

# Transitivity of Sylow permutability, the converse of Lagrange's theorem, and mutually permutable products

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*Dedicated to Professor Leonid Aleksandrovich Shemetkov on the occasion of his seventieth birthday*

## Abstract

This paper is devoted to the study of mutually permutable products of finite groups. A factorised group  $G = AB$  is said to be a mutually permutable product of its factors  $A$  and  $B$  when each factor permutes with every subgroup of the other factor. We prove that mutually permutable products of  $\mathcal{Y}$ -groups (groups satisfying a converse of Lagrange's theorem) and SC-groups (groups whose chief factors are simple) are SC-groups, by means of a local version. Next we show that the product of pairwise mutually permutable  $\mathcal{Y}$ -groups is supersoluble. Finally, we give a local version of the result stating that when a mutually permutable product of two groups is a PST-group (that is, a group in which every subnormal subgroup permutes with all Sylow subgroups), then both factors are PST-groups.

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In this paper we will deal only with finite groups.

Many group theorists have been worried about what can be said of a group  $G = G_1G_2 \cdots G_m$  which is a product of some pairwise permutable subgroups if some properties of the factors are known. For instance, a well-known theorem of Kegel and Wielandt [26, 32] says that a product of two nilpotent groups is soluble. The fact that a product of two supersoluble groups is not necessarily supersoluble, even if both factors are normal in the group, motivates the restriction of this question to factorised groups in which both factors are connected by certain stronger permutability properties. The first author and Shaalan introduced in [6] the notion of *mutually permutable product*  $G = AB$  of two subgroups  $A$  and  $B$ : in a mutually permutable product, each factor permutes with every subgroup of the other factor. In particular, this situation holds when both factors are normal in the group. Some results about normal products of supersoluble groups were extended to mutually permutable products in [6], for instance, a mutually permutable product  $G = AB$  of two supersoluble groups  $A$  and  $B$  is supersoluble whenever  $G'$  is nilpotent or one of the factors is nilpotent. They also showed that totally permutable products (that is, every subgroup of each factor permutes with every subgroup of the other factor) of supersoluble groups are supersoluble. Of course, central products and direct products are instances of totally

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permutable products. Mutually and totally permutable products have been considered as well in [2, 12, 17, 18, 21, 22].

On the other hand, Kegel [27] proved that all subgroups of a group  $G$  which permute with all the Sylow subgroups of  $G$  are subnormal. We call these subgroups  $S$ -permutable. This motivates the definition of the class of PST-*groups* or groups in which every subnormal subgroup is  $S$ -permutable. Agrawal [1] obtained a characterisation of soluble PST-groups as the groups  $G$  in which the nilpotent residual  $L$  is an abelian normal Hall subgroup of  $G$  and all elements of  $G$  induce power automorphisms in  $L$ . Some interesting subclasses of the class of all PST-groups are the class of all PT-groups (groups in which permutability is a transitive relation, or in which every subnormal subgroup is permutable) and the class of all T-groups (groups in which normality is a transitive relation). These classes of groups have been studied by several authors (for instance, [3, 5, 8, 9, 11, 13, 14, 15, 19, 20, 21, 22, 23, 25, 29, 33]).

As a consequence of the theorem of Agrawal [1], soluble PST-groups are supersoluble. Robinson [29] showed that, in the general finite universe, PST-groups have all their chief factors simple, or, as he says, they are SC-*groups*. The classification of finite simple groups and the truth of the Schreier conjecture yields the following description of SC-groups:

**Theorem 1** ([29, Proposition 2.4]). *A group  $G$  is an SC-group if and only if there is a perfect normal subgroup  $D$  such that  $G/D$  is supersoluble,  $D/Z(D)$  is a direct product of  $G$ -invariant simple groups, and  $Z(D)$  is supersolubly embedded in  $G$  (i.e., there is a  $G$ -admissible series of  $Z(D)$  with cyclic factors).*

