Vector-valued sequences and multipliers

DEPARTAMENT D'ANÀLISI MATEMÀTICA

UNIVERSITAT DE VALÈNCIA

TESI DOCTORAL REALITZADA PER:

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per a l'obtenció del títol de Doctor en Matemàtiques.

BURJASSOT, 2015

Als meus pares.

Contents

Agradecimientos

Cuando me paro a contemplar mi estado
siguiendo las pisadas de mi vida
encuentro, que aun estando perdida,
finalmente una tesis he sacado.
Y, dado que a mí me han enseñado
que una debe ser agradecida,
previo a leer y emprender la huida
dispongo estas palabras con cuidado

Bravo hidalgo de lanza en astillero,
de mil libros empedernido amante,
agradezco hasta el último estante
de tu atención, tu gracia y tu salero.
Sé que te has dedicado con esmero
pues criarme no es labor cualquiera
pero has tenido buena compañera
-poniendo las cartas sobre el tablero-
que confiaba en mí, y en que supiera,
llegado el día, tapizar sillones.
Tratando de entender mis decisiones
me dejasteis contar a mi manera.

El clon de un Ángel ha contribuido,
junto con su Princesa Primorosa,
preparando excursiones montañosas,
a calmar a la loca que yo he sido.
Fanática del mundanal ruido,
escandalosa por naturaleza,
tener tranquilidad me da pereza;
¿la descansada vida? Un sinsentido.
Reticle aportó su gentileza
obrando con paciencia y una caña.
No, no es que me amansara (¡seria hazaña!)
pescando, sino a base de cerveza.

Para atender cualquiera de mis dramas
siempre encontré alguna oreja amiga
capaz de no ceder a la fatiga
causada por mi mente *encesa en flama*.
Que si Quevedo a mi encefalograma
quisiera dedicarle un solo verso
antes debiera en todo el universo
hallar algo más plano que una cama.

En espacios de Banach siempre inmerso
no puede el seso sino derretirse
mas risa compañera le hace enchirse
volviendo a despertar al día, terso.

Han puesto aquí su granito de arena
San Piteras, compañeros de piso,
Murcis, Predocs y todo aquel que quiso
verme llegar a la siguiente escena.
Por si mi playa no quedara llena,
Ionie trajo el tiempo que tenía
lo dedicó conmigo cada día
a descubrir dónde hacen buena cena.
Estando atiborrada de alegría
el resto del camino es más sencillo.
Caminante con un ligero hatillo
estelas en la mar pronto abriría.

Llegué así a México lindo y querido
donde tuve calurosa acogida.
La UNAM tiene culpa compartida
con Candela del éxito obtenido.
La neta, el viaje estuvo chido
(si la mejor parte de la aventura
es la vuelta, ésta se antojó dura).
Ya conmigo se quedará el sonido
del país que me cuidó con ternura,
donde tuve madre atenta y risueña,
compañeros como una sólo sueña
y un equipo cargado de dulzura.

Bien rodeada estuve desde el inicio
-que en esto España en nada envidie a México-.
Arranqué la etapa con un auténtico
hijo de Valdivia (en cuanto a oficio).
Me cobijó en singular edificio
e hizo del camino algo más llano.
Ahí conocí a Miss Cádiz y a un murciano:
dos buenas amistades, a mi juicio.
Lo que en el IUMPA comenzó un verano
se convierte, la tesis ya plasmada,
en tierra, en humo, en polvo, en sombra, en nada.
Cierro una etapa, escapa de mi mano.

Yo pensé que no hallara consonante
que me sacara de este vericueto
y descubrí al superar el boceto
que no depende tanto de Violante.
Es más cuestión de ser perseverante;
mi director sabe bien de qué hablo,
pues me ayudó a conseguir el vocablo
que calzaba la rima como un guante.
Pareciera el que sigue del diablo
un texto, y ya veis que es fortuna
de amigos, familia, tutor y una
misma que se metió al retablo.

Por dedicar tiempo de vuestra vida,
por ser conmigo siempre comprensivos,
por estos y otros tantos motivos,
no puedo sino estar agradecida.

Juegos de palabras aparte, me gustaría agradecer de manera explícita a mi director de tesis, Óscar Blasco, su paciencia e inestimable ayuda. Su guía ha sido un elemento imprescindible para llegar a buen puerto.

A Pepe Bonet, su empeño por que la juventud siga abriendo el camino que otros empezaron, siempre buscando oportunidades para todos aquellos que están decididos a investigar, sean o no sus alumnos.

A Salvador Pérez Esteva y a los compañeros de la UNAM en general, su cuidado matemático y personal: se trabaja mejor cuando te hacen sentir como en casa.

A Juan Monterde, su disponibilidad y atención. La burocracia resulta con él un poquito menos farragosa.

Por supuesto, a toda mi familia y a mis amigos, quienes con su compañía han hecho del doctorado un proceso más ligero. Con especial cariño agradezco a mis padres, mi hermano, Isa y Quique el apoyo recibido durante estos años. Sin vosotros no sería lo mismo. *Vos estime, us estimo, us estim.*

Resum

El text que es presenta a continuació tracta d'establir un marc teòric al que poder desenvolupar dos nous conceptes (extensió d'altres ja coneguts): els multiplicadors per coeficients a través d'una aplicació bilineal i el producte projectiu tensorial de Hadamard. Ambdós espais es veuen sempre com espais de succesions a valors vectorials, és a dir, a un espai de Banach qualsevol. Posteriorment, s'estudia la relació entre ells i s'aporten alguns exemples.

El punt de partida del projecte són les classes d'espais introduïdes per O. Blasco i M. Pavlović al treball "Coefficient multipliers on spaces of analytic functions" (Revista Mat. Iberoamericana, 2011), on es formalitza el problema de multiplicadors i se'l relaciona amb certs productes tensorials clàssics, definint les mínimes propietats en espais de Banach de funcions analítiques per a poder desenvolupar la teoria de multiplicadors, tenint com a **objectiu** donar la versió vectorial dels mateixos.

Amb aquest objetiu en ment, s'ha decidit dividir el treball en quatre capítols diferenciats. Els tres primers fixen el context recalcant alguns dels aspectes que poden donar al lector una idea més profona de l'ús i les aplicacions d'aquesta teoria. L'últim capítol és el colofó que uneix els tres anteriors, conferint un sentit únic al text.

De manera més específica, al primer capítol es donen els preliminars necessaris per poder abordar el problema que ens ocupa. Es presenten les ferramentes precises per comprendre l'escrit i els seus exemples: els espais de sucessions amb valors vectorials, $\mathcal{S}(E)$, i els espais de funcions analítiques al disc, també amb valors vectorials, $\mathcal{H}(\mathbb{D}, E)$.

Al segon capítol es determinen les condicions mínimes que s'exigiràn als espais de treball, anomenats espais $\mathcal{S}(E)-$admissibles seguint la notació de [**16**]. Es dona el cas concret dels multiplicadors amb valors a l'espai d'operadors, germe de l'espai de multiplicadors mitjançant una aplicació bilineal, B. Per posar de manifest l'importància d'aquests espais, per una banda es dona la relació dels mateixos amb els espais sòlids i per altra, es desenvolupa l'exemple dels espais de norma mixta generalitzats.

El tercer és un breu capítol on es donen condicions específiques per al cas en què els espais de successions siguen una forma de representació d'espais de funcions analítiques (a través dels seus coeficients de Taylor). De nou, seguint la notació introduïda per [**16**], aquests espais seran notats com a $\mathcal{H}(E)-$admissibles. A més s'aporten nous resultats que seràn aplicats a espais de funcions amb valors vectorials.

Per últim, al quart capítol es detallen les dos construccions dalt mentades: els multiplicadors a través d'una aplicació bilineal i el producte tensorial projectiu de Hadamard. Es veu la relació que existeix entre les dos i finalment es mostren casos particulars del còmput del producte tensorial projectiu de Hadamard i s'aplica al càlcul de multiplicadors.

Com a **conclusió** podriem dir que el cas vectorial es troba lluny de derivar-se de manera directa de l'escalar. No obstant això, aconseguim trobar els mecanismes per salvar les diferències i relacionar els espais de multiplicadors a valors vectorials amb el producte tensorial projectiu de Hadamard. Així, veiem com es pot resoldre un problema complicat, dividint-lo en problemes més senzills o prenent camins alternatius, sempre recolzats per un marc teòric que ens garantisca la veracitat de les nostres passes.

La **metodologia** seguida durant la realització del treball ha sigut la següent: en primer lloc es va procedir a l'estudi de distints espais i la seua teoria bàsica per mitjà de la lectura de bibliografia de referència ja siguen els texts clàssics de Duren ([**24**]), Zhu ([**40**]), Axler ([**6**]) com alguns dels últims anys ([**27**]).

Una vegada coneguts els espais clàssics, es va estudiar la teoria de funcions analítiques vectorials desenvolupada al cas d'espais de Hardy i Bergman als treballs del director i col·laboradors.

Finalment es va atacar l'estudi de les tècniques de l'article abans esmentat.

Resumen

El texto que se presenta a continuación trata de establecer un marco teórico en el que poder manejar dos nuevos conceptos (extensión de conceptos ya conocidos): los multiplicadores por coeficientes a través de una aplicación bilineal y el producto proyectivo tensorial de Hadamard. Ambos espacios se ven siempre como espacios de sucesiones a valores vectoriales, esto es, en un espacio de Banach cualquiera. Posteriormente, se estudia la relación entre ellos y se aportan algunos ejemplos.

El punto de partida del proyecto son las clases de espacios introducidas por O.Blasco y M. Pavlović en el trabajo "Coefficient multipliers on spaces of analytic functions" (Revista Mat. Iberoamericana, 2011) donde se formaliza el problema de multiplicadores y se relaciona con ciertos productos tensoriales clásicos, definiendo las mínimas propiedades en espacios de Banach de funciones analíticas para poder desarrollar la teoría de multiplicadores, teniendo como **objetivo** dar la versión vectorial de los mismos.

Con este objetivo en mente, se ha decidido la división del trabajo en cuatro capítulos diferenciados. Los tres primeros fijan el contexto haciendo hincapié en ciertos aspectos que pueden dar al lector una idea más profunda del uso y de las aplicaciones de esta teoría. El último capítulo es el colofón que une los tres anteriores, confiriéndole un sentido único al texto.

De manera más específica, en el primer capítulo se dan los preliminares necesarios para poder abordar el problema que nos ocupa. Se presentan las herramientas precisas para comprender el escrito y sus ejemplos: los espacios de sucesiones con valores vectoriales, $\mathcal{S}(E)$ y los espacios de funciones analíticas en el disco, también con valores vectoriales, $\mathcal{H}(\mathbb{D}, E)$.

En el segunda capítulo se determinan las condiciones mínimas que se van a exigir a los espacios de trabajo, llamados espacios $\mathcal{S}(E)-$admisibles siguiendo la notación de [**16**]. Se da el caso concreto de los multiplicadores con valores en el espacio de operadores, germen del espacio de multiplicadores mediante una aplicación bilineal, B. Para poner de manifiesto la importancia de estos espacios, por un lado se da la relación de los mismos con los espacios sólidos y, por otro, se desarrolla el ejemplo de los espacios de norma mixta generalizados.

El tercero es un breve capítulo donde se dan condiciones específicas para el caso en el que los espacios de sucesiones sean una forma de representación de espacios de funciones analíticas (a través de sus coeficientes de Taylor). De nuevo, siguiendo la notación introducida por [**16**], estos espacios serán notados como $\mathcal{H}(E)-$admisibles. Además se aportan nuevos resultados que serán aplicados a espacios de funciones con valores vectoriales.

Por último, en el cuarto capítulo se detallan las dos construcciones arriba mencionadas: los multiplicadores a través de una aplicación bilineal y el producto tensorial proyectivo de Hadamard. Se ve la relación que existe entre ambas y finalmente se muestran casos particulares del cómputo del producto tensorial proyectivo de Hadamard y se aplica al cálculo de multiplicadores.

A modo de **conclusión** podríamos decir que el caso vectorial está lejos de seguirse de manera directa del escalar, sin embargo logramos encontrar los mecanismos para salvar estas diferencias y relacionar los espacios de multiplicadores a valores vectoriales con el producto tensorial proyectivo de Hadamard. Así, vemos cómo se puede resolver un problema complicado, partiéndolo en problemas más simples o tomando caminos alternativos, siempre respaladados por un marco teórico que nos asegure la veracidad de nuestros pasos.

La **metodología** seguida en la realización del trabajo ha sido la siguiente: en primer lugar, se procedió al estudio de distintos espacios y su teoría básica por medio de la lectura de bibliografía de referencia como los textos clásicos de Duren ([**24**]), Zhu ([**40**]), Axler ([**6**]) y alguno de los últimos años ([**27**]).

Una vez conocidos los espacios clásicos, se estudió la teoría de funciones analíticas vectoriales desarrollada en el caso de espacios de Hardy y Bergman en los trabajos del director y colaboradores.

Finalmente se atacó el estudio de las técnicas del artículo arriba mencionado.

Introduction

From the beginning of the Fourier Analysis, mathematicians try to describe the Fourier (or Taylor) coefficients of functions belonging to classical spaces of integrable (or analytic) functions defined in the unit circle (or disc), and to determine conditions on a sequence for the function with such a sequence of Fourier coefficients to belong to a given space.

The simplest example is Plancherel's Theorem, where the fact that the coefficients are square-sumable is equivalent to the fact that the function is square-integrable. Nevertheless, this complete description only holds for Hilbert spaces. In particular, when the space L^2 is replaced by L^p for $p \neq 2$, the situation changes and even though one can obtain partial results (for instance using interpolation), it is known that belonging to the space L^p whenever $p \neq 2$ can not be described in terms of the size of the coefficients. Other examples of interest, in which being an analytic function automatically improves the conditions on the coefficients, are the so-called Hardy and Paley inequalities, where it is stated that integrable functions in the torus whose Poisson integral on the disc is holomorphic (that is, in the current terminology, functions in the Hardy space H^1) have Fourier coefficients not only converging to zero (as Riemann-Lebesgue's Lemma says), but also verifying the stronger conditions $\sum_n \frac{|\hat{f}(n)|}{n+1} < \infty$ (Hardy's inequality) or $\sum_k |\hat{f}(2^k)|^2 < \infty$ (Paley's inequality). These are two first elementary instances which nowadays are denoted as "multipliers" (sometimes "Fourier multipliers") acting on Hardy spaces. The first steps in this direction go back to the work of Hardy, Littlewood, Paley and Flett among others.

From then on, describing the multiplier spaces between two function spaces has become an interesting object of study for many researchers in Fourier Analysis and Complex Variable. There are two different considerations in this kind of problems: On one hand, given a sequence $(\lambda_j)_j$ of complex numbers and a function f with coefficients $\hat{f}(j)$ (either the Fourier or the Taylor ones, in case f is analytic) belonging to a certain space X, one can generate a new function g with coefficients $\lambda_j \hat{f}(j)$. The aim would be to identify the space Y where this new function belongs to, using the known properties of f and $(\lambda_j)_j$. On the other hand, given two concrete Banach spaces X and Y, one can try to characterize the sequences $(\lambda_j)_j$ that allow them to go from X to Y through the Fourier multipliers.

On a more standard notation, given X and Y two Banach spaces contained continuously in the space $\mathcal{S}$ of all sequences, we want to describe the multipliers space

$$(X, Y) = \{(\lambda_j)_j \in \mathcal{S} : (\lambda_j \hat{f}(j))_j \in Y, \text{ for any } f \sim (\hat{f}(j))_j \in X\}.$$

Taking a look at the previous examples from this perspective, we have $(L^2(\mathbb{T}), L^2(\mathbb{T})) = \ell^\infty$. Also Hardy's inequality, which states that $(\frac{1}{n})_n \in (H^1(\mathbb{T}), \ell^1)$ and Paley's inequality, which says that the sequence $(\lambda_j)_j$, defined as $\lambda_j = 1$ for $j = 2^k$ for some k and $\lambda_j = 0$ otherwise, belongs to $(H^1(\mathbb{T}), \ell^2)$.

In our work, we restrict ourselves to the case of holomorphic functions where we shall identify the function with its Taylor coefficients. However our study will be done for vector-valued holomorphic functions, meaning that we shall allow the abstract situation of the coefficients belonging to another complex Banach space E.

The description of those spaces in the scalar context has been (and still is) an object of desire of a large number of researchers. The historical situation on Hardy spaces can be found in B. Osikiewicz' work (see [36]) and a collection of results and techniques to use on Bergman and mixed-norm spaces is gathered up in the works of M. Jetvić and I. Jovanović (see [29]) and O. Blasco (see [11]).

One of the most important recent results in this area that is inspiration and motivation for our study, is the one obtained in *Multipliers of H^p and BMO*, by M. Mateljeviċ and M. Pavlović ([34]), where an identification between the multipliers space $(H^1, BMOA)$ and the Bloch space, $\mathcal{B}loch$, is given. That is to say

$$(H^1, BMOA) = \mathcal{B}loch.$$

This result was extended by O. Blasco in [11] and later an alternative proof for functions taking values in a Banach space was also achieved in [12].

The interest on the study of the space of multipliers between spaces of vector-valued sequences or functions appears closely related to the geometric properties of Banach spaces. Several results on vector-valued multipliers and their connection with geometry of Banach spaces and absolutely summing operators (see [21] for the definition) can be found in [3, 4, 5, 15, 13, 14].

Let us formulate the general abstract situation we shall try to analyze. We use the notation X_E for certain space of analytic functions with values in a given complex Banach space E, regarded (via Taylor coefficients) as a subspace of the space of sequences with values in E, to be denoted $\mathcal{S}(E)$. Now, given complex Banach spaces E and F, our purpose is to study the space of multipliers between X_E and X_F, understood as a space of sequences with values in the space of bounded linear operators $\mathcal{L}(E, F)$ defined by

$$(X_E, X_F) = \{(\lambda_j)_j \in \mathcal{S}(\mathcal{L}(E, F)) : (\lambda_j(\hat{f}(j)))_j \in X_F, \text{ for any } f \sim (\hat{f}(j))_j \in X_E\}.$$

Of course the vector-valued interpretation is far from being straightforward. To realize the difference, it is enough to take a look at the results appearing in [7], where the geometry of the underlying Banach space plays an important role for the classical results to hold in the vector-valued setting, or to [12] where the expected extension of the result $(H^1, BMOA) = \mathcal{B}loch$, is shown not true in general. Actually the inclusion $(H^1(E), BMOA(F)) \subseteq \mathcal{B}loch(\mathcal{L}(E, F))$ always holds but it is proved that only under certain hypothesis over E and F (complex Banach spaces) it holds that

$$(H^1(E), BMOA(F)) = \mathcal{B}loch(\mathcal{L}(E, F)). \tag{0.1}$$

Recently O. Blasco and M.Pavlović (see [16]) have tried to systematize the study of multipliers between spaces of analytic functions (in the scalar-valued case) in an abstract context and have used some techniques based upon the Hadamard tensor product, which can be used for a big family of spaces. They introduce some classes of spaces of sequences and of analytic functions where some multiplier results can be shown (and which of course inlcude all the classical spaces such as Hardy, Bergman or mixed-norm spaces). We recall here the notion of Hadamard tensor product which was the main tool in such research. Given X and Y Banach spaces of power series we denote by $X \circledast Y$ the space of functions f that can be represented as formal power series of Hadamard products, that is $f(z) = \sum_n x_n * y_n(z) = \sum_n \sum_j x_n(j) y_n(j) z^j$, where $(x_n)_n \subseteq X$ and $(y_n)_n \subseteq Y$, verifying $\sum_n \|x_n\|_X \|y_n\|_Y < \infty$. This construction is intimately connected to the multiplier space through the formula

$$(X \circledast Y, Z) = (X, (Y, Z)). \tag{0.2}$$

In this monograph we shall study the vector-valued analogue of such result, giving even a more general approach, where the notion of vector-valued multiplier as a sequence of operators is extended to a general case where the action of operator and vector is replaced by a general bilinear map, and also the classical convolution and Hadamard tensor product is generalized for bilinear maps. Namely, given E, F, G complex Banach spaces and $B : E \times F \to G$ a bounded bilinear map, we consider a new space

$$(X_F, X_G)_B = \{(\lambda_j)_j \in \mathcal{S}(E) : (B(\lambda_j, \hat{f}(j)))_j \in X_G, \text{ for all } f \sim (\hat{f}(j))_j \in X_F\}.$$

Notice that for $E = \mathcal{L}(F, G)$ and $B(\lambda_j, \hat{f}(j)) = \lambda_j(\hat{f}(j))$, we recover the case of operator-valued multipliers. This study represents an original approach that includes the already known results on multiplier spaces and provides with some new applications.

Similarly one can generalize $X \circledast Y$ as follows: Given a bounded bilinear map $B : E \times F \to G$, we define the B-convolution between $f \in X_E$ and $g \in X_F$ as

$$f *_B g(z) = \sum_j B(\hat{f}(j), \hat{g}(j))z^j.$$

Now, what we call the Hadamard projective tensor product $X_E \circledast_B X_F$ is defined as the space of functions that can be represented as a formal sum of B- convolution products, $h(z) = \sum_n f_n *_B g_n(z)$, where $f_n \in X_E$ and $g_n \in X_F$ for any $n \in \mathbb{N}$, verifying $\sum_n \|f_n\|_{X_E}\|g_n\|_{X_F} < \infty$.

Our aim is to show the use of these constructions, getting new results for both sequence spaces and spaces of analytic functions (identified with sequence spaces) as well as to show the extension of the formula (0.2)

$$(X_E \circledast_{B_1} X_F, X_G)_{B_2} = (X_E, (X_F, X_G)_{B_3})_{B_4}$$

for bilinear maps $B_i,\ i = 1, 2, 3, 4$ that satisfy certain conditions (see Theorem 4.26).

The monograph is divided into four chapters. We are going to give some detail on what the reader is going to find in each one of them.

The first one is of preliminary character. We simply introduce the space of vector-valued sequences $\mathcal{S}(E)$ and the space of analytic functions on the disc which take values in a Banach space E, $\mathcal{H}(\mathbb{D}, E)$ and define the sequence spaces and function spaces we are going to work with, both in their scalar and vector-valued version. As particular cases to outline appear the mixed-norm sequence spaces considered with values in a Banach space (a generalization of the mixed-norm spaces $\ell(p, q)$ introduced by Kellogg). We will give a first definition of these spaces that will be extended and studied in the following chapter. We also consider the case of vector-valued functions obtained from a sequence space with its own norm, that is

$$X[E] = \{(x_j)_{j\geq 0} \in \mathcal{S}(E) : \|(\|x_j\|_E)_j\|_X < \infty\}.$$

This space is specially interesting when we consider X a space of analytic functions such as $H^p(\mathbb{D})$ (Hardy) or $A^p(\mathbb{D})$ (Bergman), with $1 \leq p < \infty$, and we compare it with the more natural vector-valued version of the space $H^p(\mathbb{D}, E)$ (resp. $A^p(\mathbb{D}, E)$)(see page 23). We will prove that $H^p(\mathbb{D})[E]$ and $A^p(\mathbb{D})[E]$ need not to coincide with them (see Proposition 1.38).

The second chapter is devoted to the notion of $\mathcal{S}(E)$-admissiblility following the notation introduced in [16]. This notion establishes the minimum conditions we need to provide on the abstract spaces to be able to work with them in the setting of multipliers. We consider the classical operator-valued case of multipliers with values in the space of linear bounded mappings, as well as the notion of "solid" spaces in the vector-valued setting, meaning $X_E = X_E^S$, where

$$X_E^S = \{(x_j)_j \in \mathcal{S}(E) : (\alpha_j x_j)_j \in X_E, \text{ for all } (\alpha_j)_j \in \ell^\infty\}.$$

Let us mention one nice connection discovered in Proposition 2.16, namely if X, Y are solid spaces, then

$$(X[E], Y[F]) = (X, Y)[\mathcal{L}(E, F)].$$

In this chapter we also develop a new family of $\mathcal{S}(E)$-admissible spaces: the generalized mixed-norm spaces, to be denoted $\ell^{\mathcal{I}}(p, q, E)$. These are spaces of sequences $(a_i)_{i\in\Lambda_{\mathcal{I}}}$, where the entries are in a Banach space and such that they verify

$$\left(\left(\sum_{j\in I_k} \|a_j\|_E^p\right)^{1/p}\right)_k \in \ell^q,$$

for $1 \le p, q \le \infty$ and where $\mathcal{I}$ is a family of pairwise disjoint intervals contained in $\mathbb{N}_0$ with the notation $I_k = \mathbb{N}_0 \cap [n_k, n_k')$ for $n_k < n_k' \le n_{k+1}$ and $\Lambda_{\mathcal{I}} = \cup_{k\in\mathbb{N}_0} I_k$. The weak version $\ell_w^{\mathcal{I}}(p, q, E)$ of these spaces, consisting on the sequences of elements in E such that

$$\left(\left(\sum_{j\in I_k} |\langle a^*, a_j\rangle|^p\right)^{1/p}\right)_k \in \ell^q$$

for any $a^* \in E^*$ is also considered.

We close the chapter with a detailed study of the space of multipliers between generalized mixed-norm spaces

$$(\ell^{\mathcal{I}}(p, q, E), \ell^{\mathcal{J}}(r, s, F)),$$

whenever $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$ for $\mathcal{I}$, $\mathcal{J}$ two families of intervals in $\mathbb{N}_0$, $1 \le p, q, r, s \le \infty$ and E, F complex Banach spaces (Theorem 2.63) and an application of the resulting characterization to the space of multipliers between $H(p, q, \rho)$ spaces, that is spaces of functions such that

$$\|F\|_{H(p,q,\rho)} = \left(\int_0^1 M_p^q(F, r)\frac{\rho(1-r^2)}{1-r^2} r dr\right)^{1/q} < \infty$$

(see [8, Definition 2]).

The third one is a brief chapter where we focus on spaces of vector-valued analytic functions. We introduce the notion of $\mathcal{H}(E)-$admissibility, again following the notation on [16]. A Banach space X_E of holomorphic functions from the unit disc into a Banach space E is called $\mathcal{H}(E)-$admissible whenever it is continuously contained into the space $\mathcal{H}(\mathbb{D}, E)$ and the inclusion $\mathcal{H}(\mathrm{R}\mathbb{D}, E)$ (where $\mathrm{R}\mathbb{D}$ stands for the disc of radius R) into X_E is also continuous for all $R > 1$. Most of the classical spaces are shown to be $\mathcal{H}(E)-$admissible. Some extra properties are also cosidered: In particular X_E is said to be "homogeneous" if it satisfies that

- (i) if $f \in X_E$ and $|z| = 1$, then $\|f_z\|_{X_E} = \|f\|_{X_E}$ and
- (ii) if $0 < r < 1$, then $\sup_{|z|=r} \|f_z\|_{X_E} \le K\|f\|_{X_E}$ where K is independent of f, r, and $f_z(w) = f(zw)$.

A special class of homogeneous spaces are those which verify the Fatou property (FP), given by the condition that there exists $A > 0$ s.t. if $(f_n)_n \in X_E$ with $\sup_n \|f_n\|_{X_E} \le 1$ verifying $f_n \longrightarrow f$ in $\mathcal{H}(\mathbb{D}, E)$, then $f \in X_E$ with $\|f\|_{X_E} \le A$. A particularly interesting construction consists in defining $\tilde{X}_E$ as the space of functions in $\mathcal{H}(\mathbb{D}, E)$ verifying

$w \mapsto f_w \in H^\infty(\mathrm{R}\mathbb{D}, X_E)$. This space turns out to be always homogeneous. We obtain in Proposition 3.24

$$X_E = \tilde{X}_E \text{ with equivalent norms } \Leftrightarrow X_E \text{ is homogeneous with } (FP).$$

Finally, in the fourth chapter we deepen into the two constructions mentioned above: multipliers through a bilinear map and Hadamard projective tensor product. We will prove that, under certain conditions on the corresponding bilinear map, both spaces keep the admissibility (either simply considered as spaces of sequences -$\mathcal{S}(E)-$admissibility- or as sequence spaces coming from a space of analytic functions -$\mathcal{H}(E)-$admissibility-). Furthermore, we analyze the relationship between the two of them through the formula

$$(X_E \circledast_{B_1} X_F, X_G)_{B_2} = (X_E, (X_F, X_G)_{B_3})_{B_4},$$

where $B_1 : E \times F \to E\hat{\otimes}_\pi F$, $B_2 : \mathcal{L}(E\hat{\otimes}_\pi F, G) \times E\hat{\otimes}_\pi F \to G$, $B_3 : \mathcal{L}(F, G) \times F \to G$ and $B_4 : \mathcal{L}(E, \mathcal{L}(F, G)) \times E \to \mathcal{L}(F, G)$ and see the particular cases that arise whenever we consider one of the Banach spaces E, F, G a field. We finish showing particular cases of the computation of the Hadamard projective tensor product, such as

$$A^1(\mathbb{D}) \circledast_{B_0} H^1(\mathbb{D}, E) = A^1(\mathbb{D}, E)$$

and

$$H^1(\mathbb{D}) \circledast_{B_0} H^1(\mathbb{D}, L^p(\mu)) = \mathfrak{B}^1(\mathbb{D}, L^p(\mu))$$

for $1 \leq p \leq 2$. If we consider $p' = \frac{p}{p-1}$, that is, the conjugate exponent of p, then the last computation together with Proposition 4.38, which says

$$(H^1(\mathbb{T}, L^p(\mu)), H^\infty(\mathbb{D}))_{\mathcal{D}} = BMOA(\mathbb{T}, L^{p'}(\mu))$$

and

$$(D^{-1}A^1(\mathbb{D}, E), H^\infty(\mathbb{D}))_{\mathcal{D}} = \mathcal{B}loch(\mathbb{D}, E^*),$$

where the $\mathcal{D}$ indicates we are using the map $B_{\mathcal{D}} : E^* \times E \longrightarrow \mathbb{K}$, $(x^*, x) \mapsto \langle x^*, x\rangle$, lets us recover results such as the one mentioned in (0.1).

The content related to the construction of new Kellogg's type sequence spaces (the scalar version) as well as the spaces of multipliers between them appear in the published paper [**17**].

The results on Chapter 2 refering to solid spaces and Köthe duals as well as the first section of Chapter 3 and almost the totality of Chapter 4 are submitted and accepted for its publication in [**18**].

CHAPTER 1

Preliminaries

1. Basic results on functional analysis

In order to make this text as self-contained as possible, this section is devoted to some of the basic definitions and concepts needed in our work as well as to fix the notation we will be using in the following.

For Z a locally convex space, we have that a collection Λ of zero neighbourhoods, $\varepsilon(0)$, is a fundamental system of zero neighbourhoods if $\forall U \in \varepsilon(0)\ \exists V \in \Lambda$ and $\epsilon > 0$ such that $\epsilon V \subseteq U$.
A family of seminorms, $(p_j)_{j\in\mathbb{N}_0}$, is called a fundamental system of seminorms if sets $U_j := \{f \in Z : p_j(f) < 1\}$ form a fundamental system of zero neighbourhoods.

We denote by $\mathcal{S}$ the space of sequences $f = (\alpha_j)_{j\in\mathbb{N}_0}$, where $\alpha_j \in \mathbb{K}$, endowed with the locally convex topology given by means of the seminorms $p_j(f) = |\alpha_j|$, $j \in \mathbb{N}_0$, where we use the notation $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We shall think of f as a formal power series, that is $f(z) = \sum_{j\in\mathbb{N}_0} \alpha_j z^j$ and most of the time we will write $\hat{f}(j)$ instead of α_j and then consider $\mathcal{S}$ as the space of all formal power series.
Any locally convex space with a countable fundamental system of zero neighbourhoods is metrizable. In particular $\mathcal{S}$ is a complete metrizable space.

A sequence $(f_n)_n \subset \mathcal{S}$ converges to $f \in \mathcal{S}$ if and only if $p_j(f - f_n) \to 0\ \forall j \geq 0$ as $n \to \infty$, if and only if $|\hat{f}(j) - \hat{f}_n(j)| \to 0$ as $n \to \infty$ for all $j \geq 0$. That is, convergence in $\mathcal{S}$ is coordinatewise convergence.

Given $X \subset \mathcal{S}$, we will write X^0 for the closure of the polynomials in the $X-$norm.

Another interesting locally convex space to consider is $\mathcal{H}(\mathbb{D})$, the space of analytic functions on the unit disc $\mathbb{D} \subset \mathbb{C}$, that is, functions $f : \mathbb{D} \to \mathbb{C}$, $f = \sum_{j\in\mathbb{N}_0} \hat{f}(j)e_j$, where $e_j(z) = z^j$, such that $\limsup_j \sqrt[j]{|\hat{f}(j)|} \leq 1$. It can be regarded as a vector subspace of $\mathcal{S}$ via the Taylor coefficients. Naturally, every sequence $(\alpha_j)_j \in \mathcal{S}$ which satisfies the condition $\limsup_j \sqrt[j]{|\alpha_j|} \leq 1$ can also be identified with an analytic function in $\mathbb{D}$.

Let $R\mathbb{D}$ denote the open disc of radius $R > 0$ centered at zero (we put $1\mathbb{D} = \mathbb{D}$). We write $\mathcal{H}(R\mathbb{D})$ for the space of all functions analytic in $R\mathbb{D}$ endowed with the '$\mathcal{H}-$topology', i.e., the topology of uniform convergence on compact subsets of $R\mathbb{D}$. This topology can be described by the family of seminorms

$$M_\infty(r, f) = \sup_{|z|=r} |f(z)|,$$

$0 < r < R$. Therefore a Banach space X is continuously contained in $\mathcal{H}(R\mathbb{D})$, $X \hookrightarrow \mathcal{H}(R\mathbb{D})$ if for any $0 < r < R$ there exists a constant $A_r > 0$ such that

$$M_\infty(r, f) \leq A_r \|f\|_X,\ f \in X.$$

Conversely, we will write $\mathcal{H}(R\mathbb{D}) \hookrightarrow X$ if there exists $s \leq R$ and $B_s > 0$ such that $\mathcal{H}(R\mathbb{D})$ is continuously contained in X, that is

$$\|f\|_X \leq B_s M_\infty(s, f)$$

for any $f \in \mathcal{H}(R\mathbb{D})$.

Since $\mathcal{H}(\mathrm{r}\mathbb{D}) \subset \mathcal{S}$, we see that, formally, there are two topologies on $\mathcal{H}(\mathrm{r}\mathbb{D})$, $\mathcal{H}-$ and $\mathcal{S}-$topology. However it is well known and easy to see that they coincide on $\mathcal{H}(\mathrm{r}\mathbb{D})$.

Recall that for X a normed space, the dual space X^* is the set of all linear continuous functionals from X into the base field $\mathbb{K}$. It is a normed vector space by means of

$$\|x^*\| = \sup_{\|x\|_X \le 1} |\langle x^*, x\rangle|,$$

where $\langle x^*, x\rangle = x^*(x)$. This norm gives rise to a new topology on X^* under which the space becomes a Banach space.

Also recall that on each Banach space X there exists a weak topology usually denoted by w. For each point $x_0 \in X$ its basis of neighbourhoods is defined as

$$U(x_0; \epsilon, x_1^*, \cdots, x_n^*) = \left\{x^* \in X^* : |\langle x_j^*, x - x_0\rangle| < \epsilon \text{ for } j = 1, \cdots, n\right\}$$

where $x_1^*, \cdots, x_n^*$ is an arbitrary finite set in X^* and ϵ is an arbitrary positive number. Obviously this defines a locally convex topology on X. It is the coarsest topology that makes a linear map from X into $\mathbb{K}$ remain continuous and it is characterized by the condition that a sequence $(x_n)_n$ converges to $x \in X$ in the $w-$topology iff $\langle x^*, x_n\rangle$ converges to $\langle x^*, x\rangle$ for every $x^* \in X^*$.

The dual space X^* can as well be endowed with the so-called weak-star topology, to be denoted w^*, that is the topology induced by the embedding $X \subseteq X^{**}$. For each point $x_0^* \in X^*$ its basis of neighbourhoods is defined as

$$U(x_0^*; \epsilon, x_1, \cdots, x_n) = \{x \in X : |\langle x^* - x_0^*, x_j\rangle| < \epsilon \text{ for } j = 1, \cdots, n\}$$

where $x_1, \cdots, x_n$ is an arbitrary finite set in X and ϵ is an arbitrary positive number. Clearly this also defines a locally convex topology on X^*. This topology is characterized by the condition that a sequence $(x_n^*)_n$ converges to $x^* \in X^*$ in the w^*-topology iff $\langle x_n^*, x\rangle$ converges to $\langle x^*, x\rangle$ for every $x \in X$.

Of course w-convergence implies w^*-convergence and, in case X is reflexive ($X = X^{**}$), the converse direction holds.

We now list some other classical results of functional analysis to be used in our work. Most of them have many different (and sometimes more general) versions, but for our purposes it is enough to consider them as below.

Theorem. *A (**Alaoglu**) (see [37])*
The closed unit ball B_{X^*} of X^* is compact in the w^*-topology.

The following theorems can be found in [23].

Theorem. *B (**Open Mapping Theorem**)*
Let X, Y be Banach spaces and $T : X \to Y$ a linear bounded operator such that $T(X) = Y$. Then T is an open map, i.e. the image of an open set is an open set.

As a consequence we have

Theorem. *C (**Closed Graph Theorem**)*
Given $T : X \to Y$ a linear map between Banach spaces, the set

$$Graph(T) := \{(x, T(x)) : x \in X\}$$

is closed in the product topology of $X \times Y$ if and only if T is continuous.

Theorem. *D (**Banach-Steinhaus**)*
Consider $(T_n)_{n\in\mathbb{N}}$ a family of linear operators between Banach spaces, $T_n : X \to Y$. Assume that for every $x \in X$ $\sup_n \|T_n(x)\|_Y < \infty$. Then $\sup_n \|T_n\| < \infty$.

In particular we get that the pointwise limit of a sequence of linear operators (if it exists everywhere) is a linear operator.

THEOREM. *E* ***(Hahn-Banach)***
Let X be a Banach space and $Y \subseteq X$. Consider $y^* \in Y^*$. Then there exists $x^* \in X^*$ such that $\|x^*\| = \|y^*\|$ and $x^*(y) = y^*(y)$ for every $y \in Y$. In particular we get

$$\|x\| = \sup_{\|x^*\| \leq 1} |\langle x^*, x\rangle|$$

for $x \in X$.

2. Multipliers and tensors: the scalar-valued case

The aim of this section is to motivate the results presented in the foregoing chapters and to provide further on some additional information.

A theory requires and feeds on its examples. Thus, let's start bringing back to our memory the most well-known sequence spaces and spaces of analytic functions. It is convenient to define them right here so as to have sufficiently many examples which will be considered in abstract terms in the following chapters. Some definitions and basic properties will be reviewed later in the appropiate sections.

Consider $0 < p \leq \infty$. The most famous sequence space might be the ℓ^p space, consisting on those sequences $a = (a_n)_{n \in \mathbb{N}_0} \in \mathcal{S}$ verifying

$$\|a\|_p = \left(\sum_n |a_n|^p\right)^{1/p} < \infty$$

if $p < \infty$, and

$$\|a_n\|_\infty = \sup_n |a_n|.$$

This is nothing but a generalization of the Euclidean norm. Indeed, it defines a norm for $1 \leq p \leq \infty$ and the space ℓ^p becomes a Banach space. For $0 < p < 1$ it doesn't define a norm, but rather a metric, $d(x, y) = \|x - y\|_p^p$.

The following inequality, called **Hölder's inequality**, is of fundamental importance in our work.

For $1 \leq p \leq \infty$ define p' to be the so-called conjugate exponent of p, that is, let p' be such that it verifies $\frac{1}{p} + \frac{1}{p'} = 1$, with the convention $1' = \infty$ and $\infty' = 1$. Then for $(a_n)_n$, $(b_n)_n \in \mathcal{S}$ it is verified

$$\sum_n |a_n b_n| \leq \left(\sum_n |a_n|^p\right)^{1/p} \left(\sum_n |b_n|^{p'}\right)^{1/p'},$$

with the natural modifications for $p = \infty$ or $p' = \infty$.

Let us introduce some notation before going on.

REMARK 1.1. Given $0 < u, v \leq \infty$ we denote

$$u \ominus v = \begin{cases} \frac{uv}{u-v}, & \text{if } v < u < \infty; \\ v, & \text{if } u = \infty; \\ \infty, & \text{if } u \leq v. \end{cases}$$

This notation was introduced in [20].

The inequality can be generalized by taking $1 \leq p, r \leq \infty$, $\frac{1}{p} + \frac{1}{p \ominus r} < 1$

$$\left(\sum_n |a_n b_n|^r\right)^{1/r} \leq \left(\sum_n |a_n|^p\right)^{1/p} \left(\sum_n |b_n|^{p\ominus r}\right)^{1/p\ominus r}.$$

It can be proved that $(\ell^p)^* = \ell^{p'}$ for $1 < p < \infty$. Moreover, the space ℓ^p is reflexive, i.e., $(\ell^p)^{**} = \ell^p$.

Another natural sequence space to consider is c_0, defined as the space of all sequences converging to zero, with norm identical to $\|\cdot\|_\infty$. It is a closed subspace of ℓ^∞, hence a Banach space. The dual of c_0 is ℓ^1 and the dual of ℓ^1 is ℓ^∞.
The space c_{00} is the space of all sequences finitely non-zero, that is, $a \in \ell^\infty$ such that $a_n = 0$ for almost every $n \in \mathbb{N}$. It is a dense space in c_0 with respect to $\|.\|_\infty$. Then, of course, its dual is ℓ^1.

The following order relationship is verified

$$c_{00} \subset \ell^1 \subset \ell^2 \subset \cdots \subset c_0 \subset \ell^\infty.$$

We can go one step further and consider the following spaces, defined by Kellogg.

Definition 1.2. (**Mixed-norm sequence spaces**) The space of mixed-norm sequences consists of the space of sequences $(a_n)_{n\in\mathbb{N}}$ verifying

$$\left(\left(\sum_{j\in I_k} |a_j|^p\right)^{1/p}\right)_k \in \ell^q,$$

under the norm

$$\|a\|_{p,q} = \left(\sum_{k=0}^{\infty}\left(\sum_{j\in I_k} |a_j|^p\right)^{q/p}\right)^{1/q}$$

with the obvious modifications for $p = \infty$ or $q = \infty$. We will write $\ell(p,q)$ for this space, where $1 \le p, q \le \infty$ and $\mathcal{I}$ is a collection of disjoint intervals in $\mathbb{N}_0$, say $I_k = [2^k, 2^{k+1}) \cup \mathbb{N}_0$. Of course $\ell(p,p) = \ell^p$.

As expected

$$(\ell(p,q))^* = \ell(p',q')$$

for $1 \le p, q < \infty$ (see [**32**]).

As for the spaces of analytic functions, recall that if Σ is a $\sigma-$algebra over a set Ω, then a function $\mu : \Sigma \to \mathbb{R}$ is called a finite measure if it satisfies that $0 \le \mu(A) < \infty$ for $A \in \Sigma$ non-empty, $\mu(\emptyset) = 0$ and is countable additive,that is $\mu(\cup_{i\in\mathbb{N}} A_i) = \sum_{i\in\mathbb{N}} \mu(A_i)$ for a collection of pairwise disjoint sets $(A_i)_{i\in\mathbb{N}} \subset \Sigma$. The triple (Ω, Σ, μ) is called a **measure space**.

Given a measure space (Ω, Σ, μ) we define $L^p(\Omega, d\mu)$, $(0 < p < \infty)$, as the space of (equivalent classes of) p-integrable functions over the set Ω with respect to the measure μ, i.e., such that

$$\|f\|_p = \left(\int_\Omega |f(\omega)|^p d\mu(\omega)\right)^{1/p} < \infty.$$

In case $p = \infty$, $L^\infty(\Omega, d\mu)$ is the space of functions such that

$$\|f\|_\infty = \sup_{\omega\in\Omega} |f(\omega)|.$$

We are mostly going to work with Lebesgue spaces on the unit disc, noted $L^p(\mathbb{D}, dA)$ or simply $L^p(\mathbb{D})$, consisting on the $p-$integrable functions with respect to the normalized area function ($dA(z) = rdr\frac{d\theta}{2\pi}$, $z = re^{i\theta} \in \mathbb{D}$), and with the spaces $L^p(\mathbb{T}, d\sigma) = L^p(\mathbb{T})$, where $\mathbb{T}$ stands for the unit circle and $d\sigma(\theta) = d\theta/2\pi$ for the normalized arc-length measure.
Since the following properties hold for every $L^p(\Omega, \mu)$, we will simply write L^p and $\|\cdot\|_p$ and we will specify $L^p(\mathbb{D})$ or $L^p(\mathbb{T})$ whenever is necessary. It is known that $L^p \subset L^q$ for $p > q$.

For $1 \leq p \leq \infty$ they are Banach spaces. If $0 < p < 1$, the triangle inequality is not satisfied, although it can be replaced by $\|f+g\|_p^p \leq \|f\|_p^p + \|g\|_p^p$ and the space is complete considering the metric given by $d(f,g) = \|f-g\|_p^p$.

