f)$ are well defined $\mathcal{A}$-invariants of f. It is immediate that $Q(f)$ is also invariant, because any map showing a quadruple point can be deformed (outside the special double fold world) into another that shows 4 triple points. It is not clear whether ST is $\mathcal{A}$-invariant or not, but it is easy to see that the numbers with indices $ST_i(f)$ and $C_i(f)$ are not. Given a double fold f, we can interchange x and y at the source and then permute the first two coordinates at the target to obtain a new double fold, say g, such that $ST_1(f) = ST_2(g)$, $ST_2(f) = N_1(g)$, $C_1(f) = C_2(g)$ and $C_2(f) = C_1(g)$. Apart from the permutation of indices 1 and 2 that this change of coordinates produces, examples suggest that changes of coordinates don't make the singularities jump from one space $D_i(f)$ to another one.

Therefore, the numbers $ST_i(f)$ and $C_i(f)$ seem to be $\mathcal{A}$-invariant, modulo a simultaneous permutation of all indices 1 and 2 (which still leaves ST invariant). However, we have only succeeded in showing it for finitely determined quasi homogeneous double folds (Corollary 5.4.6).

5.4 $\mathcal{A}$-equivalence and $\mathcal{K}^\alpha$-equivalence

The aim of this section is to mimic a result of David Mond [Mon85, Theorem 4.1:1], which shows the coincidence between the $\mathcal{A}$-equivalence of folds $f\colon (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$, $f(x,y) = (x,y^2,f_3)$ and some easier to use equivalence of the third coordinate function, f_3, defined ad hoc. This equivalence is given by a subgroup of $\mathcal{K}$ called $\mathcal{K}^T$ which behaves well with respect to the Whitney Fold $T\colon (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ given by

$$(x,y) \mapsto (x,y^2).$$

We take, instead of the Whitney Fold, any finite mapping $\alpha\colon (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$ and consider mappings

$$(\alpha, f_{n+1})\colon (\mathbb{C}^n,0) \to (\mathbb{C}^{n+1},0).$$

We define the group $\mathcal{K}^\alpha$ and the generalization of one direction of Mond's results comes easily: $\mathcal{K}^\alpha$-equivalence for f_{n+1} implies $\mathcal{A}$-equivalence for (α, f_{n+1}).

We denote by $\mathcal{R}_n$ the group of germs of biholomorphism $\varphi\colon (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$.

Definition 5.4.1. Let $\alpha\colon (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$ be a finite germ. We define $\mathcal{R}^\alpha$ as the subgroup consisting of the germs $\varphi \in \mathcal{R}_n$ such that there exists a germ $\hat{\varphi} \in \mathcal{R}_n$ such that

$$\hat{\varphi} \circ \alpha = \alpha \circ \varphi.$$

We say that two germs $g,h \in \mathcal{O}_n$ are **$\mathcal{K}^\alpha$-equivalent** if there exist a function $\kappa \in \alpha^*\mathcal{O}_2, \kappa(0) \neq 0$ and a germ of diffeomorphism $\varphi \in \mathcal{R}^\alpha$, such that

$$g = \kappa \cdot h \circ \varphi.$$

Example 5.4.2. Let $\alpha\colon (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$, as in Example 4.1.9, be the Folded Hankerchief

$$(x,y) \mapsto (x^2,y^2).$$

It is easy to see that any biholomorphism $\varphi \in \mathcal{R}^\alpha$ is of the form $\varphi(x,y) = (x\varphi_1, y\varphi_2)$ or $\varphi(x,y) = (y\varphi_1, x\varphi_2)$, for some functions $\varphi_1, \varphi_2 \in \alpha^*\mathcal{O}_2$, $\varphi_i(0,0) \neq 0$. In particular, if $g,h \in \mathbb{C}[x,y]$ are homogeneous $\mathcal{K}^\alpha$-equivalent polynomials, the factors κ and $h\circ\varphi$ are homogeneous. Hence, on one hand, κ is a constant in $\mathbb{C}^*$. On the other hand, since φ is a diffeomorphism, both h and $h \circ \varphi$ are homogeneous of the same degree. We can replace φ by its linear part without changing the composition. Thus, we can assume that φ is of the form $(x,y) \mapsto (ax,by)$ or $(x,y) \mapsto (by,ax)$.

Lemma 5.4.3. *A biholomorphism $\varphi\in\mathcal{R}_n$ belongs to $\mathcal{R}^\alpha$ if and only if the algebras $\alpha^*\mathcal{O}_n$ and $(\alpha\circ\varphi)^*\mathcal{O}_n$ are equal.*

Proof. Let $\varphi\in\mathcal{R}^\alpha$ with $\hat\varphi\circ\alpha\circ\varphi=\alpha$. Any function $h\circ\alpha\in\alpha^*\mathcal{O}_n$ is equal to $(h\circ\hat\varphi)\circ\alpha\circ\varphi\in(\alpha\circ\varphi)^*\mathcal{O}_n$. Now take $h\circ\alpha\circ\varphi\in(\alpha\circ\varphi)^*\mathcal{O}_n$. This function is equal to $h\circ\hat\varphi^{-1}\circ\hat\varphi\circ\alpha\circ\varphi=(h\circ\hat\varphi^{-1})\circ\alpha\in\alpha^*\mathcal{O}_n$.

Now suppose that the two sub-algebras above are equal, then there exist some functions $\hat\varphi_i$ such that $\alpha_i=\hat\varphi_i\circ\alpha\circ\varphi$. Take $\hat\varphi=(\hat\varphi_1,\dots,\hat\varphi_n)$. Then we have $\alpha=\hat\varphi\circ\alpha\circ\varphi$. As α is finite and φ is a biholomorphism, α and $\alpha\circ\varphi$ have the same finite multiplicity. Therefore $\hat\varphi$ must have multiplicity 1, and hence is a biholomorphism. □

Theorem 5.4.4. *Let $\alpha\colon(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$ be a finite germ and f_{n+1},g_{n+1} be two $\mathcal{K}^\alpha$-equivalent functions of $\mathcal{O}_n$, then the map germs $(\mathbb{C}^n,0)\to(\mathbb{C}^{n+1},0)$ $f=(\alpha,f_{n+1})$ and $g=(\alpha,g_{n+1})$ are $\mathcal{A}$-equivalent.*

Proof. $f\sim_{\mathcal{K}^\alpha}g$ implies that there exists $\theta_\alpha\colon(\mathbb{C}^n\times\mathbb{C},0)\to(\mathbb{C},0)$ of the form $\theta_\alpha(X,Z)=\theta(\alpha(X),Z)$ for some germ of function θ and such that $\theta_\alpha(0,\cdot)$ is a germ of biholomorphism, and there exists $\varphi\in\mathcal{R}_n^\alpha$ such that $g(X)=\theta_\alpha(X,f\circ\varphi(X))$. Since $\varphi\in\mathcal{R}_n^\alpha$, then there exists some germ of biholomorphism $\hat\varphi$ such that $\alpha=\hat\varphi\circ\alpha\circ\varphi$. We define $\psi_1\colon\mathbb{C}^{n+1}\to\mathbb{C}^n$ by $\psi_1=\hat\varphi\circ\pi_1$ and $\psi_2=\theta\circ(\psi_1,\pi_2)$, where π_i represents the projection over the i-th component of $\mathbb{C}^n\times\mathbb{C}$. Let $\psi=(\psi_1,\psi_2)\colon(\mathbb{C}^{n+1},0)\to(\mathbb{C}^{n+1},0)$ and, for every $X\in\mathbb{C}^n$, we have

$$\begin{aligned}\psi\circ(\alpha,f)\circ\varphi(X)&=\big(\hat\varphi(\alpha(\varphi(X))),\theta(\hat\varphi(\alpha(\varphi(X))),f(\varphi(X)))\big)\\&=\big(\alpha(X),\theta_\alpha(X,f(\varphi(X)))\big)\\&=(\alpha,g)(X).\end{aligned}$$

As a consequence of $\hat\varphi$ and $\theta_\alpha(X,\cdot)$ being biholomorphisms, we have that ψ is a biholomorphism. □

Examples suggest that the converse of Theorem 5.4.4 also holds: $\mathcal{A}$-equivalence of (α,f_{n+1}) and (α,g_{n+1}) implies $\mathcal{K}^\alpha$-equivalence of f_{n+1} and g_{n+1}. However we have not succeed in proving this in general. As we mentioned before, Mond has proved that it holds when α is the Whitney Fold. We have only succeeded in showing it for finitely determined quasihomogeneous double folds.

It is shown in [MNB08] that any quasihomogeneous double fold must be homogeneous. There are only two ways to obtain a homogeneous double fold $f(x,y)=(\alpha,xp_1+yp_2+xyp_3)$. One is $p_3=0$ and the other $p_1=p_2=0$. Every finitely determined double fold must have a reduced double point space, which is given by $(p_1+yp_3)(p_2+xp_3)(xp_1+yp_2)=0$. We deduce immediately that every finitely determined quasihomogeneous double fold must be, in fact, a homogeneous special double fold.