The relation between totally and mutually permutable products and SC-groups has been investigated in [10, 12, 17, 21, 22]. For instance:

**Theorem 2** ([12, Theorems 2 and 3]). *Assume that  $G$  is the mutually permutable product of its subgroups  $A$  and  $B$ . Then:*

1. *If  $G$  is an SC-group, then  $A$  and  $B$  are SC-groups.*
2. *If  $A$  and  $B$  are SC-groups, then  $G/\text{Core}_G(A \cap B)$  is an SC-group.*

Now let us pay attention to the class  $\mathcal{Y}$  of all groups  $G$  in which for every subgroup  $H$  and all primes  $q$  dividing the index  $|G : H|$  there exists a subgroup  $K$  of  $G$  such that  $H$  is contained in  $K$  and  $|K : H| = q$ . This condition is equivalent to say that for every subgroup  $H$  of  $G$  there exist intermediate subgroups of all possible orders. Hence  $\mathcal{Y}$  consists of groups satisfying the converse of Lagrange's theorem, the so-called CLT-groups. The class  $\mathcal{Y}$  has been studied in Chapter 1 and Section 6.1 of [31], and, more recently, in [7]. These groups can be characterised as follows:

**Theorem 3.** *A group  $G$  is a  $\mathcal{Y}$ -group if, and only if, the nilpotent residual  $L$  of  $G$  is a nilpotent Hall subgroup of  $G$  and for all subgroups  $H$  of  $L$ ,  $G = L N_G(H)$ .*

In [7], it is proved that the class of soluble PST-groups coincides with the class of  $\mathcal{Y}$ -groups with abelian nilpotent residual.

The theory of finite groups has benefited from the local techniques. Given a group theoretical property  $\mathcal{A}$ , we are interested in finding another weaker property  $\mathcal{A}_p$ , depending on a prime  $p$ , such that a group satisfies  $\mathcal{A}$  if and only if it satisfies  $\mathcal{A}_p$  for all primes  $p$ . For instance,  $p$ -solubility ( $p$  a prime) becomes a good "localisation" of solubility. The local method is behind the notions of local formation and composition or Baer-local formation and other generalisations of these concepts (see [16, 24, 30] for more details). Local techniques turn out very useful in the study of PST-groups and other related classes. For example, in [23] and [28] the authors have presented some interesting local characterisations of soluble T-groups. A local characterisation of soluble PT-groups appears in [19]. In [3, 13, 14], local characterisations of soluble PST-groups are studied. Let us recall some of these properties:

**Definition 4.** Let  $p$  be a prime number.

1. A  $p$ -soluble group  $G$  satisfies  $\text{PST}_p$  when every  $p'$ -perfect subnormal subgroup of  $G$  permutes with every Hall  $p'$ -subgroup of  $G$  (see [3]).
2. A group  $G$  satisfies  $\mathcal{U}_p^*$  when  $G$  is  $p$ -supersoluble and all  $p$ -chief factors of  $G$  are  $G$ -isomorphic when regarded as  $G$ -modules (see [3]).
3. A (not necessarily  $p$ -soluble) group  $G$  satisfies  $\mathcal{Y}_p$  when for every pair of  $p$ -subgroups  $H$  and  $K$  such that  $H \leq K$ ,  $H$  is  $S$ -permutable in  $N_G(K)$  ([14]).

It is shown in [3] and [14] that for  $p$ -soluble groups, all three properties are equivalent and so soluble  $\text{PST}$ -groups are exactly the groups satisfying  $\text{PST}_p$  for all primes  $p$ . Other local properties for  $\text{PST}$ -groups in the general finite universe appear in [9]:

**Definition 5.** Let  $p$  denote a prime number. A group  $G$  is said to satisfy  $N_p$  when every non-abelian chief factor of  $G$  of order divisible by  $p$  is simple and for each normal subgroup  $N$  of  $G$ ,  $p'$ -elements of  $G$  induce power automorphisms in  $O_p(G/N)$ .