Hölder's inequality also holds for integrable functions. Given any two functions f, g, $1 \leq p \leq \infty$ and p' its conjugate exponent,

$$\|fg\|_1 \leq \|f\|_p \|g\|_{p'}.$$

This result helps us to prove that $(L^p)^* = L^{p'}$ for $1 < p < \infty$.
Again, taking $1 \leq p, r \leq \infty$, one gets a generalized version

$$\|fg\|_r \leq \|f\|_p \|g\|_{p \ominus r}.$$

Many problems of analysis center upon analytic functions with restricted growth near the boundary. Thus, given a function f analytic in the unit disc $\mathbb{D}$, it is natural to consider the integral means defined as

$$M_p(r,f) := \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}$$

for $0 < p < \infty$, and

$$M_\infty(r,f) = \sup_{|z|=r} |f(z)|$$

in case $p = \infty$. It is known that the integral means are no decreasing functions of r (see [24]).
We will say that $f \in H^\infty$ if f is in $\mathcal{H}(\mathbb{D})$ and

$$\sup_{|z|<1} |f(z)| < \infty,$$

that is to say H^∞ is the space of all bounded analytic functions in $\mathbb{D}$. If among those functions, we consider those that are continuous in the torus, we will be talking of the disc algebra, $A(\mathbb{D})$. Obviously $A(\mathbb{D}) \subseteq H^\infty$.

In the case $0 < p < \infty$, if $M_p(r,f)$ stays bounded as $r \to 1$, then f is said to belong to the Hardy space $H^p(\mathbb{D})$, shortly H^p.
Each H^p class is a linear space, preserved under addition and scalar multiplication. The quantity

$$\|f\|_{H^p} = \lim_{r\to 1} M_p(r,f)$$

is called the norm of f and it is a true norm if $1 \leq p < \infty$ under which H^p becomes a Banach space.
If we consider the boundary function $f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$, the norm of f in H^p can be identified with the norm of the boundary function in $L^p(\mathbb{T})$. Thus, H^p can be identified with the closed subspace of $L^p(\mathbb{T})$, consisting of those functions that verify $\hat{f}(n) = 0,\ n < 0$, where $\hat{f}(n) = \frac{1}{2\pi}\int_{\mathbb{T}} f(e^{in\theta})e^{-in\theta}d\theta$.
Each function in H^p can be approximated in norm by polynomials. Thus H^p is characterized as the closure of polynomials in the space $L^p(\mathbb{T})$. An equivalent statement is that the **dilations** $f_r(z) = f(rz)$ tend to f in H^p−norm as r increases to 1.
Of course $H^q \subseteq H^p$ for $0 < p < q \leq \infty$.

The Bergman space $A^p(\mathbb{D})$ (shortly, A^p) consists of all functions f analytic on the unit disc for which

$$\|f\|_{A^p} = \left(\int_{\mathbb{D}} |f(z)|^p dA(z) \right)^{1/p} < \infty,$$

$0 < p < \infty$. The quantity $\|f\|_{A^p}$ is called the norm of f although it is a true norm only for $1 \leq p$. As in the Lebesgue spaces, for $0 < p < 1$ the triangle inequality is replaced by $\|f+g\|_{A^p}^p \leq \|f\|_{A^p}^p + \|g\|_{A^p}^p$ and the space is complete considering $d(f,g) = \|f-g\|_{A^p}^p$.

For $p \geq 1$, the space A^p is a closed subspace of L^p and therefore a Banach space. Notice that functions in these spaces cannot grow too rapidly near the boundary. It is also a remarkable fact that, in Bergman spaces, norm convergence implies locally uniform convergence. In other words, $A^p \subset \mathcal{H}(\mathbb{D})$, so if f_n, f are in A^p and $\|f_n - f\|_{A^p} \to 0$ as $n \to \infty$, then $f_n(z) \to f(z)$ on each compact subset of $\mathbb{D}$.

Considering the limiting cases as $p \to \infty$ on Bergman and Hardy spaces, we come to Bloch and BMOA spaces, respectively. An analytic function in $\mathbb{D}$, f, is said to belong to the Bloch space, to be denoted $\mathcal{B}loch$, if

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)| < \infty.$$

Equivalently, $f \in \mathcal{B}loch$ if

$$\sup_{z\in\mathbb{D}} (1-|z|)|f'(z)| < \infty.$$

This quantity however is not a true norm, since it identifies functions that differ by a constant. Hence we define the norm on this space to be $\|f\|_{\mathcal{B}loch} = \|f\|_{\mathcal{B}} + |f(0)|$. The $\mathcal{B}loch$ space becomes a Banach space under this norm.

The condition $\|f\|_{\mathcal{B}} < \infty$ (or its equivalent) can be replaced by

$$\sup_{z\in\mathbb{D}} (1-|z|)|Df(z)| < \infty,$$

where $Df(z) = zf'(z) + f(z) = \sum_n (n+1)\hat{f}(n)z^n$, for $f(z) = \sum_n \hat{f}(n)z^n$. Indeed, if f verifies that $\|f\|_{\mathcal{B}} < \infty$, taking into account that

$$f(z) - f(0) = \int_0^z f'(t)dt,$$

with the change $t = zs$ we obtain the inequality

$$M_\infty(r,f) \leq |f(0)| + rM_\infty(r,f')$$

and thus

$$M_\infty(r,Df) = M_\infty(r, zf'(z) + f(z)) \leq rM_\infty(r,f') + M_\infty(r,f) \leq 2rM_\infty(r,f') + |f(0)|.$$

On the other hand, if f is such that $\sup_{z\in\mathbb{D}} (1-|z|)|Df(z)| < \infty$, considering the fact that $Df(z) = (zf(z))'$ we obtain

$$f(z) = \int_0^1 Df(zs)ds,$$

therefore

$$M_\infty(r,f) \leq M_\infty(r,Df)$$

and

$$M_\infty(r,f') = \frac{1}{r}M_\infty(r,Df-f) \leq \frac{2}{r}M_\infty(r,Df).$$

For each $m \in \mathbb{N}$ it is easy to see that $f \in \mathcal{B}loch$ iff $z^m f \in \mathcal{B}loch$ and the fact that we can work with $f^{(m)}$ instead of f', namely $f \in \mathcal{B}loch$ iff $\sup_{0<r<1} (1-r)^m M_\infty(r, f^{(m)}) < \infty$. Now define

$$\mathcal{D}^m f(z) = \sum_n (n+m)\cdot \ldots \cdot (n+1)\hat{f}(n)z^n = (z^m f(z))^{(m)}. \tag{1.1}$$

Therefore $f \in \mathcal{B}loch$ iff $\sup_{0<r<1} (1-r)^m M_\infty(r, \mathcal{D}^m f) < \infty$.

The $\mathcal{B}loch$ space is the dual space of A^1, which can be identified as well with the dual of the little Bloch space, $\mathcal{B}loch_0$, (in the usual notation, $(A^1)^* = \mathcal{B}loch$ and $A^1 = (\mathcal{B}loch_0)^*$) defined as the subspace of $\mathcal{B}loch$ consisting of functions with the property that

$$\lim_{|z|\to 1^-} (1-|z|^2)|f'(z)| = 0.$$

In our notation we have that that $\mathcal{B}loch_0 = \mathcal{B}loch^0$, i.e., it is the closure of the polynomials in the $\mathcal{B}loch$ norm. It is a closed subspace, thus a Banach space.

The $BMOA$ space, is the space consisting of analytic functions of bounded mean oscillation. The mean oscillation is defined to be

$$\frac{1}{|I|}\int_I |f(e^{i\theta}) - f_I|\frac{d\theta}{2\pi}$$

where I is an interval $I \subseteq [0, 2\pi)$, $|I|$ is its normalized Lebesgue measure and $f_I = \frac{1}{|I|}\int_I f(e^{i\theta})\frac{d\theta}{2\pi}$. The space BMOA is the space of integrable functions in the torus, with Fourier coefficients $\hat{f}(n) = 0$ for $n < 0$ and such that

$$\|f\|_{BMOA} = |f(0)| + \sup_{I\subseteq[0,2\pi)} \frac{1}{|I|}\int_I |f(e^{i\theta}) - f_I|\frac{d\theta}{2\pi} < \infty.$$

As in the $\mathcal{B}loch$ space, we have added a constant to obtain a true norm so that it becomes a Banach space. It is known, by Fefferman's duality result, that $(H^1)^* = BMOA$.

Finally, the space of analytic functions of vanishing mean oscillation $VMOA$ is a separable subspace of $BMOA$, which is the closure in $BMOA$ of the set of polynomials, $VMOA = (BMOA)^0$. We say that a function $f \in L^1(\mathbb{T})$ is in the space $VMOA$ if it is analytic and

$$\lim_{|I|\to 0} \sup_{I\subseteq[0,2\pi)} \frac{1}{|I|}\int_I |f(e^{i\theta}) - f_I|\frac{d\theta}{2\pi} = 0.$$

The dual space of $VMOA$ can be identified with H^1. Thus, $BMOA$ can be identified with the second dual of $VMOA$.

The reader is referred to [**24, 25, 28**] for more information on these spaces.
As we said, all of them can be regarded as subspaces of $\mathcal{S}$ (in the case of function spaces, via their Fourier or Taylor coefficients).

To develop a general theory of analytic functions, several authors have formulated some natural conditions which hold in most classical spaces, but are too restrictive to include many other interesting spaces. That is the reason why we will focus on the conditions proposed by O. Blasco and M. Pavlović in [**16**].

Definition 1.3. ($\mathcal{S}-$**admissibility**)(see [**16**])A Banach space $X \subset \mathcal{S}$ is said to be $\mathcal{S}$-admissible if $\mathcal{P} \subseteq X$, for $\mathcal{P}$ the space of all polynomials, and $X \hookrightarrow \mathcal{S}$, i.e., $\forall j\ \exists C_j$ s.t. $|\hat{f}(j)| \leq C_j\|f\|_X\ \forall f \in X$.

Remark 1.4. Of course if X is a $\mathcal{S}-$admissible space, then the closure of polynomials in the X-norm, X^0, and its dual ,$(X^0)^*$, are also admissible ([**16**]).

Example 1.5. Some examples of $\mathcal{S}$-admissible spaces are ℓ^p for $1 \leq p \leq \infty$, c_0 and the space $\ell(p, q)$, $1 \leq p, q \leq \infty$.

The spaces of holomorphic functions considered as sequence spaces such as Hardy spaces, Bergman spaces, Bloch function spaces and so on, are $\mathcal{S}$-admissible as well.

Definition 1.6. ($\mathcal{H}-$**admissibility**)(see [**16**])A Banach space $X \subset \mathcal{S}$ will be called $\mathcal{H}$-admissible if $X \subset \mathcal{H}(\mathbb{D})$ with continuous inclusion, $\mathcal{H}(\mathrm{R}\mathbb{D}) \subset X$ for all $R > 1$ and the map $f \mapsto f|_{\mathbb{D}}$ is continuous from $\mathcal{H}(\mathrm{R}\mathbb{D})$ to X.

Clearly $\mathcal{H}-$admissible spaces are also $\mathcal{S}-$admissible.
This class of Banach spaces not only covers many of the classical function spaces, but is also well-adapted to the study of multipliers.

There are several common interpretations of the coefficient multipliers. One can see them as diagonal operators, relate them to a convolution product or to a Hadamard product. We will consider the third option.

Definition 1.7. (**Hadamard product**) Let f, g be in $\mathcal{H}(\mathbb{D})$. Then

$$f * g(z) := \sum_j \hat{f}(j)\hat{g}(j)z^j$$

is called the Hadamard product of f and g.

Definition 1.8. (**Multipliers**) Given two $\mathcal{S}-$admissible spaces, X and Y, $\lambda \in \mathcal{S}$ is said to be a (coefficient) multiplier from X to Y if

$$\lambda * f := \sum_j \lambda_j \hat{f}(j) e_j \in Y \text{ for each } f \in X.$$

We denote the set of all multipliers from X to Y by (X, Y) and define

$$\|\lambda\|_{(X,Y)} = \sup\{\|\lambda * f\|_Y : \|f\|_X \leq 1\}.$$

This space considered with the operator norm is an $\mathcal{S}-$admissible Banach space (see [16]).

Example 1.9. For $1 \leq p, r \leq \infty$ we have

$$(\ell^p, \ell^r) = \ell^{p \ominus r}. \tag{1.2}$$

This idea can be generalized for mixed-norm sequence spaces. Given $1 \leq p, q, r, s \leq \infty$,

$$(\ell(p, q), \ell(r, s)) = \ell(p \ominus r, q \ominus s). \tag{1.3}$$

The proof of (1.3) is based on the proof of (1.2) and it can be found in [32].

In Chapter 2 we will deepen into the study of multipliers between a generalized version of mixed-norm sequence spaces, considered with values in a Banach space.

Other well-known examples of multiplier spaces are multipliers related to Hardy spaces such as

$$(H^p, H^\infty) = H^{p'},$$

where $1/p + 1/p' = 1$, $1 < p < \infty$(see [36, 31]) or

$$(H^p, H^u) = \ell^\infty$$

for $0 \leq u \leq 2 \leq p \leq \infty$ (see [31]).

Also, multipliers related to Bloch spaces

$$(H^1, BMOA) = (H^1, \mathcal{B}loch) = \mathcal{B}loch$$

([34, 31]).

It is still an open problem to characterize the space (H^p, H^u) for some values of u, p, for example $0 < u \leq p < 1$ or (H^p, H^p) for $1 < p < 2$. A great survey on this topic can be found in [31].

Example 1.10. The Köthe dual of $X \subset \mathcal{S}$ is defined to be

$$X^K = \Big\{(y_j)_j \in \mathcal{S} : \sum_j |y_j x_j| < \infty, \forall (x_j)_j \in X\Big\}.$$

Thus, it can be regarded as the multiplier space (X, ℓ^1).

The concept of solid space was introduced and studied by Anderson and Shields ([2]). Let us mention some trivial facts about it related to multipliers and Hadamard tensor products.

Definition 1.11. (**Solid space**) A set $A \subseteq \mathcal{S}$ is said to be solid if for any $f \in A$ and $g \in \mathcal{S}$ with $|\hat{g}(j)| \leq |\hat{f}(j)|,\ j \geq 0$ implies that $g \in A$.

Note that, in terms of multipliers, an $\mathcal{S}-$admissible space X is said to be solid iff $\ell^\infty \subseteq (X, X)$.
The spaces ℓ^p and $\ell(p,q)$ spaces $(1 \le p, q \le \infty)$ are solid.

PROPOSITION 1.12. *(See [16]) If X or Y are solid $\mathcal{S}-$admissible Banach spaces, then so it is (X,Y).*

PROPOSITION 1.13. *(See [2],[16]) If X is an $\mathcal{S}-$admissible space, then there is a largest solid admissible space $s(X) \subset X$. Moreover, $s(X)$ is the largest solid subset of X and we have*

$$s(X) = (\ell^\infty, X).$$

DEFINITION 1.14. (**Hadamard tensor product**) The Hadamard tensor product is defined to be the space of linear combinations of Hadamard products, i.e.

$$X \circledast Y = \{h \in \mathcal{S} : h(z) = \sum_n f_n * g_n(z) \text{ with } \sum_n \|f_n\|_X \|g_n\|_Y < \infty\},$$

where $f_n \in X,\ g_n \in Y$ and the convergence is considered in $\mathcal{S}$.

This space considered with the norm given by

$$\|h\| = \inf \sum_n \|f_n\|_X \|g_n\|_Y,$$

where the infimum is taken over all possible representations, is an admissible Banach space.

EXAMPLE 1.15. For $1 \le p, r \le \infty$ we have $\ell^p \circledast \ell^{p \ominus r} = \ell^r$.
For $1 \le p \le 2$ one has (see [16], Corollary 8.1) $H^1 \circledast \ell^p = \ell(p,1)$ and $H^1 \circledast H^p = \mathfrak{B}^{p,1}$ where $\mathfrak{B}^{p,1}$ stands for the space of functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{\mathfrak{B}^{p,1}} = |f(0)| + \int_0^1 M_p(r, Df) r dr < \infty.$$

PROPOSITION 1.16. *(See [16]) If X or Y are solid $\mathcal{S}-$admissible Banach spaces, then so it is $X \circledast Y$.*

PROPOSITION 1.17. *(See [2], [20], [16]) If $X \subset \mathcal{S}$, then there is a smallest solid superset $S(X) \supset X$. Furthermore*

$$S(X) = \ell^\infty * X := \{\alpha * f : \alpha \in \ell^\infty,\ f \in X\}$$

and

$$S(X) = \{g \in \mathcal{S} : \exists f \in X \text{ such that } |\hat{f}(j)| \ge |\hat{g}(j)| \text{ for all } j\}.$$

Denote $SB(X) = \ell^\infty \circledast X$. Then of course $S(X) \subset SB(X)$.

PROPOSITION 1.18. *(See [16], Theorem 6.1) Let X be an $\mathcal{S}-$admissible Banach space. Then $SB(X)$ is the smallest solid Banach space containing X. More precisely, if Y is a solid Banach space containing X, then $SB(X) \subset Y$ with continuity.*

A basic formula connecting tensors and multipliers is given in [16] (Theorem 2.3) and states that, given X, Y, Z three $\mathcal{S}-$admissible Banach spaces,

$$(X \circledast Y, Z) = (X, (Y, Z)). \tag{1.4}$$

The previous equality is used to characterize new multiplier spaces, such as $(H^p, BMOA)$ for $1 \le p \le 2$, identified with certain class of Bloch spaces (see [16, 35, 39]) or $(\ell^p, BMOA) = \ell(p', \infty)$ again with $1 \le p \le 2$.
We include in this section a result regarding this topic that we have not found in the literature.

Consider the differential operator

$$D^m g = \sum_n (n+1)^m \hat{g}(n) e_n \tag{1.5}$$

for $g = \sum_n \hat{g}(n)e_n$, $e_n(z) = z^n$. Of course considering the operator defined in (1.1) it is clear that $\mathcal{D}^1 = D^1$. In this case we will write simply D. Denote by $\mathcal{D}^{-m}$ the preimage of this differential operator, that is

$$\begin{aligned} D^{-m}Y &= \left\{ f = (\hat{f}(n)) \in \mathcal{S} : D^m f \in Y \right\} \\ &= \left\{ f = (\hat{f}(n)) \in \mathcal{S} : \sum_n (n+1)^m \hat{f}(n) e_n \in Y \right\}. \end{aligned}$$

Notice that $D^{-1}f(z) = \sum_n \frac{\hat{f}(n)}{n+1} z^n = \frac{1}{z}\int_0^z f(w)dw$.

Lemma 1.19. *(Theorem 2.1, [9], with $\rho(t) = t$) Given a Banach space X, $T : A^1 \to X$ is continuous iff $\mathcal{D}^2(Tu)$ is an X-valued function satisfying*

$$\|\mathcal{D}^2(Tu)(z)\|_X = O\left(\frac{1}{1-|z|}\right) \ (|z| \to 1),$$

where $Tu = \sum_n T(u_n)e_n$ for a basis of X, $u = (u_n)_n$.

Remark 1.20. Note that

$$M_\infty(r, D^2 f) = O\left(\frac{1}{1-r}\right) \Leftrightarrow M_\infty(r, \mathcal{D}^2 f) = O\left(\frac{1}{1-r}\right),$$

as $r \to 1$.

Indeed, from the definition of D^2 and $\mathcal{D}^2$ one gets $\mathcal{D}^2 = DD - D = D^2 - D$.

Consider $M_\infty(r, D^2 f) = O(\frac{1}{1-r})$. Then $M_\infty(r, \mathcal{D}^2 f) = M_\infty(r, D^2 f - Df) \leq M_\infty(r, D^2 f) + M_\infty(r, Df) = O\left(\frac{1}{1-r}\right) + O\left(log\left(\frac{1}{1-r}\right)\right)$.

For the reverse direction, assume $M_\infty(r, \mathcal{D}^2 f) = O\left(\frac{1}{1-r}\right)$. Then $M_\infty(r, D^2 f) = M_\infty(r, \mathcal{D}^2 f + Df) \leq M_\infty(r, \mathcal{D}^2 f) + M_\infty(r, Df)$. Now notice that $z^2 Df = z\mathcal{D}^2 f - z^2 f$, thus

$$\begin{aligned} r^2 M_\infty(r, Df) &\leq r M_\infty(r, \mathcal{D}^2 f) + M_\infty(r, z^2 f) \\ &\leq r M_\infty(r, \mathcal{D}^2 f) + r M_\infty(r, (z^2 f)'), \end{aligned}$$

which, together with the fact that $(z^2 f)' \in \mathcal{B}loch$ by hypothesis, gives

$$M_\infty(r, Df) \leq \frac{1}{r} M_\infty(r, \mathcal{D}^2 f) + \frac{1}{r} O\left(\log\left(\frac{1}{1-r}\right)\right).$$

Lemma 1.21. *Let X, Y be $\mathcal{H}$-admissible spaces. Then*

$$(X, D^{-1}Y) = D^{-1}(X, Y).$$

Proof. Let $\lambda = (\lambda_j)_j \in (X, D^{-1}Y)$. Then $\lambda * x = \sum_j \lambda_j x_j e_j \in D^{-1}Y$ for all $x = (x_j)_j \in X$. Thus $D(\lambda * x) = \sum_j (j+1)\lambda_j x_j e_j \in Y$ for all $x \in X$, which implies $D\lambda \in (X, Y)$.
Conversely, if $\lambda \in D^{-1}(X, Y)$, apply the same argument to prove that $D\lambda * x \in Y$ for all $x \in X$ is equivalent to $\lambda * x \in D^{-1}Y$ for all $x \in X$. □

Lemma 1.22. *Let $f \in \mathcal{H}(\mathbb{D})$. Then*

$$f \in \mathcal{B}loch \Leftrightarrow \sup_{0<r,s<1} (1-r)(1-s) M_\infty(rs, D^2 f) < \infty$$

PROOF. Consider f an analytic function such that

$$\sup_{0<r,s<1}(1-r)(1-s)M_\infty(rs,D^2f)<\infty.$$

In particular, for $r=s$

$$\sup_{0<r<1}(1-r)^2M_\infty(r^2,D^2f)<\infty.$$

That is saying that $M_\infty(r,D^2f)=O(\frac{1}{(1-r)^2})$. Integrating using D^{-1}, that is $D^{-1}f(z)=\int_0^1 f(zs)ds$, we obtain $M_\infty(r,Df)=O(\frac{1}{1-r})$. Thus, $f\in\mathcal{B}loch$.

Now let $f\in\mathcal{B}loch$. Then $M_\infty(r,Df)=O(\frac{1}{1-r})$ and derivating using D, and the known fact that $M_\infty(r,f')\le C\frac{M_\infty(r,f)}{1-r}$, one concludes that

$$M_\infty(r,Df)=M_\infty(r,(zf)')\le C\frac{M_\infty(r^2,f)}{1-r}.$$

Thus $M_\infty(rs,D^2f)=O(\frac{1}{(1-rs)^2})$ and as for $0<r,s<1$ one has that $rs<r$ and $rs<s$,

$$\frac{1}{(1-rs)^2}<\frac{1}{(1-r)(1-s)}$$

and the result follows. □

Let us now present the following new result.

PROPOSITION 1.23.

$$(A^1\circledast A^1,H^\infty)=D^{-2}\mathcal{B}loch$$

PROOF. The formula (1.4) gives $(A^1\circledast A^1,H^\infty)=(A^1,(A^1,H^\infty))$. We will prove then $(A^1,(A^1,H^\infty))=D^{-2}\mathcal{B}loch$

Let us first prove that $(A^1,H^\infty)=D^{-1}\mathcal{B}loch$. By Lemma 1.19, given g an analytic function, $g\in(A^1,H^\infty)$ is equivalent to

$$\|\sum_n(n+2)(n+1)\hat{g}(n)z^ne_n\|_{H^\infty}=O\left(\frac{1}{1-|z|}\right),\ (|z|\to 1).$$

That is to say, using Remark 1.20 $\sup_{w\in\mathbb{D}}|D^2g_z(w)| = O\left(\frac{1}{1-|z|}\right)$ or equivalently $\sup_{w\in\mathbb{D}}|D^2g(rw)| = O\left(\frac{1}{1-r}\right)$. This is also equivalent to $M_\infty(r,D^2g)=O\left(\frac{1}{1-r}\right)$ iff $Dg\in\mathcal{B}loch$, i.e., if and only if $g\in D^{-1}\mathcal{B}loch$.

Then we have reduced the problem to prove $(A^1,D^{-1}\mathcal{B}loch)=D^{-2}\mathcal{B}loch$. Lemma 1.21 gives $(A^1,D^{-1}\mathcal{B}loch)=D^{-1}(A^1,\mathcal{B}loch)$, thus what we actually need to prove is $(A^1,\mathcal{B}loch)=D^{-1}\mathcal{B}loch$.

Again let g be an analytic function. Then applying Lemma 1.19, $g\in(A^1,\mathcal{B}loch)$ if and only if

$$\|D^2g_z(w)\|_{\mathcal{B}loch}=O\left(\frac{1}{1-|z|}\right),(|z|\to 1).$$

Using Remark 1.20, $\sup_{w\in\mathbb{D}}|D^3g_z(w)|(1-|w|)=O\left(\frac{1}{1-|z|}\right)$ if and only if $M_\infty(rs,D^3g)\le\frac{K}{(1-r)(1-s)}$ for $0<r,s<1$. By Lemma 1.22, this inequality holds if and only if $Dg\in\mathcal{B}loch$, that is, if and only if $g\in D^{-1}\mathcal{B}loch$, and the proposition is proved. □

3. Spaces of vector-valued sequences

In this section we develop some necessary background material in order to make the text accessible to anyone interested in the topic and give examples of spaces of vector-valued sequences to be used later on. From now on, the letter E will be used for Banach spaces.

Following the notation introduced in the scalar case, $\mathcal{S}(E)$ will be used to denote the space of sequences $f = (x_j)_{j\geq 0}$, $x_j \in E$, endowed with the locally convex topology given by the seminorms $p_j(f) = \|x_j\|_E$, $j \geq 0$. As in the scalar case, we shall think of f as a formal power series with coefficients in E, that is $f(z) = \sum_{j\geq 0} x_j z^j$ and most of the time we will write $\hat{f}(j)$ instead of x_j.

A sequence $(f_n)_n \subset \mathcal{S}(E)$ converges to $f \in \mathcal{S}(E)$ if and only if $p_j(f - f_n) \to 0 \; \forall j \geq 0$ if and only if $\|\hat{f}(j) - \hat{f}_n(j)\|_E \to 0$ as $n \to \infty$ for all $j \geq 0$. That is, convergence in $\mathcal{S}(E)$ is coordinatewise convergence.

$\mathcal{S}(E)$ is a metrizable and complete space. For the completeness, take $(f_n)_n \subseteq \mathcal{S}(E)$ a Cauchy sequence. Then $(\hat{f}_n(j))_n$ is Cauchy for any $j \in \mathbb{N}$ and since E is Banach, it converges to some $\hat{f}(j)$. Defining $f = (\hat{f}(j))_j$ we have $f_n \to f \;\in\; \mathcal{S}(E)$.

Recall the notation $e_j(z) = z^j$ for each $j \geq 0$ and write $\mathcal{P}(E)$ for the vector space of the analytic polynomials with coefficients in E, that is $\sum_j^N x_j e_j$, where $x_j \in E$.

Tensor product will play an important role in the exposition. We recall the definition and some properties before going on. For basic information concerning tensor products one can take a look at [**38, 22, 21**].

Definition 1.24. (**Tensor product.**) The algebraic tensor product between two vector spaces U, V can be seen as a linear form space over the bilinear continuous mappings from $U \times V$ into $\mathbb{K}$, $\mathcal{B}(U, V)$. Given $u \in U,\; v \in V$,

$$u \otimes v(F) = \langle F, u \otimes v\rangle = F(u, v).$$

The tensor product $U \otimes V$ is the subspace of the algebraic dual of $\mathcal{B}(U, V)$ generated by these elements. Hence, an element of $U \otimes V$ can be written as

$$x = \sum_{i=1}^{n} \alpha_i u_i \otimes v_i$$

where $n \in \mathbb{N},\; \alpha_i \in \mathbb{K},\; u_i \in U,\; v_i \in V$.

The representation of the element x is not necessarily unique, though. In fact, $\alpha(u \otimes v) = (\alpha u) \otimes v = u \otimes (\alpha v)$. This allows us to write

$$x = \sum_{i=1}^{n} u_i \otimes v_i.$$

Therefore we can define the following norm.

Definition 1.25. (**Projective norm.**) Given $x \in U \otimes V$, we define its projective norm by

$$\pi(x) = \inf \sum_i \|u_i\|_U \|v_i\|_V,$$

where the infimum is taken over all possible representations and the series converges in the sense of bilinear forms. The notation for the space $U \otimes V$ endowed with this norm will be $U \otimes_\pi V$.

The space $U \otimes V$ equipped with the projective norm might not be a complete space. Its completion is the so-called **projective tensor product**. We will write $U \hat{\otimes}_\pi V$.

The following theorem may be helpful to identify the elements of these spaces.

THEOREM 1.26. *Let X, Y, Z be Banach spaces and $B : X \times Y \longrightarrow Z$ a continuous bilinear form. There exists a unique operator $\tilde{B} : X\hat{\otimes}_\pi Y \longrightarrow Z$ satisfying*

$$\tilde{B}(x \otimes y) = B(x, y),$$

$x \in X,\ y \in Y$.

This correspondence gives us an isometric isomorphism between the Banach spaces

$$\mathcal{B}(X \times Y, Z) = \mathcal{L}(X\hat{\otimes}_\pi Y, Z) = \mathcal{L}(X, \mathcal{L}(Y, Z)).$$

Recall that for two given Banach spaces, X and Y, the space of continuous linear operators $\mathcal{L}(X, Y)$ endowed with the norm

$$\|T\| = \sup_{\|x\|_X \le 1} \|T(x)\|_Y$$

is a Banach space. Unless otherwise indicated, the convergence of operators will be understood in this norm.

We are now in conditions to list some ways of generating different vector-valued sequence spaces.

DEFINITION 1.27. Given $X \subseteq \mathcal{S}$ and E a Banach space, we denote

$$X\hat{\otimes}_\pi E$$

the tensor space previously described,

$$X[E] = \{(x_j)_{j\ge 0} \in \mathcal{S}(E) : \|(\|x_j\|_E)_j\|_X < \infty\}$$

and

$$X_{weak}(E) = \Big\{(x_j)_{j\ge 0} \in \mathcal{S}(E) : \|(x_j)_j\|_{X_{weak}(E)} = \sup_{\|x^*\|_{E^*}=1} \|(\langle x_j, x^*\rangle)_j\|_X < \infty\Big\}.$$

EXAMPLE 1.28. For $1 \le p \le \infty$, we consider

$$\ell^p\hat{\otimes}_\pi E$$

the projective tensor product space between ℓ^p and E and

$$\ell^p[E] = \Big\{(x_n)_{n\ge 0} : \|(x_n)\|_{\ell^p(E)} = \Big(\sum_{n=0}^{\infty} \|x_n\|_E^p\Big)^{1/p} < \infty\Big\}$$

which is usually denoted $\ell^p(E)$.

Then, in connection with Theorem 1.26, the space $E \otimes \ell^1$ may be seen as sequences with values in E via the identification $x \otimes a \mapsto (xa_n)_n$, where $a = (a_n)_n$. Since $\sum_n \|xa_n\|_E \le \|x\|_E \sum_n |a_n|$, the series is absolutely convergent. Therefore $x \otimes a \in \ell^1(E)$, the space of E-absolutely convergent series where the norm is defined to be $\|y\|_1 = \sum_n \|y_n\|_E$. If we extend the map

$$\begin{array}{rccc} J: & E\hat{\otimes}_\pi \ell^1 & \longrightarrow & \ell^1(E) \\ & x \otimes a & \mapsto & (xa_n)_n \end{array}$$

to an isometric isomorphism, we have an identification between both spaces (for the complete proof see [22]).

Following with the list of examples, we consider

$$\ell^p_{weak}(E) = \Big\{(x_n)_{n\ge 0} : \|(x_n)\|_{\ell^p_{weak}(E)} = \sup_{\|x^*\|_{E^*}=1} \Big(\sum_{n=0}^{\infty} |\langle x_n, x^*\rangle|^p\Big)^{1/p} < \infty\Big\},$$

with the obvious modifications for $p = \infty$.

In particular, $c_0(E) = (\ell^\infty(E))^0$ and

$$UC(E) = (\ell^1_{weak})^0(E) = \Big\{(x_n)_{n\ge 0} \in \ell^1_{weak}(E); \sum_n x_n \text{ converges unconditionally }\Big\}$$

We also have a vector-valued version for Kellogg's spaces $\ell(p,q)$.

DEFINITION 1.29. (**Vector-valued mixed-norm sequence spaces**) We denote by $\ell(p,q,E)$, $1 \le p,q \le \infty$, the vector-valued mixed-norm space of sequences $(a_j)_{j\in\mathbb{N}}$ in $\mathcal{S}(E)$ verifying

$$\left(\left(\sum_{j\in I_k} \|a_j\|_E^p\right)^{1/p}\right)_{k\in\mathbb{N}_0} \in \ell^q,$$

under the norm

$$\|a\|_{p,q,E} = \left(\sum_{k=0}^{\infty}\left(\sum_{j\in I_k} \|a_j\|_E^p\right)^{q/p}\right)^{1/q}$$

where $\mathcal{I} = \{I_k = [2^k, 2^{k+1}) \cap \mathbb{N}_0,\ k \in \mathbb{N}_0\}$ and with the obvious modifications for $p=\infty$ or $q=\infty$. Of course $\ell(p,p,E) = \ell^p(E)$ and $\ell(p,q,E)^* = \ell(p',q',E^*)$.

In Chapter 2 we will extend this version to more general families of intervals in $\mathbb{N}_0$.

Another space not coming from the above constructions is the vector-valued version of the *Rad* space, which hangs on the remarkable Rademacher functions. We will now give the basic notions to get an understanding of this space. More information on this topic can be found in [21].

DEFINITION 1.30. (**Rademacher function**) The Rademacher functions $r_n : [0,1] \to \mathbb{R}$, $n \in \mathbb{N}$ are defined by setting

$$r_n(t) := sign(\sin(2^n \pi t)).$$

It will be convenient to refer to the constant one function as the zero'th Rademacher function, r_0.

To grasp how Rademacher functions work, we have pictured the graphs of three of them.

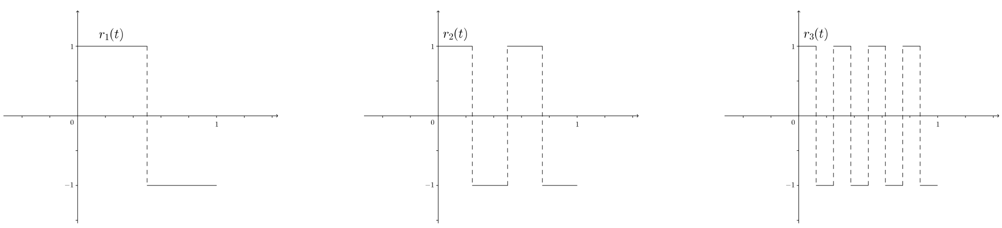


FIGURE 1. $r_1(t), r_2(t), r_3(t)$

The most important feature of the Rademacher functions is that they have nice orthogonality properties. If $0 < n_1 < n_2 < \ldots < n_k$ and $p_1, \ldots, p_k \ge 0$ are integers, then it can be easily seen from the pictures that

$$\int_0^1 r_{n_1}^{p_1} \cdot \cdots \cdot r_{n_k}^{p_k} dt = \begin{cases} 1 & \text{if each } p_j \text{ is even;} \\ 0 & \text{otherwise.} \end{cases}$$

An immediate consequence is that the Rademacher functions form an orthonormal sequence in $L^2([0,1])$ and so

$$\int_0^1 |\sum a_n r_n(t)|^2 dt = \sum |a_n|^2$$

for all $(a_n)_n \in \ell^2$. Be aware they do not form an orthonormal basis. For example, $cos(2\pi t)$ is orthogonal to all the r_n's.

The main result about the Rademacher functions is a powerful inequality.

THEOREM. *F (**Kitchin's inequality**) (see [21]) For any $0 < p < \infty$, there are positive constants A_p, B_p such that regardless of the scalar sequence $(a_n)_n \in \ell^2$ we have*

$$A_p \left(\sum_n |a_n|^2 \right)^{1/2} \leq \left(\int_0^1 |\sum a_n r_n(t)|^p dt \right)^{1/p} \leq B_p \left(\sum_n |a_n|^2 \right)^{1/2}.$$

Notice that the statement can be rephrased to say that on the span of the Rademacher functions all the L^p metrics are equivalent. The Rad_p space is defined to be this closed linear span. We will focus our attention only on Rad_2, which will be noted simply by Rad.

Then the vector-valued version will be

$$Rad(E) = \left\{ (x_j)_{j\geq 0} : \sup_N \left(\int_0^1 \| \sum_{j=0}^N x_j r_j(t) \|_E^2 dt \right)^{1/2} < \infty \right\}$$

It is well known (see [21]) that

$$\ell^1_{weak}(E) \subset Rad(E) \subset \ell^2_{weak}(E)$$

with continuous embeddings. Let us mention the interplay with the geometry of Banach spaces when comparing the space $Rad(E)$ and $Rad[E]$. We need a "Kitchin's-type" inequality for the vector valued case.

REMARK 1.31. (**Kahane's inequality**) (see [21])
Let $0 < p, q < \infty$. Then there is a constant $K(p,q) > 0$ for which

$$\left(\int_0^1 \| \sum_{k\leq n} r_k(t) x_k \|^q dt \right)^{1/q} \leq K(p,q) \left(\int_0^1 \| \sum_{k\leq n} r_k(t) x_k \|^p dt \right)^{1/p}$$

regardless of the choice of the Banach space X and the vectors $x_k \in X$.

Unlinke the situation with Kitchin's Inequality, in general infinite dimensional Banach spaces none of these quantities can be compared with $(\sum_{k\leq n} \|x_k\|^2)^{1/2}$ in a uniform way.

DEFINITION 1.32. (**Type**) A Banach space X has type p if there is a constant C such that, however we choose finitely many vectors $x_k \in X,\ k = 1, \cdots, n$,

$$\left(\int_0^1 \| \sum_{k=1}^n r_k(t) x_k \|^2 dt \right)^{1/2} \leq C \left(\sum_{k=1}^n \|x_k\|^p dt \right)^{1/p}.$$

DEFINITION 1.33. (**Cotype**) A Banach space X has cotype q if there is a constant $K \geq 0$ such that no matter how we select finitely many vectors $x_k \in X,\ k = 1, \cdots, n$,

$$\left(\sum_{k=1}^n \|x_k\|^q dt \right)^{1/q} \leq K \left(\int_0^1 \| \sum_{k=1}^n r_k(t) x_k \|^2 dt \right)^{1/2}.$$

To cover the case $q = \infty$, the left hand side should be replaced by $\max_{k\leq n} \|x_k\|$. An interesting corollary following from Kwapien's theorem (every operator from X to Y factors through a Hilbert space, [21]) tells us that the only Banach spaces which simultaneously have type and cotype 2 are the isomorphic copies of Hilbert spaces.

Notice that the notions of type 2 and cotype 2 correspond to $\ell^2(E) \subset Rad(E)$ and $Rad(E) \subset \ell^2(E)$, respectively.

PROPOSITION 1.34. *Let E be a Banach space.*
(i) $Rad(E) = Rad[E]$ if and only if E is isomorphic to a Hilbert space.
(ii) $Rad_{weak}(E) = Rad[E]$ if and only if E is finite dimensional.

PROOF. Note that, using the orthonormality of r_n, Plancherel's theorem gives that $Rad[E] = \ell^2(E)$ and $Rad_{weak}(E) = \ell^2_{weak}(E)$. Of course if E is a Hilbert space then $Rad(E) = \ell^2(E)$ and for finite dimensional spaces $Rad_{weak}(E) = \ell^2_{weak}(E) = \ell^2(E)$.

On the other hand, clearly $Rad[E] \subset Rad(E)$ if and only if E has type 2 and $Rad(E) \subset Rad[E]$ if and only if E has cotype 2 . Now use the mentioned corollary from Kwapien's theorem (see [21], 12.20, p.246) to conclude (i).

To see the direct implication in (ii), simply use that if $dim(E) = \infty$ then $\ell^2(E) \subsetneq \ell^2_{weak}(E)$ (see [21] 2.18, p.50). □

4. Spaces of vector-valued integrable and analytic functions

This section is devoted to gather the vector-valued version of some of the function spaces mentioned above and to take a look at its most basic properties. The definitions and basic properties of integrals of vector-valued functions with respect to scalar measures will be given.

Consider (Ω, Σ, μ)a finite measure space.

DEFINITION 1.35. (**Simple and $\mu-$measurable function**) A function $f : \Omega \to E$ is called simple if there exist $x_1, ..., x_n \in E$ and $A_1, \cdots, A_n \in \Sigma$ such that

$$f = \sum_{i=1}^{n} x_i \chi_{A_i},$$

where $\chi_{A_i}(\omega) = 1$ in case $\omega \in A_i$ and equals to zero otherwise.

A function $f : \Omega \to E$ is called $\mu-$measurable if there exists a sequence of simple functions $(f_n)_n$ with $\lim_n \|f_n - f\| = 0$ $\mu-$almost everywhere.

DEFINITION 1.36. (**Bochner integral**) A $\mu-$measurable function is called Bochner integrable if there exists a sequence of simple functions $(f_n)_n$ such that

$$\lim_n \int_\Omega \|f_n - f\| d\mu = 0.$$

In this case, $\int_A f d\mu$ is defined for each $A \in \Sigma$ by

$$\int_A f d\mu = \lim_n \int_A f_n d\mu.$$

If $1 \leq p \leq \infty$, then $L^p(\mu, E)$ stands for the space of all (equivalence classes of) $E-$valued Bochner integrable functions f defined on Ω with

$$\int_\Omega \|f\|^p d\mu < \infty.$$

The norm is defined by

$$\|f\|_p = \left(\int_\Omega \|f(\omega)\|_E^p d\mu(\omega) \right)^{1/p}, \ f \in L^p(\mu, E).$$

Routine computations show that $L^p(\mu, E)$ is a Banach space under $\| \cdot \|_p$. In addition, simple functions are dense in $L^p(\mu, E)$ for $1 \leq p < \infty$. For $p = \infty$ the symbol

$L^\infty(\mu, E)$ stands for the space of all (equivalence classes of) $E-$valued Bochner integrable functions defined on Ω that are essentially bounded, i.e., such that

$$\|f\|_\infty = ess\sup\{\|f(\omega)\| : \omega \in \Omega\} < \infty.$$

The space is also a Banach space under the norm $\|\cdot\|_\infty$ and the countably valued functions in $L^\infty(\mu, E)$ are dense in it. It is known that $L^p(\mu, E) \subseteq L^q(\mu, E)$ whenever $p > q$.

For $1 \le p < \infty$ it is not difficult to recognize $L^{p'}(\mu, E^*)$ isometrically as a subspace of $(L^p(\mu, E))^*$. The equality holds if and only if E^* has the Radon-Nikodym property with respect to μ, that is to say that for each $\mu-$continuous vector measure $G : \Sigma \to E$ of bounded variation there exists $g \in L^1(\mu, E)$ such that $G(A) = \int_A g d\mu$ for all $A \in \Sigma$.

Example 1.37. In a similar way we did in the Example 1.28, we can identify isometrically the spaces $E\hat{\otimes}_\pi L^1(\mu)$ and $L^1(\mu, E)$ for any measure space (Ω, Σ, μ). The proof can be found in [**24**].

Running parallel to the scalar-valued case, consider $\mathcal{H}(\mathrm{R}\mathbb{D}, E)$ to be the space of E-valued analytic functions on $\mathrm{R}\mathbb{D} \subset \mathbb{C}$ for $R > 0$. That is, functions $f : \mathrm{R}\mathbb{D} \to E$, $f(z) = \sum_{j\in\mathbb{N}_0} \hat{f}(j)z^j$ such that $\limsup_j \sqrt[j]{\|\hat{f}(j)\|_E} \le R$. It can be regarded as a vector subspace of $\mathcal{S}(E)$ via the Taylor coefficients $\hat{f}(j) \in E$. Of course every sequence $(x_j)_j \in \mathcal{S}(E)$ which satisfies the condition $\limsup_j \sqrt[j]{\|x_j\|_E} \le R$ can also be identified with an $E-$valued analytic function in $\mathrm{R}\mathbb{D}$.
We endow this space with the '$\mathcal{H}(E)-$topology', i.e., the topology of uniform convergence on compact subsets of $\mathrm{R}\mathbb{D}$. This topology can be described by the family of seminorms

$$M_\infty(r, f) = \sup_{|z|=r} \|f(z)\|_E,$$

$0 < r <$ R. Therefore, we will say a Banach space X is continuously contained in $\mathcal{H}(\mathrm{R}\mathbb{D}, E)$ if for any $0 < r <$ R there exists a constant $A_r > 0$ such that

$$M_\infty(r, f) \le A_r\|f\|_X, \ \ f \in X.$$

Conversely, we will write $\mathcal{H}(\mathrm{R}\mathbb{D}, E) \subset X$ if there exists $s \le R$ and $B_s > 0$ such that

$$\|f\|_X \le B_s M_\infty(s, f)$$

for any $f \in \mathcal{H}(\mathrm{R}\mathbb{D}, E)$.Notice that $\mathcal{H}(\mathrm{R}\mathbb{D}, E) \subseteq \mathcal{S}(E)$.

The vector-valued disc algebra and the bounded analytic functions will be denoted

$$A(\mathbb{D}, E) = \{f \in \mathcal{H}(\mathbb{D}, E), f \in \mathcal{C}(\overline{\mathbb{D}}, E)\}$$

and

$$H^\infty(\mathbb{D}, E) = \left\{f \in \mathcal{H}(\mathbb{D}, E), \sup_{|z|<1} \|f(z)\|_E < \infty\right\}$$

respectively, where we define

$$\|f\|_{A(\mathbb{D},E)} = \sup_{|z|=1} \|f(z)\|_E, \quad \|f\|_{H^\infty(\mathbb{D},E)} = \sup_{|z|<1} \|f(z)\|_E.$$

It is easy to see that $(H^\infty(\mathbb{D}, E))^0 = A(\mathbb{D}, E)$.