Theorem 5.4.5. *Let $f = (\alpha, f_3)$ and $g = (\alpha, g_3)$ be $\mathcal{A}$-equivalent finitely determined quasihomogeneous double folds , then the functions f_3 and g_3 are $\mathcal{K}^\alpha$-equivalent.*

Proof. Assume that there exist ψ and φ such that $g = \psi \circ f \circ \varphi$. Denote by φ_{i,x_j} the derivative of the i-th component with respect to the variable x_j. Taking into account that $p_1, p_2 \in \mathfrak{m}^2$, the 2-jet of the first two coordinate functions of the equality $g = \psi \circ f \circ \varphi$ gives us

$$\begin{aligned} x^2 &= \psi_{1,X}(\varphi_{1,x}^2 x^2 + \varphi_{1,x}\varphi_{1,y}xy + \varphi_{2,y}^2 y^2) \\ &\quad + \psi_{1,Y}(\varphi_{2,x}^2 x^2 + \varphi_{2,x}\varphi_{2,y}xy + \varphi_{2,y}^2 y^2), \end{aligned}$$

and also

$$\begin{aligned} y^2 &= \psi_{2,X}(\varphi_{1,x}^2 x^2 + \varphi_{1,x}\varphi_{1,y}xy + \varphi_{2,y}^2 y^2) \\ &\quad + \psi_{2,Y}(\varphi_{2,x}^2 x^2 + \varphi_{2,x}\varphi_{2,y}xy + \varphi_{2,y}^2 y^2). \end{aligned}$$

Since $d\varphi$ is invertible, we have $\varphi_{1,x}\varphi_{2,y} \neq 0$ or $\varphi_{1,y}\varphi_{2,x} \neq 0$. In the first case, from the equations we obtain $\varphi_{1,y} = \varphi_{2,x} = 0$ and, in the second case, $\varphi_{1,x} = \varphi_{2,y} = 0$. Suppose we are in the first case (the second one is analogous). Then the differential of φ is of the form $d\varphi(u, v) = (au, bv)$ for some $a, b \in \mathbb{C}^*$.

Notice that w is a source double point of g if and only if it is so for $f \circ \varphi$, if and only if $\varphi(w)$ is a source double point of f. Since f and g are finitely determined, their double point spaces are reduced and thus $\varphi|_{D(g)} : D(g) \to D(f)$ is an isomorphism between complex space germs. We claim that $\varphi|_{D_3(g)}$ is an isomorphism between $D_3(g)$ and $D_3(f)$. We proceed by reduction to the absurd: suppose there is a irreducible component R of $D_3(g)$, such that $\varphi(R) \not\subset D_3(f)$. For example, suppose $\varphi(R) \subset D_1(f)$ (the other case, $\varphi(R) \subset D_2(f)$, is analogous). Since f and g are finitely determined, their non strict double points are isolated and thus, since $R \subset D_3(g)$ and $\varphi(R) \subset D_1(f)$, we have $\varphi(i_3(R)) = i_1(\varphi(R))$. Let (u, v) be the tangent vector to the curve germ R, we have the equality $d\varphi(i_3(u, v)) = i_1(d\varphi(u, v))$, that is $(-au, -bv) = (-au, bv)$. The last equality implies (u, v) is a horizontal vector. Since g is homogeneous, the equation which defines R is also homogeneous and, thus, it is independent of x. This is implies that y divides $xq_1 + yq_2$, which in turn implies that y divides q_1. Then y^2 divides $q_1q_2(xq_1 + yq_2)$. This is a contradiction, because g is finitely determined and, thus, $D(g) = V(q_1q_2(xq_1 + yq_2))$ must be reduced.

Now we have the isomorphism of complex spaces $\varphi|_{D_3(g)} : D_3(g) \to D_3(f)$, that is, we have the equality $\langle g_3 \rangle = \varphi^* \langle f_3 \rangle$. This implies the existence of a function h, with $h(0, 0) \neq 0$, such that $g_3 = h \cdot f_3 \circ \varphi$. Since g_3 y f_3 are homogeneous, we can take the diffeomorphism $\tilde{\varphi} = d\varphi$ and the constant $\kappa = h(0, 0) \neq 0$ and get $g_3 = \kappa \cdot f_3 \circ \varphi$. Moreover, as we have seen before, $\tilde{\varphi}$ is a diagonal linear change and thus it belongs to $\mathcal{R}^\alpha$. $\square$

Notice that the $\mathcal{K}^\alpha$-equivalence of f_3 and g_3 splits into two simultaneous equivalences between P_1, P_2 and Q_1, Q_2. In the diagonal case we get an expression

$$xQ_1(x^2,y^2)+yQ_2(x^2,y^2)=\kappa axP_1(a^2x^2,b^2y^2)+\kappa byP_2(a^2x^2,b^2y^2),$$

equivalent to $Q_1(x,y)=\kappa aP_1(a^2x,b^2y)$ and $Q_2(x,y)=\kappa bP_2(a^2x,b^2y)$. In the antidiagonal case we obtain the expression

$$xQ_1(x^2,y^2)+yQ_2(x^2,y^2)=\kappa ayP_1(a^2y^2,b^2x^2)+\kappa bxP_2(a^2y^2,b^2x^2),$$

equivalent to $Q_1(x,y)=\kappa bP_2(a^2y,b^2x)$ and $Q_2(x,y)=\kappa aP_1(a^2y,b^2x)$. Now the next corollary follows immediately.

Corollary 5.4.6. *Let f and g be two $\mathcal{A}$-equivalent quasihomogeneous finitely determined special double folds, then:*

$$ST_i(f)=ST_j(g),$$

$$C_i(f)=C_j(g),$$

$$QD(f)=QD(g),$$

$$\mu(D_i(f))=\mu(D_j(g)),$$

where $j=i$ in the diagonal case, and in the antidiagonal the pairs (i,j) are $(1,2)$, $(2,1)$, $(3,3)$.

Chapter 6

Conclusion

The following is a brief summary of our accomplished goals: We have been able to unify the approaches of Gaffney and Mond for multiple point spaces. We have described effective methods to compute the spaces $D(f)$ and $M_k(f)$ (these two only for $p = n + 1$) and $D^k(f)/S_k$, under quite general conditions.

We have shown relations between algebraic properties of the double point space $D^2(f)$ and the stability and finite determinacy of f, and we have been able to find examples to justify the assumptions in our statements.

We have discarded the structure $\mathscr{H}^k(f)$ as a satisfactory multiple point structure, by giving an explicit example where it fails to satisfy the conditions M1 and M2. However, we have shown that under generic conditions, the mentioned structure is correct.

We have provided a different approach for the construction of the space $B^2(f)$, which makes very easy to compute it. We have given a proper definition of $B^2(f)$ for germs. We have shown that $B^2(f)$ is isomorphic to $D^2(f)$ if there are no points of corank ≥ 2. We have shown that $B^2(f)$ can be defined 'taking unfoldings', analogously to $D^2(f)$. We have related the properties of $B^2(f)$ to stability and finite determinacy and we have given different proofs to some properties of $B^2(f)$.

We have extended to corank 2 a characterization of finite determinacy of map germs $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ in terms of the Milnor number $\mu(D(f))$. We have also extended to corank 2 the so called Marar-Mond formulas, relating the Milnor numbers of several double point spaces and the number of cross-caps and triple points collapsed at the origin.

We have introduced a family of corank 2 map germs, the 'double folds', which contains interesting examples. Finally, in Appendix B we have introduced some techniques to compute expressions in terms of S_2-invariant funtions, where S_2 is the group of permutation of two points.

Now we list some open problems for further studies:

Open Problem 1. Give an explicit ideal sheaf $\mathscr{I}^k(f)$ defining $D^k(f)$:

This has been done for corank 1 map germs (Section 2.2) and for double points of arbitrary corank (Theorem 3.1.12). As far as we know, for $k \geq 3$ it is still an open problem to find an explicit set of generators for the k-multiple point ideal of a map germs of corank ≥ 2. As exposed in Open Problem 5, the relation between singular monogerms and the nearby strict multiple points becomes more complex if $\operatorname{corank} f \geq 3$. Consequently, it would make sense to restrict the problem to the case of corank ≤ 2.

Open Problem 2. Characterize k-multiple points set-theoretically:

The problem can be reduced to points in the small diagonal. Given $f\colon X \to Y$, a pair $(x, x) \in \Delta(X, 2)$ is a double point if and only if f is singular at x. What makes a point $(x, x, x) \in \Delta(X, 3)$ be a triple point? As in the previous open problem, and given its relation to Open Problem 5, perhaps one should restrict this problem to the case $\operatorname{corank} f \leq 2$.

Open Problem 3. Is $D^k(f)$ a Cohen Macaulay space whenever it is dimensionally correct?

We know that the claim holds for multiple points of corank one (Section 2.2) and for double points of any corank (Proposition 3.1.10). In both situations, we use explicit sets of generators to obtain the results. Thus, the problem is related to Open Problem 1

An interesting point is that it suffices to show that $D^k_S(F)$ is Cohen Macaulay for any stable multigerm F. This is true because the k-multiple point space of a given map f is given by the section at $\{s = 0\}$ of $D^k_S(F)$, for a suitable local stable unfolding $F(s, x) = (s, f_s(x)), s \in \mathbb{C}^r$ of $f = f_0$. Since both $D^k_S(F)$ and $D^k(f)$ are dimensionally correct, if we assume that $D^k_S(F)$ is Cohen Macaulay, then $s_1, \dots, s_r$ form a regular sequence for the module $\mathcal{O}_{D^k_S(F)}$ and, hence $\mathcal{O}_{D^k_S(F)}/\langle s_1, \dots, s_r\rangle \cong D^k(f)$ is a Cohen Macaulay ring, as desired.

Open Problem 4. Does the equality

$$\sqrt{\mathscr{P}(f, k) : \mathscr{I}^{\infty}_{D(X,k)}} = \mathscr{P}(f, k) : \mathscr{I}_{D(X,k)}$$

hold for stable maps?

The interest of this problem is computational. To compute multiple point spaces of corank ≥ 2 we need to compute the strict k-multiple point space of a stable unfolding (see Definition 2.1.1). It turns out that the harder computation that we have to perform are the saturations and radicals in the equality of the statement. Observe that the statement for $k = 2$ follows from Corollary 3.3.9.

Open Problem 5. Understand the relation between the different multiple point spaces and the corresponding nearby strict multiple points. For instance: When can $D^k_1(f)$ and $M_k(f)$ be computed as $D^k(f)$ in Proposition-Definition 2.1.5? To make a more precise statement. Can we determine the stable maps f satisfying the following statements?

1. $D_1^k(f)$ is reduced and equal to $\overline{\{(x \in X^k \mid |f^{-1}(f(x))| \geq k\}}$.

