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A Level Set Method with Sobolev Gradient and Haralick Edge Detection

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Abstract

Variational level set methods, which have been proposed with various energy functionals, mostly use the ordinary L^2 type gradient in gradient descent algorithm to minimize the energy functional. The gradient flow is influenced by both the energy to be minimized and the norms, which are induced from inner products, used to measure the cost of perturbation of the curve. However, there are many undesired properties related to the gradient flows due to the L^2 type inner products. For example, there is not any regularity term in the definition of this inner product that causes non-smooth flows and inaccurate results. Therefore, in this work, Sobolev gradient has been used that is more efficient than the L^2 type gradient for image segmentation and has powerful properties such as regular gradient flows, independency to parameterization of curves, less sensitive to local features and noise in the image and also faster convergence rate than the standard gradient. In addition, Haralick edge detector has been used instead of the edge indicator function in this study. Because, the traditional edge indicator function, which is the absolute of the gradient of the convolved image with the Gaussian function, is sensitive to noise in level set methods. Experimental results on real images, which are abdominal magnetic resonance images, have been obtained for spleen and kidney segmentation. Quantitative analyses have been performed by using different measurements to evaluate the performance of the proposed approach, which can ignore topological noises and detect boundaries successfully.

Keywords: haralick edge detector, variational level set method, sobolev gradient.

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1. Introduction

Level Set Methods (LSMs) have been popular in the last fifteen years. Since, it is very flexible for adaptation to different problems and has the guarantee for solutions that exist in Partial Differential Equations (PDEs). Also, numerical methods have been developed extensively by using Hamilton-Jacobi equations to obtain stable numerical schemes and to handle shocks. However, there are two important difficulties for level set based image segmentation methods. One of them is that the evolution of the curve has to be stopped efficiently (without leakage or under segmentation) when it reaches to the desired boundaries. The other difficulty is due to the contradiction between the implementation and the theory of the LSM associated to the representation of Level Set Functions (LSFs) by signed distance functions. Since LSMs do not implicitly preserve LSFs as signed distance functions in practice. The general solution for this problem is re-distancing (re-initialization), which is still a serious problem since it is not known how to re-initialize the LSF and when it should be applied to maintain the stability of the evolution [5].

Variational Level Set Methods (VLSMs) [12], [2], [1], [23], [13] have been proposed to regularize the LSF as a signed distance function and hence to avoid re-initialization of the LSF. Main advantages of these methods are easier implementation and higher computational efficiency. For instance, the VLSMs in [3], [1], combine the Chan-Vese model [4] or the piecewise smoothing algorithm [4] with variational model to avoid re-initialization. Another proposed VLSM [12] uses a penalty term which penalizes deviations of LSFs from signed distance functions. The function of the penalty term has been improved and the diffusion rate became a bounded constant in the Distance Regularized Level Set Evolution (DRLSE) method [13] to maintain the signed distance property. However, the methods in [3], [1] can only be used for LSFs which have a specific form. The penalty term in [12] leads the diffusion rate to be tend to infinity and causes undesirable effects, numerical errors in some circumstances. The DRLSE method in [13] uses Gaussian filter to remove the effect of noise. This filter causes blurred edges while reducing the image noise. Also, the steepest gradient descent process in VLSMs affects the accuracy of segmentation results.

Steepest gradient descent [18] is a widely used line search technique for finding a relative minimum or maximum of a function. Usually, the gradient descent algorithm is used to minimize a proposed energy functional for VLSMs in practice. The most standard way is to use L^2 type gradient in the gradient descent process. However, to use the ordinary L^2 gradient requires strong conditions such as having second order derivative of the unknown functions when the chosen energy function is first order. Also there are several undesired properties associated to the L^2 gradient [22]. J.W. Neuberger proposed to use Sobolev gradient in gradient descent algorithms [19]. Since the Sobolev gradient is imposed by the function space of the unknown and it is a right choice if the minimization is well-posed. Also, Sobolev gradient is satisfied by the weak solution while the L^2 gradient needs strong solution.