The $E-$valued Hardy space $H^p(\mathbb{D}, E)$ is defined as the space of $E-$valued analytic functions on the unit disc such that

$$\|f\|_{H^p(\mathbb{D},E)} = \sup_{0<r<1} M_p(r, f) < \infty,$$

where now

$$M_p(r, f) = \left(\frac{1}{2\pi}\int_0^{2\pi} \|f(re^{i\theta})\|_E^p d\theta\right)^{1/p}.$$

Also in the vector-valued case the integral means are increasing functions of r. Therefore $H^p(\mathbb{D}, E) \subseteq H^q(\mathbb{D}, E)$ whenever $p > q$.

We also have the space defined at the boundary

$$H^p(\mathbb{T}, E) = \left\{ f \in L^p(\mathbb{T}, E) : \hat{f}(n) = \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi} = 0, n \leq 0 \right\}.$$

It is not difficult to see that $H^p(\mathbb{T}, E) = (H^p(\mathbb{D}, E))^0$. Although we do not enter into this, the property ARNP is the one needed for the coincidence $H^p(\mathbb{T}, E) = H^p(\mathbb{D}, E)$ (see [7]).

Given $1 \leq p < \infty$, the E-valued Bergman space $A^p(\mathbb{D}, E)$ is defined as the space of E- valued analytic functions on the unit disc such that

$$\|f\|_{A^p(\mathbb{D},E)} = \left(\int_{\mathbb{D}} \|f(z)\|_E^p dA(z) \right)^{1/p} = \left(\int_0^1 M_p(f, r)^p r dr \right)^{1/p} < \infty.$$

It is also well-known that, for $1 \leq p < \infty$,

$$A(\mathbb{D}, E) \subset H^\infty(\mathbb{D}, E) \subset H^p(\mathbb{D}, E) \subset A^p(\mathbb{D}, E) \subseteq A^1(\mathbb{D}, E)$$

with continuous inclusions.

We define the $E-$valued Bloch space, $\mathcal{B}loch(\mathbb{D}, E)$, to be the set of E-valued holomorphic functions on the disc that verify

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) \|f'(z)\|_E < \infty$$

or, equivalently,

$$\sup_{z \in \mathbb{D}} (1 - |z|) \|f'(z)\|_E < \infty,$$

or

$$\sup_{z \in \mathbb{D}} (1 - |z|) \|Df(z)\|_E < \infty.$$

It is a Banach space under the norm

$$\|f\|_{\mathcal{B}loch(\mathbb{D},E)} = \|f(0)\|_E + \sup_{z \in \mathbb{D}} (1 - |z|) \|f'(z)\|_E.$$

The little Bloch space $\mathcal{B}loch_0(\mathbb{D}, E)$ is defined to be the subset of $f \in \mathcal{B}loch(\mathbb{D}, E)$ such that

$$\lim_{|z| \to 1^-} \sup_{z \in \mathbb{D}} (1 - |z|^2) \|f'(z)\|_E = 0$$

and turns out to be the closure of the $E-$valued polynomials in the $\mathcal{B}loch$ norm.

We will denote by $BMOA(\mathbb{T}, E)$ the space of functions in $L^1(\mathbb{T}, E)$ with Fourier coefficients $\hat{f}(n) = 0$ for $n < 0$ and such that

$$\sup \frac{1}{|I|} \int_I \|f(e^{i\theta}) - f_I\|_E \frac{d\theta}{2\pi} < \infty$$

where the supremum is taken over all intervals $I \subseteq [0, 2\pi)$, $|I|$ is normalized I's Lebesgue measure and $f_I = \frac{1}{|I|} \int_I f(e^{i\theta}) \frac{d\theta}{2\pi}$. It becomes a Banach space under the norm

$$\|f\|_{BMOA(\mathbb{T},E)} = \|f(0)\|_E + \sup \frac{1}{|I|} \int_I \|f(e^{i\theta}) - f_I\|_E \frac{d\theta}{2\pi}.$$

Finally, the space of E-valued analytic functions of vanishing mean oscillation $VMOA(\mathbb{T}, E)$ is the closure in $BMOA(\mathbb{T}, E)$ of the set of polynomials in the

$BMOA(\mathbb{T}, E)$ norm. We say that a function $f \in L^1(\mathbb{T}, E)$ is in the space $VMOA(\mathbb{T}, E)$ if it is analytic and

$$\lim_{|I|\to 0} \sup_{I\subseteq[0,2\pi)} \frac{1}{|I|}\int_I \|f(e^{i\theta}) - f_I\|_E \frac{d\theta}{2\pi} = 0.$$

We have the inclusions

$$H^\infty(\mathbb{D}, E) \subset BMOA(\mathbb{T}, E) \subset \mathcal{B}loch(\mathbb{D}, E),$$

$$A(\mathbb{D}, E) \subset VMOA(\mathbb{T}, E) \subset \mathcal{B}loch_0(\mathbb{D}, E)$$

and trivially the identification $\mathcal{B}loch(\mathbb{D}, E) = \mathcal{B}loch_{weak}(\mathbb{D}, E)$.

Let us finish this section with a new result that shows that the spaces $A^p(\mathbb{D}, E)$, $H^p(\mathbb{D}, E)$ and the vector-valued version $X[E]$ for $X = A^p(\mathbb{D}), H^p(\mathbb{D})$ are different for the particular case $E = c_0$ (see [**18**]).

Proposition 1.38. *The spaces $A^p(\mathbb{D}, c_0), H^p(\mathbb{D}, c_0)$ and the vector-valued version $X[c_0]$ for $X = A^p(\mathbb{D}), H^p(\mathbb{D})$ don't coincide.*

Proof. Let $(e_n)_n$ be the canonical basis in c_0. Consider the functions $f_N(z) = \sum_{n=0}^N e_n z^n$.

Let us analyze its norm in $H^p(\mathbb{D}, E)$ and $H^p(\mathbb{D})[E]$. We have

$$\|f_N\|_{H^p(\mathbb{D},c_0)} \le \|f_N\|_{H^\infty(\mathbb{D},c_0)} = 1, \quad p \ge 1.$$

However

$$\|f_N\|_{H^\infty(\mathbb{D})[c_0]} = N + 1,$$

$$\|f_N\|_{H^p(\mathbb{D})[c_0]} \ge \|f_N\|_{H^2(\mathbb{D})[c_0]} = (N+1)^{1/2}, \quad 2 \le p < \infty,$$

and, using Hardy's inequality for functions in H^1 (see [**24**]),

$$\|f_N\|_{H^p(\mathbb{D})[c_0]} \ge \|f_N\|_{H^1(\mathbb{D})[c_0]} \ge C\sum_{n=0}^N \frac{1}{n+1} \ge C\log(N+1), \quad 1 \le p < 2.$$

Similarly

$$A^2(\mathbb{D})[E] = \left\{ (x_j)_j \in \mathcal{S}(E) : \sum_{j=0}^\infty \frac{\|x_j\|^2}{j+1} < \infty \right\}$$

and then for $p \ge 2$

$$\|f_N\|_{A^p(\mathbb{D},c_0)} \le 1, \quad \|f_N\|_{A^p(\mathbb{D})[c_0]} \ge C(\log(N+1))^{1/2},$$

which exhibits the difference between the spaces above and the vector-valued interpretation $X[E]$. □

CHAPTER 2

New results on vector-valued sequence spaces.

We recall the reader that, during the whole text, the letter E will denote a Banach space (also when written with some natural subindex) and X_E will denote a subspace of $\mathcal{S}(E)$.

1. $\mathcal{S}(E)-$admissibility

We introduce now the basic notion which plays a fundamental role in what follows.

DEFINITION 2.1. (**$\mathcal{S}(E)$-admissibility**) Let E be a Banach space and let X_E be a subspace of $\mathcal{S}(E)$. We will say that X_E is $\mathcal{S}(E)-$admissible (or simply admissible) if

(i) $(X_E, \|\cdot\|_{X_E})$ is a Banach space,

(ii) the projection $\pi_j : X_E \longrightarrow E$, $f \mapsto \hat{f}(j)$, is continuous and

(iii) the inclusion $i_j : E \longrightarrow X_E$, $x \mapsto xe_j$ is continuous.

Here the notation e_j is used to denote a sequence where $e_j(i) = 0$ for $i \neq j$ and $e_j(j) = 1$. Hence for each $j \geq 0$ we have

$$\|\hat{f}(j)\|_E \leq \|\pi_j\|\|f\|_{X_E}, \ f \in X_E$$

$$\|xe_j\|_{X_E} \leq \|i_j\|\|x\|_E, \ x \in E.$$

To avoid misunderstandings, we will write $\|i_j\|^{X_E} = \|i_j\|$ and $\|\pi_j\|^{X_E} = \|\pi_j\|$ when we are dealing with more than one space.

Note that the third condition is the same as saying that the $E-$valued polynomials, $\mathcal{P}(E)$ are continuously embedded in the space X_E.

In the case $E = \mathbb{C}$ we would be talking of $\mathcal{S}$-admissibility, as expected.

REMARK 2.2. Let X_{E_2} be $\mathcal{S}(E_2)-$admissible and let E_1 be isomorphic to a closed subspace of E_2, say $I(E_1)$. Define

$$X_{E_1} = \{(x_j)_j \in \mathcal{S}(E_1) : x_j \in E_1, (I(x_j))_j \in X_{E_2}\}$$

and the norm

$$\|(x_j)_j\|_{X_{E_1}} = \|(I(x_j))_j\|_{X_{E_2}}.$$

Then X_{E_1} is $\mathcal{S}(E_1)-$admissible.

Also, if Z is a Banach space and $X_E \subset Z \subset Y_E$, where X_E and Y_E are $\mathcal{S}(E)$-admissible, then Z is $\mathcal{S}(E)$-admissible.

Let us give a method to generate $\mathcal{S}(E)-$admissible spaces.

PROPOSITION 2.3. *Let E be a Banach space and let X be $\mathcal{S}-$admissible. Then $X\hat{\otimes}_\pi E, X[E]$ and $X_{weak}(E)$ are $\mathcal{S}(E)-$admissible.*

PROOF. Clearly $X_{weak}(E) = \mathcal{L}(E^*, X)$ and $X\hat{\otimes}_\pi E$ have complete norms.

Due to the continuous embeddings

$$X\hat{\otimes}_\pi E \subset X[E] \subset X_{weak}(E)$$

we only need to see that $\mathcal{P}(E) \subset X\hat{\otimes}_\pi E$ with continuous injections i_j for $j \geq 0$ and that $X_{weak}(E) \subset \mathcal{S}(E)$ with continuity. Both assertions follow trivially from the facts

$$\|xe_j\|_{X\hat{\otimes}_\pi E} = \|x\|_E \|e_j\|_X \leq \|i_j\|^X \|x\|_E$$

and

$$\|x_j\|_E = \sup_{\|x^*\|_{E^*}=1} |\langle x_j, x^*\rangle| \leq \|\pi_j\|^X \|(x_k)_k\|_{X_{weak}(E)},$$

where the admissibility of X has been used in both inequalities. □

Example 2.4. Some examples of $\mathcal{S}(E)-$admissible spaces are $\ell^p(E), \ell^p_{weak}(E)$, $\ell(p,q,E)$ and $\ell^p\hat{\otimes}_\pi E$ for $1 \leq p,q \leq \infty$. In particular, $c_0(E)$ and $UC(E)$ are $S(E)-$admissible spaces.

Recall that

$$\ell^1_{weak}(E) \subset Rad(E) \subset \ell^2_{weak}(E)$$

with continuous embeddings and therefore $Rad(E)$ is $\mathcal{S}(E)$-admissible.

Definition 2.5. Let X_E be $\mathcal{S}(E)$- admissible. We define

$$X_E^K = \Big\{ f = (x_j^*)_j \in \mathcal{S}(E^*) : \sum_j |\langle x_j^*, x_j\rangle| < \infty, \forall (x_j)_j \in X_E \Big\}.$$

We denote $X_E^{KK} = (X_E^K)^K$.

The space X_E^K is nothing but the so-called **Köthe dual** of the space X_E.

Remark 2.6. The space X_E^K is $\mathcal{S}(E^*)-$admissible.

The proof is standard, taking into account the $\mathcal{S}(E)-$ admissibility of the space X_E and considering the norm defined by $\|f\|_{X_E^K} = \sup_{x\in X_E} \sum_j |\langle x_j^*, x_j\rangle|$ for $f = (x_j^*)_j \in X_E^K$.

Some well-known Köthe duals are $\ell^1(E)^K = \ell^\infty(E^*), \ell^\infty(E)^K = \ell^1(E^*)$ and $c_0(E)^K = \ell^1(E^*)$.

Definition 2.7. (**Minimal space**) Let X_E be $\mathcal{S}(E)-$admissible and recall the notation $X_E^0 = \overline{\mathcal{P}(E)}^{X_E}$. We say that X_E is minimal whenever $\mathcal{P}(E)$ is dense in X_E , that is to say $X_E^0 = X_E$.

Of course X_E^0 is $\mathcal{S}(E)-$admissible whenever X_E is.

Proposition 2.8. *Let X_E be $\mathcal{S}(E)-$admissible and let F be a Banach space. Then $\mathcal{L}(X_E, F)$ is $\mathcal{S}(\mathcal{L}(E,F))-$admissible.*
In particular $(X_E)^*$ and $(X_E^0)^*$ are $\mathcal{S}(E^*)-$admissible.

Proof. Identifying each $T \in \mathcal{L}(X_E, F)$ with the sequence $(\hat{T}(j))_j \in \mathcal{S}(\mathcal{L}(E,F))$ given by $\hat{T}(j)(x) = T(xe_j)$, we have that $\mathcal{L}(X_E, F) \hookrightarrow \mathcal{S}(\mathcal{L}(E,F))$. Moreover $\|\pi_j\|^{\mathcal{L}(X_E,F)} \leq \|i_j\|^{X_E}$ due to the estimate $\|\hat{T}(j)\|_{\mathcal{L}(E,F)} \leq \|i_j\|^{X_E}\|T\|_{\mathcal{L}(X_E,F)}$.

To show $\mathcal{P}(\mathcal{L}(E,F)) \subset \mathcal{L}(X_E, F)$ with continuity, we use that, for each $j \geq 0$ and $S \in \mathcal{L}(E,F)$, Se_j defines an operator in $\mathcal{L}(X_E, F)$ by means of

$$Se_j(f) = S(x_j), f = (x_j)_j \in X_E.$$

Moreover $\|i_j\|^{\mathcal{L}(E,F)} \leq \|\pi_j\|^{X_E}$ because $\|Se_j\|_{\mathcal{L}(X_E,F)} \leq \|\pi_j\|^{X_E}\|S\|_{\mathcal{L}(E,F)}$. □

2. Operator-valued multipliers

Recall that, given two $\mathcal{S}-$admissible Banach spaces X and Y, a coefficient multiplier $\lambda \in (X,Y)$ is a sequence whose coefficients lay in $\mathbb{K}$, where we have an inner product. Now we are dealing with spaces which have coefficients on different Banach spaces. This leads us to a change of perspective: the coefficients λ_j must "transform" coefficients in E_1 into coefficients in E_2. Thus, the most natural definition one can think of is the following.

Definition 2.9. (**Operator-valued multipliers**) Let X_{E_1}, X_{E_2} be Banach spaces. We define the multipliers space between X_{E_1} and X_{E_2} as

$$(X_{E_1}, X_{E_2}) = \{\lambda \in \mathcal{S}(\mathcal{L}(E_1,E_2)) : \ \lambda *_{\mathcal{L}} f \in X_{E_2} \ \forall f \in X_{E_1}\},$$

where

$$\lambda *_{\mathcal{L}} f = \sum_j \lambda_j(\hat{f}(j))e_j.$$

We can endow this space with the norm

$$\|\lambda\|_{(X_{E_1},X_{E_2})} = \sup_{\|f\|_{X_{E_1}}<1} \|\lambda *_{\mathcal{L}} f\|_{X_{E_2}}$$

and it becomes a Banach space.

Two particular cases worth mentioning are, on one hand, the case $E_1 = \mathbb{K}$. In this case $\lambda \in \mathcal{S}$ and we will write $\lambda *_{B_0} f = \sum_j \lambda_j \hat{f}(j)e_j$ and $(X, X_E)_{B_0}$ for the space of multipliers. On the other hand, the case $E_2 = \mathbb{K}$, where $\lambda \in \mathcal{S}(E^*)$ and naturally $\lambda_j(\hat{f}(j)) = \langle \lambda_j, \hat{f}(j)\rangle$. Here we will write $\lambda *_{\mathcal{D}} f$ for the product and $(X_E, X)_{\mathcal{D}}$ for the space of multipliers.

The notation might seem a bit strange, but we will keep it to be coherent with the following chapters.

Theorem 2.10. *If X_{E_1} and X_{E_2} are $\mathcal{S}(E_1)$, $\mathcal{S}(E_2)-$admissible Banach spaces respectively, then (X_{E_1}, X_{E_2}) is $\mathcal{S}(\mathcal{L}(E_1,E_2))-$admissible.*

Proof. Let $\lambda = (T_j)_j \in (X_{E_1}, X_{E_2})$ and $j \geq 0$. For each $x \in E_1$, using the admissibility of X_{E_1} and X_{E_2}, we have

$$\begin{aligned} \|T_j(x)\|_{E_2} &\leq \|\pi_j\|^{X_{E_2}} \|T_j(x)e_j\|_{X_{E_2}} \\ &= \|\pi_j\|^{X_{E_2}} \|\lambda *_{\mathcal{L}} xe_j\|_{X_{E_2}} \\ &\leq \|\pi_j\|^{X_{E_2}} \|\lambda\|_{(X_{E_1},X_{E_2})} \|xe_j\|_{X_{E_1}} \\ &\leq \|\pi_j\|^{X_{E_2}} \|i_j\|^{X_{E_1}} \|\lambda\|_{(X_{E_1},X_{E_2})} \|x\|_{E_1}. \end{aligned}$$

This gives $\|\pi_j\|^{(X_{E_1},X_{E_2})} \leq \|\pi_j\|^{X_{E_2}} \|i_j\|^{X_{E_1}}$ and $(X_{E_1}, X_{E_2}) \hookrightarrow \mathcal{S}(\mathcal{L}(E_1,E_2))$ with continuity.

On the other hand if $p \in \mathcal{P}(\mathcal{L}(E_1,E_2))$ and $f \in X_{E_1}$ we have $p *_{\mathcal{L}} f \in \mathcal{P}(E_2) \subset X_{E_2}$. Hence $p \in (X_{E_1}, X_{E_2})$. For each $j \geq 0$ and $T \in \mathcal{L}(E_1,E_2)$, we have to show that $\|Te_j\|_{(X_{E_1},X_{E_2})} \leq C_j\|T\|$. Now given $f \in X_{E_1}$, again by the admissibility of X_{E_1} and X_{E_2},

$$\begin{aligned} \|Te_j *_{\mathcal{L}} f\|_{X_{E_2}} &= \|T(\hat{f}(j))e_j\|_{X_{E_2}} \\ &\leq \|i_j\|^{X_{E_2}} \|T(\hat{f}(j))\|_{E_2} \\ &\leq \|i_j\|^{X_{E_2}} \|T\| \|\hat{f}(j)\|_{E_1} \\ &\leq \|i_j\|^{X_{E_2}} \|\pi_j\|^{X_{E_1}} \|T\| \|f\|_{X_{E_1}}. \end{aligned}$$

Therefore $\|i_j\|^{(X_{E_1},X_{E_2})} \leq \|i_j\|^{X_{E_2}}\|\pi_j\|^{X_{E_1}}$.

Let us now show the completeness of (X_{E_1}, X_{E_2}). Let $(\lambda_n)_n \subset (X_{E_1}, X_{E_2})$ be a Cauchy sequence of multipliers. Since the sequence of operators $\Lambda_n(f) = \lambda_n *_{\mathcal{L}} f$ is a Cauchy sequence in $\mathcal{L}(X_{E_1}, X_{E_2})$ we define $\Lambda \in \mathcal{L}(X_{E_1}, X_{E_2})$ to be its limit in the norm. Therefore

$$\|\Lambda - \Lambda_n\| \to 0 \ \Rightarrow \|\Lambda(f) - \Lambda_n(f)\|_{X_{E_2}} \to 0 \ \Rightarrow \lambda_n *_{\mathcal{L}} f \to \Lambda(f) \in \mathcal{S}(E_2).$$

On the other hand, we know $(X_{E_1}, X_{E_2}) \hookrightarrow \mathcal{S}(\mathcal{L}(E_1, E_2))$ and then there exists $\lambda \in \mathcal{S}(\mathcal{L}(E_1, E_2))$ such that

$$\lambda_n *_{\mathcal{L}} f \to \lambda *_{\mathcal{L}} f$$

in $\mathcal{S}(\mathcal{L}(E_1, E_2))$. Hence necessarily $\Lambda(f) = \lambda *_{\mathcal{L}} f$. □

We find a particular case of operator valued multipliers in solid spaces.

Definition 2.11. Let X_E be $\mathcal{S}(E)-$admissible. We define

$$X_E^S = \{f = (x_j)_j \in \mathcal{S}(E) : (\alpha_j x_j)_j \in X_E, \forall (\alpha_j)_j \in \ell^\infty\}$$

In general we have

$$X_E^S \subseteq X_E \subseteq X_E^{KK}.$$

Remark 2.12. The space X_E^S is $\mathcal{S}(E)-$admissible.

The proof is standard, taking into account the $\mathcal{S}(E)-$ admissibility of the space X_E.

Definition 2.13. (**Solid space**) We say that a Banach space, $X_E \subset \mathcal{S}(E)$, is $S(E)-$solid (or simply solid) whenever X_E is a $\mathcal{S}(E)-$admissible space verifying $(\alpha_j \hat{f}(j))_j \in X_E$ for $f \in X_E$ and $(\alpha_j)_j \in \ell^\infty$, that is to say $X_E = X_E^S$.

As in the scalar case, X_E is solid iff $\ell^\infty \subseteq (X_E, X_E)$

Proposition 2.14. *Let X_E be an $\mathcal{S}(E)-$admissible space. The largest solid subset of X_E exists and is $s(X_E) = (\ell^\infty, X_E)_{B_0}$.*

Proof. Since ℓ^∞ is a solid space and $(\ell^\infty, X_E)_{B_0}$ is an $\mathcal{S}(E)-$admissible Banach space (see Theorem 2.10), it is straightforward that $(\ell^\infty, X_E)_{B_0}$ is a solid subspace of X_E. Now let Y_E be another solid subset of X_E. If $g \in Y_E$, given $\alpha \in \ell^\infty$ we have $g *_{B_0} \alpha \in Y_E \subset X_E$. □

Remark 2.15. (a) $X[E], X_{weak}(E)$ and $X\hat{\otimes}_\pi E$ are $\mathcal{S}(E)-$solid iff X is a solid space. The proofs are quite easy taking into account the characterization $\ell^\infty \subseteq (Y, Y)$ with the corresponding space Y on each case.

In particular, $\ell^p(E), \ell^p_{weak}(E)$ and $\ell^p\hat{\otimes}_\pi E$ are $\mathcal{S}(E)$-solid for $1 \leq p \leq \infty$.

(b) $Rad(E)$ is a $\mathcal{S}(E)$-solid space. (This follows from Kahane's contraction principle).

(c) Neither $H^p(\mathbb{D}, E)$ nor $A^p(\mathbb{D}, E)$ are $\mathcal{S}(E)$-solid unless $p = 2$.

Assuming that they are $\mathcal{S}(E)$-solid, and restricting to $\phi(z)x$ for $\phi \in \mathcal{H}(\mathbb{D})$ and $x \in E$, we will have that also H^p or A^p must be solid for $p \neq 2$, which is not the case.

Proposition 2.16. *Let X, Y be a solid spaces. Then*

$$(X[E_1], Y[E_2]) = (X, Y)[\mathcal{L}(E_1, E_2)].$$

Proof. Given $\lambda = (\lambda_j)_j \in (X, Y)[\mathcal{L}(E_1, E_2)]$, note that for $x = (x_j)_j \in X[E_1]$,

$$\|\lambda_j(x_j)\|_{E_2} \leq \|\lambda_j\|\|x_j\|_{E_1}.$$

Then it is clear that $\lambda \in (X[E_1], Y[E_2])$.

For the reverse inclusion, given $\lambda = (\lambda_j)_j \in (X[E_1], Y[E_2])$, consider $\alpha = (\alpha_j)_j \in X$ and take $(\epsilon_j)_j \in (X, Y)$ such that $\epsilon_j > 0, j \in \mathbb{N}_0$. The existence of such a sequence is guaranteed by the fact that (X, Y) is solid, since X is solid. We can find $(x_j)_j \subset E_1$, $\|x_j\|_{E_1} = 1$, such that

$$\|\lambda_j(\alpha_j x_j)\| \geq \alpha_j \|\lambda_j\| - \epsilon_j \alpha_j.$$

By construction, the sequence $(\alpha_j x_j)_j \in \mathcal{S}(E_1)$ is such that $(\|\alpha_j x_j\|_{E_1})_j = (|\alpha_j|)_j$ and since X is solid and $\alpha \in X$, we get to $(|\alpha_j|)_j \in X$. Thus, $(\alpha_j x_j)_j \in X[E_1]$. Now use the fact that $\lambda \in (X[E_1], Y[E_2])$ to obtain $(\|\lambda_j(\alpha_j x_j)\|_{E_2})_j \in Y$.
Notice that, by the choice of $(\epsilon_j)_j$, we get $(\epsilon_j \alpha_j)_j \in Y$. This together with the fact that $\|\lambda_j\|\alpha_j \leq \|\lambda_j(\alpha_j x_j)\|_{E_2} + \epsilon_j \alpha_j$ and that Y is solid, gives $(\|\lambda_j\|\alpha_j)_j \in Y$. □

Making use of Remark 2.15 and Proposition 2.16,

$$(\ell^p(E_1), \ell^r(E_2)) = \ell^q(\mathcal{L}(E_1, E_2)) \tag{2.1}$$

where $1 \leq p, q \leq \infty$ and $q = p \ominus r$.

In a similar way, for $1 \leq p, q, u, v \leq \infty$

$$(\ell(p, q, E_1), \ell(u, v, E_2)) = \ell(p \ominus u, q \ominus v, \mathcal{L}(E_1, E_2)),$$

(see [32]).

Corollary 2.17. *Given X, Y solid spaces, we have $(X[E], Y)_{\mathcal{D}} = (X, Y)[E^*]$. In particular $(X[E])^K = X^K[E^*]$.*

Therefore, for $1 \leq p \leq \infty$ and p' its conjugate exponent,

$$(\ell^p(E))^K = (\ell^p(E), \ell^1)_{\mathcal{D}} = (\ell^p, \ell^1)[E^*] = \ell^{p'}(E^*)$$

and

$$(\ell^p(E))^{KK} = ((\ell^p(E))^K, \ell^1)_{\mathcal{D}} = (\ell^{p'}[E^*], \ell^1) = (\ell^{p'}, \ell^1)[E^*] = \ell^p(E^{**}).$$

Proposition 2.18. *Let X be $\mathcal{S}-$solid and E a Banach space. Then*

$$(X \hat{\otimes}_\pi E)^K = (X^K)_{weak}(E^*).$$

Proof. We first claim that $(x_j^*)_j \in (X^K)_{weak}(E^*)$ if and only if $\big(\langle x_j^*, x\rangle\big)_j \in X^K$ for all $x \in E$. We only need to see that if

$$\sup_{\|x\|_E = 1} \|\big(\langle x_j^*, x\rangle\big)_j\|_{X^K} < \infty$$

then $\big(\langle x^{**}, x_j^*\rangle\big)_j \in X^K$ for $x^{**} \in E^{**}$.

Let $x^{**} \in E^{**}$. For each $(\alpha_j)_j \in X$ with $\|(\alpha_j)_j\|_X \leq 1$ and $N \in \mathbb{N}$, there are ϵ_j with $|\epsilon_j| = 1$,

$$\begin{aligned}
\sum_{j=0}^N |\langle x^{**}, x_j^*\rangle \alpha_j| &= |\sum_{j=0}^N \langle x^{**}, x_j^*\rangle \alpha_j \epsilon_j| \\
&= |\langle x^{**}, \sum_{j=0}^N x_j^* \alpha_j \epsilon_j\rangle| \\
&\leq \|x^{**}\|_{E^{**}} \|\sum_{j=0}^N x_j^* \alpha_j \epsilon_j\|_{E^*} \\
&\leq \|x^{**}\|_{E^{**}} \sup_{\|x\|_E = 1} \sum_{j=0}^N |\langle x_j^*, x\rangle \alpha_j| \\
&\leq \|x^{**}\|_{E^{**}} \sup_{\|x\|_E = 1} \|\big(\langle x_j^*, x\rangle\big)_j\|_{X^K}.
\end{aligned}$$

This concludes the claim.

We show first $(X\hat{\otimes}_\pi E)^K \subseteq (X^K)_{weak}(E^*)$. Take $\lambda = (x_j^*)_j \in (X\hat{\otimes}_\pi E)^K$, $x \in E$ and $(\alpha_j)_j \in X$. Note that

$$\big(\langle x_j^*, x\rangle \alpha_j\big)_j \in \ell^1 \tag{2.2}$$

and then we obtain that $(x_j^*)_j \in (X^K)_{weak}(E^*)$ with $\|(x_j^*)_j\|_{(X^K)_{weak}(E^*)} \leq \|\lambda\|$ from the previous result.

Assume now that $\lambda = (x_j^*)_j \in (X^K)_{weak}(E^*)$ and let us show that $\lambda \in (X\hat{\otimes}_\pi E)^K$. If $\epsilon > 0$ and $f = \sum_n f_n \otimes x_n \in X\hat{\otimes}_\pi E$ with $\hat{f}_n(j) = \alpha_j^n$ and $\sum_n \|f_n\|_X \|x_n\|_E < \|f\|_{X\hat{\otimes}_\pi E} + \epsilon$ we have

$$\begin{aligned}
\sum_j |\langle x_j^*, \sum_n \alpha_j^n x_n\rangle| &\leq \sum_j \sum_n |\langle x_j^*, x_n\rangle \alpha_j^n| \\
&= \sum_n \sum_j |\langle x_j^*, x_n\rangle \alpha_j^n| \\
&\leq \sum_n \|x_n\|_E \Big\| \Big(\langle x_j^*, \frac{x_n}{\|x_n\|_E}\rangle\Big)_j \Big\|_{X^K} \|f_n\|_X \\
&\leq \|(x_j^*)_j\|_{(X^K)_{weak}(E^*)} \big(\sum_n \|x_n\|_E \|f_n\|_X\big) \\
&\leq \|(x_j^*)_j\|_{(X^K)_{weak}(E^*)} (\|f\|_{X\hat{\otimes}_\pi E} + \epsilon).
\end{aligned}$$

□

Remark 2.19. In general $X^K \hat{\otimes}_\pi E^* \subseteq (X_{weak}(E))^K$.

Indeed, for each $g = (\beta_j)_j \in X^K$, $x^* \in E^*$ and $f = (x_j)_j \in X_{weak}(E)$, we have that

$$\sum_j |\langle x^*, x_j\rangle \beta_j| \leq \|g\|_{X^K} \|x^*\|_{E^*} \|f\|_{X_{weak}(E)}$$

and then

$$\|g \otimes x^*\|_{(X_{weak}(E))^K} \leq \|g\|_{X^K} \|x^*\|_{E^*}$$

Now we extend using linearity and density to obtain $X^K \hat{\otimes}_\pi E^* \subseteq (X_{weak}(E))^K$.

For the case $X = \ell^p$, $1 < p < \infty$, it was shown (see [19, 26, 5]) that

$$(\ell^p_{weak}(E))^K = \ell^{p'} \hat{\otimes}_\pi E^*.$$

3. Generalized mixed-norm spaces

During the following sections the inclusion $E_1 \subset E_2$ will be considered a continuous inclusion.

We define now a vector-valued version of the spaces $\ell(p, q)$ presented in Definition 1.2 with some modifications on the support of the sequence and the way one takes the intervals:

Definition 2.20. (**Generalized vector-valued mixed-norm spaces**) Let $1 \leq p, q \leq \infty$ and let $\mathcal{I}$ be a collection of disjoint intervals in $\mathbb{N}_0$, say $I_k = \mathbb{N}_0 \cap [n_k, n'_k)$ where $n_k < n'_k \leq n_{k+1}$. We set $\Lambda_{\mathcal{I}} = \cup_{k\in\mathbb{N}_0} I_k$. We write $\ell^{\mathcal{I}}(p, q, E)$ for the space of sequences $(a_j)_{j\in\Lambda_{\mathcal{I}}} \in \mathcal{S}(E)$ verifying

$$\left(\left(\sum_{j\in I_k} \|a_j\|_E^p\right)^{1/p}\right)_k \in \ell^q.$$

This space becomes a Banach space under the norm

$$\|a\|^{\mathcal{I}}_{p,q,E} = \left(\sum_{k=0}^{\infty} \left(\sum_{j\in I_k} \|a_j\|^p_E \right)^{q/p} \right)^{1/q}$$

with the obvious modifications for $p = \infty$ or $q = \infty$.
We will simply write $\ell^{\mathcal{I}}(p,q)$ for the scalar case and $\|\cdot\|^{\mathcal{I}}_{p,q}$ either for the scalar case or for the case in which there is no possible confusion with the Banach space.

We can define the weak version as well: we will say $(a_n)_{n\in\Lambda_{\mathcal{I}}} \in \ell^{\mathcal{I}}_{weak}(p,q,E)$ (shorter, $\ell^{\mathcal{I}}_w(p,q,E)$) if

$$\left(\left(\sum_{j\in I_k} |\langle a^*, a_j\rangle|^p \right)^{1/p} \right)_k \in \ell^q$$

for every $a^* \in E^*$. This space also becomes a Banach space under the norm

$$\|a\|^{\mathcal{I},w}_{p,q,E} = \sup_{a^*\in B_{E^*}} \left(\sum_{k=0}^{\infty} \left(\sum_{j\in I_k} |\langle a^*, a_j\rangle|^p \right)^{q/p} \right)^{1/q}$$

with the obvious modifications for $p = \infty$ or $q = \infty$.

Remark 2.21. Of course

$$\ell^{\mathcal{I}}(p,p,E) = \{(a_n)_{n\in\Lambda_{\mathcal{I}}} \in \mathcal{S}(E) : (\sum_n \|a_n\|^p_E)^{1/p} < \infty\}$$

and

$$\ell^{\mathcal{I}}_w(p,p,E) = \{(a_n)_{n\in\Lambda_{\mathcal{I}}} \in \mathcal{S}(E) : \sup_{a^*\in B_{E^*}} (\sum_n |\langle a^*, a_n\rangle|^p)^{1/p} < \infty\}.$$

In particular

$$\ell^{\mathcal{I}}(p,p,E) = \ell^p(E) := \left\{ (x_k)_k \in \mathcal{S}(E);\ \left(\sum_k \|x_k\|^p_E \right)^{1/p} < \infty \right\}$$

and

$$\ell^{\mathcal{I}}_w(p,p,E) = \ell^p_{weak}(E) := \left\{ (x_k)_k \in \mathcal{S}(E);\ \sup_{a^*\in B_{E^*}} \|(\langle a^*, x_k\rangle)_k\|_{\ell^p} < \infty \right\}$$

whenever $\Lambda_{\mathcal{I}} = \mathbb{N}_0$.

Both spaces $\ell^{\mathcal{I}}(p,q,E)$ and $\ell^{\mathcal{I}}_w(p,q,E)$ are $\mathcal{S}(E)-$admissible with $\|i_j\| = \|\pi_j\| = 1$.

Proposition 2.22. *For $1 \le p,q < \infty$ and $1/p + 1/p' = 1/q + 1/q' = 1$*

$$\ell^{\mathcal{I}}(p,q,E)^* = \ell^{\mathcal{I}}(p',q',E^*).$$

Therefore, $\ell^{\mathcal{I}}(p,q,E)^*$ is an $\mathcal{S}(E^*)-$admissible space

Proof. Let $x^* = (x^*_n)_{n\in\Lambda} \in \ell^{\mathcal{I}}(p',q',E^*)$ and define $\phi_{x*} : \ell^{\mathcal{I}}(p,q,E) \to \mathbb{C}$, $\phi(x) := \sum_n \langle x^*_n, x_n\rangle$. Using Hölder's inequality twice,

$$\sum_n |\langle x^*_n, x_n\rangle| = \sum_k \sum_{n\in I_k} |\langle x^*_n, x_n\rangle| \le \|x^*\|^{\mathcal{I}}_{p',q',E^*} \|x\|^{\mathcal{I}}_{p,q,E}$$

and so ϕ_{x*} is well-defined. It is also clear that ϕ_{x^*} is a linear continuous map.
For the reverse inclusion, consider $\phi \in \ell^{\mathcal{I}}(p,q,E)^*$ and define $x^*_n : E \to \mathbb{C}$, $x^*_n(a) := \phi(ae_n)$. Then $ae_n \in \ell^{\mathcal{I}}(p,q,E)$ because it is finitely supported. Thus the map is well defined.

Now consider $x^* = (x_n^*)_{n\in\Lambda_{\mathcal{I}}}$ and let $\epsilon > 0$. Using duality in ℓ^q we can find a sequence $\beta \in \ell^q$ such that $\|\beta\|_q = 1$ supported in Λ verifying

$$\|x^*\|^{\mathcal{I}}_{p',q',E^*} = \left(\sum_{k\in\Lambda}\left(\sum_{n\in I_k}\|x_n^*\|^{p'}_{E^*}\right)^{q'/p'}\right)^{1/q'}$$
$$= \sum_{k\in\Lambda}\beta(k)\left(\sum_{n\in I_k}\|x_n^*\|^{p'}_{E^*}\right)^{1/p'}.$$

Then, for a fixed $k \in \mathbb{N}_0$, we take $\alpha_k \in \ell^p$ supported in Λ such that $\|\alpha_k\|_p = 1$ and $\left(\sum_{n\in I_k}\|x_n^*\|^{p'}_{E^*}\right)^{1/p'} = \sum_{n\in I_k}\alpha_k(n)\|x_n^*\|_{E^*}$, and let $a_k = \frac{1}{\beta(k)^{q-1}}\sum_{n\in I_k}\alpha_k(n)$. We can find a sequence $(x_n)_n \subset E$, with $\|x_n\|_E = 1$ for any n and such that $\|x_n^*\|_{E^*} = \phi(x_ne_n) + \frac{\epsilon}{a_k}$. Thus

$$\begin{aligned}
\|x^*\| &= \sum_{k\in\Lambda}\beta(k)\sum_{n\in I_k}\alpha_k(n)\|x_n^*\|_{E^*}\\
&= \sum_{k\in\Lambda}\beta(k)\sum_{n\in I_k}\alpha_k(n)\big(\phi(x_ne_n) + \frac{\epsilon}{a_k}\big)\\
&\le \sum_{k\in\Lambda}\beta(k)\sum_{n\in I_k}\alpha_k(n)\phi(x_ne_n) + \sum_{k\in\Lambda}\beta(k)\sum_{n\in I_k}\alpha_k(n)\frac{\epsilon}{a_k}\\
&= \sum_{k\in\Lambda}\beta(k)\sum_{n\in I_k}\alpha_k(n)\phi(x_ne_n) + \epsilon\sum_{k\in\Lambda}\beta(k)^q\\
&\le \sum_{k\in\Lambda}\beta(k)\sum_{n\in I_k}\alpha_k(n)\phi(x_ne_n) + \epsilon.
\end{aligned}$$

If $n \in I_k$, name $\gamma(n) = \beta(k)\alpha_k(n)$. Now notice that for $N, M \in \Lambda$, there exist $N_1, M_1 \in \mathbb{N}$ such that $N \in I_{N_1}$ and $M \in I_{M_1}$, therefore

$$\begin{aligned}
\|\sum_{n=N}^{M}\gamma(n)x_ne_n\|_{p,q,E} &\le \|\sum_{k=N_1}^{M_1}\beta(k)\sum_{n\in I_k}\alpha_k(n)x_ne_n\|_{p,q,E}\\
&\le \left(\sum_{k=N_1}^{M_1}|\beta(k)|^q\left(\sum_{n\in I_k}|\alpha_k(n)|^p\|x_n\|_E^p\right)^{q/p}\right)^{1/q} \le \left(\sum_{k=N_1}^{M_1}|\beta(k)|^q\right)^{1/q}
\end{aligned}$$

which tends to zero as $N, M \to \infty$. This together with the fact that ϕ is a continuous linear map, allow us to write

$$\|x^*\| \le \phi\big(\sum_k\beta(k)\sum_{n\in I_k}\alpha_k(n)x_ne_n\big) + \epsilon \le \|\phi\|\|(\gamma(n)x_n)_n\|_{p,q,E} + \epsilon = \|\phi\| + \epsilon,$$

by the choice of α_k, β and $(x_n)_n$. This completes the proof. □

REMARK 2.23. It is clear that $(a_j)_j \in \ell^{\mathcal{I}}(p,q,E) \Leftrightarrow (a_j^p)_j \in \ell^{\mathcal{I}}(1,q/p,E)$ in the case $p < q$ and also $(a_j)_j \in \ell^{\mathcal{I}}(p,q,E) \Leftrightarrow (a_j^q)_j \in \ell^{\mathcal{I}}(\frac{p}{q},1,E)$ in the case $p > q$. Moreover, for $a^p = (a_j^p)_j$,

$$\|a\|^{\mathcal{I}}_{p,q} = \left(\|a^p\|^{\mathcal{I}}_{1,q/p}\right)^{1/p} = \left(\|a^q\|^{\mathcal{I}}_{p/q,1}\right)^{1/q} \tag{2.3}$$

There is an analogous result for $\ell^{\mathcal{I}}_w(p,q,E)$.

REMARK 2.24. Let $a \in \ell^{\mathcal{I}}(p,q,E)$.

(i) If $\mathcal{I}'$ is a sub-collection of intervals in $\mathcal{I}$ then $\|a\|_{p,q}^{\mathcal{I}'} \leq \|a\|_{p,q}^{\mathcal{I}}$ and $\|a\|_{p,q}^{\mathcal{I}',w} \leq \|a\|_{p,q}^{\mathcal{I},w}$.

(ii) If $\mathcal{I} = \mathcal{I}' \cup \mathcal{I}''$ for two disjoint collections $\mathcal{I}'$ and $\mathcal{I}''$ then

$$\|a\|_{p,q}^{\mathcal{I}} = \Big((\|a\|_{p,q}^{\mathcal{I}'})^q + (\|a\|_{p,q}^{\mathcal{I}''})^q \Big)^{1/q}$$

and

$$\|a\|_{p,q}^{\mathcal{I},w} = \sup_{a^* \in B_{E^*}} \Big((\|(\langle a^*, a_j\rangle)_j\|_{p,q}^{\mathcal{I}'})^q + (\|(\langle a^*, a_j\rangle)_j\|_{p,q}^{\mathcal{I}''})^q \Big)^{1/q}.$$

We would like to analyze the embedding between $\ell^{\mathcal{I}}(p_1,q_1,E)$ and $\ell^{\mathcal{I}}(p_2,q_2,E)$ and $\ell^{\mathcal{I}}_w(p_1,q_1,E)$ and $\ell^{\mathcal{I}}_w(p_2,q_2,E)$.

PROPOSITION 2.25. *Let $\mathcal{I}$ be a collection of disjoint intervals in $\mathbb{N}_0$ and let $1 \leq p_1, p_2, q \leq \infty$ with $p_1 \neq p_2$. Then $\ell^{\mathcal{I}}(p_1,q,E) = \ell^{\mathcal{I}}(p_2,q,E)$ (with equivalent norms) if and only if*

$$\sup_{k\in\mathbb{N}_0} \#I_k < \infty. \tag{2.4}$$

In particular if $\sup_{k\in\mathbb{N}_0} \#I_k < \infty$ then

$$\ell^{\mathcal{I}}(p,q,E) = \left\{ (a_n)_{n\in\Lambda_{\mathcal{I}}} : \left(\sum_n \|a_n\|_E^q \right)^{1/q} < \infty \right\}.$$

PROOF. $\Longrightarrow$) Assume, for instance, $p_1 < p_2$ and that $\|a\|_{p_1,q}^{\mathcal{I}} \approx \|a\|_{p_2,q}^{\mathcal{I}}$ for all a supported in $\Lambda_{\mathcal{I}}$. Hence taking $a\chi_{I_k}$ where $\|a_i\|_E = 1$ $\forall i \in I_k$ one concludes that $(n_k' - n_k)^{1/p_1 - 1/p_2} \leq C$ for any k. Hence $\sup_k \#I_k < \infty$.

$\Longleftarrow$) Note that $\#I_k = (n_k' - n_k)$ and assume $M = \sup_k (n_k' - n_k)$. Then

$$\|a\|_{p_1,q}^{\mathcal{I}} = \left(\sum_{k=0}^{\infty} \left(\sum_{j\in I_k} \|a_j\|_E^{p_1} \right)^{q/p_1} \right)^{1/q} \approx \left(\sum_{k=0}^{\infty} \left(\sum_{j\in I_k} \|a_j\|_E^{p_2} \right)^{q/p_2} \right)^{1/q} = \|a\|_{p_2,q}^{\mathcal{I}}$$

since $\|\cdot\|_{p_1} \approx \|\cdot\|_{p_2}$ in $\mathbb{C}^M$.

