On a class of supersoluble groups

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Abstract

A subgroup H of a finite group G is said to be *S-semipermutable* in G if H permutes with every Sylow q-subgroup of G for all primes q not dividing $|H|$. A finite group G is an *MS-group* if the maximal subgroups of all the Sylow subgroups of G are S-semipermutable in G. The aim of the present paper is to characterise the finite MS-groups.

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1 Introduction

In the following, G always denotes a finite group. Recall that subgroups H and K of G is said to *permute* if HK is a subgroup of G and that a subgroup H of G is said to be *permutable* in G if H permutes with all subgroups of G.

Various generalisations of permutability have been defined and studied and, in particular, we mention the S-semipermutability. A subgroup H is said to be *S-semipermutable* in G if H permutes with every Sylow q-subgroup of G for all primes q not dividing $|H|$. This subgroup embedding property

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has been extensively studied recently (see for instance [1, 4, 7, 9]). Most of these papers concern situations where many subgroups (for instance all maximal subgroups of the Sylow subgroups) have the stated property. Thus we say that a group G is an MS-group if the maximal subgroups of all the Sylow subgroups of G are S-semipermutable in G.

The main aim of this paper is to characterise the MS-groups.

2 Preliminary results

In this section, we collect the definitions and results which are needed to prove our main theorems.

We shall adhere to the notation used in [2]: this book will be the main reference for terminology and results on permutability.

A subgroup H is permutable in a group G if and only if H permutes with every p-subgroup of G for every prime p (see for instance [2, Theorem 1.2.2]). A less restrictive subgroup embedding property is the S-permutability introduced by Kegel in 1962 [5] and defined in the following way:

Definition 1. A subgroup H of G is said to be *S-permutable* in G if H permutes with every Sylow p-subgroup of G for every prime p.

Note that we are not considering all p-subgroups, but just the maximal ones, that is, the Sylow p-subgroups.

In recent years there has been widespread interest in the transitivity of normality, permutability and S-permutability.

Definition 2.
1. A group G is a *T-group* if normality is a transitive relation in G, that is, if every subnormal subgroup of G is normal in G.
2. A group G is a *PT-group* if permutability is a transitive relation in G, that is, if H is permutable in K and K is permutable in G, then H is permutable in G.
3. A group G is a *PST-group* if S-permutability is a transitive relation in G, that is, if H is S-permutable in K and K is S-permutable in G, then H is S-permutable in G.

If H is S-permutable in G, it is known that H must be subnormal in G ([2, Theorem 1.2.14(3)]). Therefore, a group G is a PST-group (respectively, a PT-group) if and only if every subnormal subgroup is S-permutable (respectively, permutable) in G.

Note that T implies PT and PT implies PST. On the other hand, PT does not imply T (non-Dedekind modular p-groups) and PST does not imply PT (non-modular p-groups).

A less restrictive class of groups is the class of T_0-groups which has been studied in [3, 6, 8].

Definition 3. A group G is called a *T_0-group* if the Frattini factor group $G/\Phi(G)$ is a T-group.

The group in Example 13 below is a soluble T_0-group which is not a PST-group. Soluble T_0-groups are closely related to PST-groups as the following result shows.

Theorem 4 ([6, Theorems 5 and 7 and Corollary 3])**.** *Let G be a soluble T_0-group with nilpotent residual $L = \gamma_\infty(G)$. Then:*

1. *G is supersoluble.*
2. *L is a nilpotent Hall subgroup of G.*
3. *If L is abelian, then G is a PST-group.*

Here the *nilpotent residual* $\gamma_\infty(G)$ of a group G is the smallest normal subgroup N of G such that G/N is nilpotent, that is, the limit of the lower central series of G defined by $\gamma_1(G) = G$, $\gamma_{i+1}(G) = [\gamma_i(G), G]$ for $i \geq 1$.

It is known that S-semipermutability is not transitive. Hence it is natural to consider the following class of groups:

Definition 5. A group G is called a *BT-group* if S-semipermutability is a transitive relation in G, that is, if H is S-semipermutable in K and K is S-semipermutable in G, then H is S-semipermutable in G.

This class was introduced and characterised by Wang, Li and Wang in [9]. Further contributions were presented in [1].

Theorem 6 ([9, Theorem 3.1])**.** *Let G be a group. The following statements are equivalent:*

1. *G is a soluble BT-group.*
2. *Every subgroup of G is S-semipermutable.*
3. *G is a soluble PST-group and if p and q are distinct prime divisors of the order of G not dividing the order of the nilpotent residual of G, then $[G_p, G_q] = 1$, where $G_p \in \mathrm{Syl}_p(G)$ and $G_q \in \mathrm{Syl}_q(G)$.*

The group presented in Example 12 below is an MS-group which is not a soluble BT-group. Furthermore, Example 13 shows that the classes of T_0-groups and MS-groups are not closed under taking subgroups.