The paper [9] characterises  $\text{PST}$ -groups as the groups satisfying  $N_p$  for all primes  $p$ . If we fix a prime  $p$ , it is rather clear that a  $p$ -soluble group  $G$  satisfying property  $N_p$  has all  $p$ -chief factors  $G$ -isomorphic when regarded as  $G$ -modules by conjugation. Hence  $G$  is a  $\text{PST}_p$ -group. Conversely, Assume that  $G$  is a  $p$ -soluble  $\text{PST}_p$ -group. Consider a normal subgroup  $N$  of  $G$  and take a subgroup  $L/N$  of  $O_p(G/N)$ . By [3, Lemma 2],  $G/N$  is a  $\text{PST}_p$ -group as well. Then  $L/N$  is a subnormal  $p'$ -perfect subgroup of  $G/N$ , and so  $L/N$  permutes with all Hall  $p'$ -subgroups of  $G/N$ . Let  $H$  be a Hall  $p'$ -subgroup of  $G$ . Then  $HN/N$  is a Hall  $p'$ -subgroup of  $G/N$  and  $L/N$  is a subnormal Sylow  $p$ -subgroup of  $(L/N)(HN/N)$ . In particular,  $L/N$  is normalised by  $HN/N$ . This implies that all elements of  $H$  normalise  $L$ . It follows that  $G$  is an  $N_p$ -group.

Therefore we have:

**Lemma 6.** *Let  $p$  be a prime number. If a group  $G$  is  $p$ -soluble, then  $G$  satisfies  $N_p$  if and only if  $G$  satisfies  $\text{PST}_p$ .*

The local method has also been successfully applied to the study of  $\mathcal{Y}$ -groups in [7] with the definition of the property  $\mathcal{Z}_p$  ( $p$  a prime):

**Definition 7.** We say that  $G$  satisfies  $\mathcal{Z}_p$  when for every  $p$ -subgroup  $X$  of  $G$  and for every power  $q^m$  of a prime  $q$  dividing  $|G : X O_{p'}(G)|$ , there exists a subgroup  $K$  of  $G$  containing  $X O_{p'}(G)$  such that  $|K : X O_{p'}(G)| = q^m$ .

In [7, Theorem 13], it is proved that property  $\mathcal{Z}_p$  is equivalent to the following one:

**Theorem 8.** *Let  $G$  be a group and let  $p$  be a prime. If  $G$  is  $p$ -soluble, then  $G$  satisfies  $\mathcal{Z}_p$  if and only if  $G$  satisfies either of the following conditions:*

1.  $G$  is  $p$ -nilpotent, or
2.  $G(p)/O_{p'}(G(p))$  is a Sylow  $p$ -subgroup of  $G/O_{p'}(G(p))$  and for every  $p$ -subgroup  $H$  of  $G(p)$ , we have that  $G = N_G(H)G(p)$ .

Here  $X(p)$  denotes the  $p$ -nilpotent residual of a group  $X$ , that is, the smallest normal subgroup  $N$  of  $X$  such that  $X/N$  is  $p$ -nilpotent.

**Theorem 9** ([7, Theorem 15]). *A soluble group satisfies  $\mathcal{Y}$  if and only if it satisfies  $\mathcal{Z}_p$  for all primes  $p$ .*

In [4] we prove some results on mutually permutable products whose factors belong to some of the above classes. We start with a localisation of SC-groups.

**Definition 10.** Let  $p$  be a prime number. A group  $G$  is said to be an  $SC_p$ -group whenever every chief factor of  $G$  whose order is divisible by  $p$  is simple.