□

REMARK 2.26. In the same conditions of Proposition 2.25, the proof may be adapted for the weak case, that is, $\ell^{\mathcal{I}}_w(p_1,q,E) = \ell^{\mathcal{I}}_w(p_2,q,E)$ (with equivalent norms) if and only if $\sup_{k\in\mathbb{N}_0} \#I_k < \infty$.

PROPOSITION 2.27. *Let $1 \leq p_1, q_1, p_2, q_2 \leq \infty$ and let $\mathcal{I}$ be a collection of disjoint intervals in $\mathbb{N}_0$ with $\sup_k \#I_k = \infty$.*
Then $\ell^{\mathcal{I}}(p_1,q_1) \subseteq \ell^{\mathcal{I}}(p_2,q_2)$ if and only if $p_1 \leq p_2$ and $q_1 \leq q_2$.

PROOF. $\Longrightarrow$) Assume that there exists $C > 0$ such that $\|a\|_{p_2,q_2}^{\mathcal{I}} \leq C\|a\|_{p_1,q_1}^{\mathcal{I}}$ for all a supported in $\Lambda_{\mathcal{I}}$. Hence taking $k \in \mathbb{N}_0$ and $a = \chi_{I_k}$ one concludes that $(\#I_k)^{1/p_2 - 1/p_1} \leq C$. Hence $p_1 \leq p_2$. Let $N \in \mathbb{N}_0$ and consider $a = \sum_{k=1}^N \chi_{n_k}$. Applying the above inequality we obtain $N^{1/q_2 - 1/q_1} \leq C$. Therefore $q_1 \leq q_2$.

$\Longleftarrow$) Let us denote

$$\ell^q(\ell^p) = \left\{ (x_k)_{k\in\mathbb{N}_0} : x_k \in \ell^p, \left(\sum_{k=0}^{\infty} \|x_k\|_{\ell^p}^q \right)^{1/q} < \infty \right\}.$$

Hence the mapping

$$(a_n)_{n\in\Lambda_{\mathcal{I}}} \to ((a_j)_{j\in I_k})_{k\in\mathbb{N}_0}$$

is an isometric embedding from $\ell^{\mathcal{I}}(p,q)$ into $\ell^q(\ell^p)$. Notice that the index that don't belong to $\Lambda_{\mathcal{I}}$ are not considered in any case. Taking into account that $\ell^{r_1}(E) \subseteq \ell^{r_2}(E)$ for any Banach space E and $r_1 \le r_2$ we conclude that

$$\ell^{\mathcal{I}}(p,q_1) \subseteq \ell^{\mathcal{I}}(p,q_2) \text{ and } \ell^{\mathcal{I}}(p_1,q) \subseteq \ell^{\mathcal{I}}(p_2,q).$$

Therefore

$$\ell^{\mathcal{I}}(p_1,q_1) \subseteq \ell^{\mathcal{I}}(p_2,q_1) \subseteq \ell^{\mathcal{I}}(p_2,q_2).$$

□

COROLLARY 2.28. *Let $1 \le p_1, q_1, p_2, q_2 \le \infty$, E_1, E_2 be Banach spaces and let $\mathcal{I}$ be a collection of disjoint intervals in $\mathbb{N}_0$ with $sup_k \# I_k = \infty$.*
Then $\ell^{\mathcal{I}}(p_1,q_1,E_1) \subseteq \ell^{\mathcal{I}}(p_2,q_2,E_2)$ if and only if $p_1 \le p_2$, $q_1 \le q_2$ and $E_1 \subseteq E_2$.

PROOF. $\Longrightarrow$) Let us start proving that $E_1 \subseteq E_2$ with continuity. Consider $x \in E_1$ and take the sequence xe_n for some $n \in \Lambda_{\mathcal{I}}$. It is straightforward to see that $xe_n \in \ell^{\mathcal{I}}(p_1,q_1,E_1) \subseteq \ell^{\mathcal{I}}(p_2,q_2,E_2)$ and $\|x\|_{E_2} = \|xe_n\|^{\mathcal{I}}_{p_2,q_2} \le \|xe_n\|^{\mathcal{I}}_{p_1,q_1} = \|x\|_{E_1}$. Now for the condition on p_i, q_i $(i = 1,2)$ take $(\alpha_i x)_i \in \ell^{\mathcal{I}}(p_1,q_1,E_1)$ for a fixed $x \in E_1$. Then $\|(\alpha_i x)_i\|^{\mathcal{I}}_{p_2,q_2,E_2} = \|(\alpha_i)_i\|^{\mathcal{I}}_{p_2,q_2}\|x\|_{E_2} \le \|(\alpha_i x)_i\|^{\mathcal{I}}_{p_1,q_1,E_1} = \|(\alpha_i)_i\|^{\mathcal{I}}_{p_1,q_1}\|x\|_{E_1}$ and the result follows from the scalar case.
$\Longleftarrow$) Let us denote

$$\ell^q(\ell^p(E)) = \left\{ (x_k)_{k\in\mathbb{N}_0} \in \mathcal{S}(E) : x_k \in \ell^p(E), \left(\sum_{k=0}^{\infty} \|x_k\|^q_{\ell^p(E)} \right)^{1/q} < \infty \right\}.$$

Hence the mapping

$$(a_n)_{n\in\Lambda_{\mathcal{I}}} \to ((a_j)_{j\in I_k})_{k\in\mathbb{N}_0}$$

is an isometric embedding from $\ell^{\mathcal{I}}(p,q,E)$ into $\ell^q(\ell^p(E))$. Now use the fact that

$$E_1 \subset E_2 \Rightarrow \ell^p(E_1) \subset \ell^p(E_2)$$

and use the same ideas above to get the desired result. □

REMARK 2.29. For the weak version, consider the scalar case for every $x^* \in B_{E_1^*}$ whenever $p_1 \le p_2$, $q_1 \le q_2$ and $E_1 \subseteq E_2$, which implies $E_2^* \subseteq E_1^*$. Then take the supremum and

$$\ell^{\mathcal{I}}_w(p_1,q_1,E_1) \subseteq \ell^{\mathcal{I}}_w(p_2,q_2,E_2).$$

For the other direction,we use the same argument above based in the scalar case, taking into account that $\|x\|_E = \sup_{x^*\in B_E} |\langle x^*, x\rangle|$ for E a Banach space.

We would like to analyze the embedding between $\ell^{\mathcal{I}}(p,q,E_1)$ and $\ell^{\mathcal{J}}(p,q,E_2)$ for $\mathcal{I} \neq \mathcal{J}$ whenever $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$.

PROPOSITION 2.30. *Let $E_1 \subseteq E_2$ be Banach spaces. Let $\mathcal{I} = \{I_l : l \in \mathbb{N}_0\}$ and $\mathcal{J} = \{J_k : k \in \mathbb{N}_0\}$. If $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$, $p \le q$ and $\sup_k \# J_k < \infty$ then*

$$\ell^{\mathcal{I}}(p,q,E_1) \subseteq \ell^{\mathcal{J}}(p,q,E_2)$$

and

$$\ell^{\mathcal{I}}_w(p,q,E_1) \subseteq \ell^{\mathcal{J}}_w(p,q,E_2).$$

In an analogue way, if $q \le p$ and $\sup_l \# I_l < \infty$,

$$\ell^{\mathcal{J}}(p,q,E_1) \subseteq \ell^{\mathcal{I}}(p,q,E_2))$$

and

$$\ell^{\mathcal{J}}_w(p,q,E_1) \subseteq \ell^{\mathcal{I}}_w(p,q,E_2).$$

PROOF. We will only prove the first two inclusions, as the other are obtained reasoning in an analogue way. Let $p \le q$ and $\sup_k \# J_k < \infty$. Then Proposition 2.25 gives $\ell^{\mathcal{J}}(p, q, E_2) = \ell^{\mathcal{J}}(q, q, E_2)$ and clearly $\ell^{\mathcal{J}}(q, q, E_2) \supseteq \ell^{\mathcal{I}}(q, q, E_1)$. Then the result follows using Corollary 2.28, which states $\ell^{\mathcal{I}}(p, q, E_1) \subseteq \ell^{\mathcal{I}}(q, q, E_1)$. For the weak case consider the scalar case for $x^* \in E_2^* \subseteq E_1^*$, $\|x^*\|_{E_2} = 1$ fixed. Then take the supremum. □

Let us mention a particular case where they coincide.

PROPOSITION 2.31. *Let $\mathcal{I}$ be such that $I_k = [n_k, n'_k) \cap \mathbb{N}_0$ with $n'_{2k} = n_{2k+1}$ and define*

$$\mathcal{J} = \{J_k = I_{2k} \cup I_{2k+1} : k \in \mathbb{N}_0\}.$$

Then $\ell^{\mathcal{I}}(p, q, E) = \ell^{\mathcal{J}}(p, q, E)$ and $\ell^{\mathcal{I}}_w(p, q, E) = \ell^{\mathcal{J}}_w(p, q, E)$.

PROOF. Note that $J_k = I_{2k} \cup I_{2k+1}$ is again an interval in $\mathbb{N}_0$. Using that

$$(a + b)^\alpha \le C_\alpha(a^\alpha + b^\alpha) \tag{2.5}$$

for $a, b, \alpha > 0$ then

$$\begin{aligned}
\|a\|^{\mathcal{J}}_{p,q} &= \left(\sum_{k=0}^{\infty} \left(\sum_{j \in J_k} \|a_j\|_E^p \right)^{q/p} \right)^{1/q} \\
&= \left(\sum_{k=0}^{\infty} \left(\sum_{j \in I_{2k}} \|a_j\|_E^p + \sum_{j \in I_{2k+1}} \|a_j\|_E^p \right)^{q/p} \right)^{1/q} \\
&\le C \left(\sum_{k=0}^{\infty} \left(\sum_{j \in I_{2k}} \|a_j\|_E^p \right)^{q/p} + \sum_{k=0}^{\infty} \left(\sum_{j \in I_{2k+1}} \|a_j\|_E^p \right)^{q/p} \right)^{1/q} \\
&\le C \|a\|^{\mathcal{I}}_{p,q}.
\end{aligned}$$

On the other hand, using now $(a^\beta + b^\beta) \le C_\beta(a + b)^\beta$ for $a, b, \beta > 0$,

$$\begin{aligned}
\|a\|^{\mathcal{I}}_{p,q} &= \left(\sum_{k=0}^{\infty} \left(\sum_{j \in I_{2k}} \|a_j\|_E^p \right)^{q/p} + \left(\sum_{j \in I_{2k+1}} \|a_j\|_E^p \right)^{q/p} \right)^{1/q} \\
&\le C' \left(\sum_{k=0}^{\infty} \left(\sum_{j \in I_{2k} \cup I_{2k+1}} \|a_j\|_E^p \right)^{q/p} \right)^{1/q} \\
&= C' \|a\|^{\mathcal{J}}_{p,q}.
\end{aligned}$$

For weak spaces just take the supremum over all the elements of B_{E^*} and use (2.5) when necessary. □

The previous idea is easily generalized using the following definition.

DEFINITION 2.32. Let $\mathcal{I} := \{I_l : l \in \mathbb{N}_0\}$ and $\mathcal{J} := \{J_k : k \in \mathbb{N}_0\}$. We say that $\mathcal{I} \le \mathcal{J}$ if the following conditions hold:

(i) $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$,
(ii) $F_k = F_k(\mathcal{I}, \mathcal{J}) := \{l \in \mathbb{N}_0 : I_l \subseteq J_k\} \ne \emptyset$ for all $k \in \mathbb{N}_0$,
(iii) $J_k = \cup_{l \in F_k} I_l$ for all $k \in \mathbb{N}_0$.

Proposition 2.33. *Let $1 \le p, q \le \infty$, $E_1 \subseteq E_2$ and $\mathcal{I} \le \mathcal{J}$. Then*

(i) $\ell^{\mathcal{J}}(p,q,E_1) \subseteq \ell^{\mathcal{I}}(p,q,E_2)$ *and* $\ell^{\mathcal{J}}_w(p,q,E_1) \subseteq \ell^{\mathcal{I}}_w(p,q,E_2)$ *for* $p \le q$.
(ii) $\ell^{\mathcal{I}}(p,q,E_1) \subseteq \ell^{\mathcal{J}}(p,q,E_2)$ *and* $\ell^{\mathcal{I}}_w(p,q,E_1) \subseteq \ell^{\mathcal{J}}_w(p,q,E_2)$ *for* $q \le p$.

Proof. (i) Case $q = \infty$: Let $a \in \ell^{\mathcal{J}}(p,\infty,E_1)$ and $l \in \mathbb{N}_0$. We know that there is k such that $I_l \subseteq J_k$. Hence

$$\left(\sum_{n\in I_l} \|a_n\|^p_{E_2}\right)^{1/p} \le C\left(\sum_{n\in J_k} \|a_n\|^p_{E_1}\right)^{1/p} \le \|a\|^{\mathcal{J}}_{p,\infty,E_1}.$$

This gives $\|a\|^{\mathcal{I}}_{p,\infty,E_2} \le \|a\|^{\mathcal{J}}_{p,\infty,E_1}$.

Now for the weak case use that, in the scalar case, $\|(\langle x^*, a_n\rangle)_n\|^{\mathcal{I}}_{p,\infty} \le \|(\langle x^*, a_n\rangle)_n\|^{\mathcal{J}}_{p,\infty}$ and then take supremums.

The case $p = 1$: Let $a \in \ell^{\mathcal{J}}(1,q,E_1)$ and $q \ge 1$. Therefore

$$\left(\|a\|^{\mathcal{I}}_{1,q,E_2}\right)^q = \sum_k \sum_{l\in F_k}\left(\sum_{n\in I_l}\|a_n\|_{E_2}\right)^q \le C\sum_k\left(\sum_{l\in F_k}\sum_{n\in I_l}\|a_n\|_{E_1}\right)^q = C\left(\|a\|^{\mathcal{J}}_{1,q,E_1}\right)^q$$

The case $1 < p \le q < \infty$ follows using (2.3) and the previous one.

Again for the weak case use the scalar case and take supremums.

(ii) The case $p = \infty$: Let $a \in \ell^{\mathcal{I}}(\infty,q,E_1)$. Then

$$\|a\|^{\mathcal{J}}_{\infty,q,E_2} = \Big(\sum_k \sup_{l\in F_k}(\sup_{n\in I_l}\|a_n\|_{E_2})^q\Big)^{1/q} \le C\Big(\sum_k\sum_{l\in F_k}(\sup_{n\in I_l}\|a_n\|_{E_1})^q\Big)^{1/q} = \|a\|^{\mathcal{I}}_{\infty,q,E_1}.$$

To cover the remaning cases, from (2.3), we simply need to show that $\ell^{\mathcal{I}}(p,1,E_1) \subseteq \ell^{\mathcal{J}}(p,1,E_2)$ for $p \ge 1$. Now observe that

$$\begin{aligned}\|a\|^{\mathcal{J}}_{p,1,E_2} &= \sum_k\left(\sum_{l\in F_k}\sum_{n\in I_l}\|a_n\|^p_{E_2}\right)^{1/p} = \sum_k\left(\sum_{l\in F_k}\|a\chi_{I_l}\|^p_{\ell^p(E_2)}\right)^{1/p}\\ &\le C\sum_k\sum_{l\in F_k}\|a\chi_{I_l}\|_{\ell^p(E_1)} = C\sum_l\left(\sum_{n\in I_l}\|a_n\|^p_{E_1}\right)^{1/p} = C\|a\|^{\mathcal{I}}_{p,1,E_1}.\end{aligned}$$

The proof for the weak case is easily adapted given that $E_2^* \subset E_1^*$ using the scalar case ($E_1 = E_2 = \mathbb{K}$) and then taking supremums. □

Theorem 2.34. *Let $\mathcal{I} \le \mathcal{J}$ and $1 \le p, q \le \infty$ with $p \ne q$. Then $\ell^{\mathcal{I}}(p,q,E) = \ell^{\mathcal{J}}(p,q,E)$ and $\ell^{\mathcal{I}}_w(p,q,E) = \ell^{\mathcal{J}}_w(p,q,E)$ (with equivalent norms) if and only if $\sup_k \#F_k < \infty$.*

Proof. Case: $\ell^{\mathcal{I}}(p,q,E) = \ell^{\mathcal{J}}(p,q,E)$. $\Longrightarrow$) Assume that $\|a\|^{\mathcal{J}}_{p,q} \approx \|a\|^{\mathcal{I}}_{p,q}$ for all a finitely supported. Take $e \in E$ such that $\|e\|_E = 1$. Now define

$$a^{(k)} = \sum_{l\in F_k}(\#I_l)^{-1/p} e\chi_{I_l}$$

for $k \in \mathbb{N}_0$. Then $\|a^{(k)}\|^{\mathcal{J}}_{p,q} = (\#F_k)^{1/p}$ and $\|a^{(k)}\|^{\mathcal{I}}_{p,q} = (\#F_k)^{1/q}$. One concludes that $C_2 \le (\#F_k)^{1/p-1/q} \le C_1$ which implies, in the case $p \ne q$, $\sup_{k\in\mathbb{N}_0}(\#F_k) < \infty$.

$\Longleftarrow$) Case $p < q$: From Proposition 2.33 we only need to show $\ell^{\mathcal{I}}(p,q,E) \subseteq \ell^{\mathcal{J}}(p,q,E)$. Using now Hölder's inequality for $q/p > 1$

$$\left(\sum_{n\in J_k}\|a_n\|_E^p\right)^{1/p} \le \left(\sum_{l\in F_k}\sum_{n\in I_l}\|a_n\|_E^p\right)^{1/p} \le \left(\sum_{l\in F_k}\left(\sum_{n\in I_l}\|a_n\|_E^p\right)^{q/p}\right)^{1/q} (\#F_k)^{\frac{1}{p\ominus q}}.$$

Therefore, if $M = \sup_k \#F_k$, we have

$$\|a\|_{p,q}^{\mathcal{J}} = \left(\sum_{k=0}^{\infty}\left(\sum_{n\in J_k}\|a_n\|_E^p\right)^{q/p}\right)^{1/q} \le M^{\frac{1}{p\ominus q}}\left(\sum_{k\in\mathbb{N}_0}\sum_{l\in F_k}\left(\sum_{n\in I_l}\|a_n\|_E^p\right)^{q/p}\right)^{1/q}$$

$$= M^{\frac{1}{p\ominus q}}\left(\sum_{l\in\mathbb{N}_0}\left(\sum_{n\in I_l}\|a_n\|_E^p\right)^{q/p}\right)^{1/q} = M^{\frac{1}{p\ominus q}}\|a\|_{p,q}^{\mathcal{I}}.$$

Case $p > q$: Using again Proposition 2.33 we shall show $\ell^{\mathcal{J}}(p,q,E) \subseteq \ell^{\mathcal{I}}(p,q,E)$. Using $1/q = 1/q\ominus p + 1/p$

$$\|a\|_{p,q}^{\mathcal{I}} = \left(\sum_{l}\|a\chi_{I_l}\|_p^q\right)^{1/q} = \left(\sum_{k}\sum_{l\in F_k}\|a\chi_{I_l}\|_p^q\right)^{1/q}$$

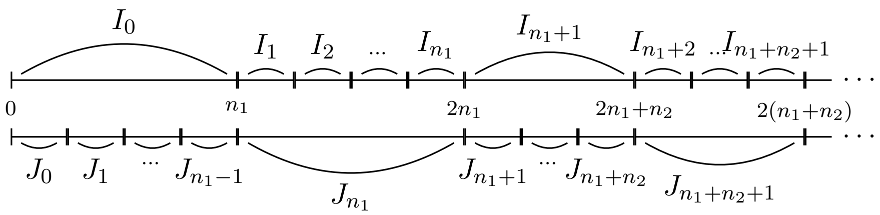
$$\le \left(\sum_{k}\left(\sum_{l\in F_k}\|a\chi_{I_l}\|_p^p\right)^{q/p}(\#F_k)^{q/q\ominus p}\right)^{1/q}$$

$$\le M^{\frac{1}{q\ominus p}}\left(\sum_{k}\left(\sum_{n\in J_k}\|a_n\|_E^p\right)^{q/p}\right)^{1/q} \le M^{\frac{1}{q\ominus p}}\|a\|_{p,q}^{\mathcal{J}}.$$

For the weak case fix $x^* \in E^*$ and argue in an analogue way making use of the scalar case. Then take the supremum in the unit ball of E^*. □

Let us now exhibit an example where neither $\ell^{\mathcal{I}}(p,q,E) \subseteq \ell^{\mathcal{J}}(p,q,E)$ nor $\ell^{\mathcal{J}}(p,q,E) \subseteq \ell^{\mathcal{I}}(p,q,E)$.

Example 2.35. Let $1 \le p < q < \infty$ and take $\mathcal{I},\mathcal{J}$ as shown below:



with:

$card(I_0) = n_1$	$card(J_0) = ... = card(J_{n_1}) = 1$
$card(I_1) = ... = card(I_{n_1}) = 1$	$card(J_{n_1}) = n_1$
$card(I_{n_1+1}) = n_2$	$card(J_{n_1+1}) = ... = card(J_{n_1+n_2}) = 1$
$card(I_{n_1+2}) = ... = card(I_{n_1+n_2+1}) = 1$	$card(J_{n_1+n_2+1}) = n_2, ...$
$card(I_{n_1+n_2+2}) = n_3, ...$	...

Let us see that neither $\ell^{\mathcal{J}}(p,q,E) \subset \ell^{\mathcal{I}}(p,q,E)$ nor $\ell^{\mathcal{I}}(p,q,E) \subset \ell^{\mathcal{J}}(p,q,E)$. Taking

$$a = (\overbrace{\beta_1 e, ..., \beta_1 e}^{n_1}, \overbrace{0, ..., 0}^{n_1}, \overbrace{\beta_2 e, ..., \beta_2 e}^{n_2}, 0, ...)$$

and

$$b = (\underbrace{0, ..., 0}_{n_1}, \underbrace{\beta_1 e, ..., \beta_1 e}_{n_1}, \underbrace{0, ..., 0}_{n_2}, \beta_2 e, ...)$$

where $e \in E$ with $\|e\|_E = 1$ we have:

$$\|a\|^{\mathcal{I}}_{p,q} = \|b\|^{\mathcal{J}}_{p,q} = (\sum_j \beta_j^q n_j^{q/p})^{1/q}$$

$$\|a\|^{\mathcal{J}}_{p,q} = \|b\|^{\mathcal{I}}_{p,q} = (\sum_j \beta_j^q n_j)^{1/q}.$$

Then it is enough to consider $q > p$ and $\beta_j = n_j^{-1/p} j^{-1/q}$. Now

$$(\sum_j \beta_j^q n_j^{q/p})^{1/q} = (\sum_j j^{-1})^{1/q} = \infty$$

and, since $n_j \geq j$,

$$(\sum_j \beta_j^q n_j)^{1/q} = (\sum_j j^{-1} n_j^{1-q/p})^{1/q} \leq (\sum_j j^{-q/p})^{1/q} < \infty.$$

Hence we have $a \in \ell^{\mathcal{J}}(p,q,E) \backslash \ell^{\mathcal{I}}(p,q,E)$ and $b \in \ell^{\mathcal{I}}(p,q,E) \backslash \ell^{\mathcal{J}}(p,q,E)$.

This procedure may be applied in the weak case (all proofs may be repeated fixing $x^* \in E^*$ and then taking the supremum). Thus, we can find examples where neither $\ell^{\mathcal{I}}_w(p,q,E) \subseteq \ell^{\mathcal{J}}_w(p,q,E)$ nor $\ell^{\mathcal{J}}_w(p,q,E) \subseteq \ell^{\mathcal{I}}_w(p,q,E)$.

We would like to explain a procedure to analyze the general case $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$. We will give the results for the weak case at the very end of the section as a remark to avoid becoming repetitive.

Definition 2.36. Let $\mathcal{I}$ and $\mathcal{J}$ be two families of disjoint intervals in $\mathbb{N}_0$ with $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$. For each $k \in \mathbb{N}_0$ we use the notation, as above, $F_k = \{l \in \mathbb{N}_0 : I_l \subseteq J_k\}$ which now might be empty. We also define

$$\tilde{F}_k = \{l \in \mathbb{N}_0 : J_k \cap I_l \neq \emptyset\}.$$

We write ϕ and Φ for the mappings given by

$$\phi(k) = \min \tilde{F}_k \text{ and } \Phi(k) = \max \tilde{F}_k.$$

Similarly, interchanging $\mathcal{I}$ and $\mathcal{J}$, we define G_l, $\tilde{G}_l$, $\psi(l)$ and $\Psi(l)$.

Definition 2.37. We define the "left" and "right" part of the interval J_k by

$$\check{J}_k = J_k \cap I_{\phi(k)} \text{ and } \hat{J}_k = J_k \cap I_{\Phi(k)}$$

and, denoting $J'_k = \cup_{l \in F_k} I_l$ and $\tilde{J}_k = \cup_{l \in \tilde{F}_k} I_l$, we have

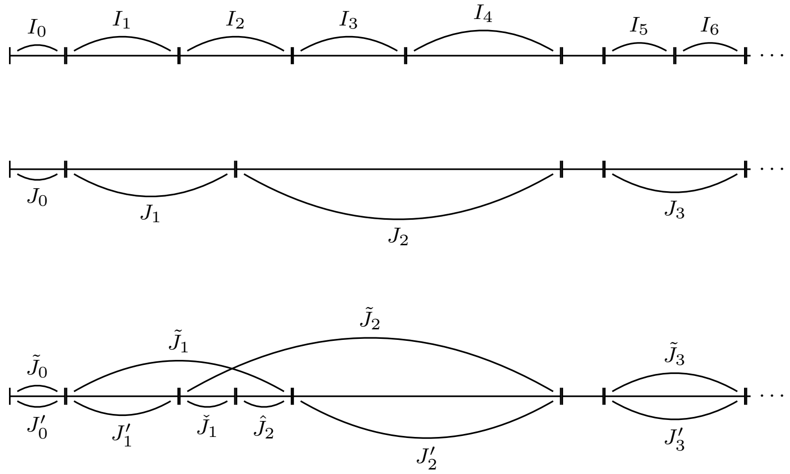
$$J'_k \subseteq J_k \subseteq \tilde{J}_k \tag{2.6}$$

and

$$J_k = J'_k \cup \hat{J}_k \cup \check{J}_k, \tag{2.7}$$

where $J'_k = \emptyset$ whenever $F_k = \emptyset$. Similarly, interchanging $\mathcal{I}$ and $\mathcal{J}$ we consider $\check{I}_l, \hat{I}_l, I'_l$ and $\tilde{I}_l$.

In the following picture one grasps the idea easily.



With this notation out of the way we can classify intervals in $\mathcal{J}$ into four different types (according to $\mathcal{I}$). Note that for each interval $J \in \mathcal{J}$ there are four possibilities: J coincides with I for some $I \in \mathcal{I}$, J can be written as a union of at least two intervals in $\mathcal{I}$, J is strictly contained into some interval $I \in \mathcal{I}$ or there exists $I \in \mathcal{I}$ which overlaps with J and its complement J^c. Therefore we decompose $\mathbb{N}_0$ into four disjoint sets defined as follows:

Definition 2.38. Let $\mathcal{I}$ and $\mathcal{J}$ families of disjoint intervals in $\mathbb{N}_0$ with $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$. We introduce

$$N^{\mathcal{J}}_{equal} = \{k \in \mathbb{N}_0 : \#(\tilde{F}_k \setminus F_k) = 0, \#\tilde{F}_k = 1\}, \tag{2.8}$$

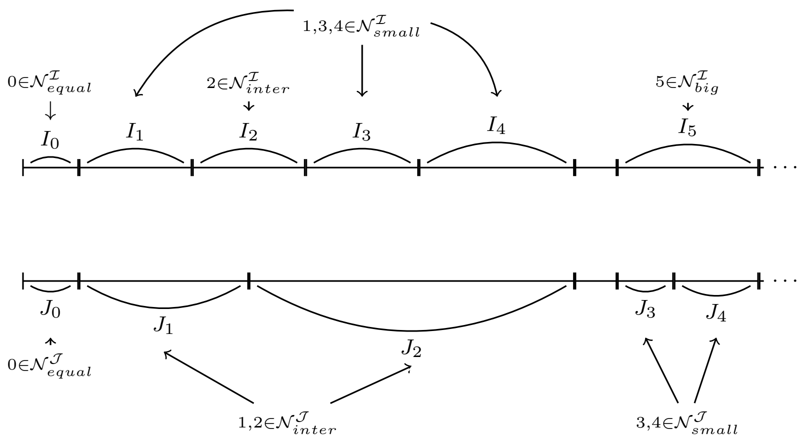
$$N^{\mathcal{J}}_{big} = \{k \in \mathbb{N}_0 : \#(\tilde{F}_k \setminus F_k) = 0, \#\tilde{F}_k \geq 2\}, \tag{2.9}$$

$$N^{\mathcal{J}}_{small} = \{k \in \mathbb{N}_0 : \#(\tilde{F}_k \setminus F_k) > 0, \#\tilde{F}_k = 1\}, \tag{2.10}$$

$$N^{\mathcal{J}}_{inter} = \{k \in \mathbb{N}_0 : \#(\tilde{F}_k \setminus F_k) > 0, \#\tilde{F}_k \geq 2\}. \tag{2.11}$$

We define the sets $N^{\mathcal{I}}_{equal}, N^{\mathcal{I}}_{big}, N^{\mathcal{I}}_{small}$ and $N^{\mathcal{I}}_{inter}$ similarly.

Since the notation may be a bit confusing, we will illustrate the idea. Let $\mathcal{I}, \mathcal{J}$ be different partitions of $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$, then:



Remark 2.39. Using (2.7) we can also give a description of the sets above in terms of ϕ and Φ:

$N^{\mathcal{J}}_{equal} = \{k : \phi(k) = \Phi(k), J_k = I_{\phi(k)}\}$.

$N^{\mathcal{J}}_{big} = \{k : \phi(k) < \Phi(k), J_k = \tilde{J}_k\}$.
$N^{\mathcal{J}}_{small} = \{k : \phi(k) = \Phi(k), J_k \subsetneq I_{\phi(k)}\}$.
$N^{\mathcal{J}}_{inter} = \{k : \phi(k) < \Phi(k), J_k \subsetneq \tilde{J}_k\}$.

Using the above decomposition we can generalize Proposition 2.30, Proposition 2.33 and Theorem 2.34. We will generalize the weak version of the theorems at the end of the section.

Note that $\sup_k \# J_k < \infty$ implies $\sup_k \# \tilde{F}_k < \infty$ and also that $\mathcal{I} \leq \mathcal{J}$ corresponds to the case where $N^{\mathcal{J}}_{inter} \cup N^{\mathcal{J}}_{small} = \emptyset$ or equivalently $\#\tilde{G}_l = 1$ for any $l \in \mathbb{N}_0$.

THEOREM 2.40. *Let $1 \leq p < q \leq \infty$ and $\mathcal{I}, \mathcal{J}$ with $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$. Then*

$$\ell^{\mathcal{I}}(p,q,E) \subseteq \ell^{\mathcal{J}}(p,q,E) \Longleftrightarrow \sup\{\#\tilde{F}_k; k \in \mathbb{N}_0\} < \infty.$$

PROOF. $\Longrightarrow$) Arguing as in Theorem 2.34, for $k \in \mathbb{N}_0$ we consider

$$a^{(k)} = \sum_{l \in \tilde{F}_k} (\#(I_l \cap J_k))^{-1/p} e \chi_{I_l \cap J_k},$$

where $\|e\|_E = 1$. Hence

$$\|a^{(k)}\|^{\mathcal{J}}_{p,q} = \left(\sum_{n \in J_k} \|a_n\|^p_E \right)^{1/p} = \left(\sum_{l \in \tilde{F}_k} \sum_{n \in I_l \cap J_k} \|a_n\|^p_E \right)^{1/p} = (\#\tilde{F}_k)^{1/p}$$

and

$$\|a^{(k)}\|^{\mathcal{I}}_{p,q} = \left(\sum_{l \in \tilde{F}_k} \left(\sum_{n \in I_l \cap J_k} \|a_n\|^p_E \right)^{q/p} \right)^{1/q} = (\#\tilde{F}_k)^{1/q}.$$

Therefore using that $\|a^{(k)}\|^{\mathcal{J}}_{p,q} \leq C \|a^{(k)}\|^{\mathcal{I}}_{p,q}$ and $p < q$ we conclude that $\sup\{\#\tilde{F}_k; k \in \mathbb{N}_0\} < \infty$.

$\Longleftarrow$) Denote $\sup_k(\#F_k) = M \geq 0$ and let $k \in \mathbb{N}_0$. Case $q = \infty$: If $k \in N^{\mathcal{J}}_{small} \cup N^{\mathcal{J}}_{equal}$ then

$$\left(\sum_{n \in J_k} \|a_n\|^p_E \right)^{1/p} \leq \left(\sum_{n \in I_{\phi(k)}} \|a_n\|^p_E \right)^{1/p} \leq \|a\|^{\mathcal{I}}_{p,\infty}.$$

If $k \in N^{\mathcal{J}}_{big} \cup N^{\mathcal{J}}_{inter}$ we have

$$\begin{aligned} \left(\sum_{n \in J_k} \|a_n\|^p_E \right)^{1/p} &= \left(\sum_{l \in F_k} \sum_{n \in I_l} \|a_n\|^p_E + \sum_{n \in \check{J}_k \cup \hat{J}_k} \|a_n\|^p_E \right)^{1/p} \\ &\leq \left(\sum_{l \in F_k} \sum_{n \in I_l} \|a_n\|^p_E \right)^{1/p} + \left(\sum_{n \in I_{\phi(k)}} \|a_n\|^p_E \right)^{1/p} + \left(\sum_{n \in I_{\Phi(k)}} \|a_n\|^p_E \right)^{1/p} \\ &\leq C \left(\sup_{l \in F_k} \left(\sum_{n \in I_l} \|a_n\|^p_E \right)^{1/p} (\#F_k)^{1/p} + 2\|a\|^{\mathcal{I}}_{p,\infty} \right). \end{aligned}$$

This shows $\ell^{\mathcal{I}}(p,\infty,E) \subseteq \ell^{\mathcal{J}}(p,\infty,E)$. Case $q<\infty$: Arguing as in Proposition 2.33 we simply show that $\ell^{\mathcal{I}}(1,q,E) \subseteq \ell^{\mathcal{J}}(1,q,E)$ for $q>1$. Observe that

$$\begin{aligned}
\sum_{k\in N^{\mathcal{J}}_{small}} \left(\sum_{n\in J_k} \|a_n\|_E\right)^q &\leq \sum_{l\in N^{\mathcal{I}}_{big}\cup N^{\mathcal{I}}_{inter}} \sum_{\phi(k)=l} \left(\sum_{n\in J_k} \|a_n\|_E\right)^q \\
&\leq \sum_{l\in N^{\mathcal{I}}_{big}\cup N^{\mathcal{I}}_{inter}} \left(\sum_{\phi(k)=l} \sum_{n\in J_k} \|a_n\|_E\right)^q \\
&= \sum_{l\in N^{\mathcal{I}}_{big}\cup N^{\mathcal{I}}_{inter}} \left(\sum_{n\in I_l} \|a_n\|_E\right)^q \\
&\leq \left(\|a\|^{\mathcal{I}}_{1,q}\right)^q.
\end{aligned}$$

Also we have

$$\begin{aligned}
\sum_{k\in N^{\mathcal{J}}_{equal}\cup N^{\mathcal{J}}_{big}} \left(\sum_{n\in J_k} \|a_n\|_E\right)^q &\leq \sum_{k\in N^{\mathcal{J}}_{equal}\cup N^{\mathcal{J}}_{big}} \left(\sum_{l\in F_k}\sum_{n\in I_l} \|a_n\|_E\right)^q \\
&\leq \sum_{k\in N^{\mathcal{J}}_{equal}\cup N^{\mathcal{J}}_{big}} (\#F_k)^{q-1} \sum_{l\in F_k} \left(\sum_{n\in I_l} \|a_n\|_E\right)^q \\
&\leq M^{q-1} \sum_{k\in N^{\mathcal{J}}_{equal}\cup N^{\mathcal{J}}_{big}} \sum_{l\in F_k} \left(\sum_{n\in I_l} \|a_n\|_E\right)^q \\
&\leq M^{q-1}\left(\|a\|^{\mathcal{I}}_{1,q}\right)^q.
\end{aligned}$$

Finally

$$\begin{aligned}
\sum_{k\in N^{\mathcal{J}}_{inter}} \left(\sum_{n\in J_k} \|a_n\|_E\right)^q &\leq \sum_{k\in N^{\mathcal{J}}_{inter}} \left(\sum_{l\in F_k}\sum_{n\in I_l} \|a_n\|_E + \sum_{n\in \check{J}_k} \|a_n\|_E + \sum_{n\in \hat{J}_k} \|a_n\|_E\right)^q \\
&\leq C \sum_{k\in N^{\mathcal{J}}_{inter}} (\#F_k)^{q-1} \sum_{l\in F_k} \left(\sum_{n\in I_l} \|a_n\|_E\right)^q \\
&+ C \sum_{k\in N^{\mathcal{J}}_{inter}} \left(\sum_{n\in \check{J}_k} \|a_n\|_E\right)^q + C \sum_{k\in N^{\mathcal{J}}_{inter}} \left(\sum_{n\in \hat{J}_k} \|a_n\|_E\right)^q \\
&\leq CM^{q-1} \sum_{l\in N^{\mathcal{I}}_{inter}\cup N^{\mathcal{I}}_{small}} \left(\sum_{n\in I_l} \|a_n\|_E\right)^q \\
&+ C \sum_{k\in N^{\mathcal{J}}_{inter}} \left(\sum_{n\in I_{\phi(k)}} \|a_n\|_E\right)^q + \sum_{k\in N^{\mathcal{J}}_{inter}} \left(\sum_{n\in I_{\Phi(k)}} \|a_n\|_E\right)^q \\
&\leq C\left(\|a\|^{\mathcal{I}}_{1,q}\right)^q.
\end{aligned}$$

Combining the above estimates we conclude this implication. □

Corollary 2.41. *Let $1 \leq p < q \leq \infty$ and $\mathcal{I}, \mathcal{J}$ with $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$. Then*

$$\ell^{\mathcal{J}}(p,q,E) \subseteq \ell^{\mathcal{I}}(p,q,E) \Longleftrightarrow \sup\{\#\tilde{G}_l; l \in \mathbb{N}_0\} < \infty.$$

Theorem 2.42. *Let $1 \leq q < p \leq \infty$ and $\mathcal{I}, \mathcal{J}$ with $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$. Then*

$$\ell^{\mathcal{J}}(p,q,E) \subseteq \ell^{\mathcal{I}}(p,q,E) \Longleftrightarrow \sup\{\#\tilde{F}_k; k \in \mathbb{N}_0\} < \infty.$$

Proof. $\Longrightarrow$) The argument is similar to the one presented in the direct implication of Theorem 2.40.

$\Longleftarrow$) Denote again $\sup_k(\#F_k) = M$. Case $p = \infty$: Observe first that if $l \in N^{\mathcal{I}}_{big} \cup N^{\mathcal{I}}_{equal}$ we have

$$(\sup_{n \in I_l} \|a_n\|_E)^q = |a_{n(l)}|^q \leq (\sup_{n \in J_k} \|a_n\|_E)^q$$

for some $k = k(l) \in N^{\mathcal{J}}_{small} \cup N^{\mathcal{J}}_{equal}$. Since $k(l) \neq k(l')$ for $l \neq l' \in N^{\mathcal{I}}_{big} \cup N^{\mathcal{I}}_{equal}$ we obtain

$$\sum_{l \in N^{\mathcal{I}}_{big} \cup N^{\mathcal{I}}_{equal}} (\sup_{n \in I_l} \|a_n\|_E)^q \leq \sum_{k \in N^{\mathcal{J}}_{small} \cup N^{\mathcal{J}}_{equal}} (\sup_{n \in J_k} \|a_n\|_E)^q.$$

Also if $l \in N^{\mathcal{I}}_{inter}$ then $(\sup_{n \in I_l} \|a_n\|_E)^q = |a_{n(l)}|^q$ where $n(l) \in I'_l \cup \hat{I}_l \cup \check{I}_l$. Note that $n(l) \in J_k$ for some $k \in N^{\mathcal{J}}_{small} \cup N^{\mathcal{J}}_{inter}$ and

$$1 \leq \#(\{l \in N^{\mathcal{I}}_{inter} : n(l) \in J_k\}) \leq 2.$$

Hence

$$\sum_{l \in N^{\mathcal{I}}_{inter}} (\sup_{n \in I_l} \|a_n\|_E)^q \leq 2 \sum_{k \in N^{\mathcal{J}}_{small} \cup N^{\mathcal{J}}_{inter}} (\sup_{n \in J_k} \|a_n\|_E)^q.$$

On the other hand

$$\begin{aligned}
\sum_{l \in N^{\mathcal{I}}_{small}} (\sup_{n \in I_l} \|a_n\|_E)^q &\leq \sum_{k \in N^{\mathcal{J}}_{big} \cup N^{\mathcal{J}}_{inter}} \sum_{\psi(l)=k} (\sup_{n \in I_l} \|a_n\|_E)^q \\
&\leq \sum_{k \in N^{\mathcal{J}}_{big} \cup N^{\mathcal{J}}_{inter}} (\sup_{n \in J_k} \|a_n\|_E)^q (\#F_k)^q \\
&\leq M^q (\|a\|^{\mathcal{J}}_{p,\infty})^q.
\end{aligned}$$

Combining the previous cases we get $\ell^{\mathcal{J}}(\infty,q,E) \subseteq \ell^{\mathcal{I}}(\infty,q,E)$.

Case $p < \infty$: Arguing as in Proposition 2.33 we simply show that $\ell^{\mathcal{J}}(p,1,E) \subseteq \ell^{\mathcal{I}}(p,1,E)$ for $p > 1$.

$$\begin{aligned}
\|a\|^{\mathcal{I}}_{p,1} &= \sum_l \Big(\sum_{n \in I_l} \|a_n\|^p_E \Big)^{1/p} \\
&\leq \sum_{l \in N^{\mathcal{I}}_{small}} \Big(\sum_{n \in I_l} \|a_n\|^p_E \Big)^{1/p} \\
&+ \sum_{l \in N^{\mathcal{I}}_{equal} \cup N^{\mathcal{I}}_{big}} \Big(\sum_{k \in G_l} \sum_{n \in J_k} \|a_n\|^p_E \Big)^{1/p} \\
&+ \sum_{l \in N^{\mathcal{I}}_{inter}} \Big(\sum_{k \in G_l} \sum_{n \in J_k} \|a_n\|^p_E + \sum_{n \in \check{I}_l} \|a_n\|^p_E + \sum_{n \in \hat{I}_l} \|a_n\|^p_E \Big)^{1/p} \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

Now observe that

$$I_1 \leq \sum_{k\in N^{\mathcal{J}}_{big}\cup N^{\mathcal{J}}_{inter}} \sum_{l\in F_k} \Big(\sum_{n\in I_l} \|a_n\|^p_E\Big)^{\frac{1}{p}} \leq \sum_{k\in N^{\mathcal{J}}_{big}\cup N^{\mathcal{J}}_{inter}} \Big(\sum_{n\in J_k} \|a_n\|^p_E\Big)^{\frac{1}{p}} \#(F_k) \leq M\|a\|^{\mathcal{J}}_{p,1}.$$

Also note, since $p > 1$,

$$I_2 \leq \sum_{l\in N^{\mathcal{I}}_{equal}\cup N^{\mathcal{I}}_{big}} \sum_{k\in G_l} \Big(\sum_{n\in J_k} \|a_n\|^p_E\Big)^{1/p} \leq \|a\|^{\mathcal{J}}_{p,1}.$$

Finally

$$\begin{aligned} I_3 &\leq \sum_{l\in N^{\mathcal{I}}_{inter}} \Big(\sum_{k\in G_l}\sum_{n\in J_k} \|a_n\|^p_E\Big)^{1/p} + \Big(\sum_{n\in \check{I}_l} \|a_n\|^p_E\Big)^{1/p} + \Big(\sum_{n\in \hat{I}_l} \|a_n\|^p_E\Big)^{1/p} \\ &\leq \sum_{k\in N^{\mathcal{J}}_{inter}\cup N^{\mathcal{J}}_{small}} \Big(\sum_{n\in J_k} \|a_n\|^p_E\Big)^{1/p} + \sum_{l\in N^{\mathcal{I}}_{inter}} \Big(\sum_{n\in J_{\psi(l)}} \|a_n\|^p_E\Big)^{1/p} \\ &+ \sum_{l\in N^{\mathcal{I}}_{inter}} \Big(\sum_{n\in J_{\Psi(l)}} \|a_n\|^p_E\Big)^{1/p} \\ &\leq C\sum_k \Big(\sum_{n\in J_k} \|a_n\|^p_E\Big)^{1/p} = C\|a\|^{\mathcal{J}}_{p,1}. \end{aligned}$$

The converse implication is now complete. □

COROLLARY 2.43. *Let $1 \leq q < p \leq \infty$ and $\mathcal{I}, \mathcal{J}$ with $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$. Then*

$$\ell^{\mathcal{I}}(p, q, E) \subseteq \ell^{\mathcal{J}}(p, q, E) \iff \sup\{\#\tilde{G}_l; l \in \mathbb{N}_0\} < \infty.$$

COROLLARY 2.44. *Let $1 \leq p, q \leq \infty$ with $p \neq q$ and $\mathcal{I}, \mathcal{J}$ with $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$. Then*

$$\ell^{\mathcal{J}}(p, q, E) = \ell^{\mathcal{I}}(p, q, E) \iff \sup\{(\#\tilde{F}_k)(\#\tilde{G}_l); k, l \in \mathbb{N}_0\} < \infty.$$

PROOF. It suffices to show the case $p < q$. Note that $\ell^{\mathcal{I}}(p, q, E) \subseteq \ell^{\mathcal{J}}(p, q, E)$ and $\ell^{\mathcal{J}}(p, q, E) \subseteq \ell^{\mathcal{I}}(p, q, E)$ are equivalent, due to Theorem 2.40 and Corollary 2.41, to the facts $sup_k(\#\tilde{F}_k) < \infty$ and $sup_l(\#\tilde{G}_l) < \infty$, or equivalently

$$\sup\{(\#\tilde{F}_k)(\#\tilde{G}_l); k, l \in \mathbb{N}_0\} = \sup_k(\#\tilde{F}_k)\sup_l(\#\tilde{G}_l) < \infty.$$

□

REMARK 2.45. For $1 \leq p < q \leq \infty$ and $\mathcal{I}, \mathcal{J}$ with $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$ we have

$$\ell^{\mathcal{I}}_w(p, q, E) \subseteq \ell^{\mathcal{J}}_w(p, q, E) \iff \sup\{\#\tilde{F}_k; k \in \mathbb{N}_0\} < \infty$$

and

$$\ell^{\mathcal{J}}_w(p, q, E) \subseteq \ell^{\mathcal{I}}_w(p, q, E) \iff \sup\{\#\tilde{G}_l; l \in \mathbb{N}_0\} < \infty.$$

On the other side, for $1 \leq q < p \leq \infty$ and $\mathcal{I}, \mathcal{J}$ with $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$ we get

$$\ell^{\mathcal{J}}_w(p, q, E) \subseteq \ell^{\mathcal{I}}_w(p, q, E) \iff \sup\{\#\tilde{F}_k; k \in \mathbb{N}_0\} < \infty$$

and

$$\ell^{\mathcal{I}}_w(p, q, E) \subseteq \ell^{\mathcal{J}}_w(p, q, E) \iff \sup\{\#\tilde{G}_l; l \in \mathbb{N}_0\} < \infty.$$

So if $1 \leq p, q \leq \infty$ with $p \neq q$ and $\mathcal{I}, \mathcal{J}$ with $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$, then

$$\ell^{\mathcal{J}}_w(p, q, E) = \ell^{\mathcal{I}}_w(p, q, E) \iff \sup\{(\#\tilde{F}_k)(\#\tilde{G}_l); k, l \in \mathbb{N}_0\} < \infty.$$

4. Multipliers between generalized mixed-norm spaces

In this section we consider $1 \leq r, s, u, v \leq \infty$ and $\mathcal{I}, \mathcal{J}$ such that $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$. We define

$$(\ell^{\mathcal{I}}(r,s,E_1), \ell^{\mathcal{J}}(u,v,E_2)) =$$
$$\{\lambda = (\lambda_n)_{n\in\Lambda_{\mathcal{I}}\cap\Lambda_{\mathcal{J}}} \in \mathcal{S}(\mathcal{L}(E_1,E_2)) : \|(\lambda_n a_n)_{n\in\Lambda_{\mathcal{J}}}\|^{\mathcal{J}}_{u,v,E_2} \leq C\|(a_n)_{n\in\Lambda_{\mathcal{I}}}\|^{\mathcal{I}}_{r,s,E_1}\}.$$

The case $\mathcal{I} = \mathcal{J}$ can be easily determined adapting the proof of (1.3) (see [32, Theorem 1]) considering $\mathcal{I} = \{I_k : k \in \mathbb{N}_0\}$ where $I_k = [2^k - 1, 2^{k+1} - 1) \cap \mathbb{N}_0$ and the norms $\|\cdot\|_{E_1}$, $\|\cdot\|_{E_2}$ instead of the modulus.