2. $M_k(f)$ is reduced and equal to $\overline{\{(y \in Y^k \mid |f^{-1}(y)| \geq k\}}$.

It follows from the work of Damon and Galligo [DG83] that not all stable maps satisfy the second condition. They show that there are stable map germs of corank ≥ 3, satisfying $M^k(f) \neq \emptyset$, and such that the origin is not in the closure of the strict k-multiple points!

Open Problem 6. Do the spaces $D_1^k(f)$ and $N_k(f) = f^{-1}(M_k(f))$ agree? We have only been able to show the equality set-theoretically and for double points. Some work in this direction can be found in [KLU92]. An example of the scheme-theorical equality can be found in Example 2.3.3, compared to the source double point given in Section 5.3

Open Problem 7. These are some questions regarding 'blowing-up k-multiple points':

1. Define spaces $B^k(f)$, for $k \geq 3$.

2. Is $B^k(f) \cong D^k(f)$ in corank 1?

3. Does the smoothness of $B^k(f)$ characterize the stability of f?

Open Problem 8. Extend the work for double folds to arbitrary 'reflection maps':

Let G be a reflection group and let $\alpha\colon \mathbb{C}^n \to \mathbb{C}^n$ be its invariant map. A G-map is any map of the form $(\alpha, f_{n+1})\colon \mathbb{C}^n \to \mathbb{C}^{n+1}$. Do we know how to mimic the results for double folds for these maps? For instance, it is not difficult to show the set-theoretical equality between $D(f)$ and the zero set of

$$(\prod_{g \in G \setminus \{1\}} f_{n+1} - g f_{n+1})/J_\alpha.$$

Observe that for double folds the equality holds as schemes.

Appendix A

Complex spaces

The aim of this appendix is to fix the notations and describe some needed results about complex spaces. Our general references are [GR84] and [dJP00]. The only notion we have not been able to find in the literature is that of a germ of a complex space along a subset (Section A.2), a straightforward generalization of the usual notion of a germ at a point.

A **model complex space** is a locally ringed space $(X, \mathcal{O}_X)$ of the form

$$X = \{p \in U \mid f_1(p) = \ldots, f_r(p) = 0\},$$

for some holomorphic functions $f_1, \ldots, f_r$ defined at some open neighbouhood $U \subseteq \mathbb{C}^n$, and

$$\mathcal{O}_X = (\mathcal{O}_{\mathbb{C}^n}/\mathscr{I})|_X,$$

where $\mathscr{I}$ is the ideal sheaf in $\mathcal{O}_U$ generated by $f_1, \ldots, f_r$. A **complex space** is a Hausdorff locally ringed space, which is locally isomorphic to some model complex space (the model may change with the point, obviously). A morphism of complex spaces is just a morphism of locally ringed spaces between two complex spaces.

A complex space X is called **reduced (resp. regular, Cohen-Macaulay, complete intersection, irreducible, normal) at a point** $x \in X$ if the stalk $\mathcal{O}_{X,x}$ is reduced (resp. regular, Cohen-Macaulay, complete intersection, a domain, integral closed in its quotient ring). We say that $(X, \mathcal{O}_X)$ is reduced, (resp. regular, Cohen-Macaulay, complete intersection, normal) if it is so at any point $x \in X$.

We say a complex space X is **unmixed** (resp. **equidimensional**) at a point $x \in X$ if all the associated primes (resp. minimal associated primes) in $\mathcal{O}_{X,x}$ have the same dimension. We say that X is unmixed if it is unmixed at any point and the dimension of X is the same at every point. We say that X is equidimensional if the dimension of X is the same at every point. Observe that a space is unmixed if and only if it is equidimensional and does not contain embeeded components. Moreover, a connected space is unmixed (resp. equidimensional) if and only if it is unmixed (resp. equidimensional) at every point.

We say that $(Y, \mathcal{O}_Y)$ is a **complex subspace** of $(X, \mathcal{O}_X)$ if $Y \subseteq X$ and there exists a morphism of complex spaces

$$(i, i^{\#})\colon (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X),$$

where $i\colon Y \hookrightarrow X$ is the inclusion and $i^{\#}$ is surjective. We say that $(Y, \mathcal{O}_Y)$ is a **closed complex subspace** if $i(Y)$ is closed in X.

Proposition A.0.7. *If all the non embedded irreducible components of $(X_i, x), 1 \leq i \leq k$ have codimension $\leq r_i$, then all the non-embedded irreducible components of $(\bigcap_{i=1}^k X_i, x)$ have codimension $\leq \sum_{i=1}^k r_i$.*

A.1 Projectivization of a cone over a complex space

Definition A.1.1. Let M be a manifold, a **cone over** M is a closed complex subspace $X \subseteq M \times \mathbb{C}^m$ (for some m), defined by some ideal sheaf $\mathscr{I}$ in $\mathcal{O}_{M\times\mathbb{C}}$ such that, for all $x \in M$, there exists an open neighbourhood U of x, such that $\mathscr{I}|_{U\times\mathbb{C}^n} = \langle h_1, \ldots, h_r\rangle$, for some homogeneous polynomials

$$h_1 \ldots, h_r \in \mathcal{O}_M(U)[y_1, \ldots, y_m].$$

Observe that, if $X \subseteq M \times \mathbb{C}^n$ is a cone over M, then, for any point $(x, y), x \in M, y \in \mathbb{C}^n \setminus \{0\}$, the space X contains the 'line' $(x, \lambda y), \lambda \in \mathbb{C}$.

To fix notation, let

$$\theta\colon \mathbb{C}^m \setminus \{0\} \to \mathbb{P}^{m-1}$$

be the class map to the projective space.

Let $U_i = \{[y_0 : \cdots : y_m] \in \mathbb{P}^{m-1} \mid x_i \neq 0\}$ and let $\varphi_i\colon M \times U_i \to M \times \mathbb{C}^{m-1}$ be the biholomorphism given by

$$(x, [y_0, \ldots, y_m]) \mapsto (x, \frac{y_0}{y_i}, \ldots, \frac{\hat{y}_i}{y_i}, \ldots, \frac{y_m}{y_i}).$$

where the hat means ommision.

Proposition-Definition A.1.2. [Fis76, P. 45] Let $X \subseteq M \times \mathbb{C}^m$ be a cone over M defined by a coherent ideal sheaf $\mathscr{I}$. Then the subset of $M \times \mathbb{P}^{m-1}$

$$\tilde{X} = \{(x, \theta(y)) \mid (x, y) \in X, y \neq 0\}$$

has structure of closed complex subspace of $M \times \mathbb{P}^{m-1}$, defined by a coherent ideal sheaf $\tilde{\mathscr{I}}$ in $\mathcal{O}_{M\times\mathbb{P}^{m-1}}$ given locally as follows:

For any open subset $V \subseteq X$, if $\mathscr{I}|_{V\times\mathbb{C}^n} = \langle h_1,\dots,h_r\rangle$, for some homogeneous polynomials $h_1\dots,h_r \in \mathcal{O}_M(V)[y_1,\dots,y_m]$, then $\tilde{\mathscr{I}}(V\cap U_i)$ is the preimage by φ^* of the ideal in $\mathcal{O}_{V\times\mathbb{C}^{m-1}}$ generated by the functions

$$h_j((x),(z_1,\dots,z_{i-1},1,z_i,\dots,z_{m-1})),$$

with the value 1 at the i-th position and being $z_1,\dots,z_{m-1}$ the variables in $\mathbb{C}^{m-1}$.

The diagonal blowing-up

Let X be a complex manifold, we denote by $B(X)$ the blowing-up of $X\times X$ along the diagonal $\Delta(X,2)=\{(x,x)\in X\times X\}$. It is well known that $B(X)$ and X are birationally equivalent as complex spaces. More precisely, $B(X)$ is endowed with a surjection

$$\pi_X\colon B(X)\to X\times X,$$

which is an isomorphism when restricted to $B(X)\setminus\pi_X^{-1}(\Delta(X,2))$. We denote π_X by π if there is no risk of confusion. Let $n=\dim X$. Since $\Delta(X,2)$ is a n-dimensional smooth subspace of the $2n$-dimensional smooth space $X\times X$, it follows that $B(X)$ is smooth of dimension $2n$ and the fiber $\pi^{-1}(x,x)$ of any diagonal point is isomorphic to the projective space $\mathbb{P}^{n-1}$ over $\mathbb{C}$. Indeed, there is a canonical identification between the fiber $\pi^{-1}(x,x)$ and the projectivised of the tangent space T_xX of X at x. The fiber $\pi^{-1}(x,x)$ can be seen as an element of the projectivised of the normal bundle

$$\frac{T_{(x,x)}(X\times X)}{T_{(x,x)}\Delta(X,2)}.$$

We have an isomorphism

$$\frac{T_{(x,x)}(X\times X)}{T_{(x,x)}\Delta(X,2)}\to T_xX,$$

given by

$$(u,v)+T_{(x,x)}\Delta(X,2)\mapsto u-v.$$

The induced map between the corresponding projectivisations

$$\pi^{-1}(x,x)\to\mathbf{P}(T_xX)$$

is the mentioned identification.

In order to know $B(X)$ locally around the fibers $\pi^{-1}(x,x')$ of points $(x,x')\in X\times X$, it suffices to know $B(\mathbb{C}^n)$: Let $(x,x')\in X\times X\setminus\Delta(X,2)$, then $B(X)$ at $\pi^{-1}(x,x')$ is locally isomorphic to $X\times X$ at (x,x'). Now let $(x,x)\in\Delta(X,2)$ and take a local chart $\psi\colon V\to W\subseteq\mathbb{C}^n$ of X around the point x. Since the blowing-up is a local construction, the preimage $\pi^{-1}(V)$ is isomorphic to $B(\mathbb{C}^n)\cap\pi_{\mathbb{C}^n}^{-1}(W)$. This isomorphism, and the local chart ψ, yield an $\mathcal{A}$-equivalence between the restriction $\pi|_{\pi^{-1}(V)}$ and the restriction of $\pi_{\mathbb{C}^n}: B(\mathbb{C}^n)\to\mathbb{C}^n\times\mathbb{C}^n$ to $B(\mathbb{C}^n)\cap\pi_{\mathbb{C}^n}^{-1}(W)$.