Recently, different works showed that Sobolev gradient is more efficient than L^2 gradient especially for image segmentation [22], [8], [24]. Sobolev gradient flows are not local that means a point of the curve depends on all other points on the curve. Deformation of the Sobolev gradient flow is local after global motions can no longer optimize the energy. Sobolev gradient based curve evolution, which avoids many local minima, is more global due to the different deformation of gradient flows to reach steady state [22].

In this study, the energy functional proposed in [13] has been chosen as a model and it has been reformulated by using the Sobolev gradient instead of the L^2 gradient and also Haralick edge detector has been implemented instead of the traditional edge indicator function, which is based on Gaussian kernel. Since, the Gaussian filter causes blurred image edges and determination of the optimal standard deviation is a dilemma for image segmentation in case of unclear boundaries and noise.

2. Sobolev gradient in active contours

To use Sobolev gradient flows in active contours has been firstly proposed in [22], which shows that the gradient flows during a contour evolution is affected by both the energy, which will be minimized, and the way how the norm of the perturbation of the curve is measured. A short overview about the Sobolev gradient with the terms from [22] is presented in this section.

Let assume that M is a differentiable manifold and denotes a space of immersed curves in R^2 . Let T_cM be the tangent space of the manifold M at a curve c for $c \in M$. The tangent space shows valid smooth perturbations of curves by the function $h: S^1 \rightarrow R^2$, where $S^1 \subset R^2$ is the circle. Also, assume an energy functional is given by $E: M \rightarrow R$. For a curve $c \in M$ and $h \in T_cM$, the variation (or differentiation at c) of the energy function E in the direction h is given by

$$E'(c) \cdot h = \left. \frac{d}{dt} E(c + th) \right|_{t=0} \quad (1)$$

Here, $(c + th)(\theta) := c(\theta) + th(\theta)$ and $\theta \in S^1$. Let $\langle \cdot, \cdot \rangle_c$ is an inner product on the tangent space T_cM at c . The gradient of the energy function E is a vector field $\nabla E(c) \in T_cM$ which satisfies that

$$E'(c) \cdot h = \langle h, \nabla E(c) \rangle_c \quad (2)$$

for all $h \in T_cM$. The proposed inner products in [22] are

- i) $\langle h, k \rangle_{H^0} := \frac{1}{L} \int_0^L h(s)k(s)ds$
- ii) $\langle h, k \rangle_{H^n} := \langle h, k \rangle_{H^0} + \lambda L^{2n} \langle h^{(n)}, k^{(n)} \rangle_{H^0}$
- iii) $\langle h, k \rangle_{\tilde{H}^n} := \text{avg}(h) \cdot \text{avg}(k) + \lambda L^{2n} \langle h^{(n)}, k^{(n)} \rangle_{H^0}$

where $k, h \in T_cM$, $\lambda > 0$ and L is the length of curve c . The term $\text{avg}(h)$ is defined as

$$\text{avg}(h) := \frac{1}{L} \int_0^L h(s)ds \quad (3)$$

and all the derivatives are computed with respect to arclength parameter. The proposed inner products use scaling factors which depend on the length. Therefore, rescaling of the curve does not affect the inner products and their corresponding norms does not change, which is a desired property.

Definition of inner products on the tangent space T_cM is important due to the reduced distance between any two points on M . Since, the commonly used L^2 type inner product is pathological that means the distance between any two points is equal to zero [14].