Theorem 2.46. $(\ell^{\mathcal{I}}(r,s,E_1), \ell^{\mathcal{I}}(u,v,E_2)) = \ell^{\mathcal{I}}(r \ominus u, s \ominus v, \mathcal{L}(E_1,E_2))$.

Corollary 2.47. $\ell^{\mathcal{I}}(r,s,E)^K = \ell^{\mathcal{I}}(r', s', E^*)$.

There are some other cases where the set of multipliers can be easily determined.

Proposition 2.48.
(i) If $\sup_{k\in\mathbb{N}_0} \#J_k < \infty$ *then* $(\ell^{\mathcal{I}}(r,s,E_1), \ell^{\mathcal{J}}(u,v,E_2)) = \ell^{\mathcal{I}}(r\ominus v, s\ominus v, \mathcal{L}(E_1,E_2))$.
(ii) If $\sup_{l\in\mathbb{N}_0} \#I_l < \infty$ *then* $(\ell^{\mathcal{I}}(r,s,E_1), \ell^{\mathcal{J}}(u,v,E_2)) = \ell^{\mathcal{J}}(s\ominus u, s\ominus v, \mathcal{L}(E_1,E_2))$.
(iii) If $\sup\{(\#\tilde{F}_k)(\#\tilde{G}_l); k,l \in \mathbb{N}_0\} < \infty$ *then*

$$(\ell^{\mathcal{I}}(r,s,E_1), \ell^{\mathcal{J}}(u,v,E_2)) = \ell^{\mathcal{J}}(r\ominus u, s\ominus v, \mathcal{L}(E_1,E_2)) = \ell^{\mathcal{I}}(r\ominus u, s\ominus v, \mathcal{L}(E_1,E_2)).$$

Proof. To prove (i) take into account that using Proposition 2.25 and Corollary 2.44 one easily obtains that if $\sup_{k\in\mathbb{N}_0} \#J_k < \infty$ then $\ell^{\mathcal{J}}(u,v,E_2) = \ell^{\mathcal{J}}(v,v,E_2) = \ell^{\mathcal{I}}(v,v,E_2)$. Then use Theorem 2.46.
The other proofs are similar. □

Also as a direct consequence of Theorem 2.40 we obtain:

Proposition 2.49. *If* $r \leq u$, $s \leq v$ *and* $u < v$ *and* $\sup\{\#\tilde{F}_k; k \in \mathbb{N}_0\} < \infty$ *then*

$$(\ell^{\mathcal{I}}(r,s,E_1), \ell^{\mathcal{J}}(u,v,E_2)) = \{(\lambda_n)_{n\in\Lambda_{\mathcal{I}}} \in \mathcal{S}(\mathcal{L}(E_1,E_2)) : \sup_n \|\lambda_n\| < \infty\}.$$

Proof. It is obvious that, if $(\lambda_n)_{n\in\Lambda_{\mathcal{I}}}$ is a multiplier, it necessarily is a bounded sequence. For the inclusion

$$\{(\lambda_n)_{n\in\Lambda_{\mathcal{I}}} \in \mathcal{S}(\mathcal{L}(E_1,E_2)) : \sup_n \|\lambda_n\| < \infty\} \subseteq (\ell^{\mathcal{I}}(r,s,E_1), \ell^{\mathcal{J}}(u,v,E_2))$$

let $\lambda \in \mathcal{S}(\mathcal{L}(E_1,E_2))$ be such that $\sup_n \|\lambda_n\| < \infty$ and consider $a \in \ell^{\mathcal{I}}(r,s,E_1)$. Then $\|\lambda_n(a_n)\|_{E_2} \leq \|\lambda_n\|\|a_n\|_{E_1}$, thus $(\lambda_n(a_n))_{n\in\Lambda_{\mathcal{I}}} \in \ell^{\mathcal{I}}(r,s,E_2)$. Now use the embedding

$$\ell^{\mathcal{I}}(r,s,E_2) \subseteq \ell^{\mathcal{I}}(u,v,E_2)$$

and argue as in Theorem 2.40 to conclude $\ell^{\mathcal{I}}(u,v,E_2) \subseteq \ell^{\mathcal{J}}(u,v,E_2)$. □

Definition 2.50. If $\mathcal{I}, \mathcal{J}$ with $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$. We define the collection of pairwise disjoint intervals in $\mathbb{N}_0$

$$\widetilde{\mathcal{I}\cap\mathcal{J}} = \{I_l \cap J_k : k \in \mathbb{N}_0, l \in \tilde{F}_k\}.$$

It coincides with $\{I_l \cap J_k : l \in \mathbb{N}_0, k \in \tilde{G}_l\}$.

Proposition 2.51. *Let* $1 \leq r,s,u,v \leq \infty$.
(i) If $r \leq s, v \leq u$ *then* $(\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{J}}(u,v)) \subseteq \ell^{\widetilde{\mathcal{I}\cap\mathcal{J}}}(r\ominus u, s\ominus v)$.
In particular, if $\sup_k \#\tilde{F}_k < \infty$ *then*

$$(\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{J}}(u,v)) \subseteq \ell^{\mathcal{J}}(r\ominus u, s\ominus v).$$

(ii) If $s \leq r, u \leq v$ *then* $\ell^{\widetilde{\mathcal{I}\cap\mathcal{J}}}(r \ominus u, s \ominus v) \subseteq (\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{J}}(u,v))$.
In particular, if $\sup_l \#\tilde{G}_l < \infty$ *then*

$$\ell^{\mathcal{I}}(r \ominus u, s \ominus v) \subseteq (\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{J}}(u,v)).$$

PROOF. (i) Note that $\widetilde{\mathcal{I}\cap\mathcal{J}} \leq \mathcal{I}$ and $\widetilde{\mathcal{I}\cap\mathcal{J}} \leq \mathcal{J}$. Hence, from Proposition 2.33,

$$\ell^{\widetilde{\mathcal{I}\cap\mathcal{J}}}(p,q,E_1) \subseteq \ell^{\mathcal{I}}(p,q,E_1),\ \ p \geq q \tag{2.12}$$

and

$$\ell^{\mathcal{J}}(p,q,E_2) \subseteq \ell^{\widetilde{\mathcal{I}\cap\mathcal{J}}}(p,q,E_2),\ \ p \leq q. \tag{2.13}$$

Now using (2.12), (2.13) and Theorem 2.46 we obtain

$$(\ell^{\mathcal{I}}(r,s,E_1), \ell^{\mathcal{J}}(u,v,E_2)) \subseteq (\ell^{\widetilde{\mathcal{I}\cap\mathcal{J}}}(r,s,E_1), \ell^{\widetilde{\mathcal{I}\cap\mathcal{J}}}(u,v,E_2)) = \ell^{\widetilde{\mathcal{I}\cap\mathcal{J}}}(r\ominus u, s\ominus v, \mathcal{L}(E_1,E_2)).$$

Also we have

$$F_k(\widetilde{\mathcal{I}\cap\mathcal{J}}, \mathcal{J}) = \{(k,l) : l \in \tilde{F}_k\}$$

and

$$F_l(\widetilde{\mathcal{I}\cap\mathcal{J}}, \mathcal{I}) = \{(k,l) : k \in \tilde{G}_l\}.$$

Therefore, $\#F_k(\widetilde{\mathcal{I}\cap\mathcal{J}}, \mathcal{J}) = \#\tilde{F}_k$ and $F_l(\widetilde{\mathcal{I}\cap\mathcal{J}}, \mathcal{I}) = \#\tilde{G}_l$. Using now Theorem 2.34

$$\ell^{\widetilde{\mathcal{I}\cap\mathcal{J}}}(p,q,\mathcal{L}(E_1,E_2)) = \ell^{\mathcal{J}}(p,q,\mathcal{L}(E_1,E_2)) \Longleftrightarrow \sup_k \#\tilde{F}_k < \infty. \tag{2.14}$$

$$\ell^{\widetilde{\mathcal{I}\cap\mathcal{J}}}(p,q,\mathcal{L}(E_1,E_2)) = \ell^{\mathcal{I}}(p,q,\mathcal{L}(E_1,E_2)) \Longleftrightarrow \sup_l \#\tilde{G}_l < \infty. \tag{2.15}$$

The particular case follows now applying (2.14).
(ii) The proof is similar and left to the reader. The particular case follows applying (2.15). □

Our purpose is to get a final description of multipliers $(\ell^{\mathcal{I}}(r,s,E_1), \ell^{\mathcal{J}}(u,v,E_2))$ for arbitrary families $\mathcal{I}$, $\mathcal{J}$. We shall deal first with the case $\mathcal{I} \leq \mathcal{J}$ and get a reduction to this situation in the remaining cases.

4.1. The case $\mathcal{I} \leq \mathcal{J}$.

In this section we consider $\mathcal{I}$ and $\mathcal{J}$ such that $\Lambda_{\mathcal{J}} = N^{\mathcal{J}}_{big} \cup N^{\mathcal{J}}_{equal}$. This means that $\tilde{F}_k = F_k \neq \emptyset$ and $J_k = \cup_{l\in F_k} I_l$ for all k. Notice that $l \in F_k$ means $I_l \subseteq J_k$ and we have

$$F_k = \{l \in \mathbb{N}_0 : \phi(k) \leq l \leq \Phi(k)\}.$$

We use the notation $\mathcal{J}/\mathcal{I} = \{F_k : k \in \mathbb{N}_0\}$.
We shall need the following well known fact.

LEMMA 2.52. *Let* $1 \leq u, r \leq \infty$*,* E, F *Banach spaces,* $A \subseteq \mathbb{N}_0$ *and* $\lambda_i \in \mathcal{L}(E,F)$*,* $i \in A$*. Given* $\epsilon > 0$ *there exists* $(a_i)_{i\in A}$*,* $a_i \in E$ *such that*

$$\left(\sum_{i\in A} \|a_i\|_E^r\right)^{1/r} = 1 \ \ and\ \ \left(\sum_{i\in A} \|\lambda_i\|^{r\ominus u}\right)^{1/r\ominus u} \leq \left(\sum_{i\in A} \|\lambda_i(a_i)\|_F^u\right)^{1/u} + \epsilon$$

(with the obvious modifications whenever u, r *or* $r \ominus u$ *equals* ∞*).*

Proof. We will prove the different cases, depending on the value of r, u. In all of them, consider $C = \left(\sum_{i\in A} 2^{-ui}\right)^{1/u}$ and $\sharp A \geq 1$ the number of indices contained in A. For $r = \infty$ (then $r \ominus u = u$) it suffices to take $a_i = \frac{1}{\sharp A}\tilde{a}_i$ for $\tilde{a}_i$ such that $\|\tilde{a}_i\|_E = 1$ and $\|\lambda_i(\tilde{a}_i)\|_F$ is arbitrarily close to $\|\lambda_i\|$, that is, given $\epsilon > 0$

$$\|\lambda_i\| \leq \|\lambda_i(\tilde{a}_i)\|_F + \frac{\epsilon}{C2^i}.$$

Then $\left(\sum_{i\in A} \|a_i\|_E^r\right)^{1/r} = 1$ and

$$\begin{aligned}
\left(\sum_{i\in A} \|\lambda_i\|^u\right)^{1/u} &\leq \left(\sum_{i\in A} \left(\|\lambda_i(\tilde{a}_i)\|_F + \frac{\epsilon}{C2^i}\right)^u\right)^{1/u} \\
&\leq \left(\sum_{i\in A} \|\lambda_i(\tilde{a}_i)\|_F^u\right)^{1/u} + \left(\sum_{i\in A} \left(\frac{\epsilon}{C2^i}\right)^u\right)^{1/u} \\
&\leq \left(\sum_{i\in A} \sharp A^u \|\lambda_i(\tilde{a}_i)\|_F^u\right)^{1/u} + \left(\sum_{i\in A} \left(\frac{\epsilon}{C2^i}\right)^u\right)^{1/u} \\
&\leq \left(\sum_{i\in A} \|\lambda_i(a_i)\|_F^u\right)^{1/u} + \left(\sum_{i\in A} \left(\frac{\epsilon}{C2^i}\right)^u\right)^{1/u} \\
&\leq \left(\sum_{i\in A} \|\lambda_i(a_i)\|_F^u\right)^{1/u} + \epsilon
\end{aligned}$$

where we have made use of the triangle inequality when necessary.

If $r < \infty$ and $u \geq r$ (hence $r \ominus u = \infty$) it suffices to take $a_{i(A)}$, $\|a_{i(A)}\|_E = 1$ such that $\|\lambda_i(a_{i(A)})\|_F$ is arbitrarily close to $\|\lambda_{i(A)}\|$ for $i(A)$ such that $\sup_{i\in A}\|\lambda_i\| = \|\lambda_{i(A)}\|$, and $a_i = 0$ for $i \neq i(A)$. That is, given $\epsilon > 0$ it is enough to take a_i verifying $\|\lambda_{i(A)}\| \leq \|\lambda_i(a_{i(A)})\|_F + \epsilon$. Then

$$\sup_{i\in A}\|\lambda_i\| = \|\lambda_{i(A)}\| \leq \|\lambda_i(a_{i(A)})\|_F + \epsilon = \left(\sum_{i\in A} \|\lambda_i(a_i)\|_F^u\right)^{1/u} + \epsilon.$$

If $u < r < \infty$ take

$$\alpha_i = \|\lambda_i\|^{r\ominus u/r} \left(\sum_{n\in A} \|\lambda_n\|^{r\ominus u}\right)^{-1/r}, \ i \in A$$

and $a_i = \alpha_i\tilde{a}_i$, where $\|\tilde{a}_i\| = 1$ and $\|\lambda_i(\tilde{a}_i)\|_F$ is arbitrarily close to $\|\lambda_i\|$, in this case given $\epsilon > 0$ we need $\|\lambda_i\| = \|\lambda_i(\tilde{a}_i)\|_F + \frac{\epsilon}{C2^i\alpha_i}$. Notice that, with this choice, one gets

$$\left(\sum_{i\in A} \|a_i\|_E^r\right)^{1/r} = 1 \text{ and } \left(\sum_{i\in A} \alpha_i^u \|\lambda_i\|^u\right)^{1/u} = \left(\sum_{i\in A} \|\lambda_i\|^{r\ominus u}\right)^{1/r\ominus u},$$

by simply using that $\frac{r\ominus u}{r}u = r\ominus u - u$. Making use of these equalities together with the triangle inequality we get to

$$\begin{aligned}\left(\sum_{i\in A}\|\lambda_i\|^{r\ominus u}\right)^{1/r\ominus u} &= \left(\sum_{i\in A}\alpha_i^u\|\lambda_i\|^u\right)^{1/u}\\ &\leq \left(\sum_{i\in A}\left(\alpha_i\|\lambda_i(\tilde{a}_i)\|_F+\frac{\epsilon\alpha_i}{C2^i\alpha_i}\right)^u\right)^{1/u}\\ &\leq \left(\sum_{i\in A}\alpha_i^u\|\lambda_i(\tilde{a}_i)\|_F^u\right)^{1/u}+\left(\sum_{i\in A}\left(\frac{\epsilon}{C2^i}\right)^u\right)^{1/u}\\ &\leq \left(\sum_{i\in A}\|\lambda_i(a_i)\|_F^u\right)^{1/u}+\left(\sum_{i\in A}\left(\frac{\epsilon}{C2^i}\right)^u\right)^{1/u}\\ &\leq \left(\sum_{i\in A}\|\lambda_i(a_i)\|_F^u\right)^{1/u}+\epsilon.\end{aligned}$$

□

REMARK 2.53. Note than in the scalar case ($a_i \in \mathbb{K}, \lambda_i \in \mathbb{K}$) arguing in a similar way we obtain the equality:

$$\left(\sum_{i\in A}|\lambda_i|^{r\ominus u}\right)^{1/r\ominus u} = \left(\sum_{i\in A}|\lambda_i a_i|^u\right)^{1/u}$$

for some $a\in\mathcal{S}$ verifying $\left(\sum_{i\in A}|a_i|^r\right)^{1/r}=1$.

THEOREM 2.54. *If $\mathcal{I}\leq\mathcal{J}$ then*
$(\ell^{\mathcal{I}}(r,s,E_1),\ell^{\mathcal{J}}(u,v,E_2)) =$

$$\left\{(\lambda_n)_n\in\mathcal{S}(\mathcal{L}(E_1,E_2)) : \left(\left(\sum_{i\in I_l}\|\lambda_i\|^{r\ominus u}\right)^{1/r\ominus u}\right)_l \in \ell^{\mathcal{J}/\mathcal{I}}(s\ominus u, s\ominus v)\right\}.$$

PROOF. $\subseteq$) Assume that $(\lambda_n)_n\in(\ell^{\mathcal{I}}(r,s,E_1),\ell^{\mathcal{J}}(u,v,E_2))$ and take $\epsilon>0$.

Now, define $\beta_l = (\sum_{i\in I_l}\|\lambda_i\|^{r\ominus u})^{1/r\ominus u}$ and use Lemma 2.52 in the scalar version with $A=F_k$ for each $k\in\mathbb{N}_0$, to choose $(\alpha_l)_{l\in F_k}$ verifying $(\sum_{l\in F_k}|\alpha_l|^s)^{1/s}=1$ and

$$(\sum_{l\in F_k}\beta_l^{s\ominus u})^{1/s\ominus u} = (\sum_{l\in F_k}|\beta_l\alpha_l|^u)^{1/u}.$$

Again, use Lemma 2.52 in its scalar version for $A=\mathbb{N}_0$. Take $\gamma=(\gamma_k)_k$ verifying $(\sum_k|\gamma_k|^s)^{1/s}=1$ and

$$\left(\sum_k(\sum_{l\in F_k}\beta_l^{s\ominus u})^{s\ominus v/s\ominus u}\right)^{1/s\ominus v} = \left(\sum_k|\gamma|_k^v(\sum_{l\in F_k}\beta_l^{s\ominus u})^{v/s\ominus u}\right)^{1/v}.$$

Finally, use Lemma 2.52 with $A=I_l$ to select for each $l\in\mathbb{N}_0$, a sequence $(\tilde{a}_i^{(l)})_{i\in I_l}$ such that $(\sum_{i\in I_l}\|\tilde{a}_i^{(l)}\|_{E_1}^r)^{1/r}=1$ and

$$\beta_l = (\sum_{i\in I_l}\|\lambda_i\|^{r\ominus u})^{1/r\ominus u} \leq K(u)(\sum_{i\in I_l}\|\lambda_i\tilde{a}_i^{(l)}\|_{E_2}^u)^{1/u}+\frac{\epsilon}{\alpha_l\gamma_k2^{l+k}}.$$

This procedure allows us to obtain the sequence $a = (a_i)_i$, $a_i = \gamma_k \alpha_l \tilde{a}_i^{(l)}$ where $i \in I_l$, $l \in F_k$ and $k \in \mathbb{N}_0$. With this choice we get that $\|a\|^{\mathcal{I}}_{r,s,E_1} = 1$ and

$$\|\beta\|^{\mathcal{J}/\mathcal{I}}_{s\ominus u, s\ominus v} \le K(u,v)\|\lambda * a\|^{\mathcal{J}}_{u,v,E_2} + K\epsilon \le C\|\lambda\|.$$

$\supseteq$) Let $a = (a_i)_i \in \ell^{\mathcal{I}}(r,s,E_1)$ and $\lambda = (\lambda_i)_i$ such that $(\beta_l)_l \in \ell^{\mathcal{J}/\mathcal{I}}(s\ominus u, s\ominus v)$ where

$$\beta_l = \left(\sum_{i\in I_l} \|\lambda_i\|^{r\ominus u}\right)^{1/r\ominus u}.$$

Fix $k \in \mathbb{N}_0$

$$\begin{aligned}
\left(\sum_{i\in J_k} \|\lambda_i(a_i)\|^u_{E_2}\right)^{1/u} &= \left(\sum_{l\in F_k}\sum_{i\in I_l} \|\lambda_i(a_i)\|^u_{E_2}\right)^{1/u} \\
&\le \left(\sum_{l\in F_k}\left(\sum_{i\in I_l}\|\lambda_i\|^{r\ominus u}\right)^{u/r\ominus u}\left(\sum_{i\in I_l}\|a_i\|^r_{E_1}\right)^{u/r}\right)^{1/u} \\
&\le \left(\sum_{l\in F_k}\left(\sum_{i\in I_l}\|\lambda_i\|^{r\ominus u}\right)^{s\ominus u/r\ominus u}\right)^{1/s\ominus u}\left(\sum_{l\in F_k}\left(\sum_{i\in I_l}\|a_i\|^r_{E_1}\right)^{s/r}\right)^{1/s}.
\end{aligned}$$

Taking the v-norm, we get to:

$$\begin{aligned}
\left(\sum_k\left(\sum_{i\in J_k}\|\lambda_i(a_i)\|^u_{E_2}\right)^{\frac{v}{u}}\right)^{\frac{1}{v}} &\le \left(\sum_k\left(\sum_{l\in F_k}\beta_l^{s\ominus u}\right)^{\frac{v}{s\ominus u}}\left(\sum_{l\in F_k}\left(\sum_{i\in I_l}\|a_i\|^r_{E_1}\right)^{\frac{s}{r}}\right)^{\frac{v}{s}}\right)^{\frac{1}{v}} \\
&\le \left(\sum_k\left(\sum_{l\in F_k}\beta_l^{s\ominus u}\right)^{\frac{s\ominus v}{s\ominus u}}\right)^{\frac{1}{s\ominus v}}\left(\sum_k\sum_{l\in F_k}\left(\sum_{i\in I_l}\|a_i\|^r_{E_1}\right)^{\frac{s}{r}}\right)^{\frac{1}{s}} \\
&= \left(\sum_k\left(\sum_{l\in F_k}\beta_l^{s\ominus u}\right)^{\frac{s\ominus v}{s\ominus u}}\right)^{\frac{1}{s\ominus v}}\left(\sum_l\left(\sum_{i\in I_l}\|a_i\|^r_{E_1}\right)^{\frac{s}{r}}\right)^{\frac{1}{s}}.
\end{aligned}$$

Hence $(\lambda_n)_n \in (\ell^{\mathcal{I}}(r,s,E_1), \ell^{\mathcal{J}}(u,v,E_2))$ and $\|\lambda\| \le \|\beta\|^{\mathcal{J}/\mathcal{I}}_{s\ominus u, s\ominus v}$.

□

Corollary 2.55. *Let $\mathcal{J} \le \mathcal{I}$, $E_1 = E_2 = \mathbb{K}$ and $1 \le r,s,u,v \le \infty$. Then*
$(\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{J}}(u,v)) =$

$$\left\{(\lambda_n)_n \in \mathcal{S}(\mathcal{L}(E_1,E_2)) : \left(\left(\sum_{i\in J_k}|\lambda_i|^{r\ominus u}\right)^{1/r\ominus u}\right)_k \in \ell^{\mathcal{I}/\mathcal{J}}(r\ominus v, s\ominus v)\right\}.$$

Proof. Recall that $\tilde{G}_l = G_l = \{k \in \mathbb{N}_0 : J_k \subseteq I_l\}$ and $I_l = \cup_{k\in G_l} J_k$: We now denote $\mathcal{I}/\mathcal{J} = \{G_l : l \in \mathbb{N}_0\}$. Using Köthe duals we actually have

$$(\ell^{\mathcal{I}}(r,s), \ell^{\mathcal{J}}(u,v)) = (\ell^{\mathcal{J}}(u',v'), \ell^{\mathcal{I}}(r',s')).$$

Taking into account that $q' \ominus p' = p \ominus q$ for all p,q the result follows from Theorem 2.54. □

Corollary 2.56. *Let $\mathcal{J} \leq \mathcal{I}$, E_1, E_2 Banach spaces and $1 \leq r, s, u, v \leq \infty$. Then* $(\ell^{\mathcal{I}}(r,s,E_1), \ell^{\mathcal{J}}(u,v,E_2)) =$

$$\left\{ (\lambda_n)_n \in \mathcal{S}(\mathcal{L}(E_1,E_2)) : \left(\left(\sum_{i \in J_k} \|\lambda_i\|^{r \ominus u} \right)^{1/r \ominus u} \right)_k \in \ell^{\mathcal{I}/\mathcal{J}}(r \ominus v, s \ominus v) \right\}.$$

Proof. Let $\lambda = (\lambda_i)_i$ be in $(\ell^{\mathcal{I}}(r,s,E_1), \ell^{\mathcal{J}}(u,v,E_2))$. Consider the adjoint λ' defined as $\langle \lambda' *_{\mathcal{L}} b', a \rangle = \langle b', \lambda *_{\mathcal{L}} a \rangle$ where $b' \in \ell^{\mathcal{J}}(u', v', E_2^*)$ and $a \in \ell^{\mathcal{I}}(r,s,E_1)$. Then $\lambda' = (\lambda_i')_i \in (\ell^{\mathcal{J}}(u', v', E_2^*), \ell^{\mathcal{I}}(r', s', E_1^*))$ is well-defined.
Taking into account $\mathcal{J} < \mathcal{I}$ and applying Theorem 2.54, we have that λ' verifies

$$\left(\left(\sum_{i \in J_k} \|\lambda_i'\|^{u' \ominus r'} \right)^{1/u' \ominus r'} \right)_k \in \ell^{\mathcal{I}/\mathcal{J}}(v' \ominus r', v' \ominus s').$$

Since $\|\lambda_i'\| = \|\lambda_i\|$ and $q' \ominus p' = p \ominus q$ for all p, q , we have the inclusion.
For the other inclusion, let

$$\lambda \in \left\{ (\lambda_n)_n \in \mathcal{S}(\mathcal{L}(E_1,E_2)) : \left(\left(\sum_{i \in J_k} \|\lambda_i\|^{r \ominus u} \right)^{1/r \ominus u} \right)_k \in \ell^{\mathcal{I}/\mathcal{J}}(r \ominus v, s \ominus v) \right\}.$$

Now it is enough to consider the inequality $\|\lambda_i(a_i)\|_{E_2} \leq \|\lambda_i\| \|a_i\|_{E_1}$ and apply the previous Corollary to the scalar sequence $(\|\lambda_i\|)_i$ to get the result. □

4.2. The case $\widetilde{\mathcal{I} \cap \mathcal{J}} \subseteq \mathcal{I} \cup \mathcal{J}$.
Let $\mathcal{I} = \{I_l : l \in \mathbb{N}_0\}$ and $\mathcal{J} = \{J_k : k \in \mathbb{N}_0\}$ such that $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$. We assume in this section that $N^{\mathcal{I}}_{inter} = \emptyset$ and $N^{\mathcal{J}}_{inter} = \emptyset$, that is to say for a given $l \in \mathbb{N}_0$ either there exists k such that $I_l \subseteq J_k$ or there exist k' such that $J_{k'} \subseteq I_l$. In other words each interval in $\widetilde{\mathcal{I} \cap \mathcal{J}}$ belongs either to $\mathcal{I}$ or to $\mathcal{J}$.

To extend the result on multipliers to this setting we shall use the following lemma whose easy proof is left to the reader.

Lemma 2.57. *Let $\mathcal{I} = \{I_l : l \in \mathbb{N}_0\}$ and $\mathcal{J} = \{J_k : k \in \mathbb{N}_0\}$ such that $\Lambda_{\mathcal{I}} = \Lambda_{\mathcal{J}}$ and let $\mathcal{I}_i$ (respect. $\mathcal{J}_i$) for $i = 1, \cdots, m$ sub-collections of $\mathcal{I}$ (respect. $\mathcal{J}$) with $\mathcal{I} = \cup_{i=1}^m \mathcal{I}_i$ (respect. $\mathcal{J} = \cup_{i=1}^m \mathcal{J}_i$) satisfying $\Lambda_{\mathcal{I}_i} = \Lambda_{\mathcal{J}_i}$ for $i = 1, \cdots, m$. Then*

$$\lambda = (\lambda_n)_{n \in \Lambda_{\mathcal{I}}} \in (\ell^{\mathcal{I}}(r,s,E_1), \ell^{\mathcal{J}}(u,v,E_2))$$

if and only if

$$\lambda^{(i)} = (\lambda_n)_{n \in \Lambda_{\mathcal{I}_i}} \in (\ell^{\mathcal{I}_i}(r,s,E_1), \ell^{\mathcal{J}_i}(u,v,E_2)), i = 1, \cdots, m.$$

Moreover $\|\lambda\| \approx \sum_{i=1}^m \|\lambda^{(i)}\|$.

Theorem 2.58. *Let $\widetilde{\mathcal{I} \cap \mathcal{J}} \subseteq \mathcal{I} \cup \mathcal{J}$. Then $(\lambda_n)_n \in (\ell^{\mathcal{I}}(r,s,E_1), \ell^{\mathcal{J}}(u,v,E_2))$ if and only if it satisfies the conditions*

$$\left(\left(\sum_{i \in J_k} \|\lambda_i\|^{r \ominus u} \right)^{1/r \ominus u} \right)_{k \in N^{\mathcal{J}}_{equal}} \in \ell^{s \ominus v}(\mathcal{L}(E_1,E_2)), \tag{2.16}$$

$$\left(\left(\sum_{i \in I_l} \|\lambda_i\|^{r \ominus u} \right)^{1/r \ominus u} \right)_{l \in N^{\mathcal{I}}_{small}} \in \ell^{\mathcal{F}}(s \ominus u, s \ominus v, \mathcal{L}(E_1,E_2)), \tag{2.17}$$

(2.18) $$\left(\left(\sum_{i\in J_k}\|\lambda_i\|^{r\ominus u}\right)^{1/r\ominus u}\right)_{k\in N^{\mathcal{J}}_{small}} \in \ell^{\mathcal{G}}(r\ominus v, s\ominus v, \mathcal{L}(E_1,E_2)),$$

where $\mathcal{F}=\{F_k : k\in N^{\mathcal{J}}_{big}\}$ *and* $\mathcal{G}=\{G_l : l\in N^{\mathcal{I}}_{big}\}$.

Proof. Let us consider the following collection of intervals

$$\mathcal{J}_b=\{J_k : k\in N^{\mathcal{J}}_{big}\},\quad \mathcal{J}_e=\{J_k : k\in N^{\mathcal{J}}_{equal}\},\ \text{and}\ \mathcal{J}_s=\{J_k : k\in N^{\mathcal{J}}_{small}\}$$

and similarly for $\mathcal{I}$.

If $J_k\in\mathcal{J}_b$ (respect. $I_l\in\mathcal{I}_b$) we have $F_k=\{l\in\mathbb{N}_0 : I_l\subsetneq J_k\}\neq\emptyset$ (respect. $G_l=\{k\in\mathbb{N}_0 : J_k\subsetneq I_l\}\neq\emptyset$) and

(2.19) $$J_k=\cup_{l\in F_k}I_l, I_l\in\mathcal{I}_s \quad (\text{ respect. } I_l=\cup_{l\in G_l}J_k, J_k\in\mathcal{J}_s).$$

Hence $\mathcal{J}=\mathcal{J}_e\cup\mathcal{J}_b\cup\mathcal{J}_s$, $\mathcal{I}=\mathcal{I}_e\cup\mathcal{I}_b\cup\mathcal{I}_s$ and

$$\mathcal{J}_e=\{J_k : k\in N^{\mathcal{J}}_{equal}\}=\{I_l : l\in N^{\mathcal{I}}_{equal}\}=\mathcal{I}_e.$$

Observe that $\mathcal{I}_s\le\mathcal{J}_b$ and $\mathcal{J}_s\le\mathcal{I}_b$ and, in particular, $\mathcal{G}=\mathcal{I}_b/\mathcal{J}_s$ and $\mathcal{F}=\mathcal{J}_b/\mathcal{I}_s$.

We use Lemma 2.57 and observe that, denoting $\Lambda_0=\Lambda_{\mathcal{J}_e}$, $\Lambda_1=\Lambda_{\mathcal{J}_b}=\Lambda_{\mathcal{I}_s}$ and $\Lambda_2=\Lambda_{\mathcal{J}_s}=\Lambda_{\mathcal{I}_b}$,

$$(\lambda_n)_{n\in\Lambda_0}\in(\ell^{\mathcal{I}_e}(r,s,E_1),\ell^{\mathcal{J}_e}(u,v,E_2))$$

corresponds to (2.16) invoking Theorem 2.46, also that

$$(\lambda_n)_{n\in\Lambda_1}\in(\ell^{\mathcal{I}_s}(r,s,E_1),\ell^{\mathcal{J}_b}(u,v,E_2))$$

corresponds to (2.17) invoking Theorem 2.54 and, finally,

$$(\lambda_n)_{n\in\Lambda_2}\in(\ell^{\mathcal{I}_b}(r,s,E_1),\ell^{\mathcal{J}_s}(u,v,E_2))$$

corresponds to (4.3) invoking Corollary 2.56. □

4.3. The general case.

In this section we assume that there exist $k\in\mathbb{N}_0$ and $l\in\tilde{F}_k$ such that $I_l\cap J_k\in\widetilde{\mathcal{I}\cap\mathcal{J}}$ and $I_l\cap J_k\notin\mathcal{I}\cup\mathcal{J}$.

The situation we are handling now corresponds to $N^{\mathcal{J}}_{inter}\neq\emptyset$ (and hence $N^{\mathcal{I}}_{inter}\neq\emptyset$).

Definition 2.59.

$$\mathcal{J}'=\{J'_k=\cup_{l\in F_k}I_l : k\in\mathbb{N}_0, \#F_k>0\},$$

$$\mathcal{H}=\widetilde{\mathcal{I}\cap\mathcal{J}}\setminus(\mathcal{I}\cup\mathcal{J}),$$

$$\mathcal{J}_s=\{J_k : k\in N^{\mathcal{J}}_{small}\}.$$

Denote $\mathcal{J}''=\mathcal{J}'\cup\mathcal{J}_s$ and $\mathcal{J}_{new}=\mathcal{J}''\cup\mathcal{H}$.

We use similar notations for $\mathcal{I}$.

Recall that $\phi(k)=\min\tilde{F}_k$ and $\Phi(k)=\max\tilde{F}_k$ for $k\in\mathbb{N}_0$. We easily observe that $\phi(N^{\mathcal{J}}_{equal})\subseteq N^{\mathcal{I}}_{equal}$, $\phi(N^{\mathcal{J}}_{big})\subseteq N^{\mathcal{I}}_{small}$, $\phi(N^{\mathcal{J}}_{small})\subseteq N^{\mathcal{I}}_{big}\cup N^{\mathcal{I}}_{inter}$ and $\phi(N^{\mathcal{J}}_{inter})\subseteq N^{\mathcal{I}}_{small}\cup N^{\mathcal{I}}_{inter}$. Same results hold for Φ.

Lemma 2.60.

$$\mathcal{H}=\{\hat{J}_k : k\in N^{\mathcal{J}}_{inter}, \phi(k)\in N^{\mathcal{I}}_{inter}\}\cup\{\check{J}_k : k\in N^{\mathcal{J}}_{inter}, \Phi(k)\in N^{\mathcal{I}}_{inter}\}.$$

PROOF. $\subseteq$) Let $I \in \mathcal{H}$. Since $I \in \widetilde{\mathcal{I} \cap \mathcal{J}}$ then there exist $k \in \mathbb{N}_0$ and $l \in \tilde{F}_k$ such that $I = I_l \cap J_k$. On the other hand, since $I \notin \mathcal{I} \cup \mathcal{J}$ we have that $I \subsetneq I_l$ and $I \subsetneq J_k$. Hence either $\phi(k) = l$ and $\Psi(l) = k$ or $\Phi(k) = l$ and $\psi(l) = k$. This gives either $k \in N^{\mathcal{J}}_{inter}$ and $\phi(k) \in N^{\mathcal{I}}_{inter}$ (and hence $I = \hat{J}_k$) or $k \in N^{\mathcal{J}}_{inter}$ and $\Phi(k) \in N^{\mathcal{I}}_{inter}$ (and hence $I = \check{J}_k$).

$\supseteq$) Let $k \in N^{\mathcal{J}}_{inter}$ with $\phi(k) \in N^{\mathcal{I}}_{inter}$ and consider $\hat{J}_k = J_k \cap I_{\phi(k)} \in \widetilde{\mathcal{I} \cap \mathcal{J}}$. Then $\hat{J}_k \subsetneq J_k$ (hence $\hat{J}_k \notin \mathcal{J}$) and $\hat{J}_k \subsetneq I_\phi(k)$ (hence $\hat{J}_k \notin \mathcal{I}$). Similarly for $\check{J}_k$ in the case $k \in N^{\mathcal{J}}_{inter}$ with $\Phi(k) \in N^{\mathcal{I}}_{inter}$

□

REMARK 2.61. Note that $\hat{J}_k = J_k \cap I_l$ if and only if $\check{I}_l = I_l \cap J_k$. Therefore

$$\mathcal{H} = \{\hat{I}_l : l \in N^{\mathcal{I}}_{inter}, \psi(l) \in N^{\mathcal{J}}_{inter}\} \cup \{\check{I}_l : k \in N^{\mathcal{I}}_{inter}, \Psi(l) \in N^{\mathcal{J}}_{inter}\}.$$

LEMMA 2.62.

$$\widetilde{\mathcal{I}'' \cap \mathcal{J}''} \subseteq \mathcal{I}_s \cup \mathcal{J}_s \cup \mathcal{I}_e \subseteq \mathcal{I}'' \cup \mathcal{J}''.$$

PROOF. Let $I \in \mathcal{I}' \cup \mathcal{I}_s$ and $J \in \mathcal{J}' \cup \mathcal{J}_s$ with $I \cap J \neq \emptyset$. The case $I \in \mathcal{I}_s$ and $J \in \mathcal{J}_s$ can not hold. If $I \in \mathcal{I}_s$ and $J \in \mathcal{J}'$ then $I \cap J = I \in \mathcal{I}_s$. Similarly if $I \in \mathcal{I}'$ and $J \in \mathcal{J}_s$ then $I \cap J = J \in \mathcal{J}_s$. Finally if $I \in \mathcal{I}'$ and $J \in \mathcal{J}'$ then $I = J \in \mathcal{I}_e = \mathcal{J}_e$. □

THEOREM 2.63. $\lambda \in (\ell^{\mathcal{I}}(r, s, E_1), \ell^{\mathcal{J}}(u, v, E_2))$ *if and only if* $(\lambda_n)_n$ *satisfies*

$$\left(\Big(\sum_{i \in J_k} \|\lambda_i\|^{r \ominus u}\Big)^{1/r \ominus u}\right)_{k \in N^{\mathcal{J}}_{equal}} \in \ell^{s \ominus v}(\mathcal{L}(E_1, E_2)) \tag{2.20}$$

$$\left(\Big(\sum_{i \in I_l} \|\lambda_i\|^{r \ominus u}\Big)^{1/r \ominus u}\right)_{l \in N^{\mathcal{I}}_{small}} \in \ell^{\mathcal{F}}(s \ominus u, s \ominus v, \mathcal{L}(E_1, E_2)) \tag{2.21}$$

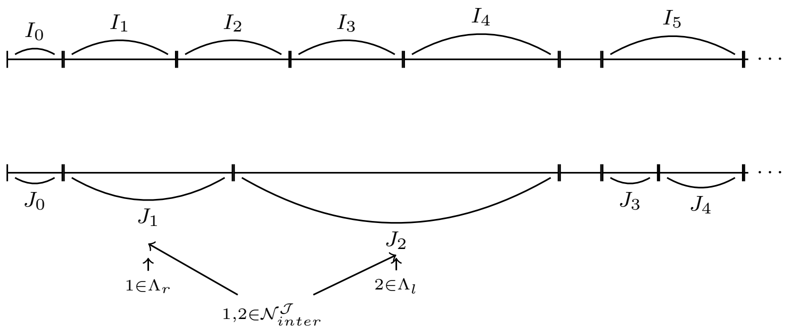
$$\left(\Big(\sum_{i \in J_k} \|\lambda_i\|^{r \ominus u}\Big)^{1/r \ominus u}\right)_{k \in N^{\mathcal{J}}_{small}} \in \ell^{\mathcal{G}}(r \ominus v, s \ominus v, \mathcal{L}(E_1, E_2)) \tag{2.22}$$

$$\left(\Big(\sum_{i \in \check{J}_k} \|\lambda_i\|^{r \ominus u}\Big)^{1/r \ominus u}\right)_{k \in \Lambda_r} + \left(\Big(\sum_{i \in \hat{J}_k} \|\lambda_i\|^{r \ominus u}\Big)^{1/r \ominus u}\right)_{k \in \Lambda_l} \in \ell^{s \ominus v}(\mathcal{L}(E_1, E_2)) \tag{2.23}$$

where

$$\Lambda_r = \{k \in N^{\mathcal{J}}_{inter}, \Phi(k) \in N^{\mathcal{I}}_{inter}\} \text{ and } \Lambda_l = \{k \in N^{\mathcal{J}}_{inter}, \phi(k) \in N^{\mathcal{I}}_{inter}\},$$

for example,



$$\mathcal{G} = \{G_l : l \in N^{\mathcal{I}}_{big} \cup N^{\mathcal{I}}_{inter}, \#G_l > 0\}$$

and

$$\mathcal{F} = \{F_k : k \in N^{\mathcal{J}}_{big} \cup N^{\mathcal{J}}_{inter}, \#F_k > 0\}.$$

PROOF. Using $J_k = J'_k \cup \hat{J}_k \cup \check{J}_k$ and Lemma 2.60 one obtains $\mathcal{J}_{new} \leq \mathcal{J}$ and $\mathcal{I}_{new} \leq \mathcal{I}$. Clearly $\#F_l(\mathcal{I}_{new}, \mathcal{I}) \leq 3$ and $\#F_k(\mathcal{J}_{new}, \mathcal{J}) \leq 3$ for all k. Therefore, using Theorem 2.34, we have $\ell^{\mathcal{J}_{new}}(p, q, E) = \ell^{\mathcal{J}}(p, q, E)$ and $\ell^{\mathcal{I}_{new}}(p, q, E) = \ell^{\mathcal{I}}(p, q, E)$, which gives

$$(\ell^{\mathcal{I}}(r, s, E_1), \ell^{\mathcal{J}}(u, v, E_2)) = (\ell^{\mathcal{I}_{new}}(r, s, E_1), \ell^{\mathcal{J}_{new}}(u, v, E_2)). \tag{2.24}$$

Taking into account Lemma 2.60 and Remark 2.61 we observe that $\Lambda_{\mathcal{H}} = \Lambda_r \cup \Lambda_l$ and $\Lambda_{\mathcal{I}''} = \Lambda_{\mathcal{J}''}$.