Example A.1.3. The blowing up of $\mathbb{C}^n \times \mathbb{C}^n$ along the diagonal can be realized as the space

$$B(\mathbb{C}^n) = \{(x, x', [u]) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{P}^{n-1} \mid \exists \lambda \in \mathbb{C}, \lambda(x - x') = u\},$$

together with the projection

$$\pi\colon \begin{array}{ccc} B(\mathbb{C}^n) & \to & \mathbb{C}^n \times \mathbb{C}^n \\ (x, x', [u]) & \mapsto & (x, x') \end{array} .$$

We label the homogeneous coordinates in $\mathbb{P}^{n-1}$ as $u_1 : \cdots : u_n$. It is immediate that $B(\mathbb{C}^n)$ can be regarded as the smooth complex subspace of $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{P}^{n-1}$ given by the following equations homogeneous in the u_i coordinates:

$$(x_i - x_i')u_j - (x_j - x_j')u_i = 0, \quad 1 \leq i < j \leq 0.$$

To produce an atlas of $B(\mathbb{C}^n)$, we use a slight modification of the usual affine covering of the projective space: Let $U_i = \{(x, x', [u]) \in B(\mathbb{C}^n) \mid u_i \neq 0\}$. It is obvious that $B(\mathbb{C}^n)$ is covered by the open subsets $U_i, i = 1, \ldots, n$. Just to fix notation, we write $A = \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^{n-1}$ and label the coordinates of a point $(x, \lambda, a) \in A$ as $x_1, \ldots, x_n, \lambda, a_1, \ldots, a_{n-1}$. We define the local chart $\phi_i\colon U_i \to A$ as the map given by

$$(x, x', [u]) \mapsto (x, x_i - x_i', \frac{\widehat{u}}{u_i}),$$

where $\widehat{u}$ is obtained deleting the i-th coordinate of u. The inverse map $\phi^{-1}\colon A \to U_i$ is given by

$$(x, \lambda, a) \mapsto (x, x + \lambda\widetilde{a}, [\widetilde{a}]),$$

where $\widetilde{a} \in \mathbb{C}^n$ is obtained adding a new coordinate, with value 1 and at the i-th position, to the point $a \in \mathbb{C}^{n-1}$. The transition map $\tau_{ij}\colon \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is given by $(x, \lambda, a) \mapsto (x, \lambda, a/a_j)$. It is a biholomorphism because $\phi_i(U_i \cap U_j)$ is precisely the subset of points $(x, \lambda, a) \in A$, such that $a_j \neq 0$.

Note also that, via ϕ_i, points in $\pi^{-1}(\Delta(\mathbb{C}^n, 2)) \cap U_i$ correspond exactly to the points $(x, \lambda, a) \in A$ with $\lambda = 0$.

A.2 Germs of complex spaces

Definition A.2.1. We say that the complex subspaces X_1, X_2 of a complex space Z define the same germ of complex space along S if

1. The germs of subsets (X_1, S) and (X_2, S) agree.

2. The restricted structures $\mathcal{O}_{X_1}|_S$ and $\mathcal{O}_{X_2}|_S$ agree.

This defines clearly an equivalence relation whose classes are called **germs of complex space along** S. Every germ of complex space X along S defines a pair, consisting of the germ of subset (X,S) and the corresponding structure sheaf

$$\mathcal{O}_{X,S} = \mathcal{O}_X|_S = \lim_{U \supseteq S} \mathcal{O}_X(U).$$

Observe that, if S is a finite set, then the sheaf $\mathcal{O}_{X,S}$ can be identified with the direct sum of stalks $\bigoplus_{x\in S} \mathcal{O}_{X,x}$.

For any morphism of complex spaces $(f, f^{\#})\colon X \to Y$ and every $S \subseteq X$, we will denote by

$$f^{\#}_S : \mathcal{O}_{Y,f(S)} \to \mathcal{O}_{X,S}$$

the morphism of sheaves obtained by the composition

$$\mathcal{O}_{Y,f(S)} \to (f_*\mathcal{O}_X)|_{f(S)} \to \mathcal{O}_{X,S}.$$

The first arrow is $f_*|_{f(S)}$. The second arrow is the canonical morphism between the corresponding limits, given by the inclusion $\{f^{-1}(f(U)) \mid S \subseteq U, U \text{ open in } X\} \subseteq \{U \mid S \subseteq U\}$).

Definition A.2.2. Let X and Y be complex spaces and $S \subseteq X$. Let U_1, U_2 be open neighbourhoods of S in X and denote by X_i the complex subspace $(X \cap U_i, \mathcal{O}_X|_{U_i})$. We say that two morphisms of complex spaces $(f, f^{\#})\colon X_1 \to Y$ and $(g, g^{\#})\colon X_2 \to Y$ define the same germ along S if

1. The map germs (f,S) and (g,S) agree.

2. The induced morphisms $f^{\#}_S$ and $g^{\#}_S : \mathcal{O}_{Y,f(S)} \to \mathcal{O}_{X,S}$ agree.

This defines an equivalence relation and every class yields a pair consisting of a map germ (f,S) and a morphism of sheaves $f^{\#}_S$.

Definition A.2.3. A morphism of germs of complex spaces

$$\varphi\colon (X,S) \to (Y,T)$$

is just the germ along S of a morphism of complex spaces $(f, f^{\#})\colon X' \to Y$ satisfying $f(S) \subseteq T$, where X' is a complex space of the form $(X \cap U, \mathcal{O}_X|_U)$, for some open neighboohood U of S in X. Observe that φ induces a morphism of sheaves

$$\varphi^{\#} : \mathcal{O}_{Y,T} \to \mathcal{O}_{X,S},$$

given by the composition $\mathcal{O}_{Y,T} \to \mathcal{O}_{Y,f(S)} \to \mathcal{O}_{X,S}$. The first arrow is given by restricion to $f(S)$ and the second arrow is $f^{\#}_S$.

Of course, if S and T are finite sets, then the morphism of sheaves $f^{\#}_S$ and $\varphi^{\#}$ are just morphisms of rings between the direct sum of the corresponding stalks. If $f(S) = T$ (in particular, if T is a single point and $S \neq \emptyset$), then the morphisms $f^{\#}_S$ and $\varphi^{\#}$ agree.

A.3 Fittings and multiple points

Definition A.3.1. Let $\mathscr{M}$ be a coherent $\mathcal{O}_X$-module. Since every coherent $\mathcal{O}_X$ is locally finitely presented, we can produce local presentations of $\mathscr{M}$ and define local Fitting ideal presheaves as in Definition B.1.1, for all $k \in \mathbb{Z}$. From the fact that these ideals do not depend on the chosen presentation, it follows that the sheaves associated to these presheaves can be glued together to a sheaf on $\mathcal{F}_k(\mathscr{M})$ defined on X. The ideal sheaf (defined up to isomorphism) $\mathcal{F}_k(\mathscr{M})$ is called the k**-th Fitting ideal sheaf**, and satisfies

$$(\mathcal{F}_k(\mathscr{M}))_x = F_k(\mathscr{M}_x),$$

for all $x \in X$, where $\mathscr{M}_x$ is regarded as an $\mathcal{O}_{X,x}$-module.

Proposition A.3.2. *[MP89, Prop. 1.5] Let $f\colon X \to Y$ be a finite morphism of analytic spaces. We have the following set-theoretical equality:*

$$V(\mathcal{F}_k(f_*\mathcal{O}_X)) = \{y \in Y \mid \sum_{x\in f^{-1}(y)} \dim_{\mathbb{C}} \frac{\mathcal{O}_{X,x}}{f^*\mathfrak{m}_y} > k\}.$$

Lemma A.3.3. *Let $f\colon (X,x) \to (\mathbb{C}^{n+1},0)$ be a finite morphism, where (X,x) is an n-dimensional Cohen-Macaulay complex space germ. Then $V(F_0(f))$ is reduced if and only if (X,x) is reduced and f is generically one-to-one.*

Proof. We take representatives X of (X,x) and V of $(\mathbb{C}^{n+1},0)$ such that $f\colon X \to V$ is a finite morphism of complex spaces. First we show that for any $y \in f(X)$, $V(\mathcal{F}_0(f))$ is smooth at y if and only if $f^{-1}(y) = \{x\}$, X is smooth at x and f is regular at x. Indeed, suppose $V(\mathcal{F}_0(f))$ is smooth at y and let $f^{-1}(y) = \{x_1,\dots,x_r\}$. Then the stalk at y is he product

$$\mathcal{F}_0(f)_y = F_0(f_{x_1})\dots F_0(f_{x_r}),$$

where $f_{x_i}\colon (X,x_i) \to (V,y)$ is the germ of f at x_i. Since each $F_0(f_{x_i}) \subset \mathfrak{m}_{V,y}$, we must have $r = 1$ and we write $f^{-1}(y) = \{x\}$.

Let $q = \dim_{\mathbb{C}} \mathcal{O}_{X,x}/f^*\mathfrak{m}_{V,y}$. By [?], a minimal presentation of $\mathcal{O}_{X,x}$ has the form

$$\mathcal{O}_{V,y}^q \xrightarrow{\ \lambda\ } \mathcal{O}_{V,y}^q \xrightarrow{\ \varphi\ } \mathcal{O}_{X,x} \longrightarrow 0.$$

Then $\mathcal{F}_0(f)_y$ is generated by $\det(\lambda) \in \mathfrak{m}_{V,y}^q$ so that necessarily $q = 1$. The exactness of the sequence implies $\mathcal{F}_0(f)_y = \mathrm{Im}(\lambda) = \mathrm{Ker}(\varphi)$ and thus, $\mathcal{O}_{X,x}$ is isomorphic to $\mathcal{O}_{V,y}/\mathcal{F}_0(f)_y$, which is regular. On the other hand, since $q = 1$ we have $f^*\mathfrak{m}_{V,y} = \mathfrak{m}_{X,x}$ and f has rank n. The proof of the converse is analogous.