It is clear that  $G$  is an SC-group (i.e., all its chief factors are simple) if and only if  $G$  is an  $SC_p$ -group for all primes  $p$ . In what follows,  $p$  will denote a fixed prime number. The proofs of Theorem 2 can be adapted to prove:

**Lemma 11.** Assume that  $G$  is a mutually permutable product of its subgroups  $A$  and  $B$ .

1. If  $G$  is an  $SC_p$ -group, then  $A$  and  $B$  are  $SC_p$ -groups.
2. If  $A$  and  $B$  are  $SC_p$ -groups, then  $G/\text{Core}_G(A \cap B)$  is an  $SC_p$ -group.

Mutually permutable products of  $SC_p$ -groups and  $p$ -soluble  $\mathcal{Z}_p$ -groups are the object of the next result:

**Theorem 12.** Let  $G = AB$  be a mutually permutable product of its subgroups  $A$  and  $B$ . Assume that  $A$  is an  $SC_p$ -group and that  $B$  is a  $p$ -soluble  $\mathcal{Z}_p$ -group. Then  $G$  is an  $SC_p$ -group.

The following corollaries follow immediately from Theorem 12:

**Corollary 13.** If  $G$  is a mutually permutable product of an SC-group  $A$  and a  $\mathcal{Y}$ -group  $B$ , then  $G$  is an SC-group. In particular, if  $G$  is a mutually permutable product of a supersoluble group  $A$  and a  $\mathcal{Y}$ -group  $B$ , then  $G$  is supersoluble.

Let  $\mathfrak{X}$  be a class of groups. A class of groups  $\mathfrak{F}$  is called the *Fitting core* of  $\mathfrak{X}$  provided that whenever if  $A \in \mathfrak{X}$  and  $B \in \mathfrak{F}$ , and  $A$  and  $B$  are normal subgroups of a group  $G$ , then  $AB \in \mathfrak{X}$  (see [20]). From Corollary 2 of [20] it follows that the class of soluble PST-groups belongs to the Fitting core of the formation of supersoluble groups. In fact from Corollary 13 we obtain a more general statement, mainly: the class  $\mathcal{Y}$  is contained in the Fitting core of both the formation of supersoluble groups and hence the formation of SC-groups.

**Corollary 14.** If  $G$  is a mutually permutable product of two  $p$ -soluble  $\mathcal{Z}_p$ -groups, then  $G$  is  $p$ -supersoluble.

**Corollary 15.** If  $G$  is a mutually permutable product of two  $\mathcal{Y}$ -groups, then  $G$  is supersoluble.

Corollary 15 admits the following generalisation:

**Theorem 16.** Let  $G = G_1G_2 \cdots G_r$  be a group such that  $G_1, G_2, \dots, G_r$  are pairwise mutually permutable subgroups of  $G$ . If all  $G_i$  are  $\mathcal{Y}$ -groups, then  $G$  is supersoluble.

The proof of Theorem 16 depends on the following lemma:

**Lemma 17.** Let  $G$  be a  $\mathcal{Y}$ -group with an abelian normal Sylow  $p$ -subgroup  $P$  for a prime  $p$ . Then every subgroup of  $P$  is normal in  $G$ .

We do not know whether a local version of Theorem 16 is true, namely, if all  $G_i$  are  $\mathcal{Z}_p$ -groups, then  $G$  is  $p$ -supersoluble.

In [22, Theorem 5], the following result is proved:

**Theorem 18.** Let  $G = AB$  be a mutually permutable product of the subgroups  $A$  and  $B$ . If  $G$  is a PST-group, then  $A$  is a PST-group.

Our proof of Theorem 20 depends on the following:

**Lemma 19.** *Let  $N$  be a normal subgroup of  $G$  such that  $G/N$  satisfies  $N_p$ . If either  $N$  is non-abelian and simple or  $N$  is a  $p'$ -group, then  $G$  satisfies  $N_p$ .*

We conclude with a local version of Theorem 18, from which it follows immediately:

**Theorem 20.** *Let  $G$  be a mutually permutable product of its subgroups  $A$  and  $B$ . If  $G$  is a SC-group and satisfies  $N_p$ , then  $A$  satisfies  $N_p$ .*

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