Since $\mathcal{J}_{new} = \mathcal{J}'' \cup \mathcal{H}$ and $\mathcal{I}_{new} = \mathcal{I}'' \cup \mathcal{H}$ we can apply Lemma 2.57 to conclude that $\lambda \in (\ell^{\mathcal{I}}(r, s, E_1), \ell^{\mathcal{J}}(u, v, E_2))$ if and only if $(\lambda_n)_{n\in\Lambda_{\mathcal{H}}} \in (\ell^{\mathcal{H}}(r, s, E_1), \ell^{\mathcal{H}}(u, v, E_2))$ and $(\lambda_n)_{n\notin\Lambda_{\mathcal{H}}} \in (\ell^{\mathcal{I}''}(r, s, E_1), \ell^{\mathcal{J}''}(u, v, E_2))$.

Now apply Theorem 2.46 to obtain $(\lambda_n)_{n\in\Lambda_{\mathcal{H}}} \in \ell^{\mathcal{H}}(r \ominus u, s \ominus v, \mathcal{L}(E_1, E_2))$ which corresponds to (2.23).

On the other hand, comparing $\mathcal{I}''$ and $\mathcal{J}''$ we notice that $I \in \mathcal{I}''_{big}$ corresponds to $I = I'_l$ for some $l \in N^{\mathcal{I}}_{big} \cup N^{\mathcal{I}}_{inter}$ and $\#G_l \geq 1$. Hence we obtain that $\mathcal{G} = \{G_I : I \in \mathcal{I}''_{big}\}$ and similarly $\mathcal{F} = \{F_J : J \in \mathcal{J}''_{big}\}$.

We now use Lemma 2.62 together with Theorem 2.58 to obtain the equivalence with (2.20), (2.21) and (2.22) and $(\lambda_n)_{n\notin\Lambda_{\mathcal{H}}} \in (\ell^{\mathcal{I}''}(r, s, E_1), \ell^{\mathcal{J}''}(u, v, E_2))$.

□

4.4. An application.

Let us apply the previous ideas to a particular case. Consider $E_1 = E_2 = \mathbb{C}$. Let $\rho : [0, 1) \to [0, \infty)$ be a non-decreasing function such that $\rho(0) = 0$ and $\rho(t)/t \in L^1([0, 1))$. We define the weighted Bergman-Besov space $B^1(\rho)$ as those analytic functions F in the unit disk such that

$$\int_{\mathbb{D}} |F'(z)| \frac{\rho(1 - |z|)}{1 - |z|} dA(z) < \infty.$$

An analytic function F is called lacunary if $F(z) = \sum_{n\in\Lambda_{\mathcal{L}}} a_n z^n$ where $\mathcal{L} = \{\{n_k\} : k \in \mathbb{N}_0\}$ for some (n_k) such that $\inf_k n_{k+1}/n_k > 1$.

Recently weights with the following condition had been considered in [33]: There exist $C_1, C_2 > 0$ and $K(n,\rho)$ such that

$$C_1 \int_0^1 r^{2^n-1}\frac{\rho(1-r)}{1-r}dr \le K(n,\rho) \le C_2 \int_{1-2^{-n}}^{1-2^{-(n+1)}} r^{2^{n+1}-1}\frac{\rho(1-r)}{1-r}dr \tag{2.25}$$

and the following result has been shown.

Theorem 2.64. *(see [33]) Let $F(z) = \sum_{n\in\Lambda_{\mathcal{L}}} a_n z^n$ be a lacunary function and let ρ be a weight satisfying (2.25). Then F belongs to $B^1(\rho)$ if and only if*

$$\sum_{k=0}^{\infty}\Big(\sum_{n\in J_k}|a_n|^2\Big)^{1/2} 2^k K(k,\rho) < \infty \tag{2.26}$$

where $J_k = \{n : 2^k - 1 \le n < 2^{k+1} - 1\}$.

We shall extend the previous result for more general classes of weight functions and families of intervals $\mathcal{J}$.

Definition 2.65. Let $0 < q < \infty$, $\mathcal{J}$ be a collection of disjoint intervals in $\mathbb{N}_0$, say $J_k = \mathbb{N}_0 \cap [m_k, m_{k+1})$ where $m_0 = 0$ and (m_k) is some increasing sequence in $\mathbb{N}_0$. and let $\rho : [0,1) \to [0,\infty)$ be a measurable function such that $\rho(t)/t \in L^1([0,1))$.

We say that ρ is q-adapted to $\mathcal{J}$ whenever there exists $C > 0$ depending on m_n, q and ρ such that

$$\int_0^1 r^{qm_n}\frac{\rho(1-r)}{1-r}dr \le C\int_{A_n} r^{qm_{n+1}}\frac{\rho(1-r)}{1-r}dr \tag{2.27}$$

for all $n \ge 0$ where $A_0 = [0, 1-\frac{1}{m_1})$ and $A_n = [1-\frac{1}{m_n}, 1-\frac{1}{m_{n+1}})$ for $n \ge 1$.

We denote

$$\mu_\rho(s) = \int_0^1 r^s \frac{\rho(1-r)}{1-r}dr, s \ge 0. \tag{2.28}$$

In particular, from condition (2.27) if ρ is q-adapted to $\mathcal{J}$ we get that

$$\mu_\rho(qm_n) \approx \mu_\rho(qm_{n+1}). \tag{2.29}$$

Note also that condition (2.25) means that ρ is 1/2-adapted for $\mathcal{J}$ where $m_n = 2^n - 1$.

Proposition 2.66. *Let $\rho_\alpha(t) = t^\alpha$ with $\alpha > 0$ and $\mathcal{J} = \{[m_n, m_{n+1})\cap\mathbb{N}_0 : n \in \mathbb{N}_0\}$. The following statements are equivalent:*

(i) ρ_α is q-adapted to $\mathcal{J}$ for all $q > 0$.

(ii) ρ_α is q-adapted to $\mathcal{J}$ for some $q > 0$.

(iii) $\sup_n m_{n+1}/m_n < \infty$.

Proof. $(i) \implies (ii)$ Obvious.

$(ii) \implies (iii)$ It is well known that $B(n+1,\alpha) = \int_0^1 r^n(1-r)^{\alpha-1}dr \approx n^{-\alpha}$ and therefore $\mu_{\rho_\alpha}(qm_n) \approx m_n^{-\alpha}$.

Hence it follows from (2.29) that $m_{n+1} \approx m_n$. therefore $\sup m_{n+1}/m_n < \infty$.

$(iii) \Longrightarrow (i)$ Let $\sup m_{n+1}/m_n = \delta$ and take $q > 0$. Now observe that

$$\begin{aligned}
\int_{1-\frac{1}{m_n}}^{1-\frac{1}{m_{n+1}}} r^{qm_{n+1}}(1-r)^{\alpha-1}dr &\geq \left(1-\frac{1}{m_n}\right)^{qm_{n+1}} \int_{\frac{1}{m_{n+1}}}^{\frac{1}{m_n}} s^{\alpha-1}ds \\
&\geq \frac{1}{\alpha}\left(1-\frac{1}{m_n}\right)^{qm_{n+1}} m_n^{-\alpha}\left(1-\left(\frac{m_n}{m_{n+1}}\right)^{\alpha}\right) \\
&\geq \frac{1}{\alpha}\left(\left(1-\frac{1}{m_n}\right)^{m_n}\right)^{\delta q} m_n^{-\alpha}\left(1-\frac{1}{\delta^{\alpha}}\right) \\
&\geq C\mu_{\rho_\alpha}(qm_n).
\end{aligned}$$

□

We now modify the proof of Lemma 3 in [8] to obtain the following result.

LEMMA 2.67. *Let $0 < q \leq 1$, let $\mathcal{J}$ be a collection of disjoint intervals in $\mathbb{N}_0$ and assume ρ is a weight q-adapted to $\mathcal{J}$. If $(\alpha_n) \geq 0$ then*

$$\int_0^1 \left(\sum_{n=0}^{\infty}\alpha_n r^n\right)^q \frac{\rho(1-r)}{1-r}dr \approx \sum_{n=0}^{\infty}\left(\sum_{k\in J_n}\alpha_k\right)^q \mu_\rho(qm_n)$$

where $J_n = \{k : m_n \leq k < m_{n+1}\}$

PROOF. As above $A_0 = [0, 1-\frac{1}{m_1})$ and $A_n = [1-\frac{1}{m_n}, 1-\frac{1}{m_{n+1}})$ for $n \geq 1$. Then

$$\begin{aligned}
\int_0^1 \left(\sum_{n=0}^{\infty}\alpha_n r^n\right)^q \frac{\rho(1-r)}{1-r}dr &= \sum_{n=0}^{\infty}\int_{A_n}\left(\sum_{n=0}^{\infty}\alpha_n r^n\right)^q \frac{\rho(1-r)}{1-r}dr \\
&\geq \sum_{n=0}^{\infty}\int_{A_n}\left(\sum_{k\in J_n}\alpha_k r^k\right)^q \frac{\rho(1-r)}{1-r}dr \\
&\geq \sum_{n=0}^{\infty}\int_{A_n}\left(\sum_{k\in J_n}\alpha_k\right)^q r^{qm_{n+1}}\frac{\rho(1-r)}{1-r}dr \\
&\geq C^{-1}\sum_{n=0}^{\infty}\left(\sum_{k\in J_n}\alpha_k\right)^q \mu_\rho(qm_n).
\end{aligned}$$

Conversely, since $q \leq 1$,

$$\begin{aligned}
\int_0^1 \left(\sum_{n=0}^{\infty}\alpha_n r^n\right)^q \frac{\rho(1-r)}{1-r}dr &\leq \int_0^1 \sum_{n=0}^{\infty}\left(\sum_{k\in J_n}\alpha_k r^k\right)^q \frac{\rho(1-r)}{1-r}dr \\
&\leq \sum_{n=0}^{\infty}\left(\sum_{k\in J_n}\alpha_k\right)^q \left(\int_0^1 r^{qm_n}\frac{\rho(1-r)}{1-r}dr\right) \\
&\leq \sum_{n=0}^{\infty}\left(\sum_{k\in J_n}\alpha_k\right)^q \mu_\rho(qm_n).
\end{aligned}$$

□

We first note that for lacunary functions F and $0 < p < \infty$ we have (see [41])

$$M_p(r,F) = \left(\int_0^{2\pi} |F(re^{i\theta})|^p \frac{d\theta}{2\pi}\right)^{1/p} \approx M_2(r,F) = \left(\int_0^{2\pi} |F(re^{i\theta})|^2 \frac{d\theta}{2\pi}\right)^{1/2}.$$

Therefore for lacunary functions F one has that $F \in B^1(\rho)$ if and only if

$$\int_0^1 M_2(r,F')\frac{\rho(1-r)}{1-r}dr < \infty.$$

Therefore invoking Plancherel's theorem and Lemma 2.67 we recover Theorem 2.64.

Recall that an analytic function $F : \mathbb{D} \to \mathbb{C}$ with $F(z) = \sum_{n=0}^\infty a_n z^n$ is said to belong to $H(p,q,\rho)$ (see [8, Definition 2]) whenever

$$\|F\|_{H(p,q,\rho)} = \left(\int_0^1 M_p^q(r,F)\frac{\rho(1-r^2)}{1-r^2} r dr\right)^{1/q} < \infty.$$

We use the notation $H(p,q,\alpha)$ if $\rho(t) = t^\alpha$.

A consequence of Lemma 2.67 is the following result.

Corollary 2.68. *Let $0 < q \leq 2$, let $\mathcal{J}$ be a collection of disjoint intervals in $\mathbb{N}_0$ and ρ be a weight $q/2$-adapted to $\mathcal{J}$. Then*

$$\|F\|_{H(2,q,\rho)} \approx \left(\sum_{n=0}^\infty \left(\sum_{k\in J_n} |a_k|^2\right)^{q/2} \mu_\rho((qm_n)/2)\right)^{1/q}.$$

Moreover if F is lacunary and $0 < p < \infty$ then

$$\|F\|_{H(p,q,\rho)} \approx \left(\sum_{n=0}^\infty \left(\sum_{k\in J_n\cap\Lambda_{\mathcal{L}}} |a_k|^2\right)^{q/2} \mu_\rho((qm_n)/2)\right)^{1/q}.$$

Theorem 2.69. *Let $0 < q < \infty$, let $\mathcal{J}$ be a collection of disjoint intervals in $\mathbb{N}_0$ and assume ρ is a weight q-adapted to $\mathcal{J}$. Define $\lambda = (\lambda_k)_k$ such that*

$$\lambda_k = \left(\int_0^1 r^{qm_n}\frac{\rho(1-r)}{1-r}\right)^{1/q}, k \in J_n$$

and $\lambda_k = 0$ otherwise. Then $(\lambda_k)_k \in (H(1,q,\rho), \ell^{\mathcal{J}}(\infty,q))$.

Proof. We shall show that

$$\left(\sum_{n=0}^\infty (\sup_{k\in J_n} |a_k|)^q \mu_\rho(qm_n)\right)^{1/q} \leq C\|F\|_{H(1,q,\rho)}.$$

Recall that

$$\sup_{k\in J_n} |a_k| r^k \leq M_1(r,F)$$

and therefore, if $A_0 = [0, 1 - \frac{1}{m_1})$ and $A_n = [1 - \frac{1}{m_n}, 1 - \frac{1}{m_{n+1}})$ for $n \geq 1$ then

$$\begin{aligned}\sum_{n=0}^{\infty}(\sup_{k\in J_n}|a_k|)^q \mu_\rho(qm_n) &\leq C\sum_{n=0}^{\infty}(\sup_{k\in J_n}|a_k|)^q \int_{A_n} r^{qm_{n+1}}\frac{\rho(1-r)}{1-r}dr \\ &\leq C\sum_{n=0}^{\infty}\int_{A_n}(\sup_{k\in J_n}|a_k|r^k)^q\frac{\rho(1-r)}{1-r}dr \\ &\leq C\sum_{n=0}^{\infty}\int_{A_n}M_1^q(r,F)\frac{\rho(1-r)}{1-r}dr \\ &= C\|F\|^q_{H(1,q,\rho)}.\end{aligned}$$

□

THEOREM 2.70. *Let $1 \leq q_2 < q_1 \leq 2$ and let $\mathcal{J}$ and $\mathcal{I}$ be collections of disjoint intervals in $\mathbb{N}_0$, generated by sequences m_k and n_k respectively, such that $\mathcal{I} \leq \mathcal{J}$. Assume that ρ_1 is a weight $q_1/2$-adapted to $\mathcal{I}$ and ρ_2 is a weight $q_2/2$-adapted to $\mathcal{J}$. Denote*

$$\mu_{\rho_1,\rho_2}(k) = \Big((\mu_{\rho_2}((q_2m_k)/2))^{1/q_2}(\mu_{\rho_1}((q_1n_k)/2))^{-1/q_1}\Big)^{1/q_1\ominus q_2}$$

Then

$$(H(2,q_1,\rho_1), H(2,q_2,\rho_2)) = \{(\lambda_n)_n; (\sup_{k\in I_n}\mu_{\rho_1,\rho_2}(k)|\lambda_k|) \in \ell^{\mathcal{J}/\mathcal{I}}(\infty, q_1\ominus q_2)\}.$$

PROOF. Let

$$F_{\mathcal{I}}(z) = \sum_{k=0}^{\infty}(\mu_{\rho_1}(q_1n_k/2))^{1/q_1}\left(\sum_{j\in I_k} z^j\right),$$

$$\tilde{F}_{\mathcal{I}}(z) = \sum_{k=0}^{\infty}(\mu_{\rho_1}(q_1n_k/2))^{-1/q_1}\left(\sum_{j\in I_k} z^j\right),$$

and

$$G_{\mathcal{J}}(z) = \sum_{k=0}^{\infty}(\mu_{\rho_2}(q_2m_k/2))^{1/q_2}\left(\sum_{j\in J_k} z^j\right)$$

Using Corollary 2.68 one has that $f \in H(2,q_1,\rho_1)$ if and only if $f * F_{\mathcal{I}} \in \ell^{\mathcal{I}}(2,q_1)$ and $g \in H(2,q_2,\rho_2)$ if and only if $g * G_{\mathcal{J}} \in \ell^{\mathcal{J}}(2,q_2)$

We use that $\lambda \in (H(2,q_1,\rho_1), H(2,q_2,\rho_2))$ is equivalent to $\lambda * G_{\mathcal{J}} \in (H(2,q_1,\rho_1), \ell^{\mathcal{J}}(2,q_2))$ and also equivalent to $\lambda * G_{\mathcal{J}} * \tilde{F}_{\mathcal{I}} \in (\ell^{\mathcal{I}}(2,q_1), \ell^{\mathcal{J}}(2,q_2))$.

Making use of Theorem 2.54 we have

$$(\ell^{\mathcal{I}}(2,q_1), \ell^{\mathcal{J}}(2,q_2)) = \{(\gamma_n)_n; (\sup_{k\in I_n}|\gamma_k|)_n \in \ell^{\mathcal{J}/\mathcal{I}}(\infty, q_1\ominus q_2)\}.$$

This concludes the result. □

Let us finish by observing some examples to apply the above results.

EXAMPLE 2.71. Let $\lambda > 1$ and denote $m_0(\lambda) = 0$ and $m_k(\lambda) = [\lambda^k]$ for $k \in \mathbb{N}_0$ and $\mathcal{J}(\lambda)$ the partition of intervals $J_k(\lambda) = [m_k(\lambda), m_{k+1}(\lambda)) \cap \mathbb{N}_0$. In this case $\mu_{\rho_\alpha}(qm_n) \approx \lambda^{-\alpha n}$, and then, from Proposition 2.66, ρ_α is q-adapted to $\mathcal{J}(\lambda)$ for any value of $q > 0$.

Let $\lambda > \gamma > 1$ with $\lambda = \gamma^N$ with $N \in \mathbb{N}_0$. Then $\mathcal{J}(\gamma) \leq \mathcal{J}(\lambda)$ because

$$m_k(\lambda) = [\lambda^k] = [\gamma^{Nk}] = m_{Nk}(\gamma)$$

and therefore

$$J_k(\lambda) = \cup_{l \in F_k} J_l(\gamma)$$

where $F_k = \{l : Nk \leq l < Nk + N\}$. Hence $\mathcal{J}(\lambda)/\mathcal{J}(\gamma) = \mathcal{I}$ where $I_k = [Nk, N(k+1)) \cap \mathbb{N}_0$, that is $m_k(\mathcal{I}) = Nk$.

CHAPTER 3

New results on spaces of vector-valued analytic functions.

1. $\mathcal{H}(E)-$admissibility

Definition 3.1. A Banach space $X_E \subseteq \mathcal{S}(E)$ is called $\mathcal{H}(E)-$ *admissible* if

(i) $X_E \hookrightarrow \mathcal{H}(\mathbb{D}, E)$ with continuous inclusion

(ii) $\mathcal{H}(\mathrm{R}\mathbb{D}, E) \subseteq X_E$ $\forall R > 1$ and $f \mapsto f|_{\mathbb{D}}$ is continuous from $\mathcal{H}(\mathrm{R}\mathbb{D}, E)$ to X_E.

Remark 3.2. We already mentioned that $\mathcal{H}(\mathrm{R}\mathbb{D}, E) \subseteq \mathcal{S}(E)$ with continuous inclusion. Now note that the definition implies $\mathcal{P}(E) \hookrightarrow X_E$ and therefore $\mathcal{H}(E)-$admissibility implies $\mathcal{S}(E)-$admissibility.

Indeed, the only thing left to prove is $i_j : E \to X_E, i_j(x) = xe_j$ is continuous for every $j \in \mathbb{N}_0$. Thus, recall $e_j(z) = z^j$ and take $x \in E$. Obviously $xe_j \in \mathcal{H}(\mathrm{R}\mathbb{D}, E)$. By condition (ii), we know that there exists $0 < r < R$ such that

$$\|i_j(x)\|_{X_E} = \|xe_j\|_{X_E} \leq C \sup_{|z|=r} \|xz^j\|_{X_E} \leq Cr^j \|x\|_E.$$

Most of the vector-valued version of classical spaces are $\mathcal{H}(E)-$admissible.

Example 3.3. The spaces $A(\mathbb{D}, E)$, $BMOA(\mathbb{T}, E)$, $VMOA(\mathbb{T}, E)$, $\mathcal{B}loch(\mathbb{D}, E)$, $\mathcal{B}loch_0(\mathbb{D}, E)$ $H^p(\mathbb{D}, E)$ and $A^p(\mathbb{D}, E)$ (for $1 \leq p \leq \infty$) are $\mathcal{H}(E)-$admissible.

Indeed take $1 \leq p < \infty$, then

$$A(\mathbb{D}, E) \subset H^\infty(\mathbb{D}, E) \subset H^p(\mathbb{D}, E) \subset A^p(\mathbb{D}, E) \subset A^1(\mathbb{D}, E) \tag{3.1}$$

with continuous inclusions. The $\mathcal{H}(E)-$admissibility, follows from the facts that $A^1(\mathbb{D}, E) \hookrightarrow \mathcal{H}(\mathbb{D}, E)$ and $\mathcal{H}(\mathrm{R}\mathbb{D}, E) \subseteq A(\mathbb{D}, E)$ $\forall R > 1$ with continuous restriction.

In the case of $BMOA(\mathbb{T}, E)$ and $\mathcal{B}loch(\mathbb{D}, E)$ use that

$$H^\infty(\mathbb{D}, E) \subset BMOA(\mathbb{T}, E) \subset \mathcal{B}loch(\mathbb{D}, E)$$

and

$$A(\mathbb{D}, E) \subset VMOA(\mathbb{T}, E) \subset \mathcal{B}loch_0(\mathbb{D}, E)$$

For the $\mathcal{H}(E)-$admissibility, the only thing left to prove is the continuity of the inclusion $\mathcal{B}loch(\mathbb{D}, E) \subseteq \mathcal{H}(\mathbb{D}, E)$. Taking into account

$$f(z) - f(0) = \int_0^1 f'(tz)z\,dt$$

for f in any Banach space, we have that for $f \in \mathcal{B}loch(\mathbb{D}, E)$

$$\begin{aligned}\|f(z)\|_E &\leq \|f(0)\|_E + \int_0^1 \|f'(tz)\|_E |z| dt \\ &\leq \|f\|_{\mathcal{B}loch(\mathbb{D},E)} + \|f\|_{\mathcal{B}loch(\mathbb{D},E)} \int_0^1 \frac{|z|}{1-t|z|} dt \\ &= \|f\|_{\mathcal{B}loch(\mathbb{D},E)} \left(1 + \log\left(\frac{1}{1-|z|}\right)\right)\end{aligned}$$

Therefore $M_\infty(r, f) \leq C \log\left(\frac{1}{1-r}\right) \|f\|_{\mathcal{B}loch(\mathbb{D},E)}$.

We give now an easy way to generate new $\mathcal{H}(E)-$admissible spaces from other admissible spaces.

EXAMPLE 3.4. (*i*) For X $\mathcal{H}-$admissible, $X\hat{\otimes}_\pi E$, $X[E]$ and $X_{weak}(E)$ are $\mathcal{H}(E)-$admissible. In particular the spaces $\ell^p\hat{\otimes}_\pi E$, $\ell^p[E]$ and $\ell^p_{weak}(E)$ are $\mathcal{H}(E)-$admissible for $1 \le p \le \infty$. Also $A(\mathbb{D})\hat{\otimes}_\pi E$ and $A^1_{weak}(\mathbb{D}, E)$ are $\mathcal{H}(E)-$admissible and

$$A(\mathbb{D})\hat{\otimes}_\pi E \subset A(\mathbb{D}, E) \subset A^1(\mathbb{D}, E) \subset A^1_{weak}(\mathbb{D}, E)$$

(*ii*) The space $Rad(E)$ is $\mathcal{H}(E)-$admissible, since $\ell^1_{weak}(E) \subseteq Rad(E) \subseteq \ell^2_{weak}(E)$ with continuous inclusions.

(*iii*) For X_E $\mathcal{H}(E)-$admissible, the spaces X^S_E, X^K_E are $\mathcal{H}(E)-$ and $\mathcal{H}(E^*)-$admissible, respectively.

There are other ways to obtain $\mathcal{H}(E)-$admissible spaces, but we will need some new notions and tools first.

Recall the notation $i_j : E \to X_E,\ i_j(x) = xe_j$ and $\pi_j : X_E \to E,\ \pi_j(f) = \hat{f}(j)$.

PROPOSITION 3.5. *Let X_E be $\mathcal{H}(E)-$admissible. Then:*

(*i*) $C^E_X(z) = \sum_n i_n z^n \in \mathcal{H}(\mathbb{D}, \mathcal{L}(E, X_E))$

(*ii*) $C^X_E(z) = \sum_n \pi_n z^n \in \mathcal{H}(\mathbb{D}, \mathcal{L}(X_E, E)))$

(*iii*) *The mapping $f \mapsto F$ where $F(w) = f_w$ (recall that$f_w(z) = f(wz)$) defines a continuous inclusion $X_E \subseteq \mathcal{H}(\mathbb{D}, X^0_E)$*

PROOF. (*i*) Since X_E is $\mathcal{H}(E) - admissible$ we have that given $f \in X_E$, $\forall 0 < r < 1\ \exists A_r$ such that

$$M_\infty(r, f) \le A_r \|f\|_{X_E}$$

Concretely, for $f = xe_n$, where $x \in E$ we get that

$$M_\infty(r, xe_n) \le A_r \|xe_n\|_{X_E}$$

Equivalently

$$r^n \|x\|_E \le A_r \|xe_n\|_{X_E}$$

This implies $r^n \le A_r \|i_n\|$.

On the other hand, if $f \in \mathcal{H}(\mathbb{D}, E) \Rightarrow f_r \in \mathcal{H}(r^{-1}\mathbb{D}, E) \hookrightarrow X_E$ for any $0 < r < 1$ and it exists $\mathrm{R}_1 < r^{-1}$ verifying

$$\|f_r\|_{X_E} \le C_{\mathrm{R}_1} \sup_{|z| \le \mathrm{R}_1} \|f_r(z)\|_E \le B_r \sup_{|z| < r^{-1}} \|f_r(z)\|_E = B_r \sup_{|z|<1} \|f(z)\|_E$$

Again, for $f = xe_n$ we get $r^n \|xe_n\|_{X_E} \le B_r \|x\|_E$ and $\|i_n\| \le B_r r^{-n}$. From these estimates one deduces $\lim_{n\to\infty} \sqrt[n]{\|i_n\|} = 1$ Therefore (*i*) follows.

(*ii*) Let us prove that the series converges.

Observe that $1 = \|id_E\| = \|\pi_n \circ i_n\| \le \|\pi_n\| \|i_n\|$. Also

$$\begin{aligned}\|\pi_n\| &= \sup_{\|f\|_{X_E} \le 1} \|\pi_n(f)\|_E = \sup_{\|f\|_{X_E} \le 1} \|\hat{f}(n)\|_E \\ &= r^{-n} \sup_{\|f\|_{X_E} \le 1} \|r^n \hat{f}(n)\|_E = r^{-n} \sup_{\|f\|_{X_E} \le 1} \|\hat{f}_r(n)\|_E \\ &\le r^{-n} \sup_{\|f\|_{X_E} \le 1} M_\infty(r, f) \le r^{-n} \sup_{\|f\|_{X_E} \le 1} A_r \|f\|_{X_E} = r^{-n} A_r\end{aligned}$$

These two conditions together give us $\lim_{n\to\infty} \sqrt[n]{\|\pi_n\|} = 1$. Therefore (*ii*) follows.

(iii) The mapping is well defined as for $|w|<1$ the series $f_w = \sum_j^N \hat{f}(j) w^j e_j = \sum_j i_j(\hat{f}(j)) w^j$ is absolutely convergent in X_E (use the fact that $\lim_{n\to\infty} \sqrt[n]{\|i_n\|} = 1$). Hence $f_w = \lim_{N\to\infty} (f_w)_N \in X_E$. It is also clear that if $f \in X_E$, then $F(w) = f_w$ is holomorphic. We will only prove that the defined mapping gives us a continuous inclusion. Let $r \in (0,1)$:

$$\begin{aligned} M_\infty(r,F) &= \sup_{|z|=r} \|f_z\|_{X_E} = \sup_{|z|=r} \|\sum i_n(\pi_n(f)) z^n\|_{X_E} \\ &\leq \sup_{|z|=r} \sum \|i_n\| \|\pi_n(f)\|_E |z|^n \leq \|f\|_{X_E} \sum \|i_n\| \|\pi_n\| r^n \\ &= \|f\|_{X_E} C_r \end{aligned}$$

□

Remark 3.6. Let $0<r<1$. If X_E is $\mathcal{H}(E)-$admissible and $f \in X_E$, then $f_r \in X_E^0$. Indeed, if $f \in X_E$ and $0<r<1$, we have $f_r = \sum_j \hat{f}(j) r^j e_j = \sum_j i_j(\hat{f}(j)) r^j \in \mathcal{H}(r^{-1}\mathbb{D}, E) \hookrightarrow X_E$. Also, $\sum_j \|i_j(\hat{f}(j)) r^j\|_{X_E} = \sum_j \|i_j\| \|\hat{f}(j)\| r^j < \infty$ by the previous result. Thus, the series converges absolutely and $f_r = \lim_{N\to\infty} \sum_j^N \hat{f}(j) r^j e_j$.

Definition 3.7. Let X_E be a $\mathcal{H}(E)-$admissible space. Define

$$M_{X_E}(r,f) = \sup_{|w|=r} \|f_w\|_{X_E}$$

for $0<r<1$.

Proposition 3.8. *Let X_E be a $\mathcal{H}(E)-$admissible space. Then*

(i) *$M_{X_E}(r,f)$ is increasing*
(ii) *$M_\infty(r,f) \leq A_{X_E}(r)\|f\|_{X_E}$, where $A_{X_E}(r) = \|(C_E^X)_r\|_{\mathcal{C}(\mathbb{T},\mathcal{L}(X_E,E))}$ and $r \in (0,1)$.*
(iii) *If $r \in (0,1)$ and $f \in A(\mathbb{D},E)$, $M_{X_E}(r,f) \leq B_{X_E}(r)\|f\|_\infty$, where $B_{X_E}(r) = \|(C_X^E)_r\|_{L^1(\mathbb{T},\mathcal{L}(E,X_E))}$*

Proof.

(i) Since F is holomorphic, using Cauchy formula we have $w \mapsto \|F(w)\|_{X_E}$ is subharmonic. Then using the maximum modulus principle, for $s>r$ we can write

$$M_{X_E}(r,f) = \sup_{|w|=r} \|F(w)\|_{X_E} \leq \sup_{|w|=s} \|F(w)\|_{X_E} = M_{X_E}(s,f)$$

and the function is increasing.

(ii) Let $r \in (0,1)$. We can write

$$f_r(z) = \sum_n \hat{f}_n r^n z^n = (C_E^X)_r(z)[f]$$

Taking norms $\|f_r(z)\|_E \leq \|(C_E^X)_r(z)\|_{\mathcal{L}} \|f\|_{X_E}$ and since the series converges absolutely for every $z \in r^{-1}\mathbb{D}$, we have the result.

(iii) Write

$$f_w = \int_0^{2\pi} (C_X^E)_w(e^{i\theta})[f(e^{-i\theta})] \frac{d\theta}{2\pi}$$

Now for $|w| = r$

$$\begin{aligned} \|f_w\|_{X_E} &\le \int_0^{2\pi} \|(C_X^E)_w(e^{i\theta})[f(e^{-i\theta})]\|_{X_E}\frac{d\theta}{2\pi} \\ &\le \|f\|_\infty \int_0^{2\pi} \|(C_X^E)_w(e^{i\theta})\|_{\mathcal{L}}\frac{d\theta}{2\pi} \\ &= \|f\|_\infty \int_0^{2\pi} \|(C_X^E)_r(e^{i\theta})\|_{\mathcal{L}}\frac{d\theta}{2\pi} \end{aligned}$$

Taking the supremum we obtain the desired result.

□

Let $\nu : \mathbb{D} \mapsto [0,\infty)$ be a continuous weight. Define $H^\infty_\nu(\mathbb{D}, E)$ the subspace of $f \in \mathcal{H}(\mathbb{D}, E)$ such that $\sup_{z\in\mathbb{D}} \nu(z)\|f(z)\|_E < \infty$. Hence (ii) in the previous proposition shows the following fact.

Corollary 3.9. *Let X_E be $\mathcal{H}(E)-$admissible. Define $\nu^{-1}(z) = A_{X_E}(|z|)$. Then $X_E \subseteq H^\infty_\nu(\mathbb{D}, E)$ with continuous inclusion.*

Consider $f = \sum_n \hat{f}(n)e_n \in \mathcal{H}(\mathbb{D})$ and $g = \sum_n \hat{g}(n)e_n \in \mathcal{H}(\mathbb{D}, E)$. We recall the definition of the convolution product $f *_{B_0} g$

$$f *_{B_0} g = \sum_n \hat{f}(n)\hat{g}(n)e_n.$$

Lemma 3.10. *Let $X_E \subseteq \mathcal{H}(\mathbb{D}, E)$ be an $\mathcal{H}(E)-$admissible Banach space. If $f \in \mathcal{H}(\mathbb{D})$, $g \in \mathcal{H}(\mathbb{D}, E)$,then:*

$$M_{X_E}(rs, f *_{B_0} g) \le M_1(r, f)M_{X_E}(s, g)$$

Proof. Let $0 \le r, s < 1$, $|v| = r$ and $|w| = s$. Notice that

$$\begin{aligned} (f *_{B_0} g)_{vw} &\le \int_0^{2\pi} \Big(\sum_n \hat{f}(n)v^n e^{-in\theta}\hat{g}(n)w^n e^{in\theta}e_n\Big)\frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} \Big(\sum_n (\sum_j \hat{f}(j)v^j e^{-ij\theta})\hat{g}(n)w^n e^{in\theta}e_n\Big)\frac{d\theta}{2\pi} \end{aligned}$$

Hence

$$\begin{aligned} \|(f *_{B_0} g)_{vw}\| &\le \int_0^{2\pi} \Big(\sum_n |f(ve^{-i\theta})|\|g_{we^{i\theta}}\|_{X_E}\Big)\frac{d\theta}{2\pi} \\ &\le M_{X_E}(|w|, g)\int_0^{2\pi} |f(ve^{-i\theta})|\frac{d\theta}{2\pi} \\ &\le M_{X_E}(s, g)M_1(r, f) \end{aligned}$$

□

Lemma 3.11. *Let $X_E \subseteq \mathcal{H}(\mathbb{D}, E)$ be a $\mathcal{H}-$admissible Banach space and $f \in \mathcal{H}(\mathbb{D}, E)$, then*

$$M_{X_E}(rs, Df) \le \frac{1}{1-r^2}M_{X_E}(s, f) \tag{3.2}$$

$$M_{X_E}(r, f) \le \int_0^1 M_{X_E}(rs, Df)ds \tag{3.3}$$

PROOF. For the first inequality, recall that $De_n = (n+1)e_n$ and $Df = K *_{B_0} f$ where $K(z) = \frac{1}{(1-z)^2}$. Now use Lemma 3.10 to obtain the result.
For the second one simply use that, for each $0 \leq r < 1$ and $|\xi| = 1$, one has

$$rf_{r\xi} = \int_0^r (Df)_{s\xi} ds$$

as X_E-valued function. Hence, by Minkowski's inequality,

$$rM_{X_E}(r,f) \leq \int_0^r M_{X_E}(s, Df)ds = r\int_0^1 M_{X_E}(rs, Df)ds$$

□

We are now ready to define a new $\mathcal{H}(E)-$admissible space.

DEFINITION 3.12. If X_E is an $\mathcal{H}(E)-$admissible space, we define $\tilde{X}_E$ as the space of functions in $\mathcal{H}(\mathbb{D}, E)$ such that $w \mapsto f_w \in H^\infty(\mathbb{D}, X_E)$. We write

$$\|f\|_{\tilde{X}_E} = \sup_{0<r<1} M_{X_E}(r,f).$$

For instance

$$\widetilde{A(\mathbb{D},E)} = H^\infty(\mathbb{D},E), \widetilde{A^p(\mathbb{D},E)} = A^p(\mathbb{D},E),$$

$$\widetilde{H^p(\mathbb{D},E)} = H^p(\mathbb{D},E), \widetilde{\mathcal{B}loch(\mathbb{D},E)} = \mathcal{B}loch(\mathbb{D},E)$$

and

$$\widetilde{BMOA(\mathbb{T},E)} = BMOA(\mathbb{T},E).$$

Indeed for the first equality notice that

$$\sup_{0<r<1} M_{A(\mathbb{D},E)}(r,f) = \sup_{0<r<1}\sup_{|z|=r} \|f(z)\|_E = \|f\|_{H^\infty(\mathbb{D},E)}.$$

For Bergman and Hardy spaces we have

$$\sup_{0<r<1} M_{A^p(\mathbb{D},E)}(r,f) = \sup_{0<r<1}\left(\int_{r\mathbb{D}} \|f(z)\|_E^p\right)^p = \|f\|_{A^p(\mathbb{D},E)}$$

and

$$\sup_{0<r<1} M_{H^p(\mathbb{D},E)}(r,f) = \sup_{0<r<1} M_p(r,f) = \|f\|_{H^p(\mathbb{D},E)}$$

respectively, for the integral means are increasing functions of r.
The same result can be used for the case of $\mathcal{B}loch$ spaces. As $M_\infty(r, f')$ are increasing functions of r, we obtain

$$\begin{aligned}\sup_{0<r<1} M_{\mathcal{B}loch(\mathbb{D},E)}(r,f) &= \|f(0)\|_E + \sup_{0<r<1}\sup_{z\in\mathbb{D}}(1-|z|)\|f_r'(z)\|_E \\ &= \|f(0)\|_E + \sup_{z\in\mathbb{D}}(1-|z|)M_\infty(r,f') = \|f\|_{\mathcal{B}loch(\mathbb{D},E)}.\end{aligned}$$

Finally, consider $BMOA(\mathbb{T},E)$. We can embed $BMOA(\mathbb{T},E)$ isometrically in $H^1(\mathbb{T},E^*)^*$. Then

$$\sup_{0<r<1}\|f_r\|_{BMOA(\mathbb{T},E)} = \sup_{0<r<1}\sup_{\|g\|_{H^1(\mathbb{T},E^*)}=1}\lim_{s\to 1}\left|\int \langle f_{sr}, g\rangle\right|.$$

PROPOSITION 3.13. *Let X_E be $\mathcal{H}(E)-$admissible. Then:*

(i) $\tilde{X}_E$ is $\mathcal{H}(E)-$admissible.

(ii) $(\tilde{X}_E)^0 \subseteq X_E^0$ and $\tilde{X}_E = \widetilde{(X_E)^0} = \tilde{\tilde{X}}_E$.

PROOF.

(i) The fact that $\|\cdot\|_{\tilde{X}_E}$ is a complete norm is standard. Due to (i) in Proposition 3.8 one has that for $0 < r < 1$

$$M_{\tilde{X_E}}(r, f) = \|f_r\|_{\tilde{X}_E} = M_{X_E}(r, f).$$

From this one easily shows that $\tilde{X}_E$ is also $\mathcal{H}(E)-$admissible.

(ii) Note that for $0 < r < 1$

$$\|f_r\|_{\tilde{X}_E} = M_{X_E^0}(r, f) = M_{\tilde{X}_E}(r, f)$$

which gives that $\tilde{X}_E = \widetilde{(X_E)^0}$. On the other hand if $f \in \mathcal{P}(E)$ then

$$\|f\|_{X_E} = \lim_{r\to 1} \|f_r\|_{X_E} \le \sup_{0<r<1} M_{X_E}(r, f) = \|f\|_{\tilde{X}_E}$$

and we obtain the first inclusion.

□

Then

$$\widetilde{\mathcal{B}loch_0(\mathbb{D}, E)} = (\widetilde{\mathcal{B}loch(\mathbb{D}, E)})^0 = \mathcal{B}loch(\mathbb{D}, E)$$

and

$$\widetilde{VMOA(\mathbb{T}, E)} = (\widetilde{BMOA(\mathbb{T}, E)})^0 = BMOA(\mathbb{T}, E).$$

2. Homogeneous spaces and Fatou property

Definition 3.14. (**Homogeneous space**) Let X_E be $\mathcal{H}(E)-$admissible. We will say X_E is homogeneous if

(i) for $f \in X_E$ and $|\xi| = 1$ it is verified $\|f_\xi\|_{X_E} = \|f\|_{X_E}$, and

(ii) for $f \in X_E$ and $0 < r < 1$ it is verified $M_{X_E}(r, f) \le K\|f\|_{X_E}$ for some K independent of f, r.

Proposition 3.15. *Let X_E be a homogeneous Banach space.*

(i) *If $f \in X_E$ then $w \mapsto f_w \in H^\infty(\mathbb{D}, X_E^0)$.*

(ii) *If $f \in X_E^0$ then $w \mapsto f_w \in A(\mathbb{D}, X_E^0)$.*

Proof.

(i) Note that $\mathcal{H}(E)-$admissibility guarantees

$$F(w) = f_w \in \mathcal{H}(\mathbb{D}, X_E^0)$$

(see Proposition 3.5) and given that X_E is homogeneous,

$$M_{X_E}(r, f) = \sup_{|\xi|=1} \|f_{r\xi}\|_{X_E} = \|F_r\|_{H^\infty(\mathbb{D}, X_E)} < K\|f\|_{X_E}.$$

Hence $F \in H^\infty(\mathbb{D}, X_E)$.

(ii) It is clear that if $f \in X_E^0$ then $\lim_{r\to 1} \|f_r - f\|_{X_E} = 0$. Now use that $\|F - F_r\|_{H^\infty(\mathbb{D}, X_E)} = \|f - f_r\|_{X_E}$ (because $F_r \in A(\mathbb{D}, X_E)$ for each $0 < r < 1$) to conclude the result.

□

Remark 3.16. If $Y_E \subset X_E$ and X_E is homogeneous, then so it is Y_E. In particular X_E^0 is homogeneous for X_E homogeneous.

Proposition 3.17. *Let X_E be a $\mathcal{H}(E)-$admissible Banach space. Then the space $\tilde{X_E}$ is homogeneous.*

Proof. To show that $\tilde{X_E}$ is homogeneous use that $M_{X_E}(r,f)$ is increasing and the facts, for $|\xi|=1$ and $0<r,s<1$,

$$M_{X_E}(r,f_\xi)=M_{X_E}(r,f) \text{ and } M_{X_E}(s,f_r)=M_{X_E}(sr,f).$$

□

Note that X_E homogeneous implies $X_E \subseteq \tilde{X}_E$ with continuity. Now the previous proposition gives us the following.

Corollary 3.18. *Let X_E be an $\mathcal{H}(E)$-admissible space such that $X_E=\tilde{X}_E$ with equality of norms. Then X_E is homogeneous.*

Corollary 3.19. *The spaces $Bloch(\mathbb{D},E)$, $BMOA(\mathbb{T},E)$, $A^p(\mathbb{D},E)$, $H^p(\mathbb{D},E)$ and $H^\infty(\mathbb{D},E)$ ($1\le p<\infty$) are homogeneous.*

Proposition 3.20. *Let X_E be an $\mathcal{H}(E)-$admissible homogeneous space. Then X_E^K is also homogeneous.*

Proof. Let $f\in X_E^K$ and $|\xi|=1$. Then

$$\|f_\xi\|_{X_E^K}=\sup_{\|g\|_{X_E}=1}\sum_j|\langle \xi^j\hat{f}(j),\hat{g}(j)\rangle|=\sup_{\|g\|_{X_E}=1}\sum_j|\langle \hat{f}(j),\hat{g}(j)\rangle|=\|f\|_{X_E^K}.$$

Consider again $f\in X_E^K$ and take $0<r<1$. Then making use of the hypothesis

$$\begin{aligned} M_{X_E^K}(r,f) &= \sup_{|w|=r}\sup_{\|g\|_{X_E}=1}\sum_j|\langle w^j\hat{f}(j),\hat{g}(j)\rangle| \\ &= \sup_{|w|=r}\sup_{\|g\|_{X_E}=1}\sum_j|\langle \hat{f}(j),w^j\hat{g}(j)\rangle| \\ &\le K\|f\|_{X_E^K}\sup_{|w|=r}\sup_{\|g\|_{X_E}=1}\|g_w\|_{X_E}\le K'\|f\|_{X_E^K} \end{aligned}$$

□

Definition 3.21. (**Fatou property**) Let $X_E\subseteq\mathcal{H}(\mathbb{D},E)$ be a homogeneous Banach space. X_E is said to satisfy the Fatou property, to be denoted (FP), if there exists $A>0$ such that for any sequence $(f_n)_n\in X_E$ with $\sup_n\|f_n\|_{X_E}\le 1$ and $f_n\longrightarrow f$ in $\mathcal{H}(\mathbb{D},E)$ one has that $f\in X_E$ and $\|f\|_{X_E}\le A$.