To show the relation between the Sobolev inner product and the L^2 type inner product, let define a differential operator that consists of two elements; ϕ and two partial derivatives of ϕ as

$$D\phi = \begin{pmatrix} \phi \\ \nabla \phi \end{pmatrix} \quad (4)$$

where $D: H^{1,2}(\Omega) \rightarrow L^2(\Omega)^3$. The inner product on $H^{1,2}(\Omega)$ with $k, h \in H^{1,2}(\Omega)$ is written as

$$\langle h, k \rangle_{H^{1,2}(\Omega)} = \int_{\Omega} hk + \langle \nabla h, \nabla k \rangle_{L^2(\Omega)^2} = \langle Dh, Dk \rangle_{L^2(\Omega)^3} \quad (5)$$

3. Comparison of Sobolev steepest descent and Euclidean steepest descent

Gradients are based on Sobolev space instead of Euclidean space in Sobolev steepest descent technique. Descent based on different spaces causes the required number of iterations and also computation time to be different. Table 1 [15] presents the difference between Sobolev gradient and Euclidean gradient based steepest descent process for a simple linear differential equation that is $u' = u$ on the interval [1]. Accuracy is computed by $\|u_{k+1} - u_k\| < \varepsilon$. The value of Maximum Absolute Error (MAE) is computed by

$$MAE = \max \left\{ |u_k^{true} - u_k^{approximate}| : k = 1, 2, 3, \dots, n+1 \right\} \quad (6)$$

The value of Average Absolute Error (AAE) is computed by

$$AAE = \frac{1}{n+1} \sum_{i=1}^{n+1} |u_k^{true} - u_k^{approximate}| \quad (7)$$

Table 1. Comparison of Sobolev and Euclidean Gradients [15].

Gradient Type	ε	Number of division	Number of iterations	Required Time(second)	MAE	AAE
Euclidean	0,01	100	1845	3	0,92	0,59
Sobolev	0,01	100	4	1	0,0074	0,0056
Euclidean	0,0001	1000	1477	27	0,092	0,066
Sobolev	0,0001	1000	6	1	0,00001	0,000087

The solution can be found by minimizing the function $\|u_0 - cE\|^2$ for Euclidean Steepest descent and the function $\|u_0 - cE\|_s^2$ for Sobolev steepest descent. Since all solutions are at the form $c * E$ where $E(t) = e^t$ on [0,1].

Sobolev gradient based descent requires less computation time than Euclidean gradient based descent according to the values in Table1. Also, Euclidean gradient based descent requires more iteration for convergence than Sobolev gradient based descent when the number of division is increased from 100 to 1000. When we look at the computed error values, it is observed that Sobolev gradient yields error than Euclidean gradient. Therefore, Sobolev gradient based steepest descent is more efficient and it gives more accurate results in shorter time than Euclidean gradient based steepest descent method.

4. The traditional edge indicator function and Haralick edge detection

The traditional edge indicator that allows the level set evolution to stop at the boundaries in LSMs is a positive, regular and decreasing edge stopping function [20], [16], [2], [21], [12], [13] which is written as

$$g(\|\nabla I\|) = \frac{1}{1 + |\nabla G_\sigma * I|^2} \quad (8)$$

where G is a Gaussian kernel with the standard deviation σ . The LSF stops when the value of the edge indicator function approaches to zero. In other words, the contour will be halt where the

gradient of the image is big enough for making $g \rightarrow 0$. However, to use the traditional edge indicator function g is not appropriate for many images. Since, the Gaussian filter can not preserve edge information while removing the noise effect. Therefore, it causes blurred image edges. To choose a small value of standard deviation may lead to sensitivity to noise. In this case, the curve evolution will not be stable. To choose a big value of standard deviation may cause boundary leakage problem, which results inaccurately segmented image. Therefore, determination of the optimal standard deviation is a dilemma for image segmentation in case of unclear boundaries and noise. This problem can be solved by using a different function instead of the traditional edge stopping function.

Many alternative solutions have been used in the literature instead of the traditional edge stopping function. For instance, the stopping term in C-V model [4] is based on a result which is obtained from a segmentation method proposed by Mumford-Shah algorithm [17]. This segmentation method uses intensity values from different region for energy minimization. Liu et al [13] proposed a method which uses the differences between estimated and actual probability densities of intensity values from different regions. However, these methods are not successful for images with low contrast and unclear boundaries. Some edge based level set methods have been proposed to use a balloon force to expand or shrink the active contour. However, to design the balloon force is not easy. Because, the contour passes over weak edges if the force is large. The active contour might not pass through the narrow region of the object, if the force is not large enough. Therefore, Haralick edge detection has been proposed in this work.