Now we prove the lemma. If $V(F_0(f))$ is reduced then it is generically smooth. Thus, X is also generically smooth and f is generically one-to-one. Since (X,x) is Cohen-Macaulay and generically smooth, we conclude that (X,x) is reduced.

Conversely, if (X, x) is reduced and f is generically one-to-one, then (X, x) is generically smooth. Thus, $V(F_0(f))$ is also generically smooth. Again, $V(F_0(f))$ is Cohen-Macaulay (it is a hypersurface in $(\mathbb{C}^{n+1}, 0)$) and geerically reduced, and hence reduced. $\square$

Appendix B

Algebra

Here we describe some necessary results about commutative algebra and invariant functions under the action of groups. As general references for commutative algebra we recommend the books of Matsumura [Mat89, Mat80] and also [GP02]. The results about invariant theory can be found in [Sta79]. The only original work in this appendix is contained in Section B.3.

B.1 Fitting Ideals

Definition B.1.1. We say that a module M over a ring R is **finitely presented** if it exists an exact sequence of the form

$$R^p \to R^q \to M \to 0.$$

The $q \times p$ matrix λ, with entries in R, which represents the homomorphism $R^p \to R^q$ is called a **presentation matrix** of M. For $0 \leq k \leq \min(p,q)-1$, the k-th Fitting ideal of M is the ideal $F_k(M)$ in R generated by the minors of size $\min(p,q) - k$ of any presentation matrix of M. By convention, we define $F_k(M) = R$, for all $k \geq \min(p,q)$, and $F_k(M) = 0$, for all $k \leq 0$. These ideals do not depend on the chosen presentation and matrix. Isomorphic modules yield the same ideal.

Proposition B.1.2. *With the notations above, the following hold:*

1. $F_0(M) \subseteq \mathrm{Ann}(M)$,

2. $\mathrm{Ann}(M)^q \subseteq M$.

In particular, $\sqrt{F_0(M)} = \sqrt{\mathrm{Ann}(M)}$.

B.2 Invariants

Through this section, G represents a finite subgroup of the general linear group $GL(\mathbb{C}^m)$ of all linear bijections $\mathbb{C}^m \to \mathbb{C}^m$. We denote by $\mathfrak{X}$ the set of irreducible characters of G and χ_0 the trivial character with constant value 1. The action of G on $\mathbb{C}^m$ induces a linear action on $\mathcal{O}_m$, given by $(gh)(x) = h(gx)$, for all $g \in G, h \in \mathcal{O}_m, x \in \mathbb{C}^m$.

Definition B.2.1. For any irreducible character $\chi \in \mathfrak{X}$ we define a map $\rho_\chi \colon \mathcal{O}_m \to \mathcal{O}_m$, by $a \mapsto a^\chi$, where

$$a^\chi = \frac{\deg(\chi)}{|G|} \sum_{g \in G} \overline{\chi(g)} ga.$$

We write $\mathcal{O}_m^\chi = \operatorname{Im} \rho_\chi$ and call it the **χ-isotypical component** of $\mathcal{O}_m$. The elements $a \in \mathcal{O}_m^\chi$ are called χ-invariant. We denote by $M^G = M^{\chi_0}$ and call $\rho_\sharp = \rho_{\chi_0}$ the Reynold operator. The image of an element by the Reynold operator is denoted by $a^\sharp = \rho_\sharp(a) =$ and we say that the elements $a^\sharp$ are G-invariant (or just invariant).

Theorem B.2.2. *In the situation above:*

- *If $\chi \in \mathfrak{X}$ is a linear character (i.e. if $\deg(\chi) = 1$), then the χ-invariant elements are exactly those $h \in \mathcal{O}_m$ satisfying $gh = \chi(g)h$, for all $g \in G$. In particular, $\mathcal{O}_m^G$ is a ring, $\mathcal{O}_m$ and all $\mathcal{O}_m^\chi, \chi \in \mathfrak{X}$, are $\mathcal{O}_m^G$-modules and the maps ρ_χ are $\mathcal{O}_m^G$-module homomorphisms.*
- *The ring $\mathcal{O}_m$ admits the following $\mathcal{O}_m^G$-module decomposition*

$$\mathcal{O}_m = \bigoplus_{\chi \in \mathfrak{X}} \mathcal{O}_m^\chi.$$

- *The homomorphisms $\rho_\chi \colon \mathcal{O}_m \to \mathcal{O}_m^\chi$ are projections (that is, $h^\chi = 0$ for all $h \in \bigoplus_{\chi' \neq \chi} \mathcal{O}_m^{\chi'}$, and $h^\chi = h$, for all $h \in \mathcal{O}_m^\chi$). Every $h \in \mathcal{O}_m$ is of the form $h = \sum_{\chi \in \mathfrak{X}} h^\chi$.*

Theorem B.2.3. *There exist m algebraically independent invariant germs $\alpha_1, \dots, \alpha_m \in \mathcal{O}_m^G$ such that, if we let α be the map germ with coordinates α_i, then each isotypical component is a finitely generated free $\alpha^*\mathcal{O}_m$-module of the form*

$$\mathcal{O}_m^\chi = \bigoplus_{i=1}^{s_\chi} \beta_i \alpha^* \mathcal{O}_m,$$

for some $\beta_i \in \mathcal{O}_m^\chi$. Moreover, the germs α_i can be chosen homogeneous.

Corollary B.2.4. *$\mathcal{O}_m^G$ is a Cohen Macaulay ring of dimension m.*

Theorem B.2.5 (Shephard-Todd theorem [ST54])**.** *With the notations above, G is a reflection group if and only if its invariant ring is $\mathcal{O}_m^G = \alpha^*\mathcal{O}_m$.*

Definition B.2.6. Given an ideal I in $\mathcal{O}_m$, we define the following ideals of $\mathcal{O}_m^G$.

- $I^G = I \cap \mathcal{O}_m^G$.
- $I^\chi = \{a^\chi \mid a \in I\}$, for all $\chi \in \mathfrak{X}$. We write $I^\sharp = I_0^\chi$.

We say that I is G**-invariant** if $GI = I$.

In general, we have $I^G \subseteq I^\sharp$.

Proposition B.2.7. *Given an ideal I in $\mathcal{O}_m$, consider the following conditions If I is a G-invariant ideal of R, then:*

1. *I is G-invariant.*
2. *$I \cap R^\chi = I^\chi$, for all $\chi \in \mathfrak{X}$. In particular $I^G = I^\sharp$.*
3. *I can be generated over R^G by χ-invariant elements, $\chi \in \mathfrak{X}$.*

Then (1) implies (2), and (2) implies (3). If all the characters of G are linear, then (1), (2) and (3) are equivalent.

Proof. Assume that I is G-invariant and let $a \in I$. All the terms of the sum defining a^χ belong to I, and therefore we have $a^\chi \in I \cap R^\chi$. This shows (2).

Assume (2) and let $a \in I$. We have $a = \sum_{\chi \in \mathfrak{X}} a^\chi$, where by hypothesis each of the terms a^χ belongs to I. Therefore I is generated by the union of all its χ-invariant elements, $\chi \in \mathfrak{X}$.

Now assume that I is generated over R^G by χ-invariant elements. Then every element $a \in I$ is of the form $a = \sum_i h_i a_i$, with $h_i \in R^G$ and $a_i \in I^{\chi_i}$, $\chi_i \in \mathfrak{X}$. If all characters χ are linear, then we have $ga = \sum_i h_i \chi_i(g) a_i \in I$, for any $g \in G$. $\square$

The second item of the previous shows that, for any G-invariant ideal I, the ideal I^G of $\mathcal{O}_m^G$ is just $I^\sharp$, which we know how to compute: Let $\alpha_1, \dots, \alpha_r \in \mathcal{O}_m^G$ be some invariant germs and write

$$\alpha = (\alpha_1, \dots, \alpha_r)\colon \mathbb{C}^m \to \mathbb{C}^s.$$

Let $\beta_1, \dots, \beta_s \in \mathcal{O}_m$ satisfying

$$\mathcal{O}_m = \bigoplus_{i=1}^{s} \beta_i \alpha^* \mathcal{O}_r.$$

Observe that, by Theorem B.2.3, we can always find such germs α_i, β_l. However, we are not asking α_i to be algebraically independent or β to belong to any specific isotypical component.

Lemma B.2.8. *With the notations above, let* $I = \langle f_j \mid 1 \leq j \leq n \rangle$*, be an ideal of* $\mathcal{O}_m$*, then* $I^\sharp$ *is generated by the elements*

$$(\beta_i f_j)^\sharp,$$

with $0 \leq i \leq s$ *and* $1 \leq j \leq n$*.*

Proof. $I^\sharp$ is generated by the elements $f^\sharp$, $f \in I$. Any $f \in I$ is of the form

$$f = \sum_{j=1}^{n} a_j f_j = \sum_{j=1}^{n} (\sum_{i=0}^{s} \beta_i h_j^i) f_j,$$

for some elements $h_j^i \in \mathcal{O}_m^G$. Thus, we have

$$f^\sharp = \sum_{j=1}^{n} (\sum_{i=0}^{s} (\beta_i f_j)^\sharp h_j^i).$$

□

B.3 Invariants of the permutation group S_2

The permutation group S_k acts on $\mathbb{C}^{kn}$ by

$$(x^{(1)}, \ldots, x^{(k)}) \mapsto (x^{(\sigma(1))}, \ldots, x^{(\sigma(k))}),$$

for any $\sigma \in S_k$. The character table of the group S_2 is

S_2	1	(1,2)
χ_0	1	1
χ_1	1	-1

.