Proposition 3.22. *Let X_E be $\mathcal{H}(E)-$admissible. Then $\tilde{X}_E$ has (FP).*

Proof. Let $(f_n)_n\subseteq\tilde{X}_E$ with $\|f_n\|_{\tilde{X}_E}\le 1$ and $f_n\longrightarrow f$. Using that

$$\lim_n M_{X_E}(r,f_n)=M_{X_E}(r,f)$$

one concludes $f\in\tilde{X}_E$ and $\|f\|_{\tilde{X}_E}\le 1$. □

Corollary 3.23. *The spaces $\mathcal{B}loch(\mathbb{D},E)$, $BMOA(\mathbb{T},E)$, $A^p(\mathbb{D},E)$, $H^p(\mathbb{D},E)$ and $H^\infty(\mathbb{D},E)$ have the (FP) ($1\le p<\infty$).*

Proposition 3.24. *Let X_E be homogeneous. TFAE:*

- *(i) X_E has (FP).*
- *(ii) If $f\in\mathcal{H}(\mathbb{D},E)$ and $\sup_{w\in\mathbb{D}}\|f_w\|_{X_E}<\infty$, then $f\in X_E$.*
- *(iii) $X_E=\tilde{X}_E$ with equivalent norms.*

Proof.

$(i)\to(ii)$ Take $f\in\mathcal{H}(\mathbb{D},E)$ with $0<\sup_{0\le r<1}M_{X_E}(r,f)=A<\infty$. Select a sequence r_n converging to 1 and define $f_n=A_nf_{r_n}$ where $A_n^{-1}=M_{X_E}(r_n,f)$. Then of course $f_n\longrightarrow A^{-1}f\in\mathcal{H}(\mathbb{D},E)$ and $\|f_n\|_{X_E}\le 1$. Applying the assumption, one gets that $f\in X_E$.

$(ii) \to (iii)$ The homogeneity of X_E gives us the inclusion $X_E \subset \tilde{X}_E$ with continuity. The assumption means that $\tilde{X}_E \subseteq X_E$. The continuity follows from the closed graph theorem.

$(iii) \to (i)$ Follows directly from Proposition 3.22.

□

COROLLARY 3.25. *In the same conditions of Proposition 3.24, if X_E^0 has (FP), then $X_E = X_E^0$.*

Then we can assure, for example, that $H^p(\mathbb{T}, E)$ doesn't have (FP). Otherwise it would be $H^p(\mathbb{T}, E) = H^p(\mathbb{D}, E)$. Also the spaces $\mathcal{B}loch_0(\mathbb{D}, E)$ and $VMOA(\mathbb{T}, E)$ do not have this property. Indeed $\mathcal{B}loch_0(\mathbb{D}, E) = (\mathcal{B}loch(\mathbb{D}, E))^0$ and $VMOA(\mathbb{T}, E) = (BMOA(\mathbb{T}, E))^0$, then is enough to consider a sequence in $\mathcal{B}loch_0(\mathbb{D}, E)$ and another one in $VMOA(\mathbb{T}, E)$ converging to an element in $\mathcal{B}loch(\mathbb{D}, E) \backslash \mathcal{B}loch_0(\mathbb{D}, E)$ and $BMOA(\mathbb{T}, E) \backslash VMOA(\mathbb{T}, E)$, respectively.

CHAPTER 4

Vector-valued multipliers associated to bilinear maps and B-Hadamard product

1. Vector-valued multipliers associated to bilinear maps

Our aim in this section is to generalize the notion of coefficient multipliers and Hadamard tensor product through continuous bilinear maps.

DEFINITION 4.1. ($B-$**convolution product**) Let $B : E_1 \times E_2 \longrightarrow E_3$ be a bounded bilinear map.
We define the $B-$convolution product as the continuous bilinear map $\mathcal{S}(E_1)\times\mathcal{S}(E_2) \to \mathcal{S}(E_3)$ given by $(\lambda, f) \to \lambda *_B f$ where

$$\widehat{\lambda *_B f}(j) = B(\hat{\lambda}(j), \hat{f}(j)), \quad j \geq 0.$$

Thus, given $f \in \mathcal{S}(E_2)$ and $\lambda \in S(E_1)$,

$$\lambda *_B f(z) = \sum_j B(\hat{\lambda}(j), \hat{f}(j))z^j.$$

REMARK 4.2. Notice that, if $\lambda \in A(\mathbb{D}, E)$ and $f \in A(\mathbb{D}, E_1)$, one can write

$$\lambda *_B f(z) = \int_0^{2\pi} B(\lambda(ze^{-i\theta}), f(e^{i\theta}))\frac{d\theta}{2\pi}.$$

We have already used the following bilinear maps:

- For $B_0 : E \times \mathbb{K} \longrightarrow E$, $(x, \alpha) \mapsto \alpha x$ we get
$$\lambda *_{B_0} f = (\lambda_j \hat{f}(j))_j.$$
- For $B_{\mathcal{D}} : E^* \times E \longrightarrow \mathbb{K}$, $(x^*, x) \mapsto \langle x^*, x\rangle$ we get
$$\lambda *_{\mathcal{D}} f = (\langle \lambda_j^*, \hat{f}(j)\rangle)_j.$$
- For $B_{\mathcal{L}} : \mathcal{L}(E_1, E_2) \times E_1 \longrightarrow E_2$, $(T, x) \mapsto T(x)$ we get
$$\lambda *_{\mathcal{L}} f = (\lambda_j(\hat{f}(j)))_j$$

Of course many maps could be used, but we will only mention two more.

- For $B_\pi : E_1 \times E_2 \longrightarrow E_1\hat{\otimes}_\pi E_2$, $(x, y) \mapsto x \otimes y$ we get
$$f *_\pi g = (\hat{f}(j) \otimes \hat{g}(j))_j$$
- For a Banach algebra $(A, .)$ and $P : A \times A \longrightarrow A$, $(a, b) \mapsto ab$ we get
$$\lambda *_P f = (\lambda_j \hat{f}(j))_j.$$

REMARK 4.3. Notice that Lemma 3.10 can be generalized now in terms of bilinear maps as follows:
Let $X_{E_i} \subseteq \mathcal{H}(\mathbb{D}, E_i)$ be an $\mathcal{H}(E_i)-$admissible Banach space, $i = 1, 2, 3$. If $f \in \mathcal{H}(\mathbb{D}, E_1)$, $g \in \mathcal{H}(\mathbb{D}, E_2)$ and $B : E_1 \times E_2 \longrightarrow E_3$ a bilinear function. Then:

$$M_{X_{E_3}}(rs, f *_B g) \leq \|B\|M_1(r, f)M_{X_{E_2}}(s, g).$$

Associated to a bilinear convolution we have the space of multipliers.

Definition 4.4. (**B-multipliers**) Let $B : E \times E_1 \longrightarrow E_2$ be a bounded bilinear map. Let X_{E_1} and X_{E_2} be Banach spaces. We define the multipliers space between X_{E_1} and X_{E_2} through the bilinear map B as

$$(X_{E_1}, X_{E_2})_B = \{\lambda \in \mathcal{S}(E) : \ \lambda *_B f \in X_{E_2} \ \forall f \in X_{E_1}\}$$

with the norm

$$\|\lambda\|_{(X_{E_1}, X_{E_2})_B} = \sup_{\|f\|_{X_{E_1}} \leq 1} \|\lambda *_B f\|_{X_{E_2}}.$$

In the particular case $E = \mathcal{L}(E_1, E_2)$ and $B = B_{\mathcal{L}}$ we are in the case of the operator-valued multipliers. For this case, we will keep on writing simply (X_{E_1}, X_{E_2}).

It is easy to prove that $\|\cdot\|_{(X_{E_1}, X_{E_2})_B}$ is a norm on $(X_{E_1}, X_{E_2})_B$ whenever B satisfies the condition

$$B(e, x) = 0 \ \forall x \in E_1 \Longrightarrow e = 0. \tag{4.1}$$

In other words, the mapping $E \to \mathcal{L}(E_1, E_2)$ given by $e \to T_e$ where $T_e(x) = B(e, x)$ is injective. The previous mappings satisfy this condition.

Theorem 4.5. *Let $B : E \times E_1 \longrightarrow E_2$ be a bounded bilinear map satisfying condition (4.1) and for which there exists $C > 0$ such that*

$$\|e\|_E \leq C \sup_{\|x\|_{E_1} = 1} \|B(e, x)\|_{E_2}, \quad e \in E. \tag{4.2}$$

If X_{E_1} and X_{E_2} are $\mathcal{S}(E_1)$, $\mathcal{S}(E_2)-$admissible Banach spaces respectively, then $(X_{E_1}, X_{E_2})_B$ is $\mathcal{S}(E)-$admissible.

Proof. We have proved the case where $E = \mathcal{L}(E_1, E_2)$ and $B = B_{\mathcal{L}}$ in Theorem 2.10. For the general case assumption (4.2) allows to use Remark 2.2 where the isomorphism is given by $e \in E \to T_e \in \mathcal{L}(E_1, E_2)$ where $T_e(x) = B(e, x)$ for each $e \in E$ and $x \in E_1$. Just note that

$$(X_{E_1}, X_{E_2})_B = \{(\hat{\lambda}(j))_j \in \mathcal{S}(E) : (T_{\hat{\lambda}(j)})_j \in (X_{E_1}, X_{E_2})\}.$$

□

Notice that condition (4.2) together with condition (4.1) say that we have $E \to \mathcal{L}(E_1, E_2)$ with equivalence of norms.
With this theorem we can recover some previous results on $\mathcal{S}(E)-$admissibility. For example, recall the definition of X_E^S, X_E^K and X_E^{KK} given in Definition 2.11 and Definition 2.5. For X_E an E-valued sequence Banach space, we can write the spaces as follows:

$$X_E^S = (\ell^\infty, X_E)_{B_0} = s(X_E),$$

$$X_E^K = (X_E, \ell^1)_{\mathcal{D}}$$

and

$$X_E^{KK} = (X_E^K, \ell^1)_{\mathcal{D}}.$$

It is easy to check condition (4.2) on B_0 and $B_{\mathcal{D}}$. Then Theorem 4.5 gives us the $\mathcal{S}(E)-$ and $\mathcal{S}(E^*)-$admissibility automatically.

We've seen so far how $\mathcal{S}(E)-$admissibility remains stable under the construction of the multipliers space through bilinear maps under certain conditions. This also works for the notion of $\mathcal{H}(E)-$admissibility.

THEOREM 4.6. *Let X_{E_1}, X_{E_2} be $\mathcal{H}(E_1)$ and $\mathcal{H}(E_2)-$admissible respectively. Consider $B : E \times E_1 \longrightarrow E_2$ such that there exists $C > 0$ s.t.*

$$\|e\|_E \leq C \sup_{\|x\|_{X_{E_1}}=1} \|B(e, x)\|_{E_2}, \ e \in E.$$

Then $(X_{E_1}, X_{E_2})_B$ is $\mathcal{H}(E)-$admissible.

PROOF. $S(\cdot)-$admissibility guarantees we are dealing with Banach spaces. Need only check the continuous inclusion conditions.

Let λ be in $(X_{E_1}, X_{E_2})_B$ and $r \in (0, 1)$.

$$\begin{aligned}
M_\infty(r^2, \lambda) &= \sup_{|z|=r^2} \|\lambda(z)\|_E \leq C \sup_{\|x\|_{E_1}\leq 1} \sup_{|z|=r^2} \|B(\lambda(z), x)\|_{E_2} \\
&= C \sup_{\|x\|_{E_1}\leq 1} \sup_{|z|=r} \|B(\lambda_r(z), x)\|_{E_2} \\
&\leq C \sup_{\|x\|_{E_1}\leq 1} \sup_{|z|=r} \|\lambda *_B (C^X_{E_1})_r[x](z)\|_{E_2} \\
&= \sup_{\|x\|_{E_1}\leq 1} M_\infty(r, \lambda *_B (C^X_{E_1}[x])_r) \\
&\leq A_{X_{E_2}}(r) \sup_{\|x\|_{E_1}\leq 1} \|\lambda *_B (C^X_{E_1}[x])_r\|_{X_{E_2}} \\
&\leq A_{X_{E_2}}(r)\|\lambda\|_{(X_{E_1},X_{E_2})_B} \sup_{\|x\|_{E_1}\leq 1} \|(C^X_{E_1}[x])_r\|_{X_{E_1}} \\
&\leq A_{X_{E_2}}(r)\|\lambda\|_{(X_{E_1},X_{E_2})_B} \sum_n \|i_n\| r^n
\end{aligned}$$

where we have used the previous theorem and the fact that $(C^X_E)_r[x] = \sum_n r^n x e_n \in X_{E_1}$.

On the other hand, let $\tilde{\lambda} \in \mathcal{H}(\mathbb{D}, E)$ and take $r \in (0, 1)$.

$$\begin{aligned}
\|\tilde{\lambda}_{r^2}\|_{(X_{E_1},X_{E_2})_B} &= \sup_{\|f\|_{X_{E_1}}\leq 1} \|(\tilde{\lambda} *_B f)_{r^2}\|_{X_{E_2}} \\
&\leq \sup_{\|f\|_{X_{E_1}}\leq 1} \sup_{|w|=r} \|((\tilde{\lambda} *_B f)_r)_w\|_{X_{E_2}} \\
&= \sup_{\|f\|_{X_{E_1}}\leq 1} M_{X_{E_2}}(r, (\tilde{\lambda} *_B f)_r) \\
&\leq B_{X_{E_2}}(r) \sup_{\|f\|_{X_{E_1}}\leq 1} M_\infty(r, \tilde{\lambda} *_B f) \\
&\leq B_{X_{E_2}}(r)\|\tilde{\lambda}\|_\infty \sup_{\|f\|_{X_{E_1}}\leq 1} M_\infty(r, f) \\
&\leq B_{X_{E_2}}(r) A_{X_{E_1}}(r)\|\tilde{\lambda}\|_\infty.
\end{aligned}$$

Therefore, if $\lambda \in \mathcal{H}(\mathrm{R}\mathbb{D}, E)$, we can write $\lambda = (\lambda_{1/r^2})_{r^2}$. Consider $\tilde{\lambda} = \lambda_{1/r^2}$ for $R > 1/r^2 > 1$. Then

$$\|\lambda\|_{(X_{E_1},X_{E_2})_B} \leq B_{X_{E_2}}(r) A_{X_{E_1}}(r)\|\lambda_{1/r^2}\|_\infty < \infty.$$

□

THEOREM 4.7. *Let $B : E \times E_1 \longrightarrow E_2$ be a bounded bilinear map satisfying (4.2). Define $B_* : E \times E_2^* \to E_1^*$ given by*

$$\langle B_*(e, y'), x\rangle = \langle y', B(e, x)\rangle, \quad e \in E, x \in E_1, y' \in E_2^*.$$

If $X_{E_1} \subseteq \mathcal{S}(E_1)$ and $X_{E_2} \subseteq \mathcal{S}(E_2)$ are $\mathcal{S}(E_1)$ and $\mathcal{S}(E_2)-$admissible spaces respectively and $X_{E_2} = X_{E_2}^{KK}$, then

$$(X_{E_1}, X_{E_2})_B = (X_{E_2}^K, X_{E_1}^K)_{B_*}$$

PROOF. From the definition we can write for $\lambda \in \mathcal{S}(E)$, $f \in \mathcal{S}(E_1)$, $g \in \mathcal{S}(E_2^*)$ and $j \geq 0$,

$$\langle \hat{g}(j), \widehat{\lambda *_B f}(j)\rangle = \langle \widehat{\lambda *_{B_*} g}(j), \hat{f}(j)\rangle.$$

Assume now that $\lambda \in (X_{E_1}, X_{E_2})_B$ and $g \in X_{E_2}^K$. We have

$$\begin{aligned}
\|\lambda *_{B_*} g\|_{X_{E_1}^K} &= \sup\Big\{\sum_j |\langle \widehat{\lambda *_{B_*} g}(j), \hat{f}(j)\rangle| : \|f\|_{X_{E_1}} \leq 1\Big\} \\
&= \sup\Big\{\sum_j |\langle \hat{g}(j), \widehat{\lambda *_B f}(j)\rangle| : \|f\|_{X_{E_1}} \leq 1\Big\} \\
&\leq \|g\|_{X_{E_2}^K} \sup\{\|(\lambda *_B f)\|_{X_{E_2}} : \|f\|_{X_{E_1}} \leq 1\} \\
&\leq \|\lambda\|_{(X_{E_1}, X_{E_2})_B} \|g\|_{X_{E_2}^K}.
\end{aligned}$$

Using the assumption $X_{E_2} = X_{E_2}^{KK}$ one can argue as above for $\lambda \in (X_{E_2}^K, X_{E_1}^K)_{B_*}$ and $f \in X_{E_1}$ to obtain

$$\begin{aligned}
\|\lambda *_B f\|_{X_{E_2}} &= \sup\Big\{\sum_j |\langle \hat{g}(j), \widehat{\lambda *_B f}(j)\rangle| : \|g\|_{X_{E_2}^K} \leq 1\Big\} \\
&= \sup\Big\{\sum_j |\langle \widehat{\lambda *_{B_*} g}(j), \hat{f}(j)\rangle| : \|g\|_{X_{E_2}^K} \leq 1\Big\} \\
&\leq \|f\|_{X_{E_1}} \sup\Big\{\|(\lambda *_{B_*} g)\|_{X_{E_1}^K} : \|g\|_{X_{E_2}^K} \leq 1\Big\} \\
&\leq \|\lambda\|_{(X_{E_2}^K, X_{E_1}^K)_{B_*}} \|f\|_{X_{E_1}}.
\end{aligned}$$

□

Let us see under which conditions this new space we have generated becomes a solid space, a homogeneous space or a space with (FP).
The following result has already been used in a weaker version ($B = B_0$) in Proposition 2.14. We give now a general version.

PROPOSITION 4.8. *If either $X_{E_1} \subset \mathcal{S}(E_1)$ or $X_{E_2} \subset \mathcal{S}(E_2)$ is an $\mathcal{S}(E_1)-$ or $\mathcal{S}(E_2)-$admissible solid space, then so it is $(X_{E_1}, X_{E_2})_B$.*

PROOF. Let $\alpha = (\hat{\alpha}(j))_j \in \ell^\infty$ and $\lambda \in (X_{E_1}, X_{E_2})_B$. Then $(\alpha *_{B_0} \lambda) *_B f = \alpha *_{B_0} (\lambda *_B f) = \lambda *_B (\alpha *_{B_0} f)$ for $f \in X_{E_1}$. So $\alpha *_{B_0} \lambda \in (X_{E_1}, X_{E_2})_B$ whenever X_{E_1} or X_{E_2} is solid. □

PROPOSITION 4.9. *Let X_{E_1} and X_{E_2} be $\mathcal{H}(E_1)-$ and $\mathcal{H}(E_2)-$admissible Banach spaces, respectively. Let B be a bilinear form defined as in Theorem 4.6. If X_{E_2} is homogeneous then $(X_{E_1}, X_{E_2})_B$ is homogeneous.*

PROOF. The vector-valued $\mathcal{H}$-admissibility has already been proved in Theorem 4.6. Given $\lambda \in (X_{E_1}, X_{E_2})_B$ and $f \in X_{E_1}$ one has that

$$\|\lambda\|_{(X_{E_1}, X_{E_2})_B} = \sup_{\|f\|_{X_{E_1}}=1} \|\lambda *_B f\|_{X_{E_2}} \text{ and } \lambda_w *_B f = (\lambda *_B f)_w$$

what trivially gives the result using the homogeneity of X_{E_2}. □

PROPOSITION 4.10. *Let X_{E_1} and X_{E_2} be $\mathcal{H}(E_1)$ and $\mathcal{H}(E_2)-$admissible Banach spaces respectively and B defined as in Theorem 4.6. If X_{E_2} is homogeneous with (FP), then $(X_{E_1}, X_{E_2})_B$ has the (FP).*

PROOF. Let $(\lambda_n)_n \subseteq (X_{E_1}, X_{E_2})_B$ such that $\|\lambda_n\|_{(X_{E_1},X_{E_2})_B} \leq 1$ and $\lambda_n \longrightarrow \lambda$ in $\mathcal{H}(\mathbb{D}, E)$. Hence for a given $f \in X_{E_1}$ with $\|f\|_{X_{E_1}} = 1$ we have $\lambda_n *_B f \in X_{E_2}$ with $\|\lambda_n *_B f\|_{X_{E_2}} \leq 1$ and $\lambda_n *_B f \longrightarrow \lambda *_B f$ in $\mathcal{H}(\mathbb{D}, E)$. Since X_{E_2} has (FP), $\lambda *_B f \in X_{E_2}$ and $\|\lambda *_B f\|_{X_{E_2}} \leq A$. Therefore $\lambda \in (X_{E_1}, X_{E_2})_B$ with $\|\lambda\|_{(X_{E_1},X_{E_2})_B} \leq A$. □

Let us see some other examples of multiplier spaces.

DEFINITION 4.11. Let X_E be $\mathcal{H}(E)-$admissible. Define

$$X_E^\sharp = \{f = (x_j^*)_j \in \mathcal{S}(E^*) : \sum_j \langle x_j^*, x_j\rangle e_j \in A(\mathbb{D}) \text{ for } (x_j)_j \in X_E\}$$

$$X_E^\star = \{f = (x_j^*)_j \in \mathcal{S}(E^*) : \sum_j \langle x_j^*, x_j\rangle e_j \in H^\infty(\mathbb{D}) \text{ for } (x_j)_j \in X_E\}$$

That is to say $X_E^\sharp = (X_E, A(\mathbb{D}))_\mathcal{D}$ and $X_E^\star = (X_E, H^\infty(\mathbb{D}))_\mathcal{D}$. In general, we will use the multipliers notation, as it makes easier to keep in mind where we are working. Note that from Proposition 2.16, if X is a solid space, $(X[E])^\sharp = X^\sharp[E^*]$ and $(X[E])^\star = X^\star[E^*]$.

COROLLARY 4.12. *The spaces $(X_E, A(\mathbb{D}))_\mathcal{D}$ and $(X_E, H^\infty(\mathbb{D}))_\mathcal{D}$ are $\mathcal{H}(E^*)-$admissible.*

It is clear that $(X_E, A(\mathbb{D}))_\mathcal{D} \subseteq (X_E, H^\infty(\mathbb{D}))_\mathcal{D}$. Notice that $(X_E, A(\mathbb{D}))_\mathcal{D} \subseteq X_E^*$ by means of $f \mapsto \lambda *_\mathcal{D} f(1)$. Therefore we have the following chain of continuous inclusions:

$$X_E^K \subseteq X_E^\sharp \subseteq X_E^*.$$

There are other inclusions that is worth studying.

PROPOSITION 4.13. *Let $X_E \subseteq \mathcal{H}(\mathbb{D}, E)$ be $\mathcal{H}(E)-$admissible. Then*

$$(X_E, A(\mathbb{D}))_\mathcal{D} \subseteq (X_E, H^\infty(\mathbb{D}))_\mathcal{D} \subseteq (X_E^0, A(\mathbb{D}))_\mathcal{D} \subseteq (\tilde{X}_E, H^\infty(\mathbb{D}))_\mathcal{D}$$

with continuous inclusions. In particular $(X_E^0, H^\infty(\mathbb{D}))_\mathcal{D} = (X_E^0, A(\mathbb{D}))_\mathcal{D}$.

PROOF. The first inclusion is immediate. For the second one note that $(H^\infty(\mathbb{D}))^0 = A(\mathbb{D})$ and that $(X, Y) \subseteq (X^0, Y^0)$. For the third one, let $g \in (X_E^0, A(\mathbb{D}))_\mathcal{D}$ and take $f \in \tilde{X}_E$. Then take $0 < s < 1$. By the definition of $\tilde{X}_E$, $f_{\sqrt{s}} \in X_E$. Now apply that $h_r \in X_E^0$ for any $h \in X_E$ to $h = (f_{\sqrt{s}})$ and $r = \sqrt{s}$ to get $f_s \in X_E^0$

$$\|\sum_n \langle \hat{g}(n), \hat{f}(n)\rangle s^n e_n\|_{A(\mathbb{D})} \leq C\|f_s\|_{X_E} \leq C\|f\|_{\tilde{X}_E}.$$

Therefore $g \in (X_E, H^\infty(\mathbb{D}))_\mathcal{D}$. □

Let us now give some information on the dual of homogeneous Banach spaces.

PROPOSITION 4.14. *Let $X_E \subseteq \mathcal{S}(E)$ be a homogeneous Banach space. Then $(X_E, A(\mathbb{D}))_\mathcal{D} \subseteq (X_E^0)^* \subseteq (X_E^0, A(\mathbb{D}))_\mathcal{D}$ with continuity.*

PROOF. For the first inclusion let $\lambda \in (X_E, A(\mathbb{D}))_\mathcal{D}$. Now define $\gamma_\lambda(g) = \sum_n \langle \hat{\lambda}(n), \hat{g}(n)\rangle$, for $g \in X_E^0$. Then $\gamma_\lambda \in (X_E^0)^*$ and $\|\gamma_\lambda\| \leq \|\lambda\|_\sharp$.

For the second one, given $\gamma \in (X_E^0)^*$define $\lambda_\gamma \in (X_E^0)^\sharp = (X_E^0, A(\mathbb{D}))_{\mathcal{D}}$ as follows: $\lambda_\gamma = \sum_n \hat{\lambda}(n)e_n$ where $\langle \hat{\lambda}(n), x\rangle = \gamma(xe_n)$ for any $x \in E$. Then given $f \in X_E^0$

$$\begin{aligned} \|\sum_n \langle \hat{\lambda}(n), \hat{f}(n)\rangle e_n\|_{A(\mathbb{D})} &= \sup_{|w|<1} \sum_n \gamma(\hat{f}(n)e_n)w^n \\ &= \sup_{|w|<1} |\gamma(f_w)| = \sup_{|w|<1} |\gamma(F(w))| \\ &\leq \|\gamma\| \|F\|_{A(\mathbb{D}, X_E^0)} \end{aligned}$$

where $F(w) = f_w$ and we have used Proposition 3.15. □

Corollary 4.15. *If $X_E \subseteq \mathcal{S}(E)$ is an homogeneous Banach space then*

$$(X_E, H^\infty(\mathbb{D}))_{\mathcal{D}} = (X_E^0, H^\infty(\mathbb{D}))_{\mathcal{D}} = (X_E^0, A(\mathbb{D}))_{\mathcal{D}} = (X_E^0)^*$$

with equivalent norms.

Proof. Since $X_E \subseteq \tilde{X}_E$, we have $\tilde{X}_E^\star \subseteq X_E^\star$. Then it follows from Proposition 4.13 (taking $\tilde{X}_E$ as X_E) and Proposition 3.11 ($\tilde{\tilde{X}}_E = \tilde{X}_E$) that $(\tilde{X}_E, H^\infty(\mathbb{D}))_{\mathcal{D}} = (X_E, H^\infty(\mathbb{D}))_{\mathcal{D}}$. Clearly, for X_E homogeneous,

$$(\tilde{X}_E, H^\infty(\mathbb{D}))_{\mathcal{D}} = (X_E^0, A(\mathbb{D}))_{\mathcal{D}} = (X_E^0)^\star.$$

Now X_E homogeneous implies X_E^0 homogeneous. Applying Proposition 4.13 we get $(X_E^0, A(\mathbb{D}))_{\mathcal{D}} = (X_E^0)^*$. □

Proposition 4.16. *Let $X_E \subseteq \mathcal{S}(E)$ be homogeneous and recall the notation $X_E^\star = (X_E, H^\infty(\mathbb{D}))_{\mathcal{D}}$. Then $X_E^0 \subseteq X_E^{\star\star}$ and there exists $K > 0$ such that*

$$\|f\|_{X_E^{\star\star}} \leq \|f\|_{X_E} \leq K\|f\|_{X_E^{\star\star}}, \;\; f \in X_E^0.$$

In particular X_E^0 is isomorphically contained in $(X_E^{\star\star})^0$.

Proof. The inclusion and first inequality are straightforward. Let now $f \in X_E^0$. From the previous results and Hahn-Banach theorem

$$\begin{aligned} \|f\|_{X_E} &\leq \sup\{|\gamma(f)| : \gamma \in (X_E^0)^*, \|\gamma\| = 1\} \\ &\leq K \sup\{|g *_{\mathcal{D}} f(1)| : g \in (X_E^0)^\sharp, \|g\|_\sharp \leq 1\} \\ &\leq K \sup\{\|g *_{\mathcal{D}} f\|_\infty : g \in (X_E^0)^\sharp, \|g\|_\sharp \leq 1\} \\ &= K \sup\{\|g *_{\mathcal{D}} f\|_\infty : g \in (X_E)^\star, \|g\|_\star \leq 1\} \\ &\leq K\|f\|_{\star\star} \end{aligned}$$

□

Again, recall the notation $X_E^\star = (X_E, H^\infty(\mathbb{D}))_{\mathcal{D}}$

Proposition 4.17. *Let $X_E \subseteq \mathcal{S}(E)$ be homogeneous. Then X_E has (FP) if and only if $X_E = \overline{X_E^0}^{X_E^{\star\star}}$.*

Proof. For the direct implication, recall $X_E^{\star\star} = (X_E^\star, H^\infty(\mathbb{D}))_{\mathcal{D}}$. Since $H^\infty(\mathbb{D})$ is homogeneous with (FP), Proposition 4.10 and the fact $\overline{X_E^0}^{X_E^{\star\star}}$ is a closed subspace of $X_E^{\star\star}$ give us the desired result. For the reverse direction it is enough to check $f \in \tilde{X}_E$ (see Corollary 3.24). Consider $f \in \overline{X_E^0}^{X_E^{\star\star}}$. Then $f_r \in X_E^0$. Hence using Proposition 4.16

$$M_{X_E}(r, f) \leq K M_{X_E^{\star\star}}(r, f) \leq K'\|f\|_{X_E^{\star\star}}.$$

This gives $f \in \tilde{X}_E$. □

2. $B-$Hadamard product

In Chapter 2, we mentioned that the space

$$X_{E_1} *_B X_{E_2} = \{f *_B g: \ f \in X_{E_1}, \ g \in X_{E_2}\}$$

was not necessarily a Banach space for $B = B_0$. But if we consider the space of infinite sums of these elements endowed with a proper norm, it becomes a Banach space.

DEFINITION 4.18. (**Hadamard projective tensor product**) Let $B : E_1 \times E_2 \longrightarrow E_3$ be a bounded bilinear map and let X_{E_1}, X_{E_2} be $\mathcal{S}(E_1)-$, $\mathcal{S}(E_2)-$admissible, respectively. We define the Hadamard projective tensor product $X_{E_1} \circledast_B X_{E_2}$ as the space of elements $h \in \mathcal{S}(E_3)$ that can be represented as

$$h = \sum_n f_n *_B g_n$$

where the convergence of $\sum_n f_n *_B g_n$ is considered in $\mathcal{S}(E_3)$, being $f_n \in X_{E_1}, g_n \in X_{E_2}$ and

$$\sum_n \|f_n\|_{X_{E_1}} \|g_n\|_{X_{E_2}} < \infty.$$

The particular case $E_3 = E_1 \hat{\otimes}_\pi E_2$ and $B_\pi : E_1 \times E_2 \to E_3$ will be simply denoted $X_{E_1} \circledast X_{E_2}$

PROPOSITION 4.19. *Let E_1, E_2 and E_3 be Banach spaces and let $B : E_1 \times E_2 \longrightarrow E_3$ be a bounded bilinear map. Let $h \in X_{E_1} \circledast_B X_{E_2}$ and define*

$$\|h\|_B = \inf \sum_n \|f_n\|_{X_{E_1}} \|g_n\|_{X_{E_2}}$$

where the infimum is taken over all possible representations of $h = \sum_n f_n *_B g_n$.

Then $(X_{E_1} \circledast_B X_{E_2}, \|\cdot\|_B)$ is a Banach space.

PROOF. Let $\|h\|_B = 0$ and $\epsilon > 0$. Thus there exists a representation $h(z) = \sum_n f_n *_B g_n(z)$ such that $\sum_n \|f_n\|_{X_{E_1}} \|g_n\|_{X_{E_2}} < \epsilon$. Since the series converges in $\mathcal{S}(E_3)$ we conclude that $\hat{h}(j) = \sum_n B(\hat{f}_n(j), \hat{g}_n(j))$. Using the admissibility of X_{E_1} and X_{E_2}

$$\begin{aligned}
\|\hat{h}(j)\|_{E_3} &\leq \sum_n \|B(\hat{f}_n(j), \hat{g}_n(j))\|_{E_3} \\
&\leq \|B\| \sum_n \|\hat{f}_n(j)\|_{E_1} \|\hat{g}_n(j)\|_{E_2} \\
&\leq \|B\| \|\pi_j\|^{X_{E_1}} \|\pi_j\|^{X_{E_2}} \sum_n \|\hat{f}_n\|_{X_{E_1}} \|\hat{g}_n\|_{X_{E_2}} < \epsilon.
\end{aligned}$$

Consequently $\hat{h}(j) = 0$ for all $j \geq 0$.
Of course $\|\alpha h\|_B = |\alpha| \|h\|_B$ for any $\alpha \in \mathbb{K}$ and $h \in X_{E_1} \circledast_B X_{E_2}$.

The triangle inequality follows using that if $h_1 \sim (f_n^1 *_B g_n^1)_n$ and $h_2 \sim (f_n^2 *_B g_n^2)_n$ such that

$$\sum_n \|f_n^i\|_{X_{E_1}} \|g_n^i\|_{X_{E_2}} < \|h_i\|_B + \frac{\epsilon}{2}, \ i = 1, 2.$$

Then $h_1 + h_2 = \sum_n f_n^1 *_B g_n^1 + \sum_m f_m^2 *_B g_m^2$ and

$$\|h_1 + h_2\|_B \leq \sum_n \|f_n^1\|_{X_{E_1}} \|g_n^1\|_{X_{E_2}} + \sum_m \|f_m^2\|_{X_{E_1}} \|g_m^2\|_{X_{E_2}} < \|h_1\|_B + \|h_2\|_B + \epsilon.$$

Finally, let us see that $X_{E_1} \circledast_B X_{E_2}$ is complete. Let $\sum_n h_n$ be an absolute convergent series in $X_{E_1} \circledast_B X_{E_2}$ with $h_n \in X_{E_1} \circledast_B X_{E_2}$. For each $n \in \mathbb{N}$ select a decomposition $h_n = \sum_k f_k^n *_B g_k^n$ such that

$$\sum_k \|f_k^n\|_{X_{E_1}} \|g_k^n\|_{X_{E_2}} < 2\|h_n\|_B.$$

Let us now show that $\sum_n h_n = \sum_n \sum_k f_k^n *_B g_k^n$ in $\mathcal{S}(E_3)$. Indeed, for each $j \geq 0$ we have

$$\begin{aligned}\sum_n \sum_k \|B(\widehat{f_k^n}(j), \widehat{g_k^n}(j))\|_{E_3} &\leq \|B\| \|\pi_j\|^{X_{E_1}} \|\pi_j\|^{X_{E_2}} \sum_n \sum_k \|f_k^n\|_{X_{E_1}} \|g_k^n\|_{X_{E_2}} \\ &< 2\|B\| \|\pi_j\|^{X_{E_1}} \|\pi_j\|^{X_{E_2}} \sum_n \|h_n\|_B\end{aligned}$$

and since E_3 is complete we get the result.

Moreover $h = \sum_n h_n \in X_{E_1} \circledast_B X_{E_2}$ because $\sum_n \sum_k \|f_k^n\|_{X_{E_1}} \|g_k^n\|_{X_{E_2}} < \infty$. Now use that

$$\| \sum_{n=N}^{\infty} h_n \|_B \leq \sum_{n=N}^{\infty} \sum_k^{\infty} \|f_k^n\|_{X_{E_1}} \|g_k^n\|_{X_{E_2}} < 2 \sum_{n=N}^{\infty} \|h_n\|_B$$

to conclude that the series $\sum_n h_n$ converges to h in $X_{E_1} \circledast_B X_{E_2}$. □

Remark 4.20. If $h = \sum_n f_n *_B g_n \in X_{E_1} \circledast_B X_{E_2}$ then $\sum_n \|f_n *_B g_n\|_B < \infty$ and $h = \sum_n f_n *_B g_n$ converges in $X_{E_1} \circledast_B X_{E_2}$.

Indeed, simply use that

$$\|f *_B g\|_B \leq \|f\|_{X_{E_1}} \|g\|_{X_{E_2}}$$

for $f \in X_{E_1}$ and $g \in X_{E_2}$ and that for $M > N$

$$\| \sum_{n=N}^{M} f_n *_B g_n \|_B \leq \sum_{n=N}^{M} \|f_n *_B g_n\|_B \leq \sum_{n=N}^{M} \|f_n\|_{X_{E_1}} \|g_n\|_{X_{E_2}}.$$

Theorem 4.21. *Let E_1, E_2 and E be Banach spaces and let $B : E_1 \times E_2 \longrightarrow E$ be a bounded bilinear map satisfying that there exists $C > 0$ such that for each $e \in E$ there exists $(x_n, y_n) \in E_1 \times E_2$ such that*

$$e = \sum_n B(x_n, y_n), \quad \sum_n \|x_n\|_{E_1} \|y_n\|_{E_2} \leq C\|e\|_E. \tag{4.3}$$

If X_{E_1} and X_{E_2} are admissible spaces, then $X_{E_1} \circledast_B X_{E_2}$ is $\mathcal{S}(E)-$admissible.

In particular $X_{E_1} \circledast X_{E_2}$ is $\mathcal{S}(E_1 \hat{\otimes}_\pi E_2)-$admissible.

Proof. We show first that $X_{E_1} \circledast_B X_{E_2} \subset \mathcal{S}(E)$ with continuity. For $\epsilon > 0$ we can find a representation $h = \sum_n f_n *_B g_n$ such that $\sum_n \|f_n\|_{X_{E_1}} \|g_n\|_{X_{E_2}} < \|h\|_B + \epsilon$. Therefore, for each $j \geq 0$,

$$\begin{aligned}\|\hat{h}(j)\|_E &\leq \sum_n \|B(\hat{f}_n(j), \hat{g}_n(j))\|_E \\ &\leq \|B\| \sum_n \|\hat{f}_n(j)\|_{E_1} \|\hat{g}_n(j)\|_{E_2} \\ &\leq \|B\| \|\pi_j\|^{X_{E_1}} \|\pi_j\|^{X_{E_2}} \sum_n \|\hat{f}_n\|_{X_{E_1}} \|\hat{g}_n\|_{X_{E_2}} \leq C_j \|h\|_B + \epsilon.\end{aligned}$$

To show that $\mathcal{P}(E) \subset X_{E_1} \circledast_B X_{E_2}$, it suffices to see that $ee_j \in X_{E_1} \circledast_B X_{E_2}$ for each $j \geq 0$ and $e \in E$. Now use condition (4.3) to write $e = \sum_n B(x_n, y_n) \in E$ and therefore

$$ee_j = \sum_n (x_n e_j) *_B (y_n e_j)$$

and

$$\sum_n \|x_n e_j\|_{X_{E_1}} \|y_n e_j\|_{X_{E_2}} \leq \|i_j\|^{X_{E_1}} \|i_j\|^{X_{E_2}} \sum_n \|x_n\|_{E_1} \|y_n\|_{E_2} \leq C_j \|e\|_E. \tag{4.4}$$

Hence $ee_j \in X_{E_1} \circledast_B X_{E_2}$ and $\|ee_j\|_B \leq C\|i_j\|^{X_{E_1}} \|i_j\|^{X_{E_2}} \|e\|_E$. □

REMARK 4.22. If E_1, E_2 and E are Banach spaces and $B : E_1 \times E_2 \longrightarrow E$ is a surjective bounded bilinear map such that there exists $C > 0$ s.t. for every $e \in E$ there exists $(x, y) \in E_1 \times E_2$ verifying

$$e = B(x, y), \quad \|x\|_{E_1} \|y\|_{E_2} \leq C\|e\|_E \tag{4.5}$$

then we can apply Theorem 4.21.

A simple application of (4.5) gives the following cases.

COROLLARY 4.23. *(i) If X and X_E are $\mathcal{S}$ and $\mathcal{S}(E)$-admissible spaces respectively, then $X \circledast_{B_0} X_E$ is $\mathcal{S}(E)-$admissible.*

(ii) Let $(\mathbb{D}, \Sigma, \mu)$ be a measure space, $1 \leq p_i \leq \infty$ for $i = 1, 2, 3$ and $1/p_3 = 1/p_1 + 1/p_2$. Let $B : L^{p_1}(\mu) \times L^{p_2}(\mu) \to L^{p_3}(\mu)$ be given by $(f, g) \to fg$. If $X_{L^{p_1}}$ and $X_{L^{p_2}}$ are admissible spaces then $X_{L^{p_1}(\mu)} \circledast_B X_{L^{p_2}(\mu)}$ is admissible.

(iii) Let A be a Banach algebra with identity and $P : A \times A \to A$ given by $(a, b) \to ab$. If X_A and Y_A are admissible spaces then $X_A \circledast_P Y_A$ is admissible.

Recall the concept of minimal space, that is, a space such that $X_E^0 = X_E$. The new space we have built preserves minimality.

PROPOSITION 4.24. *Let E_1, E_2 and E be Banach spaces and let $B : E_1 \times E_2 \longrightarrow E$ be a bounded bilinear map satisfying (4.3). Let X_{E_1}, X_{E_2} be $\mathcal{S}(E_1)$ and $\mathcal{S}(E_2)$-admissible Banach spaces respectively, such that either X_{E_1} or X_{E_2} are minimal spaces, then $X_{E_1} \circledast_B X_{E_2}$ is a minimal $\mathcal{S}(E)$-admissible space.*

PROOF. We shall prove the case $X_{E_1}^0 = X_{E_1}$. Let $h \in X_{E_1} \circledast_B X_{E_2}$. From Remark 4.20, there exist $f_n \in X_{E_1}$, $g_n \in X_{E_2}$ and $N \in \mathbb{N}$ such that

$$\|h - \sum_{n=0}^{N} f_n *_B g_n\|_B < \frac{\epsilon}{2}.$$

By density choose polynomials p_n with coefficients in E_1 such that

$$\|f_n - p_n\|_{X_{E_1}} \leq \frac{\epsilon}{2(N+1)\|g_n\|_{X_{E_2}}}$$

Then $\sum_{n=0}^{N} p_n *_B g_n \in \mathcal{P}(E)$ and

$$\begin{aligned} \|h - \sum_{n=0}^{N} p_n *_B g_n\|_B &\leq \|h - \sum_{n=0}^{N} f_n *_B g_n\|_B + \|\sum_{n=0}^{N} (f_n - p_n) *_B g_n\|_B \\ &\leq \frac{\epsilon}{2} + \sum_{n=0}^{N} \|f_n - p_n\|_{X_{E_1}} \|g_n\|_{X_{E_2}} \leq \frac{\epsilon}{2} + \sum_{n=0}^{N} \frac{\epsilon}{2(N+1)} = \epsilon \end{aligned}$$

□

With the same conditions on the bilinear map, the $\mathcal{H}(E)-$admissibility also remains stable under $B-$Hadamard products.

THEOREM 4.25. *Let X_{E_1}, X_{E_2} be $\mathcal{H}(E_1)-$ and $\mathcal{H}(E_2)-$admissible, respectively. Let $B : E_1 \times E_2 \longrightarrow E$ be a bilinear map verifying (4.3). Then $X_{E_1} \circledast_B X_{E_2}$ is $\mathcal{H}(E)-$admissible.*

Proof. We have already checked the Banach space condition in Proposition 4.19. Need only check the continuous inclusion conditions. Take $h = \sum f_n *_B g_n \in X_{E_1} \circledast_B X_{E_2}$ such that $\|h\|_B = \sum_n \|f_n\|_{X_{E_1}} \|g_n\|_{X_{E_2}} - \epsilon$. For $r \in (0,1)$

$$\begin{aligned} M_\infty(r^2, h) &\le \|B\| \sum_n M_\infty(r, f_n) M_\infty(r, g_n) \\ &\le \|B\| A_{X_{E_1}}(r) A_{X_{E_2}}(r) \sum_n \|f_n\|_{X_{E_1}} \|g_n\|_{X_{E_2}} \\ &= \|B\| A_{X_{E_1}}(r) A_{X_{E_2}}(r)(\|h\|_B + \epsilon) \end{aligned}$$

Also let $h(z) = \sum_j \hat{h}(j) z^j \in \mathcal{H}(R\mathbb{D}, E)$ with $\hat{h}(j) = \sum_n B(\hat{f}_n(j), \hat{g}_n(j))$ for some $f_n \in X_{E_1}$, $g_n \in X_{E_2}$ verifying $\sum_n \|\hat{f}_n(j)\|_{E_1} \|\hat{g}_n(j)\|_{E_2} < C\|\hat{h}(j)\|_E$. Then arguing as in (4.4) and using the $\mathcal{S}(E_1)$ and $\mathcal{S}(E_2)$-admissibility of X_{E_1}, X_{E_2} respectively, we obtain

$$\begin{aligned} \|h\|_B &= \| \sum_j \sum_n (\hat{f}_n(j) e_j) *_B (\hat{g}_n(j) e_j) \|_B \\ &\le \sum_j \|\hat{h}(j) e_j\|_B \\ &\le \sum_j \sum_n \|\hat{f}_n(j) e_j\|_{X_{E_1}} \|\hat{g}_n(j) e_j\|_{X_{E_2}} \\ &\le \sum_j \|i_j\|^{X_{E_1}} \|i_j\|^{X_{E_2}} \sum_n \|\hat{f}_n(j)\|_{E_1} \|\hat{g}_n(j)\|_{E_2} \\ &\le C \sum_j \|i_j\|^{X_{E_1}} \|i_j\|^{X_{E_2}} \|\hat{h}(j)\|_E \\ &\le C \|h_S\|_\infty \sum_j \|i_j\|^{X_{E_1}} \|i_j\|^{X_{E_2}} S^{-j} \end{aligned}$$

for any $S \in (1, R)$. □

The $B-$Hadamard product can help us determine some spaces of multipliers and vice-versa. Let us see how this two concepts are related.