The first remarkable fact concerning the structure of an MS-group can be found in [7]. It is proved there that every MS-group is supersoluble.

Theorem 7 ([7, Corollary 9])**.** *Let G be an MS-group. Then G is supersoluble.*

More recently, the second and fourth authors proved the followign theorem.

Theorem 8 ([4, Theorems A, B and C])**.** *Let G be an MS-group with nilpotent residual $L = \gamma_\infty(G)$. Then:*

1. *If N is a normal subgroup of G, then G/N is an MS-group;*
2. *L is a nilpotent Hall subgroup of G ;*
3. *G is a soluble T_0-group.*

It is well-known that the nilpotent residual of a supersoluble group is nilpotent. Hence the nilpotency of L in Theorem 8 is a consequence of Theorem 7.

Let G be a group whose nilpotent residual $L = \gamma_\infty(G)$ is a Hall subgroup of G. Let $\pi = \pi(L)$ and let $\theta = \pi'$, the complement of π in the set of all prime numbers. Let θ_N denote the set of all primes p in θ such that if P is a Sylow p-subgroup of G, then P has at least two maximal subgroups. Further, let θ_C denote the set of all primes q in θ such that if Q is a Sylow q-subgroup of G, then Q has only one maximal subgroup, or equivalently, Q is cyclic.

Throughout this paper we will use the notation presented above concerning π, $\theta = \pi'$, θ_N, and θ_C.

3 The main results

Our first main result is a characterisation theorem.

Theorem 9. *Let G be a group with nilpotent residual $L = \gamma_\infty(G)$. Then G is an MS-group if and only if G satisfies the following properties.*

1. *G is a T_0-group.*
2. *L is a nilpotent Hall subgroup of G.*

3. *If $p \in \pi$ and $P \in \mathrm{Syl}_p(G)$, then a maximal subgroup of P is normal in G.*

4. *Let p and q be distinct primes with $p \in \theta_N$ and $q \in \theta$. If $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$, then $[P, Q] = 1$.*

5. *Let p and q be distinct primes with $p \in \theta_C$ and $q \in \theta$. If $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ and M is the maximal subgroup of P, then $QM = MQ$ is a nilpotent subgroup of G.*

Proof. Let G be an MS-group. By Theorems 7 and 8, G is a supersoluble T_0-group whose nilpotent residual L is a nilpotent Hall subgroup of G. Thus properties 1 and 2 hold.

Let $\pi = \pi(L)$ and let $p \in \pi$. Further, let P be a Sylow p-subgroup of G and let M be a maximal subgroup of P. Then $M \leq P \trianglelefteq L$ and M is normal in L and subnormal in G. Let $q \in \theta = \pi'$ and note that MQ is a subgroup of G for a given Sylow q-subgroup Q of G. Moreover M is a Sylow p-subgroup of MQ and so M is a normal subgroup of MQ. Consequently M normalises P and each Sylow q-subgroup Q of G, so M is a normal subgroup of G and property 3 holds.

Let X be a Hall θ-subgroup of G and note that $G = L \rtimes X$, the semidirect product of L by X, and X is nilpotent. Let t be a prime from θ_N and r be a prime from θ. Also let $T \in \mathrm{Syl}_t(G)$ and $R \in \mathrm{Syl}_r(G)$. Let M_1 and M_2 be two distinct maximal subgroups of $T = \langle M_1, M_2 \rangle$. Since G is an MS-group, $M_1 R = RM_1$ and $M_2 R = RM_2$. Applying [2, Theorem 1.2.2], we have $RT = TR$. Observe that TR is a θ-subgroup of G and so TR is nilpotent since TR is a subgroup of some conjugate of X. Therefore, $[T, R] = 1$ and property 4 holds.

Let p and q be distinct primes with $p \in \theta_C$ and $q \in \theta$. Further, let $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$. If M is the maximal subgroup of P, then $QM = MQ$ is a nilpotent θ-subgroup of G. Thus property 5 holds.

Let G be a group satisfying properties 1–5. We are to show that G is an MS-group. By properties 1 and 2, G is a soluble T_0-group, and by Theorem 4, G is thus supersoluble.

Let $p \in \pi = \pi(L)$, let P be a Sylow p-subgroup of G, and let M be a maximal subgroup of P. Then M is a normal subgroup of G by property 3 and clearly P is a normal subgroup of G. This means that M permutes with every Sylow subgroup of G and P permutes with every maximal subgroup of any Sylow subgroup of G.