Haralick edge detection [6] is a well known edge detection approach which describes the edge as a location where the value of the gradient magnitude of the image is maximized along the image gradient direction. In other word, Haralick edge detector detects the positions in the image where $|\nabla I|$ has a local maximum value along the gradient orientation. The gradient direction vector field for a gray level image $I(x,y)$ is identified as

$$\vec{\xi}(x, y) = \frac{\nabla I}{|\nabla I|} = \frac{\{I_x, I_y\}}{|\nabla I|} \quad (9)$$

and the orthogonal vector field is written as

$$\vec{\eta}(x, y) = \frac{\{-I_y, I_x\}}{|\nabla I|} \quad (10)$$

Since $\langle \vec{\eta}, \vec{\xi} \rangle = 0$. The Haralick edge detector, $I_{\xi\xi}$, detects edges where $|\nabla I|$ is greater than a threshold and $I_{\xi\xi} = 0$, which is the second derivative of the image along the gradient direction and defined as

$$I_{\xi\xi} = \nabla^2 I - I_{\eta\eta} = 0 \quad (11)$$

Here, the first term $\nabla^2 I$ is the Laplacian form of the image, $I_{\eta\eta} = |\nabla I| \operatorname{div} \left(\frac{\nabla I}{|\nabla I|} \right) = \kappa |\nabla I|$ and κ is the curvature [9]. The Haralick edge detector might be thought as optimal edge contour whose normal best align with the image gradient field. Also it satisfies a topological uniformity measure inside a closed contour [10-11].

5. The proposed VLSM

In this paper, image segmentation by using a VLSM that is based on both Sobolev gradient and Haralick edge detector has been proposed. For this purpose, the DRLSE [13], which has been recently published and mostly known method, has been chosen as a model in this work. The Haralick operator has been implemented to avoid blurring effects of the Gaussian filter in the DRLSE. The level set evolution in the DRLSE method was formulated with L^2 type inner product for the gradient definition

by the authors. In this work, the Sobolev gradient has been used instead of the standard L^2 type gradient for both the regularization term and the external energy to obtain better convergence rate for the problem of minimizing energy functional.

5.1. Obtaining the Sobolev gradient of the energy functional used in the DRLSE method

The Gateaux derivative of a given energy functional $E(\phi)$ is written by

$$E'(\phi) \cdot h = \lim_{\alpha \rightarrow 0} \left(\frac{E(\phi + \alpha h) - E(\phi)}{\alpha} \right) \text{ for } h \in H_0^1(\Omega) \quad (12)$$

The energy functional in the DRLSE method (Li et al 2010) is

$$E(\phi) = \mu R_p(\phi) + \lambda L_g(\phi) + \alpha A_g(\phi) \quad (13)$$

where μ , λ and α are constant coefficients and g is the traditional edge indicator function. $R_p(\phi)$ is a regularization term defined as $R_p(\phi) = \int_{\Omega} p(|\nabla\phi|) dx$ with a potential function $p: [0, \infty) \rightarrow \mathfrak{R}$. The term $L_g(\phi)$ is the length of the active contour and expressed as $L_g(\phi) = \int_{\Omega} g\delta(\phi)|\nabla\phi| dx$ by using the dirac delta function $\delta(\cdot)$. The term $A_g(\phi)$ is the area inside the contour computed with the Heaviside function $H(\cdot)$ as $A_g(\phi) = \int_{\Omega} gH(-\phi) dx$.