The characters of S_k can be obtained as restrictions of characters in any S'_k, $k' > k$. Indeed, for any $S_k, k \geq 2$, the characters in S_2 appear as restrictions of the trivial character χ_0 and the signature character χ_1, given by $\sigma \mapsto \operatorname{sign}(\sigma)$, where $\operatorname{sign}(\sigma)$ is the signature of $\sigma \in S_k$.

The S_k-invariant elements $h \in \mathcal{O}_{nk}^{S_k}$ are called **symmetric germs** and the χ_1-invariant elements $h \in \mathcal{O}_{nk}^{\chi_1}$ are called **antisymmetric germs**. For the particular case of S_2, a germ $h \in \mathcal{O}_{2n}$ is symmetric if and only if

$$h(x, x') = h(x', x),$$

and it is antisymmetric if and only if

$$h(x, x') = -h(x, x').$$

It is obvious that any germ $h \in \mathcal{O}_{2n}$ can be expressed as the sum of a symmetric and an antisymmetric germ. More explicitly:

$$h = h^\sharp + h^{\chi_1},$$

where $h^\sharp(x, x') = \frac{h(x,x')+h(x',x)}{2}$ and $h^{\chi_1}(x, x') = \frac{h(x,x')-h(x',x)}{2}$.

Symmetric functions

Definition B.3.1. We define the **symmetric functions** as follows:

$$s_i(x, x') = \frac{x_i + x'_i}{2}, \text{ for all } 1 \leq i \leq n.$$

$$r_{ij} = t_i t_j, \text{ for all } 1 \leq i \leq j \leq n, \text{ where } t_l(x, x') = \frac{x_l - x'_l}{2}.$$

We write

- $s = (s_1, \dots, s_n)$,
- $t = (t_1, \dots, t_n)$,
- $r = (r_{11}, r_{12}, r_{22}, \dots, r_{1n}, r_{2n}, \dots, r_{nn})$.

Let $l = n(n+1)/2$ and let $\psi\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^{n+l}$ be given by

$$(x, x') \mapsto (s(x, x'), r(x, x')),$$

that is, the coordinate functions of ψ are the symmetric functions. Finally, let $\alpha\colon \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be given by

$$(x, x') \mapsto (s_1(x, x'), \dots, s_n(x, x'), r_{11}(x, x'), \dots, r_{nn}(x, x')),$$

and define the functions

$$\beta_{i_1 \dots i_k} = t_{i_1} \dots t_{i_k}, \quad 1 \leq i_1 < \dots i_k \leq n.$$

We will need two lemmas:

Lemma B.3.2. *For every $h \in \mathcal{O}_{\mathbb{C}^2 \times \mathbb{C}^2, 0}$, with the notations above:*

1. *If h is symmetric, then $h = A(s, r)$, for some $A \in \mathcal{O}_{n+l}$.*
2. *If h is antisymmetric, then $h = \sum_{i=1}^n B_i(s, r) t_i$, for some $B_i \in \mathcal{O}_{n+l}$.*

Proof. The functions $1, t_i, 1 \leq i \leq n$ form a base of the $\mathbb{C}$-vector space $\mathcal{O}_{\mathbb{C}^n \times \mathbb{C}^n, 0}/\psi^* \mathfrak{m}_{n+l}$ and thus, by the Malgrange preparation theorem, $\mathcal{O}_{\mathbb{C}^2 \times \mathbb{C}^2, 0}$ is an $\mathcal{O}_{n+l}$-module (via ψ^*) generated by $1, t_i, 1 \leq i \leq n$. It is, for every h, there exist functions $A, B_i \in \mathcal{O}_{n+l}$, such that $h = A(s, r) + \sum_{i=1}^n B_i(s, r) t_i$. If h is symmetric, then $h = (h + \tau \cdot h)/2 = A(s, t)$, and if h is antisymmetric, then $h = (h - \tau \cdot h)/2 = \sum_{i=1}^n B_i(s, t) t_i$. □

Lemma B.3.3. *The kernel of the morphism $\psi^*\colon \mathcal{O}_{n+l} \to \mathcal{O}_{2n}$ is generated by*

$$r_{i_1, i_2} r_{i_3, i_4} - r_{i_{\sigma(1)}, i_{\sigma(2)}} r_{i_{\sigma(3)}, i_{\sigma(4)}},$$

such that $1 \leq i_1 \leq i_2 \leq n, 1 \leq i_3 \leq i_4 \leq n, i_{\sigma(1)} \leq i_{\sigma(2)}, i_{\sigma(3)} \leq i_{\sigma(4)}$.

Proof. Consider the germ of linear difeomorphism $\phi\colon (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^n \times \mathbb{C}^n, 0)$ defined by $\phi(s, t) = (s_1 + t_1, \ldots, s_n + t_n, s_1 - t_1, \ldots, s_n - t_n)$. Obviously $\operatorname{Ker} \psi^* = \operatorname{Ker}(\phi \circ \psi)^*$ and, thus, $h \in \mathcal{O}_{n+l,0}$ belongs to $\operatorname{Ker} \psi^*$ if and only if every term in the series expansion of $(\phi \circ \psi)^* h$ has coefficient 0. We consider $(\phi \circ \psi)^*$ as a morphism between $(\mathbb{C}\{s\})\{r\}$ and $(\mathbb{C}\{s\})\{t\}$. For every $h = \sum_{\beta \in \mathbb{N}^l} a_\beta(s) r_{i,j}^{\beta_{i,j}}$, the function $(\phi \circ \psi)^* h$ is obtained using the identities $r_{i,j} = t_i t_j$, it is, $\sum_{1 \leq i \leq j} a_\beta(s)(t_i t_j)^{\beta_{i,j}}$. Take the decomposition by exponents of both polynomial spaces and notice that $(\phi \circ \psi)^*$ converts terms into terms. Moreover, for any even exponent $e = (e_1, \ldots, e_n)$ for the t variable, there is a set of exponents, $S(e) = \{(\beta_{1,1}, \ldots, \beta_{n,n}) \in \mathbb{N}^l \mid \sum_{j=1}^n \beta_{i,j} + \sum_{l=1}^i \beta_{l,i} = e_i, 1 \leq i \leq n\}$, such that the terms with exponents in $S(e)$ are the ones which $(\phi \circ \psi)^*$ sends to the term of exponent e. We can decompose any $h \in \mathbb{C}_{n+l,0}$ as $h = \sum_{e \in \mathbb{N}^n} h_{S(e)}$, where $h_{S(e)}$ is a polynomial whose terms only have exponents in $S(e)$. it is obvious that $h \in \operatorname{Ker} \psi^*$ if and only if $h_{S(e)} \in \operatorname{Ker} \psi^*$ for all $e \in \mathbb{N}$. Then, it suffices to show that the subspace $(\operatorname{Ker} \psi^*)_{S(e)} = \sum_{g \in S(e)} (\operatorname{Ker} \psi^*)_g$, given by the polynomials of $\operatorname{Ker} \psi^*$ with terms of exponents in $S(e)$, is generated by the elements $r_{i_1,i_2} r_{i_3,i_4} - r_{i_{\sigma(1)},i_{\sigma(2)}} r_{i_{\sigma(3)},i_{\sigma(4)}}$ of the statement. Let K be the ideal generated by the relations above, and h be a polynomial in $(\operatorname{Ker} \psi^*)_{S(e)}$, we want to show $h + K = 0 + K$. We proceed by induction on the number of terms, k, of h. The case $k = 0$ is trivial, and the case $k = 1$ never happens, since any non zero term in $\mathcal{O}_{n+l}$ leads to a non zero term in $\mathcal{O}_{2n}$, thus h can't belong to $(\operatorname{Ker} \psi^*)$ if $k = 1$. Assume $k \geq 2$ and take $q_1 \neq 0 \neq q_2$ two terms appearing in h such that $h = h' + q_1 + q_2$ where h' has $k - 2$ terms. We only have to show that exists a term q such that $q_1 + q_2 + K = q + K$ and the induction hypothesis will do the rest. To prove the existence of q, we proceed again by induction, this time on the degree d, of q_1/g, being g the greatest common divisor of q_1, and q_2. If $d = 0$, then q_1, q_2 are just the product of the monomial g by constants c_1, c_2 respectively, thus we can take the term $q = (c_1 + c_2) g = q_1 + q_2$. Again, the case $k = 1$ doesn't has to be considered, since $q_1 = g r_{i_1,j_1}, q_2 = g r_{i_2,j_2}$ implies $i_1 = i_2$ and $j_1 = j_2$. Now we assume $d \geq 2$ and let $q_1 = g a r_{i_1,j_1}$ for some i_1, j_1. Obviously r_{i_1,j_1} does not divide q_2, but as the exponents of q_1 and q_2 are in $D(d)$, then the number of times that the indices i_1, j_1 (counting the exponents as repetitions) appear in q_1 and q_2 must be the same and, thus, exists a monomial of type $r_{i_1,i_2} r_{j_1,j_2}, r_{i_2,i_1} r_{j_1,j_2}, r_{i_1,i_2} r_{j_2,j_1}$ or $r_{i_2,i_1} r_{j_2,j_1}$ which divides q_2/g. Assume, for simplicity, that the monomial is $r_{i_1,i_2} r_{j_1,j_2}$ with $i_2 \leq j_2$, then $r_{i_1,i_2} r_{j_1,j_2} - r_{i_1,j_1} r_{i_2,j_2} \in K$ and thus, if we define $\tilde{q}_2 = q_2 r_{i_1,j_i} r_{i_2,j_2} / (r_{i_1,i_2} r_{j_1,j_2})$, we have $q_1 + q_2 + K = q_1 + \tilde{q}_2 + K$ and, since the greatest common divisor of q_1 and $\tilde{q}_2$ is strictly greater than d, by induction, exists a term q such that, $q_1 + q_2 + K = q_1 + \tilde{q}_2 + K = q + K$. $\square$

Immediately from Lemmas B.3.2 and B.3.3, we obtain the explicit form of the statement in Theorem B.2.3 for the action of S_2 in $\mathbb{C}^{2n}$.