Theorem 4.26. *Let $X_{E_1}, X_{E_2}, X_{E_3}$ be $\mathcal{S}(E_1)-$, $\mathcal{S}(E_2)-$ and $\mathcal{S}(E_3)-$admissible Banach spaces, respectively. Then*

$$(X_{E_1} \circledast X_{E_2}, X_{E_3}) = (X_{E_1}, (X_{E_2}, X_{E_3}))$$

Proof. Due to the identification between $\mathcal{L}(E_1 \hat{\otimes}_\pi E_2, E_3)$ and $\mathcal{L}(E_1, \mathcal{L}(E_2, E_3))$ where the correspondence is given by $\phi(x \otimes y) = T_\phi(x)(y)$ we obtain, in our case, that each $\lambda \in \mathcal{S}(\mathcal{L}(E_1 \hat{\otimes}_\pi E_2, E_3))$ can be identified with $\tilde{\lambda} \in \mathcal{S}(\mathcal{L}(E_1, \mathcal{L}(E_2, E_3))$ satisfying

$$\hat{\lambda}(j)(\hat{f}(j) \otimes \hat{g}(j)) = \hat{\tilde{\lambda}}(j)(\hat{f}(j))(\hat{g}(j)).$$

Let $\lambda \in (X_{E_1} \circledast X_{E_2}, X_{E_3})$. For each $f \in X_{E_1}$ and $g \in X_{E_2}$ we have

$$\lambda *_1 (f *_\pi g) = (\tilde{\lambda} *_2 f) *_3 g \tag{4.6}$$

where $*_1$ is used for multipliers in $\mathcal{S}(\mathcal{L}(E_1 \hat{\otimes}_\pi E_2), E_3)$, $*_2$ for multipliers in $\mathcal{S}(\mathcal{L}(E_1, \mathcal{L}(E_2, E_3)))$ and $*_3$ for multipliers in $\mathcal{S}(\mathcal{L}(E_2, E_3))$.

Let us now show that $\tilde{\lambda} \in (X_{E_1}, (X_{E_2}, X_{E_3}))$.
We use (4.6) to get

$$\|(\tilde{\lambda} *_2 f) *_3 g\|_{X_{E_3}} \le \|\lambda\|_{(X_{E_1} \circledast X_{E_2}, X_{E_3})} \|(f *_\pi g)\| = \|\lambda\|_{(X_{E_1} \circledast X_{E_2}, X_{E_3})} \|f\|_{X_{E_1}} \|g\|_{X_{E_2}}.$$

Therefore $\|\tilde{\lambda}\|_{(X_{E_1},(X_{E_2},X_{E_3}))} \leq \|\lambda\|_{(X_{E_1}\circledast X_{E_2},X_{E_3})}$.
For the converse, take $\tilde{\lambda} \in (X_{E_1},(X_{E_2},X_{E_3}))$ and $h \in X_{E_1} \circledast X_{E_2}$. Assume that $h = \sum_n f_n *_\pi g_n$ with $\sum_n \|f_n\|_{X_{E_1}}\|g_n\|_{X_{E_2}} < \infty$. Hence

$$\begin{aligned}\|\lambda *_1 h\|_{X_{E_3}} &\leq \sum_n \|\lambda *_1 (f_n *_\pi g_n)\|_{X_{E_3}} \\ &= \sum_n \|(\tilde{\lambda} *_2 f_n)\|_{(X_{E_2},X_{E_3})}\|g_n\|_{X_{E_2}} \\ &\leq \sum_n \|\tilde{\lambda}\|_{(X_{E_1},(X_{E_2},X_{E_3}))}\|f_n\|_{X_{E_1}}\|g_n\|_{X_{E_2}} \\ &\leq \|\tilde{\lambda}\|_{(X_{E_1},(X_{E_2},X_{E_3}))}\sum_n \|f_n\|_{X_{E_1}}\|g_n\|_{X_{E_2}},\end{aligned}$$

which gives $\|\lambda\|_{(X_{E_1}\circledast X_{E_2},X_{E_3})} \leq \|\tilde{\lambda}\|_{(X_{E_1},(X_{E_2},X_{E_3}))}$. □

PROPOSITION 4.27. *Let $B : E_1 \times E_2 \to E$ be a bounded bilinear map satisfying (4.3). Denote $B^* : E^* \times E_1 \to E_2^*$ the bounded bilinear map defined by*

$$\langle B^*(e^*,x),y\rangle = \langle e^*,B(x,y)\rangle, \quad x \in E_1, y \in E_2, e^* \in E^*.$$

If X_{E_1} and X_{E_2} are $\mathcal{S}(E_1)-$ and $\mathcal{S}(E_2)-$admissible respectively, then

$$(X_{E_1} \circledast_B X_{E_2})^K = (X_{E_1}, X_{E_2}^K)_{B^*} \text{ and}$$

$$(X_{E_1} \circledast_B X_{E_2})^* = (X_{E_1}, X_{E_2}^*)_{B^*}.$$

In particular $(X_{E_1} \circledast X_{E_2})^* = (X_{E_1}, X_{E_2}^*)$ and $(X_{E_1} \circledast X_{E_2})^K = (X_{E_1}, X_{E_2}^K)$.

PROOF. Let $\lambda \in (X_{E_1}, X_{E_2}^K)_{B^*}$ and define, for $f \in X_{E_1}$ and $g \in X_{E_2}$,

$$\tilde{\lambda}(f *_B g)\hat{}(j) = \langle(\lambda *_{B^*} f)\hat{}(j), \hat{g}(j)\rangle, j \geq 0.$$

Let us see that $\tilde{\lambda} \in (X_{E_1} \circledast_B X_{E_2})^K$.

$$\begin{aligned}\sum_j |\tilde{\lambda}(f *_B g)\hat{}(j)| &= \sum_j |\langle(\lambda *_{B^*} f)\hat{}(j), \hat{g}(j)\rangle| \\ &\leq \|\lambda *_{B^*} f\|_{X_{E_2}^K}\|g\|_{X_{E_2}} \\ &\leq \|\lambda\|_{(X_{E_1},X_{E_2}^K)_{B^*}}\|f\|_{X_{E_1}}\|g\|_{X_{E_2}}.\end{aligned}$$

By linearity we can extend the result to finite linear combinations of $*_B-$products and, by continuity, to $X_{E_1} \circledast_B X_{E_2}$, that is

$$\tilde{\lambda}(h) = \sum_n \tilde{\lambda}(f_n *_B g_n)$$

whenever $h = \sum_n f_n *_B g_n$ and $\sum_n \|f_n *_B g_n\|_B \leq \infty$. Therefore we conclude $(X_{E_1}, X_{E_2}^K)_{B^*} \subseteq (X_{E_1} \circledast_B X_{E_2})^K$.
For the other inclusion, consider $\gamma \in (X_{E_1} \circledast_B X_{E_2})^K$ and define $\tilde{\gamma}(f)\hat{}(j) \in E_2^*$ by

$$\langle\tilde{\gamma}(f)\hat{}(j), y\rangle = \gamma(f *_B ye_j)\hat{}(j), \quad f \in X_{E_1}, y \in E_2, \quad j \geq 0.$$

This gives

$$\langle\tilde{\gamma}(f)\hat{}(j), \hat{g}(j)\rangle = \gamma(f *_B g)\hat{}(j), f \in X_{E_1}, g \in X_{E_2}, \quad j \geq 0.$$

Let us see that $\tilde{\gamma} \in (X_{E_1}, X_{E_2}^K)_{B^*}$:

$$\begin{aligned}\|\tilde{\gamma}(f)\|_{X_{E_2}^K} &= \sup_{\|g\|_{X_{E_2}}=1} \sum_j |\gamma(f *_B g)\hat{}(j)| \\ &\leq \|\gamma\|_{(X_{E_1} \circledast_B X_{E_2})^K} \sup_{\|g\|_{X_{E_2}}=1} \|f *_B g\|_B \\ &\leq \|\gamma\|_{(X_{E_1} \circledast_B X_{E_2})^K} \|f\|_{X_{E_1}}.\end{aligned}$$

The argument to study the dual is similar: Let $\lambda \in (X_{E_1}, X_{E_2}^*)_{B^*}$ and define $\phi_\lambda(f *_B g) = \langle \lambda *_{B^*} f, g\rangle$. Note that $X_{E_2}^*$ is also $\mathcal{S}(E_2^*)-$admissible and

$$|\phi_\lambda(f *_B g)| \leq \|\lambda\|_{(X_{E_1}, X_{E_2}^*)_{B^*}} \|f\|_{X_{E_1}} \|g\|_{X_{E_2}}.$$

By linearity we can extend the result to finite linear combinations of $*_B-$products and then extend by continuity to $X_{E_1} \circledast_B X_{E_2}$, that is

$$\phi_\lambda(h) = \sum_n \phi_\lambda(f_n *_B g_n)$$

whenever $h = \sum_n f_n *_B g_n$ and $\sum_n \|f_n *_B g_n\|_B \leq \infty$. Therefore we conclude $(X_{E_1}, X_{E_2}^*)_{B^*} \subseteq (X_{E_1} \circledast_B X_{E_2})^*$.
For the other inclusion, consider $T \in (X_{E_1} \circledast_B X_{E_2})^*$ and define

$$\lambda_T(f)(g) = T(f *_B g).$$

Then

$$\|\lambda_T(f)\|_{X_{E_2}^*} = \sup_{\|g\|_{X_{E_2}}=1} |\lambda_T(f)(g)| \leq \sup_{\|g\|_{X_{E_2}}=1} \|T\|\|f *_B g\|_B \leq \|T\|\|f\|_{X_{E_1}}.$$

□

Let us see what happens with solid spaces, homogeneity and (FP) in our new spaces.

We first give a characterization of $\mathcal{S}(E)-$solid spaces in terms of Hadamard tensor products.

Remark 4.28. It is straightforward to see that, under the assumptions of Theorem 4.21, if either X_{E_1} or X_{E_2} are solid spaces then $X_{E_1} \circledast_B X_{E_2}$ is a $\mathcal{S}(E)-$solid space.

Consider now the set $X *_{B_0} Y_E = \{f *_{B_0} g : \ f \in X \text{ and } g \in Y_E\}$.

Proposition 4.29. *Let $X_E \subseteq \mathcal{S}(E)$. Then there is a smallest solid superset of X_E, $X_E \subset S(X_E)$. Moreover $S(X_E) = \ell^\infty *_{B_0} X_E = \{g \in \mathcal{S}(E) : \exists f \in X \text{ such that } \|\hat{f}(j)\|_E \geq \|\hat{g}(j)\|_E\}$.*

Proof. Clearly $S(X_E)$ is the intersection of all solid sets containing X_E and since $\ell^\infty *_{B_0} X_E$ is solid, we have $S(X_E) \subseteq \ell^\infty *_{B_0} X_E$. On the other hand, by definition $X_E \subset S(X_E)$, therefore

$$\ell^\infty *_{B_0} X_E \subseteq \ell^\infty *_{B_0} S(X_E) = S(X_E),$$

as $S(X_E)$ is solid.

For the second equality, name $A = \{g \in \mathcal{S}(E) : \exists f \in X_E \text{ such that } \|\hat{f}(j)\|_E \geq \|\hat{g}(j)\|_E\}$. This set is, by definition, a solid superset of X_E. Now let B be any other solid superset of X_E and let $g \in A$. Then there exists $f \in X_E$ such that $\|\hat{g}(j)\|_E \leq \|\hat{f}(j)\|_E$ for all j. As $f \in X_E \subset B$ and B is solid, we get $g \in B$. □

Denote by $SB(X_E) = \ell^\infty \circledast_{B_0} X_E$. Of course, for $S(X_E)$ the smallest solid superset of X_E, we have $S(X_E) \subseteq SB(X_E)$, but also we have the following result.

Proposition 4.30. *Let X_E be admissible. Then $\ell^\infty \circledast_{B_0} X_E$ is the smallest $\mathcal{S}(E)-$solid space which contains X_E.*

In particular X_E is $\mathcal{S}(E)-$solid if and only if $X_E = \ell^\infty \circledast_{B_0} X_E$

Proof. Of course $X_E \subseteq \ell^\infty \circledast_{B_0} X_E$ and $\ell^\infty \circledast_{B_0} X_E$ is solid (due to Remark 4.28).

Let Y_E be a solid space with $X_E \subset Y_E$. We shall see that $\ell^\infty \circledast_{B_0} X_E \subset Y_E$. Let $h \in \ell^\infty \circledast_{B_0} X_E$ be given by $h = \sum_n f_n *_{B_0} g_n$ where $f_n \in \ell^\infty$, $g_n \in X_E$ and $\sum_n \|f_n\|_\infty \|g_n\|_{X_E} < \infty$. Note that $f_n *_{B_0} g_n \in Y_E$ and $\|f_n *_{B_0} g_n\|_{Y_E} \le \|f_n\|_\infty \|g_n\|_{Y_E}$ for each n because Y_E is solid. Hence

$$\sum_n \|f_n *_{B_0} g_n\|_{Y_E} \le \sum_n \|f_n\|_\infty \|g_n\|_{Y_E} \le C \sum_n \|f_n\|_\infty \|g_n\|_{X_E} < \infty$$

and then $h \in Y_E$. □

Proposition 4.31. *Let X_{E_1} and X_{E_2} be $\mathcal{H}(E_1)-$ and $\mathcal{H}(E_2)-$admissible Banach spaces, respectively. Let B be a bilinear form defined as in Theorem 4.25. If X_{E_1} and X_{E_2} are homogeneous then $X_{E_1} \circledast_B X_{E_2}$ is homogeneous.*

Proof. The vector-valued $\mathcal{H}$-admissibility has already been proved in Theorem 4.25. Given $h = \sum_n f_n * g_n \in X_{E_1} \circledast_B X_{E_2}$ with $\sum_n \|f_n\|_{X_{E_2}} \|g_n\|_{X_{E_1}} < \infty$, using the homogeneity of X_{E_1} and X_{E_2}, as well as the properties of M_{X_E} described in Chapter 3, one has

$$M_{X_{E_1} \circledast_B X_{E_2}}(r^2, h) \le \|B\| \sum_n M_{X_{E_1}}(r, f_n) M_{X_{E_2}}(r, g_n) \le \|B\| K_1 K_2 \sum_n \|f_n\|_{X_{E_2}} \|g_n\|_{X_{E_1}}.$$

Therefore $M_{X_{E_1} \circledast_B X_{E_2}}(r, h) \le K \|h\|_B$ $(K = \|B\| K_1 K_2)$, for all $0 < r < 1$.

For the other condition, let $\epsilon > 0$. Then we can find $(f_n)_n \subset X_{E_1}$, $(g_n)_n \subset X_{E_2}$ such that $h = \sum_n f_n *_B g_n$ and

$$\sum_n \|f_n\|_{X_{E_1}} \|g_n\|_{X_{E_2}} < \|h\|_B + \epsilon$$

Taking into account that

$$h_\xi = \sum_n (f_n)_\xi *_B g_n, \ |\xi| = 1,$$

and the homogeneity of X_{E_1}, one concludes

$$\|h_\xi\|_B \le \sum_n \|(f_n)_{xi}\|_{X_{E_1}} \|g_n\|_{X_{E_2}} = \sum_n \|f_n\|_{X_{E_1}} \|g_n\|_{X_{E_2}} < \|h\|_B + \epsilon$$

for $|\xi| = 1$ and the result follows.

□

Lets see under which conditions the space we've generated has the (FP). Notice that is not enough for $X_{E_1} \circledast_B X_{E_2}$ to have this property if only one of the spaces has it (see [16], p.437).

Theorem 4.32. *Let X_{E_1} and Y_{E_2} be homogeneous with (FP) and B defined as in Theorem 4.21. Then $X_{E_1} \circledast_B X_{E_2}$ has also (FP).*

Proof. Let $(h_n)_n \in X_{E_1} \circledast_B X_{E_2}$ with $\|h_n\| \le 1$ $\forall n \in \mathbb{N}$ and such that $h_n \longrightarrow h$ in $\mathcal{H}(\mathbb{D}, E_3)$. Take a decomposition $h_n = \sum_j f_{n,j} *_B g_{n,j}$ where $\|f_{n,j}\|_{X_{E_1}} = \|g_{n,j}\|_{X_{E_2}}$ and

$$\|h_n\| \le \sum_j \|f_{n,j}\|_{X_{E_1}} \|g_{n,j}\|_{X_{E_2}} \le \|h_n\| + \frac{1}{n} \le 2.$$

Then $(\|f_{n,j}\|_{X_{E_1}})_j$, $(\|g_{n,j}\|_{X_{E_2}})_j \in \ell^2$ and if we consider $(\alpha_j)_j \in \ell^2, \|(\alpha_j)_j\|_2 = 1$, we get

$$\max\{\|\sum_j \alpha_j f_{n,j}\|_{X_{E_1}},\ \|\sum_j \alpha_j g_{n,j}\|_{X_{E_2}}\} \leq 2.$$

For $\phi_n = \sum_j \alpha_j f_{n,j}$, $\varphi = \sum_j \alpha_j g_{n,j}$, $\sup_n \|\phi_n\| \leq 2$ and $\sup_n \|\varphi_n\| \leq 2$. Taking into account $X_E \subseteq ((X_E, A(\mathbb{D}))_{\mathcal{D}})^*$, the Banach-Alaoglu theorem gives us a subsequence $(n_k)_k \in \mathbb{N}$ such that $\phi_{n_k} \longrightarrow \phi$, $\varphi_{n_k} \longrightarrow \varphi$ in the weak-star topology. Consequently $\phi_{n_k} \longrightarrow \phi \ \in \mathcal{H}(\mathbb{D}, E_1)$, $\varphi_{n_k} \longrightarrow \varphi \ \in \mathcal{H}(\mathbb{D}, E_2)$. Using (FP), $\phi \in X_{E_1}$ and $\varphi \in X_{E_2}$ with $\|\phi\|_{X_{E_1}} \leq 2$, $\|\varphi\|_{X_{E_2}} \leq 2$.

Apply the previous idea to the canonical basis $(e_i)_i \subseteq \ell^2$, then for $(\alpha_j)_j = e_i$ $\phi_n = f_{n,i}$ and $\varphi_n = g_{n,i}$ and, as before, it exists $(n_k)_k \in \mathbb{N}$ such that $f_{n_k,i} \longrightarrow f_i$ and $g_{n_k,i} \longrightarrow g_i$ in $\mathcal{H}(\mathbb{D}, E_1)$ and $\mathcal{H}(\mathbb{D}, E_2)$ respectively. Taking limits $h = \sum_i f_i *_B g_i$ in $\mathcal{S}(E_3)$.

To see $\sum_i \|f_i\|_{X_{E_1}} \|g_i\|_{X_{E_2}} < \infty$ it is enough to check $(\|f_i\|_{X_{E_1}})_i$, $(\|g_i\|_{X_{E_2}})_i \in \ell^2$. Then we have $\sum_i \|f_i\|_{X_{E_1}} \|g_i\|_{X_{E_2}} \leq 4$, and our space has the (FP). □

THEOREM 4.33. *Given Y a $\mathcal{S}-$admissible space with the (FP) and E_2 a reflexive Banach space, we have $(X_{E_1} \circledast Y[E_2]^\star)^\star = (X_{E_1}, Y[E_2])$.*

PROOF. Taking into account that if Y has the (FP), then $Y = Y^{\star\star}$ (see [**?**, **?**, Theorem 5.1,BP] Theorem 4.26 and Corollary 2.17 we get the result. □

3. Computing the $\circledast_B$−product.

Let us see some useful examples of the $\circledast_B$−product and how they can help to compute some multiplier spaces. We start computing the Hadamard projective tensor product for sequence spaces.

PROPOSITION 4.34. *Let $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\ell^p(E_1) \circledast \ell^q(E_2) = \ell^1(E_1 \hat{\otimes}_\pi E_2).$$

PROOF. Let $f \in \ell^p(E_1)$ and $g \in \ell^q(E_2)$. Since $\widehat{f *_\pi g}(j) = \hat{f}(j) \otimes \hat{g}(j)$ and

$$\|\widehat{f *_\pi g}(j)\|_{E_1 \hat{\otimes}_\pi E_2} \leq \|\hat{f}(j)\|_{E_1} \|\hat{g}(j)\|_{E_2}$$

we have, using Hölder's inequality,

$$\|f *_\pi g\|_{\ell^1(E_1 \hat{\otimes}_\pi E_2)} \leq \|f\|_{\ell^p(E_1)} \|g\|_{\ell^q(E_2)}. \tag{4.7}$$

Let $h \in \ell^p(E_1) \circledast \ell^q(E_2)$. Let $\epsilon > 0$ and take $h = \sum_n f_n *_\pi g_n$ with $f_n \in \ell^p(E_1)$ and $g_n \in \ell^q(E_2)$ and $\sum_n \|f_n\|_{\ell^p(E_1)} \|g_n\|_{\ell^q(E_2)} \leq \|h\|_{B_\pi} + \epsilon$.

From (4.7) we have that $h = \sum_n f_n *_\pi g_n$ converges in $\ell^1(E_1 \hat{\otimes}_\pi E_2)$ and $\|h\|_{\ell^1(E_1 \hat{\otimes}_\pi E_2)} \leq \|h\|_{B_\pi} + \epsilon$. This implies that $\ell^p(E_1) \circledast \ell^q(E_2) \subseteq \ell^1(E_1 \hat{\otimes}_\pi E_2)$ with inclusion of norm 1.

Take now $h \in \ell^1(E_1 \hat{\otimes}_\pi E_2)$. In particular for each $j \geq 0$ and $\epsilon > 0$ there exists $x_n^j \in E_1$ and $y_n^j \in E_2$ such that $\hat{h}(j) = \sum_n x_n^j \otimes y_n^j$ and

$$\sum_n \|x_n^j\|_{E_1} \|y_n^j\|_{E_2} < \|\hat{h}(j)\|_{E_1 \hat{\otimes}_\pi E_2} + \frac{\epsilon}{2^j}.$$

Define F_n and G_n by the formulae

$$\hat{F}_n(j) = \Big(\|x_n^j\|_{E_1} \|y_n^j\|_{E_2}\Big)^{1/p} \frac{x_n^j}{\|x_n^j\|_{E_1}}, \quad \hat{G}_n(j) = \Big(\|x_n^j\|_{E_1} \|y_n^j\|_{E_2}\Big)^{1/q} \frac{y_n^j}{\|y_n^j\|_{E_2}}.$$

Note that

$$\|F_n\|_{\ell^p(E_1)} = (\sum_j \|x_n^j\|_{E_1}\|y_n^j\|_{E_2})^{1/p}, \quad \|G_n\|_{\ell^q(E_2)} = (\sum_j \|x_n^j\|_{E_1}\|y_n^j\|_{E_2})^{1/q}$$

and

$$\sum_n \|F_n\|_{\ell^p(E_1)}\|G_n\|_{\ell^q(E_2)} = \sum_{n,j} \|x_n^j\|_{E_1}\|y_n^j\|_{E_2} \leq \|h\|_{\ell^1(E_1\hat{\otimes}_\pi E_2)} + \epsilon.$$

In such a way we have $h = \sum_n F_n *_\pi G_n \in \ell^p(E_1) \circledast \ell^q(E_2)$ with $\|h\|_{B_\pi} \leq \|h\|_{\ell^1(E_1\hat{\otimes}_\pi E_2)}$. □

To analyze the other values of p we shall make use of 2.1.

Proposition 4.35. *Let $1 \leq p, q \leq \infty$ with $0 < \frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1$. Then*

$$\ell^p(E_1) \circledast \ell^q(E_2) = \ell^r(E_1\hat{\otimes}_\pi E_2).$$

Proof. Note that same argument as in Proposition 4.34 gives $\ell^p(E_1) \circledast \ell^q(E_2) \subseteq \ell^r(E_1\hat{\otimes}_\pi E_2)$ with inclusion of norm 1.

Indeed, as above, if $f \in \ell^p(E_1)$ and $g \in \ell^q(E_2)$ then

$$\|\widehat{f *_\pi g}(j)\|_{E_1\hat{\otimes}_\pi E_2} \leq \|\hat{f}(j)\|_{E_1}\|\hat{g}(j)\|_{E_2}.$$

Hence

$$\|f *_\pi g\|_{\ell^r(E_1\hat{\otimes}_\pi E_2)} \leq \|f\|_{\ell^p(E_1)}\|g\|_{\ell^q(E_2)}. \tag{4.8}$$

For a general $h = \sum_n f_n *_\pi g_n \in \ell^p(E_1) \circledast \ell^q(E_2)$ where f_n, g_n are chosen such that $f_n \in \ell^p(E_1)$ and $g_n \in \ell^q(E_2)$ and $\sum_n \|f_n\|_{\ell^p(E_1)}\|g_n\|_{\ell^q(E_2)} \leq \|h\|_{B_\pi} + \epsilon$ we have from (4.8) that $\sum_n \|f_n *_\pi g_n\|_{\ell^r(E_1\hat{\otimes}_\pi E_2)} < \infty$. Then $h = \sum_n f_n *_\pi g_n$ converges in $\ell^r(E_1\hat{\otimes}_\pi E_2)$ and $\|h\|_{\ell^r(E_1\hat{\otimes}_\pi E_2)} \leq \|h\|_{B_\pi} + \epsilon$.

To see that they coincide it suffices to show that $(\ell^p(E_1) \circledast \ell^q(E_2))^* = (\ell^r(E_1\hat{\otimes}_\pi E_2))^*$. It is well known that for $\frac{1}{r'} = 1 - \frac{1}{r}$,

$$(\ell^r(E_1\hat{\otimes}_\pi E_2))^* = \ell^{r'}(\mathcal{L}(E_1, E_2^*)).$$

On the other hand, using Proposition 4.27 and (2.1) we have

$$(\ell^p(E_1) \circledast \ell^q(E_2))^* = (\ell^p(E_1), \ell^{q'}(E_2^*)) = \ell^{r'}(\mathcal{L}(E_1, E_2^*))$$

where $\frac{1}{q'} = 1 - \frac{1}{q}$. □

One of the purposes of our work was to get to know the $B-$Hadamard tensor product between analytic function spaces. We now compute the Hadamard tensor product in some particular cases of spaces of analytic functions. We shall analyze the case H^1 and $H^1(\mathbb{D}, E)$ at least for particular Banach spaces E following the ideas developed in [**16**].

We need some notions and lemmas before the statement of the result. Recall that for an E-valued analytic function, $F(z) = \sum_{j=0}^\infty x_j z^j$, we define $DF(z) = \sum_{j=0}^\infty (j+1)x_j z^j$.

Lemma 4.36. *Let E be a complex Banach space, $1 \leq p \leq \infty$.*

(i) There exist $A_1, A_2 > 0$ such that

$$A_1 r^m \|f\|_{H^p(\mathbb{D},E)} \leq M_p(r, f) \leq A_2 r^n \|f\|_{H^p(\mathbb{D},E)}, \quad 0 < r < 1$$

for $f \in \mathcal{P}(E)$ given by $f(z) = \sum_{j=n}^m x_j z^j, x_j \in E, n, m \in \mathbb{N}$.

(ii) If $P(z) = \sum_{k=2^{n-1}}^{2^{n+1}} \hat{P}(k)z^k$, $\hat{P}(k) \in \mathbb{C}$, then there exist constants B_1 and B_2 such that

$$B_1 2^n \|P *_{B_0} f\|_{H^p(\mathbb{D},E)} \leq \|P *_{B_0} Df\|_{H^p(\mathbb{D},E)} \leq B_2 2^n \|P *_{B_0} f\|_{H^p(\mathbb{D},E)} \tag{4.9}$$

for any $f \in H^p(\mathbb{D}, E)$.

PROOF. It is well known (see Lemma 3.1 [**34**]) that

$$r^m\|\phi\|_\infty \le M_\infty(r,\phi) \le r^n\|\phi\|_\infty, \quad 0<r<1.$$

for each scalar-valued polynomial $\phi(z) = \sum_{j=n}^m \alpha_j z^j$, where $\|\phi\|_\infty = \sup_{|z|=1}|\phi(z)|$ and $M_\infty(r,\phi) = \sup_{|z|=1}|\phi(rz)|$.

This allows us to conclude, composing with elements in the unit ball of the dual space,

$$r^m\|F\|_\infty \le M_\infty(r,F) \le r^n\|F\|_\infty, \quad 0<r<1.$$

for any $F(z) = \sum_{j=n}^m y_j z^j$ where $y_j \in Y$ where Y is a complex Banach space.

Now select $Y = H^p(\mathbb{D}, E)$ and $F(z) = f_z$ that is to say

$$F(z)(w) = \sum_{j=n}^m x_j w^j z^j.$$

Using that

$$\|F\|_\infty = \sup_{|z|=1}\|f_z\|_{H^p(\mathbb{D},E)} = \|f\|_{H^p(\mathbb{D},E)}$$

and $M_\infty(r,F) = M_p(r,f)$ we obtain the result.

To see (ii) we first use [**16**, Lemma 7.2] that guarantees the existence of constants B_1, B_2 such that

$$B_1 2^n\|P *_{B_0} \phi\|_\infty \le \|P *_{B_0} D\phi\|_\infty \le B_2 2^n\|P *_{B_0} \phi\|_\infty$$

for any $\phi \in H^\infty(\mathbb{D})$. Now apply the same argument as above to extend it to $H^p(\mathbb{D}, E)$. □

THEOREM 4.37. *Let* $D^{-1}A^1(\mathbb{D}, E)$ *denote the space of* E*-valued analytic functions* $F(z) = \sum_{j=0} x_j z^j$ *such that* $DF(z) \in A^1(\mathbb{D}, E)$ *with the norm given by*

$$\|F\|_{D^{-1}A^1(\mathbb{D},E)} = \|F(0)\|_E + \int_{\mathbb{D}} \|DF(z)\|_E dA(z).$$

Let $E = L^p(\mu)$ *for any measure* μ *and* $1 \le p \le 2$.

$$H^1(\mathbb{D}) \circledast_{B_0} H^1(\mathbb{D}, L^p(\mu)) = D^{-1}A^1(\mathbb{D}, L^p(\mu)).$$

PROOF. Let us first show that $D^{-1}A^1(\mathbb{D}, E) \subseteq H^1(\mathbb{D}) \circledast_{B_0} H^1(\mathbb{D}, E)$ for any Banach space E. We argue similarly to [**16**, Thm 7.1].

Let $\{W_n\}_0^\infty$ be a sequence of polynomials such that

$$\text{(4.10)} \quad \operatorname{supp}(\hat{W}_n) \subset [2^{n-1}, 2^{n+1}] \quad (n \ge 1), \quad \operatorname{supp}(\hat{W}_0) \subset [0,1], \quad \sup_n \|W_n\|_1 < \infty$$

and

$$\text{(4.11)} \qquad g = \sum_{n=0}^\infty W_n *_{B_0} g, \qquad g \in \mathcal{H}(\mathbb{D}, E).$$

Let $f \in D^{-1}A^1(\mathbb{D}, E)$. Note that

$$\|(W_n *_{B_0} f)_r\|_{H^1(\mathbb{D},E)} \le \|W_n\|_1 \|f_r\|_{H^1(\mathbb{D},E)} \le C\|f\|_{H^1(\mathbb{D},E)},$$

Hence, $\|W_n *_{B_0} f\|_{H^1(\mathbb{D},E)} \le C\|f\|_{H^1(\mathbb{D},E)}$.

Denoting $Q_n = W_{n-1} + W_n + W_{n+1}$ we can write

$$f = \sum_{n=0}^\infty Q_n *_{B_0} W_n *_{B_0} f.$$

Note now that Lemma 4.36 allows us to conclude

$$\begin{aligned}
\sum_{n=0}^{\infty}\|Q_n\|_{H^1}\|W_n *_{B_0} f\|_{H^1(\mathbb{D},E)} &\leq K\sum_{n=0}^{\infty}\|W_n *_{B_0} f\|_{H^1(\mathbb{D},E)}\\
&\leq K\sum_{n=0}^{\infty}\int_{1-2^{-n}}^{1-2^{-(n+1)}} 2^n r^{2^n}\|W_n *_{B_0} f\|_{H^1(\mathbb{D},E)}dr\\
&\leq K\sum_{n=0}^{\infty}\int_{1-2^{-n}}^{1-2^{-(n+1)}} r^{2^n}\|W_n *_{B_0} Df\|_{H^1(\mathbb{D},E)}dr\\
&\leq K\sum_{n=0}^{\infty}\int_{1-2^{-n}}^{1-2^{-(n+1)}} M_1(r, W_n *_{B_0} Df)dr\\
&\leq K\sum_{n=0}^{\infty}\int_{1-2^{-n}}^{1-2^{-(n+1)}} M_1(r, Df)dr\\
&= K\int_0^1 M_1(r, Df)dr\\
&\leq K\|f\|_{D^{-1}A^1(\mathbb{D},E)}.
\end{aligned}$$

To show the other inclusion between these spaces we shall use that $E = L^p(\mu)$ for $1 \leq p \leq 2$ satisfies the following vector-valued extension of a Hardy-Littlewood theorem,

$$\left(\int_0^1 (1-r)M_1^2(Df, r)dr\right)^{1/2} \leq A\|f\|_{H^1(\mathbb{D},E)} \tag{4.12}$$

for some constant $A > 0$(see [10], Definition 3.5 and Proposition 4.4).

It suffices to see that $\phi *_{B_0} g \in D^{-1}A^1(\mathbb{D}, L^p(\mu))$ for each $\phi \in H^1(\mathbb{D})$ and $g \in H^1(\mathbb{D}, L^p(\mu))$. Now taking into account that $D^2(\phi *_{B_0} g) = D\phi *_{B_0} Dg$ and

$$rD(\phi *_{B_0} g)(re^{it}) = \sum_{j=0}^{\infty}(j+1)\hat{\phi}(j)\hat{g}(j)r^{j+1}e^{itj} = \int_0^r D^2(\phi *_{B_0} g)(se^{it})ds$$

we have,

$$\begin{aligned}
\int_0^1 M_1(r, D(\phi *_{B_0} g))rdr &\leq \int_0^1\left(\int_0^r M_1(s, D^2(\phi *_{B_0} g))ds\right)rdr\\
&= \int_0^1 (1-s)M_1(s, D^2(\phi *_{B_0} g))ds\\
&\leq 2\int_0^1 (1-r^2)M_1(r, D\phi)M_1(r, Dg)rdr.
\end{aligned}$$

Now from Cauchy-Schwarz and (4.12) we obtain

$$\begin{aligned}
\int_0^1 (1-r^2)M_1(r, D\phi)M_1(r, Dg)rdr &\leq \left(\int_0^1 (1-r^2)M_1^2(r, D\phi,)rdr\right)^{1/2}\\
&\quad . \left(\int_0^1 (1-r^2)M_1^2(r, Dg)rdr\right)^{1/2}\\
&\leq K\|\phi\|_{H^1}\|g\|_{H^1(\mathbb{D},L^p(\mu))}.
\end{aligned}$$

□

It has been already mentioned that $(H^1)^* = BMOA$. In the vector-valued case, using L^p is an UMD space for $1 < p < \infty$, we have

$$(H^1(\mathbb{T}, L^p(\mu)))^* = BMOA(\mathbb{T}, L^{p'}(\mu)), 1 < p < \infty$$

(see [7]). It is also well known that $(D^{-1}A^1)^* = \mathcal{B}loch$ (see [1]) and for the vector-valued case $(D^{-1}A^1(\mathbb{D}, E))^* = \mathcal{B}loch(\mathbb{D}, E^*)$ for any complex Banach space E (see [8], Corollary 2.1) under the pairing

$$\langle F, G\rangle = \int_{\mathbb{D}} \langle DF(z), G(z)\rangle dA(z).$$

We can give a version of this duality in terms of $H^\infty(\mathbb{D})$.

Proposition 4.38. *Let* $1 < p \leq 2$.

$$(H^1(\mathbb{T}, L^p(\mu)), H^\infty(\mathbb{D}))_{\mathcal{D}} = BMOA(\mathbb{T}, L^{p'}(\mu))$$

and

$$(D^{-1}A^1(\mathbb{D}, E), H^\infty(\mathbb{D}))_{\mathcal{D}} = \mathcal{B}loch(\mathbb{D}, E^*).$$

Proof. Given $f \in BMOA(\mathbb{T}, L^{p'}(\mu))$, take $g \in H^1(\mathbb{T}, L^p(\mu))$. Then

$$f *_{\mathcal{D}} g(z) = \langle f, g_z\rangle = \int_{\mathbb{T}} \langle f(e^{i\theta}), g(ze^{-i\theta})\rangle \frac{d\theta}{2\pi}$$

for any $|z| \leq 1$. Thus, since $BMOA(\mathbb{T}, L^{p'}(\mu))$ is the topological dual of $H^1(\mathbb{T}, L^p(\mu))$,

$$\sup_{|z|<1} |f *_{\mathcal{D}} g(z)| \leq \|f\|_{BMOA(\mathbb{T},L^{p'}(\mu))}\|g_z\|_{H^1(\mathbb{T},L^p(\mu))} \leq \|f\|_{BMOA(\mathbb{T},L^{p'}(\mu))}\|g\|_{H^1(\mathbb{T},L^p(\mu))}.$$

For the reverse inclusion, we will use again a duality argument. Consider $f \in (H^1(\mathbb{T}, L^p(\mu)), H^\infty(\mathbb{D}))_{\mathcal{D}}$ and take $g \in H^1(\mathbb{T}, L^p(\mu))$. Then $|f * g(z)| = \langle f_z, g\rangle < \infty$ for $|z| \leq 1$, and so $f_r \in BMOA(E^*)$ with $\|f_r\|_{BMOA(\mathbb{T},E^*)} \leq C\|g\|_{H^1(\mathbb{T},E)}$ for any $r \in (0, 1]$. By Alaoglu's Theorem, there exists $h \in BMOA(\mathbb{T}, E^*)$ such that f_r converges in the weak-star topology to h as $r \to 1$.
To see $h = f$, consider the polynomial $xe_n(z) = xz^n$, where x is an arbitrary element of E. Then

$$\langle f_r, \overline{xe_n}\rangle = \langle \hat{f}(n), \bar{x}\rangle r^n \to \langle \hat{h}(n), \bar{x}\rangle, \text{ as } r \to 1.$$

As polynomials are dense in $H^1(\mathbb{T}, E)$ we get $\hat{f}(n) = \hat{h}(n)$ almost everywhere in $\mathbb{T}$. Therefore $h = f$ almost everywhere in the disc.

Now consider $f \in \mathcal{B}loch(\mathbb{D}, E^*)$ and $g \in D^{-1}A^1(\mathbb{D}, E)$. Then we can identify the convolution product with the following pairing:

$$f *_{\mathcal{D}} g(z) = 2\langle Df + f, Dg_z\rangle = 2\int_{\mathbb{D}} \langle (Df + f)(w), Dg_z(\bar{w})\rangle(1 - |w|^2)dA(w).$$

Notice that $\int_{\mathbb{D}} \|Dg_z(w)\|dA(w) < \int_{\mathbb{D}} \|Dg(w)\|dA(w) < \|g\|_{D^{-1}A^1}$, that $Df(z) = (zf(z))' = f(z) + zf'(z)$ and that writing $f(re^{i\theta}) = f(0) + \int_0^r f'(se^{i\theta})ds$ one deduces

$$\begin{aligned}\|f(re^{i\theta})\|_{E^*}(1 - r^2) &\leq (1 - r^2)\left(\|f(0)\|_{E^*} + \int_0^r \|f'(se^{i\theta})\|_{E^*}ds\right)\\ &\leq (1 - r^2)\left(\|f(0)\|_{E^*} + sup_{0<t<1}\|f'(te^{i\theta})\|_{E^*}r\right)\\ &\leq (1 - r^2)\|f(0)\|_{E^*} + r\|f\|_{\mathcal{B}loch}.\end{aligned}$$

Therefore for a fixed $|z| < 1$

$$\begin{aligned}|f * g(z)| &< K \sup_{|w|<1} ((Df(w) + f(w))(1 + |w|^2)) \int_{\mathbb{D}} \|Dg_z(\bar{w})\| dA(w) \\ &\leq K \sup_{|w|<1} ((2f(w) + wf'(w))(1 + |w|^2)) \int_{\mathbb{D}} \|Dg_z(\bar{w})\| dA(w) \\ &\leq K'(\|f\|_{\mathcal{B}loch(\mathbb{D},E^*)} + \|g\|_{D^{-1}A^1(\mathbb{D},E)}).\end{aligned}$$

Finally, consider $f \in (D^{-1}A^1(\mathbb{D}, E), H^\infty(\mathbb{D}))_{\mathcal{D}} = \mathcal{B}loch(\mathbb{D}, E^*)$. We know

$$f * g(z) = \langle (Df + f)(w), Dg_z(w) \rangle = \langle Df(w), Dg_z(w) \rangle + \langle f_z(w), Dg(w) \rangle < \infty$$

□

Using this together with Proposition 4.27 and Theorem 4.37 we recover the following result.

Corollary 4.39. *(See [10]) Let $1 \leq p_1 \leq 2$ and $2 \leq p_2 < \infty$. Then*

$$(H^1(\mathbb{T}, L^{p_1}), BMOA(\mathbb{T})) = \mathcal{B}loch(\mathbb{D}, L^{p_1'}) \text{ and}$$

$$(H^1(\mathbb{T}), BMOA(\mathbb{T}, L^{p_2})) = \mathcal{B}loch(\mathbb{D}, L^{p_2}).$$

Using similar techniques as in Theorem 4.37, we can compute the $\circledast_{B_0}-$product between $A^1(\mathbb{D})$ and the vector-valued Hardy space $H^1(\mathbb{D}, E)$.

Theorem 4.40. *Let E be a Banach space. Then*

$$A^1(\mathbb{D}) \circledast_{B_0} H^1(\mathbb{D}, E) = A^1(\mathbb{D}, E).$$

Proof. Let $f \in A^1(\mathbb{D}, E)$. Again take $(W_n)_{n \in \mathbb{N}_0}$ a sequence of polynomials verifying conditions (4.10) and (4.11). Again, consider $Q_n = W_{n-1} + W_n + W_{n+1}$ and write $f = \sum_n Q_n *_{B_0} (W_n *_{B_0} f)$. Let us see that

$$\sum_n \|Q_n\|_{A^1} \|W_n *_{B_0} f\|_{H^1(\mathbb{D},E)} < \infty.$$

Lemma 2.3 in [30] gives for $r \in (0, 1)$

$$r^{2^{n+1}} \|W_n\|_{H^1} \leq M_1(r, W_n) \leq r^{2^{n-1}} \|W_n\|_{H^1}.$$

Integrating the expressions

$$\frac{\|W_n\|_{H^1}}{2^{n+1} + 1} \leq \|W_n\|_{A^1} \leq \frac{\|W_n\|_{H^1}}{2^{n-1} + 1}.$$

Using $\sup_n \|W_n\|_{H^1} < \infty$ we get to $\|W_n\|_{A^1} \in O(2^{-n})$ and thus $\|Q_n\|_{A^1} \in O(2^{-n})$. Therefore we need only check the convergence of $\sum_n 2^{-n} \|W_n *_{B_0} f\|_{H^1(\mathbb{D},E)}$. Note now that Lemma 4.36 allows us to conclude

$$\begin{aligned}\sum_{n=0}^{\infty}\|Q_n\|_{A^1}\|W_n *_{B_0} f\|_{H^1(\mathbb{D},E)} &\leq K\sum_{n=0}^{\infty}2^{-n}\|W_n *_{B_0} f\|_{H^1(\mathbb{D},E)}\\ &\leq K\sum_{n=0}^{\infty}\int_{1-2^{-n}}^{1-2^{-(n+1)}} r^{2^n}\|W_n *_{B_0} f\|_{H^1(\mathbb{D},E)}dr\\ &\leq K\sum_{n=0}^{\infty}\int_{1-2^{-n}}^{1-2^{-(n+1)}} M_1(r, W_n *_{B_0} f)dr\\ &\leq K\sum_{n=0}^{\infty}\int_{1-2^{-n}}^{1-2^{-(n+1)}} M_1(r, f)dr\\ &= K\int_0^1 M_1(r,f)dr\\ &\leq K\|f\|_{A^1(\mathbb{D},E)}.\end{aligned}$$

For the reverse inclusion consider $f = \sum_n g_n *_{B_0} h_n \in A^1(\mathbb{D}) \circledast_{B_0} H^1(\mathbb{D}, E)$. Since $\|B_0\| = 1$, one gets

$$M_1(r^2, \sum_n g_n *_{B_0} h_n) \leq \sum_n M_1(r, g_n)M_1(r, h_n),$$

integrating both expressions and taking into account that $h_n \in H^1(\mathbb{D}, E)$ for every n,

$$\begin{aligned}\|f\|_{A^1(\mathbb{D},E)} = \int_0^1 M_1(r^2, \sum_n g_n *_{B_0} h_n)dr &\leq \int_0^1 \sum_n M_1(r, h_n)M_1(r, g_n)dr\\ \leq \sum_n \|h_n\|_{H^1(\mathbb{D},E)}\int_0^1 M_1(r, g_n)dr &= \sum_n \|h_n\|_{H^1(\mathbb{D},E)}\|g_n\|_{A^1}.\end{aligned}$$

□

Desmayarse, atreverse, estar furioso,
áspero, tierno, liberal, esquivo,
alentado, mortal, difunto, vivo,
leal, traidor, cobarde y animoso;
no hallar fuera del bien centro y reposo,
mostrarse alegre, triste, humilde, altivo,
enojado, valiente, fugitivo,
satisfecho, ofendido, receloso;
huir el rostro al claro desengaño,
beber veneno por licor süave,
olvidar el provecho, amar el daño;
creer que un cielo en un teorema cabe,
dar la vida y el alma a un desengaño;
es la tesis, quien terminó lo sabe.

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