The Gateaux derivative of the energy functional given in (13) at ϕ is written as

$$E'(\phi) \cdot h = \int_{\Omega} \mu \left\langle \frac{p'(|\nabla\phi|)}{|\nabla\phi|} \nabla\phi, \nabla h \right\rangle + \lambda \left\langle \frac{g\delta_{\varepsilon}(\phi)\nabla\phi}{|\nabla\phi|}, \nabla h \right\rangle + \lambda g\delta'_{\varepsilon}(\phi)|\nabla\phi|h - \nu g\delta_{\varepsilon}(\phi)h \quad (14)$$

Integration by parts has been applied to the above equation (14) and it has been obtained that

$$E'(\phi) \cdot h = \int_{\Omega} \left[-\mu \operatorname{div} \left(\frac{p'(|\nabla\phi|)}{|\nabla\phi|} \nabla\phi \right) - \lambda \operatorname{div} \left(\frac{g\delta_{\varepsilon}(\phi)\nabla\phi}{|\nabla\phi|} \right) + \lambda g\delta'_{\varepsilon}(\phi)|\nabla\phi| - \nu g\delta_{\varepsilon}(\phi) \right] h \quad (15)$$

with $\nabla\phi = 0$ on the boundary of Ω . Therefore, the gradient of the energy has been obtained as

$$\nabla E(\phi) = -\mu \operatorname{div} \left(\frac{p'(|\nabla\phi|)}{|\nabla\phi|} \nabla\phi \right) - \lambda \operatorname{div} \left(\frac{g\delta_{\varepsilon}(\phi)\nabla\phi}{|\nabla\phi|} \right) + \lambda g\delta'_{\varepsilon}(\phi)|\nabla\phi| - \nu g\delta_{\varepsilon}(\phi) \quad (16)$$

$$\nabla E(\phi) = -\mu \operatorname{div} \left(\frac{p'(|\nabla\phi|)}{|\nabla\phi|} \nabla\phi \right) - \lambda \delta_{\varepsilon}(\phi) \operatorname{div} \left(\frac{g\nabla\phi}{|\nabla\phi|} \right) - \nu g\delta_{\varepsilon}(\phi) \quad (17)$$

$E'(\phi)$ is a bounded linear functional on $L^2(\Omega)$ for $\phi \in H^{2,2}(\Omega)$. Therefore, there is a unique gradient $\nabla E(\phi) \in L^2(\Omega)$ according to the Reisz representation theorem, such that

$$E'(\phi)h = \langle \nabla E(\phi), h \rangle_{L^2(\Omega)} \quad (18)$$

Also, $E'(\phi)$ is a bounded linear functional on $H^{1,2}(\Omega)$ for $\phi \in H^{1,2}(\Omega)$. Therefore, there is a unique gradient $\nabla_s E(\phi) \in H^{1,2}(\Omega)$, which is called as Sobolev gradient, according to the Reisz representation theorem, such that

$$E'(\phi)h = \langle \nabla_s E(\phi), h \rangle_{H^{1,2}(\Omega)} \quad (19)$$

The following equations can be written by using (18) and (19) as

$$\begin{aligned} \langle \nabla_s E(\phi), h \rangle_{H^{1,2}(\Omega)} &= \langle \nabla E(\phi), h \rangle_{L^2(\Omega)} \\ \langle D(\nabla_s E(\phi)), Dh \rangle_{L^2(\Omega)^3} &= \langle \nabla E(\phi), h \rangle_{L^2(\Omega)} \\ \langle D^* D(\nabla_s E(\phi)), h \rangle_{L^2(\Omega)} &= \langle \nabla E(\phi), h \rangle_{L^2(\Omega)} \end{aligned} \quad (20)$$

Here, $D^* D(\nabla_s E(\phi)) = \nabla E(\phi)$ or $\nabla_s E(\phi) = (D^* D)^{-1} \nabla E(\phi)$ for all $h \in H^{1,2}(\Omega)$, where

$$D^* D = \begin{pmatrix} I & -\nabla \\ \nabla & \end{pmatrix} = I - \Delta. \quad (21)$$

Since, $D^* = (I, -\nabla)$ is the adjoint of D . The result can be extended to $\phi \in H^{1,2}(\Omega)$ by continuity since $H^{2,2}(\Omega)$ is a dense space of $H^{1,2}(\Omega)$. Therefore, the Sobolev gradient of the energy functional is,

$$\nabla_s E(\phi) = -(I - \Delta)^{-1} \nabla E(\phi) \quad (22)$$

with the boundary condition $\frac{\delta(\phi)}{\nabla \phi} \frac{\partial \phi}{\partial n} = 0$ on $\partial\Omega$ and the initial condition $\phi(0, x, y) = \phi_0(x, y)$.