Example B.3.4. These are the two simpler cases of the action of S_2 in $\mathbb{C}^{2n}$
(1) If $n = 1$, the map $\psi\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is given by

$$(x, x') \mapsto (\frac{x + x'}{2}, \frac{(x - x')^2}{4}).$$

The pullback $\psi^*\colon \mathcal{O}_2 \to \mathcal{O}_2^{S_2}$ is an isomorphism.
(2) If $n = 2$, we have

- $s_1 = \dfrac{x + x'}{2}$, $s_2 = \dfrac{y + y'}{2}$,
- $r_{11} = \dfrac{(x - x')^2}{4}$, $r_{12} = \dfrac{(x - x')(y - y')}{4}$ and $r_{22} = \dfrac{(y - y')^2}{4}$.

Then, for every $h \in \mathcal{O}_{\mathbb{C}^2 \times \mathbb{C}^2, 0}$:

1. If h is symmetric, then $h = A_1(s_1, s_2, r_{11}, r_{22}) + r_{12}A_2(s_1, s_2, r_{11}, r_{22})$, for some $A_1, A_2 \in \mathcal{O}_4$.
2. If h is antisymmetric, then $h = (x - x')B((s_1, s_2, r_{11}, r_{22})) + (y - y')C((s_1, s_2, r_{11}, r_{22}))$, for some $B, C \in \mathcal{O}_4$.

A simple way to find this expressions is the following: rewrite $h(x, y, x', y')$ as $h(s_1 + t_1, s_2 + t_2, s_1 - t_1, s_2 - t_2)$ and in this expression make the following changes $t_1^{2k} = r_{11}^k, t_2^{2k} = r_{22}^k, t_2 t_4 = r_{12}$. Now we have an expression of the form $A_1(s_1, s_2, r_{11}, r_{22}) + r_{12}A_2(s_1, s_2, r_{11}, r_{22}) + (x - x')B((s_1, s_2, r_{11}, r_{22})) + (y - y')C((s_1, s_2, r_{11}, r_{22}))$.

The general case looks as follows:

Example B.3.5. With the notations above:

- The coordinate functions $s_1, \dots, s_n, r_{11}, \dots, r_{nn}$ of α are algebraically independent.
- The algebra of invariant functions is a free $\alpha^*\mathcal{O}_{2n}$-module of the form:
$$\mathcal{O}_{2n}^{S_2} = \alpha^*\mathcal{O}_{2n} \oplus \bigoplus_{k \text{ even}} \beta_{i_1 \dots i_k} \alpha^* \mathcal{O}_m.$$
- The antisymmetric functions form a free $\alpha^*\mathcal{O}_{2n}$-module of the form:
$$\mathcal{O}_{2n}^{\chi_1} = \bigoplus_{k \text{ odd}} \beta_{i_1 \dots i_k} \alpha^* \mathcal{O}_m.$$

The previous result reflects, via Sephard-Todd's Theorem B.2.5, that the action of S_2 on $\mathbb{C}^{2n}$ is only generated by reflections in the case $n = 1$, where there are no $\beta_{i_1, \dots, i_k}$ with k even, and thus the invariant algebra is precisely $\mathcal{O}_2^{S_2} = \alpha^*\mathcal{O}_2$.

Example B.3.6. From B.2.7 we obtain that an ideal I in $\mathcal{O}_{2n}$ is S_2-invariant if and only if it can be generated by symmetric and antisymmetric germs.

Computing quotients by S_2

Example B.3.7. Now we show how $\mathbb{C}^2 \times \mathbb{C}^2/S_2$ embeds as a complex subspace of $\mathbb{C}^5$: Let $\psi\colon \mathbb{C}^4 \to \mathbb{C}^5$ be given by

$$(x, y, x', y') \mapsto (x + x', y + y', (x - x')^2, (x - x')(y - y'), (y - y')^2).$$

Take its pull-back

$$\psi^*\colon \mathcal{O}_{\mathbb{C}^5} \to \mathcal{O}_{\mathbb{C}^2\times\mathbb{C}^2}.$$

By Lemma B.3.2, we have that $\operatorname{Im}\psi^* = \mathcal{O}^{S_2}_{\mathbb{C}^2\times\mathbb{C}^2}$. Therefore $\mathbb{C}^2 \times \mathbb{C}^2/S_2$ is isomorphic to the complex space $\operatorname{Im}\psi$ with the analytic structure defined by the ideal sheaf $\operatorname{Ker}\psi^* = \langle r_{11}r_{22} - r_{12}^2\rangle$. Eventually, the space $\mathbb{C}^2 \times \mathbb{C}^2/S_2$ is just the cone in $\mathbb{C}^5$ defined by the equation $r_{11}r_{22} = r_{12}^2$.

Let $X = V(I)$ be a germ of symmetric complex subspace of $\mathbb{C}^n \times \mathbb{C}^n$, then X/S_2 is isomorphic to the complex space $\psi(X)$ whose defining ideal sheaf is $(\psi^*)^{-1}(I^{S_2})$. As a consequence, we have:

Proposition B.3.8. *Let $X = V(I) \subseteq (\mathbb{C}^n, 0)$ be a germ of symmetric complex space, with*

$$I = \langle A_i(s, r), \sum_{j=1}^{n} B_j^k(s, r)t_j \mid 1 \le i \le m, 1 \le k \le m'\rangle,$$

as in Lemma B.3.6. Then X/S_2 is isomorphic to $V(J) \subset (\mathbb{C}^{n+l}, 0)$, where the generators of J in the variables s, r of $\mathbb{C}^{n+l}$ are:

- *The m function germs $A_i(s, r), \quad i = 1, \ldots, m$.*
- *The nm' function germs*

$$r_{1,1}B_1^k(s, r) + r_{1,2}B_2^k(s, r) + \cdots + r_{1,n}B_n^k(s, r),$$

$$r_{1,2}B_1^k(s, r) + r_{2,2}B_2^k(s, r) + \cdots + r_{2,n}B_n^k(s, r),$$

$$\vdots$$

$$r_{1,n}B_1^k(s, r) + r_{2,n}B_2^k(s, r) + \cdots + r_{n,n}B_n^k(s, r),$$

  with $i = 1, \ldots, m'$.

- *The generators of* $\ker\psi^*$

$$r_{i_1,i_2}r_{i_3,i_4} - r_{i_{\sigma(1)},i_{\sigma(2)}}r_{i_{\sigma(3)},i_{\sigma(4)}},$$

  with $1 \le i_1 \le i_2 \le n, 1 \le i_3 \le i_4 \le n, i_{\sigma(1)} \le i_{\sigma(2)}, i_{\sigma(3)} \le i_{\sigma(4)}$.

Proof. Follows directly from Proposition B.2.7 and Lemma B.2.8. □

Example B.3.9. For $n = 2$ we have: Let I be an S_2-invariant ideal in $\mathcal{O}_4$, generated by some symmetric functions $A_i(s_1, s_2, r_{11}, r_{12}, r_{22})$ and some antisymmetric functions $B_j^1(s_1, s_2, r_{11}, r_{22})(x-x')+B_j^2(s_1, s_2, r_{11}, r_{22})(y-y')$. Then, I^{S_2} is generated by:

$$A_i(s_1, s_2, r_{11}, r_{12}, r_{22}),$$

$$r_{11}B_j^1(s_1, s_2, r_{11}, r_{22}) + r_{12}B_j^2(s_1, s_2, r_{11}, r_{22}),$$

$$r_{12}B_j^1(s_1, s_2, r_{11}, r_{22}) + r_{22}B_j^2(s_1, s_2, r_{11}, r_{22}),$$

and

$$r_{11}r_{22} - r_{12}^2.$$

Expressions in the symmetric functions

Here, we show how to obtain the expressions in the symmetric functions of Lemma B.3.2. Moreover, we want to fix a canonical choice among all the possibilities for A and B_i.

We use the following multi-index notation: For any $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{N}^m$, we write $w^\lambda := \prod_{i=1}^m w_i^{\lambda_i}$, $\lambda! = \prod_{i=1}^m(\lambda_i!)$, and $|\lambda| := \sum_{i=1}^m \lambda_i$. We also say that λ is even or odd if $|\lambda|$ is so. For any fixed $n \in \mathbb{N}$, we let Λ_0 be the set of even multi-indices and, for all $1 \leq i \leq n$, Λ_i the set of odd multi-indices $\lambda \in \mathbb{N}^n$ with λ_i odd and λ_j even for all $n \geq j > i$. Obviously, the set of multi-indices in $\mathbb{N}^n$ is the disjoint union of $\Lambda_i, 0 \leq i \leq n$. We also denote by u_i the multi-index $(0, \ldots, 0, 1, 0 \ldots, 0)$ with 1 in the i-th position.

We can compute the decomposition of a polynomial in symmetric and antisymmetric polynomials, expressed in the symmetric functions as follows: Let h be a polynomial in the variables x, x'. We can use the identities $x = s + t$, $x' = s - t$ and, for any term of h, say $as^\lambda t^\mu$, we can rewrite the monomial t^μ in the following way: Every time t_i^2 divides t^μ we change the factor t_i^2 for $r_{i,i}$ and, recursively, we get to an expression with a factor in s and r and a remaining factor $t_{i_1} \ldots t_{i_k}$, with $i_1 < \cdots < i_k$. Now we rewrite $t_{i_1}t_{i_2} = r_{i_1 i_2}$, $t_{i_3}t_{i_4} = r_{i_3 i_4}$ and so on, until we are left with, at most, the factor t_{i_k}. If k is even, then we have $\mu \in \Lambda_0$, and we get a term of the form $A(s, r)$. If k is odd, then we have $\mu \in \Lambda_{i_k}$, and the term is of the form $B_{i_k}(s, r)t_{i_k}$.