Let p and q be distinct primes from θ and let $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$. We consider a maximal subgroup M of P. Note that $\theta = \theta_N \cup \theta_C$

and $\theta_N \cap \theta_C = \emptyset$, the empty set. If $p \in \theta_N$, then by property 4, $[P, Q] = 1$, so that $MQ = QM$. Hence assume $p \in \theta_C$. Then, by property 5, $MQ = QM$.

Therefore, every maximal subgroup of any Sylow subgroup of G is S-semipermutable in G and G is an MS-group. □

The second and fourth authors in [4] posed the following two questions.

1. When is a soluble PST-group an MS-group?

2. When is a soluble PST-group which is also an MS-group a BT-group?

Using Theorem 9 we are able to answer the first question and provide a partial answer to the second.

Theorem 10. *Let G be a soluble PST-group. Then G is an MS-group if and only if G satisfies 4 and 5 of Theorem 9.*

Proof. Let G be a soluble PST-group with nilpotent residual $L = \gamma_\infty(G)$. By [6, Lemma 5], $G/\Phi(G)$ is a T-group and so G is a T_0-group. Notice that 1, 2 and 3 of Theorem 9 are satisfied for the group G.

Assume that G is an MS-group. By Theorem 9, 4 and 5 are satisfied by G.

Conversely, assume that 4 and 5 of Theorem 9 are satisfied by G. By Theorem 9, G is an MS-group.

This completes the proof. □

The group given in Example 12 below is a soluble PST-group which is not an MS-group and the group given in Example 13 is an MS-group which is not a soluble PST-group.

Theorem 11. *Let G be a soluble PST-group which is also an MS-group. If θ_C is the empty set, then G is a BT-group.*

Proof. Let G be a soluble PST-group which is also an MS-group. Let $L = \gamma_\infty(G)$ be the nilpotent residual of G. By the Theorem of Agrawal [2, Theorem 2.1.8], L is an abelian Hall subgroup of G on which G acts by conjugation as a group of power automorphisms. Recall that $\theta = \pi'$, where $\pi = \pi(L)$. Moreover $\theta = \theta_N$ if θ_C is empty. Let p and q be distinct primes from θ and let $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$. Note that since G is an MS-group, we have that G satisfies properties 4 and 5 of Theorem 9. Then $[G_p, G_q] = 1$ by property 4 of that theorem. Therefore, G is a BT-group by Theorem 6. This completes the proof of Theorem 11.

We remark that if θ contains only one prime, then G is a BT-group by [9, Corollary 3.4]. □

4 Examples

The following examples appear in [4]. For the sake of completeness, we list them here.

Example 12. Let $G = \langle y, z, x \mid y^3 = z^2 = x^7 = 1, [y, z] = 1, x^y = x^2, x^z = x^{-1}\rangle$. Then $[\langle y\rangle^x, z] \neq 1$ and G is a soluble group which is not a BT-group. However, G is an MS-group.

Example 13. Let $G = \langle a, x, y \mid a^2 = x^3 = y^3 = [x, y]^3 = [x, [x, y]] = [y, [x, y]] = 1, x^a = x^{-1}, y^a = y^{-1}\rangle$. Then $H = \langle x, y\rangle$ is an extraspecial group of order 27 and exponent 3. Let $z = [x, y]$, so $z^a = z$. Then $\Phi(G) = \Phi(H) = \langle z\rangle = \mathrm{Z}(G) = \mathrm{Z}(H)$. Note that $G/\Phi(G)$ is a T-group so that G is a T_0-group. The maximal subgroups of H are normal in G and it follows that G is an MS-group. Let $K = \langle x, z, a\rangle$. Then $\langle xz\rangle$ is a maximal subgroup of $\langle x, z\rangle$, the Sylow 3-subgroup of K. However, $\langle xz\rangle$ does not permute with $\langle a\rangle$ and hence $\langle xz\rangle$ is not an S-semipermutable subgroup of K. Therefore, K is not an MS-subgroup of G. Also note that $\Phi(K) = 1$ and so K is not a T-subgroup of G and K is not a T_0-subgroup of G. Hence the class of soluble T_0-groups is not closed under taking subgroups. Note that G is a soluble group which is not a PST-group.

Example 14. Let $G = \langle y, z, x \mid y^9 = z^2 = x^{19^2} = 1, [y, z] = 1, x^y = x^{62}, x^z = x^{-1}\rangle$. Then the soluble group G is a PST-group, but G is not an MS-group since $[\langle y^2\rangle^x, z] \neq 1$.

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