5.2. Obtaining the level set evolution equation

The Sobolev gradient has been used in the variational level set model for both regularization term and the external energy. The level set evolution equation used in this study is written as

$$\frac{\partial \phi}{\partial t} = (1 - \Delta)^{-1} \left[\mu \operatorname{div} \left(\frac{p'(|\nabla \phi|)}{|\nabla \phi|} \nabla \phi \right) + \lambda \delta_\varepsilon(\phi) \operatorname{div} \left(\frac{g \nabla \phi}{|\nabla \phi|} \right) + v g \delta_\varepsilon(\phi) \right] \quad (23)$$

which is computed by using a fast Poisson solver [7]. Here, g is the image that is obtained after the Haralick edge detection. Finite difference scheme has been used to obtain numerically an approximate solution of the equation [23].

6. Results

Experimental results (Figure 1.d,h) have been obtained from real images (Figure 1.a,e), which have been chosen as T2 weighted abdominal magnetic resonance images. Since, spleen and kidney segmentation from these images is difficult due to in-homogen intensity values, unclear boundaries, noise and partial volume effects.

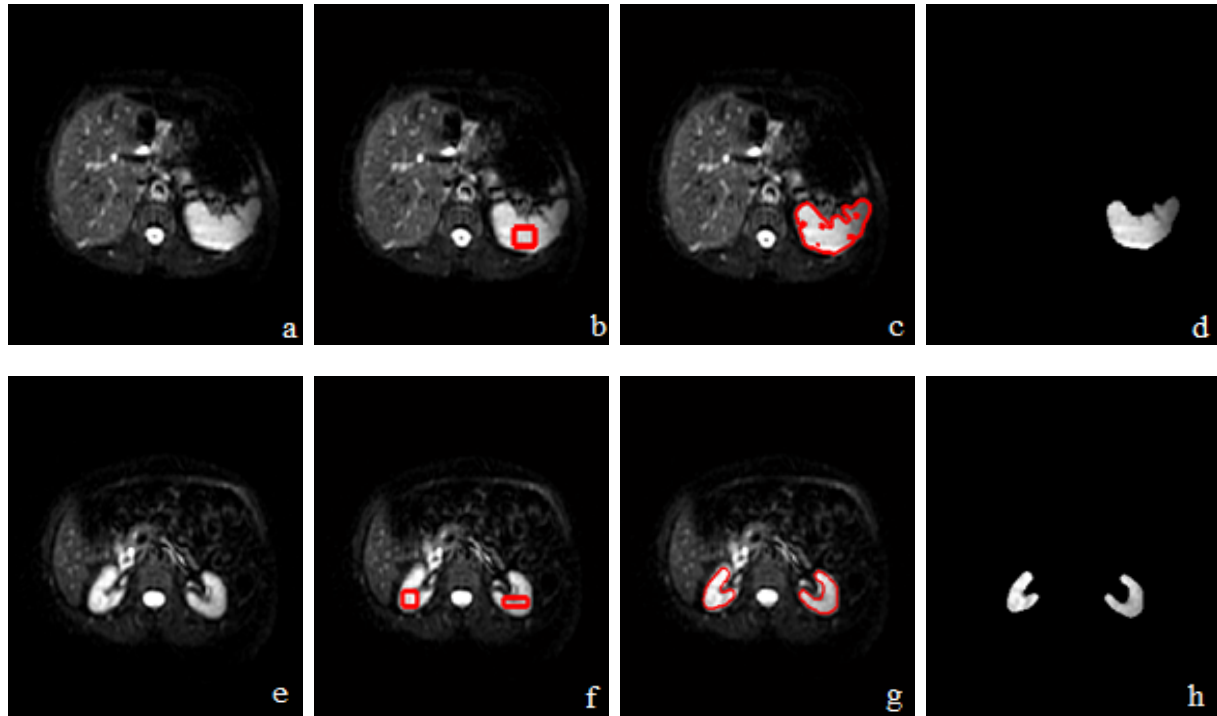


Figure 1. Original image (a, e); Initial closed curves (b, f); Results of the proposed method (c, g); Segmented images (d, h)