Thus, A and B_i can be assumed to be polynomials which monomials are of type

$$r_{1,1}^{a_1} \cdots r_{n,n}^{a_n} r_{i_1,i_2} \cdots r_{i_{k-1},i_k},$$

with $i_1 < \cdots < i_k$ and $i_k < i$ in B_i. Such a representation of A and B_i is unique. In particular, we can define an mapping E which sends every even λ to the exponent $E(\lambda)$, satisfying $t^\alpha = r^{E(\lambda)}$ and such that $r^{E(\lambda)}$ is of the form we choosen above.

The following proposition gives us a closed formula that we can use to obtain a matrix α, satisfying $f(x) - f(x') = \alpha(x, x')(x - x')$ (see Proposition-Definition 3.1.1), expressed in the symmetric functions.

Proposition B.3.10. *With the notations above, for all $h \in \mathcal{O}_n$, we have:*

$$h(x) - h(x') = \sum_{i=1}^{n} \Big(\sum_{\lambda \in \Lambda_i} \frac{2}{\lambda!} \frac{\partial^\lambda h(s)}{\partial x^\lambda} r^{E(\lambda - u_i)} \Big) t_i.$$

Proof. From the multivariate Taylor series

$$h(w) = \sum_{\lambda \in \mathbb{N}^n} \frac{1}{\lambda!} \frac{\partial^\lambda h(a)}{\partial x^\lambda} (w - a)^\lambda,$$

taking $a = s$, since $x - s = t$ and $x' - s = -t$, we obtain:

$$h(x) - h(x') = \sum_{\lambda \in \mathbb{N}^n} \frac{1}{\lambda!} \frac{\partial^\lambda h(s)}{\partial x^\lambda} (t^\lambda - (-t)^\lambda) = \sum_{\lambda \text{ odd}} \frac{1}{\lambda!} \frac{\partial^\lambda h(s)}{\partial x^\lambda} 2t^\lambda =$$
$$\sum_{i=1}^{n} \Big(\sum_{\lambda \in \Lambda_i} \frac{2}{\lambda!} \frac{\partial^\lambda h(s)}{\partial x^\lambda} t^{(\lambda - u_i)} \Big) t_i.$$

Since $\lambda - u_i$ is even for any $\lambda \in \Lambda_i$, we rewrite $t^{(\lambda - u_i)}$ as $r^{E(\lambda - u_i)}$. $\square$

Example B.3.11. For $n = 2$ the expressions in the symmetric functions of a matrix α, satisfying $f(x) - f(x') = \alpha(x, x')(x - x')$, are given by

$$\alpha_{j1} = \sum \frac{1}{(2i+1)!(2j)!} \frac{\partial f(s_1, s_3)}{\partial x_1^{2i+1} x_2^{2j}} s_2^i s_4^j$$

and

$$\alpha_{j2} = \sum \frac{1}{(2i)!(2j+1)!} \frac{\partial f(s_1, s_3)}{\partial x_1^{2i} x_2^{2j+1}} s_2^i s_4^j.$$

Bibliography

[Alt11] A. Altintas, **Multiple point spaces and finitely determined map-germs**, Ph.D. thesis, University of Warwick, UK, 2011.

[BA01] C. Bivià-Ausina, **Singularidades de Thom Boardman en deformaciones genéricas de aplicaciones y métodos para el cálculo de clausuras integrales de ideales**, Ph.D. thesis, Universitat de València, 2001.

[BG80] R. Buchweitz and G. M. Greuel, **The Milnor number and deformations of complex curve singularities**, Invent. Math. **48** (1980), no. 3, 241–282.

[dCS81] C. de Concini and E. Strickland, **On the variety of complexes**, Advances in Mathematics **41** (1981), 1981.

[DG83] J. Damon and A. Galligo, **On the Hilbert-Samuel partition of stable map-germs**, Bulletin de la S. M. F. **111** (1983), 327–358.

[dJP00] T. de Jong and G. Pfister, **Local analytic geometry, basic theory and applications**, Springer, 2000.

[Fis76] G. Fischer, **Complex analytic geometry**, Springer-Verlag, 1976.

[Gaf83] T. Gaffney, **Multiple points and associated ramification loci**, Proceedings of Symposia in Pure Mathematics (American Mathematical Society, Providence) **40** (1983), no. 1, 429–437.

[GG86] M. Golubitsky and V. Guillemin, **Stable mappings and their singularities**, Springer, 1986.

[Gib79] C. G. Gibson, **Singular points of smooth mappings**, Pitman, 1979.

[GP02] G. M. Greuel and G. Pfister, **A singular introduction to commutative algebra**, Cambridge Studies in Advanced Mathematics, 2002.

[GR84] H. Grauert and R. Remmert, **Coherent analytic sheaves**, Springer, 1984.

[GWdPL76] C. G. Gibson, K. Wirthmüller, A. A. du Plessis, and E. Looijenga, **Topological stability of smooth mappings**, Springer-Verlag, 1976.

[HMPS15] M. E. Hernandes, A. J. Miranda, and G. Peñafort-Sanchis, **A presentation matrix algorithm for $f_*\mathcal{O}_{X,x}$**, Preprint, 2015.

[Hou94] K. Houston, **Local topology of finite complex analytic maps**, Ph.D. thesis, University of Warwick, 1994.

[Hum90] J. Humphreys, **Reflection groups and coxeter groups**, vol. 29, Cambridge studies in advanced mathematics, 1990.

[Kle81] S. L. Kleiman, **Multiple-point formulas I: Iteration**, Acta mathematica (1981), no. 147, 13–49.

[KLU92] S. L. Kleiman, J. Lipman, and B. Ulrich, **The source double-point cycle of a finite map of codimension one.**, London Math. Soc. Lecture Note Ser. **179** (1992), 199–212.

[KLU96] ———, **The multiple-point schemes of a finite curvilinear map of codimension one**, Ark. Mat. **34** (1996), no. 2, 285–326.

[Lak77] D. Laksov, **Residual intersections and Todds formula for the double locus of a morphism**, Acta mathematica **140** (1977), no. 1-2, 75–92.

[Mat69] J Mather, **Stability of c^∞ mappings iv: Classification of stable germs by $\mathbb{R}$-algebras**, Publications Mathématiques de l'I.H.É.S. (1969), no. 37, 223–248.

[Mat80] H. Matsumura, **Commutative algebra**, Benjamin-Cummings, 1980.

[Mat89] ———, **Commutative ring theory**, Springer-Verlag, 1989.

[Mir] A. J. Miranda, `https://sites.google.com/site/aldicio/publicacoes`.

[MM89] W. L. Marar and D. Mond, **Multiple point schemes for corank 1 maps**, J. London Math. Soc. **2** (1989), no. 3, 553–567.

[MNB08] W. L. Marar and J. J. Nuño-Ballesteros, **A note on finite determinacy for corank 2 map germs from surfaces to 3-space**, Math. Proc. Cambridge Philos. Soc. **145** (2008), no. 1, 153–163.

[MNBPS12] W. L. Marar, J. J. Nuño-Ballesteros, and G. Peñafort-Sanchis, **Double point curves for corank 2 map germs from $\mathbb{C}^2$ to $\mathbb{C}^3$**, Topology Appl. **159** (2012), no. 2, 526–536.

[Mon85] D. Mond, **On the clasification of germs of maps from $\mathbb{R}^2$ to $\mathbb{R}^3$**, Proc. London Math. Soc. **50** (1985), no. 2, 333–369.

[Mon86] J. Montaldi, **On contact between submanifolds**, Michigan Math. J. **33** (1986), no. 2, 195–199.

[Mon87] D. Mond, **Some remarks on the geometry and classification of germs of maps from surfaces to 3-space**, Topology **26** (1987), no. 3, 361–383.

[MP89] D. Mond and R. Pellikaan, **Fitting ideals and multiple points of analytic mappings**, Algebraic geometry and complex analysis (Pátzcuaro, 1987) Lecture Notes in Mathematics, 1414 (1989), 107–161.

[NBJP09] J.J. Nuño-Ballesteros and V. H. Jorge-Pérez, **Finite determinacy and whitney equisingularity of map germs from cn to c(2n-1)**, Manuscripta mathematica (2009), no. 128, 389–410.

[PS14] G. Peñafort-Sanchis, **The geometry of double fold maps**, Journal of Singularities **10** (2014), 250–263.

[Ron72] F. Ronga, **La classe duale aux points doubles d'une application**, Compositio Mathematica **27** (1972), no. 2, 223–232.

[ST54] G. Shephard and J. A. Todd, **Finite unitary reflection groups**, Canadian J. Math **6** (1954), 274–304.

[Sta79] R. P. Stanley, **Invariants of finite groups and their applications to combinatorics**, Bull. Amer. Math. Soc. **1** (1979), no. 3, 475–511.

[WA00] R. Wik Atique, **On the classification of multi-germs of maps from $\mathbb{C}^2$ to $\mathbb{C}^3$ under $\mathcal{A}$-equivalence**, Real and complex singularities. Chapman and Hall/CRC. Research Notes in Mathematics Series, (2000).

[Wal81] C. T. C. Wall, **Finite determinacy of smooth map germs**, Bull. London Math. Soc. (1981), no. 13, 481–539.

[WGPS15] D. Wolfram, G. M. Greuel, G. Pfister, and H. Schönemann, Singular **4-0-2 — A computer algebra system for polynomial computations**, `http://www.singular.uni-kl.de`, 2015.

[Whi44] H. Whitney, **The singularities of a smooth n-manifold in $(2n-1)$-space**, Annals of Mathematics **45** (1944), no. 2, 247–293.