PRIMITIVE SUBGROUPS AND PST-GROUPS

A. BALLESTER-BOLINCHES, J. C. BEIDLEMAN, AND R. ESTEBAN-ROMERO

Abstract. All groups are finite. A subgroup H of a group G is called a primitive subgroup if it is a proper subgroup in the intersection of all subgroups of G containing H as its proper subgroup. He, Qiao and Wang [7] proved that every primitive subgroup of a group G has index a power of a prime if and only if $G/\Phi(G)$ is a solvable PST-group. Let $\mathfrak{X}$ denote the class of groups G all of whose primitive subgroups have prime power index. It is established here that a group G is a solvable PST-group if and only if every subgroup of G is an $\mathfrak{X}$-group.

1. Introduction and Statements of Results.

All groups considered here are finite. A subgroup H of a group G is called primitive if it is a proper subgroup in the intersection of all subgroups containing H as a proper subgroup. All maximal subgroups of G are primitive. Some properties of primitive subgroups are given in Lemma 2.1 and include:

(a) Every proper subgroup of G is the intersection of a set of primitive subgroups of G.
(b) If X is a primitive subgroup of a subgroup T of G, then there exists a primitive subgroup Y of G such that $X = Y \cap T$.

Johnson [10] introduced the concept of primitive subgroup of a group. He proved that a group G is supersolvable if every primitive subgroup of G has prime power index in G.

The next results on primitive subgroups of a group G indicate how such subgroups give information about the structure of G.

Theorem 1.1 ([7]). *Let G be a group. The following statements are equivalent:*

(1) *Every primitive subgroup of G containing $\phi(G)$ has prime power index.*
(2) *$G/\phi(G)$ is a solvable PST-group.*

Theorem 1.2 ([6]). *Let G be a group. The following statements are equivalent:*

(1) *Every primitive subgroup of G has prime power index.*
(2) *$G = [L]M$ is a supersolvable group, where L and M are nilpotent Hall subgroups of G, L is the nilpotent residual of G and $G = L\,\mathrm{N}_G(L \cap X)$ for every*

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primitive subgroup X of G. In particular, every maximal subgroup of L is normal in G.

Note that $G = [L]M$ in Theorem 1.2 means that G is the semidirect product of L by M.

Let $\mathfrak{X}$ denote the class of groups G such that the primitive subgroups of G have prime power index (see [5, pp. 132-137]). By (a) it is clear that $\mathfrak{X}$ consists of those groups whose subgroups are intersections of subgroups of prime-power indices.

One purpose of this paper is to characterize solvable PST-groups in terms of $\mathfrak{X}$-subgroups.

A subgroup H of a group G is said to be S-permutable in G if it permutes with the Sylow subgroups of G. Kegel [2, 1.2.14] proved that an S-permutable subgroup of G is subnormal in G. S-permutability is said to be transitive in G if H and K are subgroups of G such that H is S-permutable in K and K is S-permutable in G, then H is S-permutable in G. A group G is said to be a PST-group if S-permutability is a transitive relation in G. By Kegel's result G is a PST-group if and only if every subnormal subgroup of G is S-permutable. Agrawal [1] characterized solvable PST-groups. He proved the following theorem.

Theorem 1.3. *Let G be a solvable group. G is a PST-group if and only if it has an abelian normal Hall subgroup N such that G/N is nilpotent and G acts by conjugation on N as a group of power automorphisms.*

In Theorem 1.3 N can be taken to be the nilpotent residual of G. From Theorem 1.3 it follows that subgroups of solvable PST-groups are solvable PST-groups. Many interesting results about PST-groups can be found in Chapter 2 of [2].

Theorem A. *Let G be a group. The following statements are equivalent:*

(1) *G is a solvable PST-group.*
(2) *Every subgroup of G is an $\mathfrak{X}$-group.*

Let G be an $\mathfrak{X}$-group. It follows from Theorem A that if G is not a solvable PST-group, then G has a subgroup H which does not belong to $\mathfrak{X}$. See Examples 1 and 2.

A well-known theorem of Lagrange (see [5, Chapter 1, 1.3.6]) states that given a subgroup H of a group G, the order of G is the product of the order $|H|$ of H and the index $|G : H|$ of H in G. In particular, the order of any subgroup divides the order of the group. The converse, namely, if d divides the order of a group G, then G has a subgroup of order d, is not true in general. Groups satisfying this condition are often called CLT-groups. The alternating group of order 12, having no subgroups of order 6, is an example of a non-CLT-group.