Results of the original DRLSE method (see Figure 2) have been obtained by using the same images (Figure 1.a, e) and the initial closed curves (Figure 1.b, f) for comparative evaluation. The standard deviation of the Gaussian filter has been chosen as 3, which has been observed as the most optimal value for the experimental abdominal data set. All other weight values in the DRLSE model, which are the coefficient for the length, area and the distance regularization term, have been chosen as the same with the coefficients that have been used for the proposed approach.

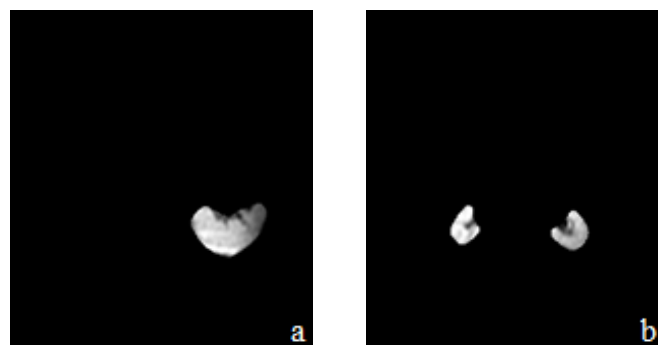


Figure 2. The results of the DRLSE method (a, b)

7. Conclusion

A variational level set based image segmentation approach by using both the Sobolev gradient and Haralick edge detection has been proposed in this work. Experimental results have been obtained from abdominal magnetic resonance images for spleen and kidney segmentation. These results have been evaluated and quantitative analyses have been performed by using the reference images shown in Figure 3.

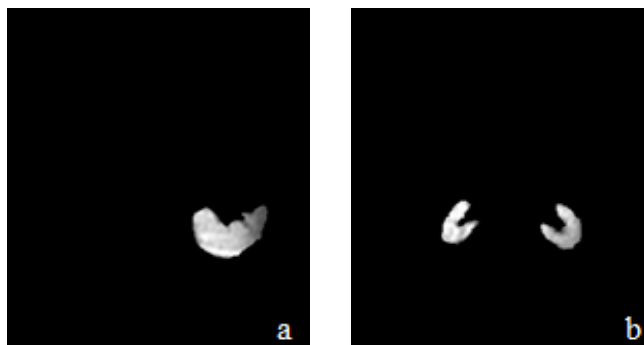


Figure 3. Reference images (a,b)

Table 2 presents numeric values of performance evaluations for the results obtained by the proposed approach and the original DRLSE method in terms of accuracy, sensitivity, specificity and also required processing time.

Table 2. Quantitative analyses for the proposed approach and the original DRLSE method

Method	Results	In percentage			Required time (in second)
		Sensitivity	Specificity	Accuracy	
Proposed Method	Spleen Image (Figure 1.d)	94.5719	99.8863	99.7803	7.460
	Kidney Image (Figure 1.e)	81.0139	99.9535	99.6628	10.048
DRLSE	Spleen Image (Figure 2.a)	95.3670	99.7624	99.6747	264.800
	Kidney Image (Figure 2.b)	72.8231	99.8866	99.4711	116.919

Preliminary results, the quantitative values can be seen in Table 2, show that the proposed segmentation approach gives successful results for spleen and kidney segmentation from magnetic resonance images in shorter time than the DRLSE method. Furthermore, one of the advantages of the method proposed, is that is not necessary to try to find an optimal standard deviation value.

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