On the other hand, abelian groups contain subgroups of every possible order, and it is not difficult to prove that a group is nilpotent if and only if it contains a normal subgroup of each possible order [8]. Ore [11] and Zappa [15] obtained a similar characterization for supersolvable groups:

Theorem 1.4. *A group G is supersolvable if and only if each subgroup $H \leq G$ contains a subgroup of order d for each divisor d of $|H|$.*

Of course, we can state Theorem 1.4 in the following equivalent way, more easily treated:

Theorem 1.5. *A group G is supersolvable if and only if each subgroup $H \leq G$ contains a subgroup of index p for each prime divisor p of $|H|$.*

A proof of this theorem can be found in [5, Chapter 1, 4.2]. It must be noted that CLT-groups are not necessarily supersolvable, as the symmetric group of order 4 shows.

The condition on a group G given in Theorem 1.5, namely

> for all $H \leq G$ and for all primes q dividing $|H|$, there exists a subgroup K of G such that $K \leq H$ and $|H : K| = q$,

has a dual formulation:

> for all $H \leq G$ and for all primes q dividing $|G : H|$, there exists a subgroup K of G such that $H \leq K$ and $|K : H| = q$.

Groups satisfying the latter condition have been studied by some authors. Following [5, Chapter 1, 4], we will call them $\mathcal{Y}$-groups.

A group G is said to be a $\mathcal{Y}$-group if for all subgroups H of G and all primes q dividing the index $|G : H|$ of H in G, there exists a subgroup K of G with $H \leq K$ and $|K : H| = q$.

Note that a group G is a $\mathcal{Y}$-group if and only if for every subgroup H of G and for every natural number d dividing $|G : H|$ there exists a subgroup K of G such that $H \leq K$ and $|K : H| = d$. The following characterization of $\mathcal{Y}$-groups appears in [5, Chapter 1, 4.3].

Theorem 1.6. *Let $L = G^{\mathfrak{N}}$ be the nilpotent residual of the group G. Then G is a $\mathcal{Y}$-group if and only if L is a nilpotent Hall subgroup of G such that for all subgroups H of L, $G = L\,\mathrm{N}_G(H)$.*

From Theorem 1.6, we see that if $G \in \mathcal{Y}$ and X is a normal subgroup of L, then X is normal in G. In particular, $\mathcal{Y}$-groups are supersolvable. Moreover, if $G \in \mathcal{Y}$, then L must have odd order.

Further results on $\mathcal{Y}$-groups can be found in [5, Chapter 4, 5.2, 5.3]. For example, a solvable group G is a $\mathcal{Y}$-group if and only if every subgroup of G can be written as an intersection of subgroups of G of coprime prime-power indices.

From Theorem 1.3 and Theorem 1.6 we obtain

Theorem 1.7. *Let G be a $\mathcal{Y}$-group with nilpotent residual L.*

(1) *G is a solvable PST-group if and only if L is abelian.*
(2) *$G/\phi(G)$ is a solvable PST-group.*

We note that the class $\mathcal{Y}$ is a subclass of the class $\mathfrak{X}$ by Theorems 1.2 and 1.7. The example of Humphreys on p. 136 of [5] (see also [9]) shows that $\mathcal{Y}$ is a proper subclass of $\mathfrak{X}$.

Theorem B. *Let G be a group. The following statements are equivalent:*
(1) *G is a solvable PST-group.*
(2) *Every subgroup of G is a $\mathcal{Y}$-group.*
(3) *Every subgroup of G is an $\mathfrak{X}$-group.*

Let $\mathfrak{F}$ be a class of groups. Denote by $\mathcal{S}\mathfrak{F}$ (resp. $\mathcal{S}(\mathfrak{F})$) the class of groups all of whose subgroups are $\mathfrak{F}$-groups (resp. solvable $\mathfrak{F}$-groups).

Theorem C. $\mathcal{S}\mathfrak{X} = \mathcal{S}\mathcal{Y} = \mathcal{S}T_0 = \mathcal{S}(T_0) = \mathcal{S}PST = \mathcal{S}(PST) = \mathcal{S}(PST_0) = \mathcal{S}(PT_0)$.

We mention that $\mathcal{S}\mathfrak{X} = \mathcal{S}\mathcal{Y}$ of Theorem C follows from Theorem B and is Theorem 5.3 of [5, p. 135]. The proof of Theorem 5.3 in [5] is much different and more difficult than the proof of Theorem B.

2. Preliminaries.

Lemma 2.1 ([6, 7, 10])**.** *Let G be a group. The following statements hold:*
(1) *For every proper subgroup H of G, there is a set of primitive subgroups $\{X_i \mid i \in I\}$ in G such that $H = \cap_{i\in I} X_i$.*
(2) *If $H \leq G$ and T is a primitive subgroup of H, then $T = H \cap X$ for some primitive subgroup X of G.*
(3) *If $K \trianglelefteq G$ and $K \leq H \leq G$, then H is a primitive subgroup of G if and only if H/K is a primitive subgroup of G/K.*
(4) *Let P and Q be subgroups of G with $(|P|, |Q|) = 1$. Suppose that H is a subgroup of G such that $HP \leq G$ and $HQ \leq G$. Then $HP \cap HQ = H$. In particular, if H is a primitive subgroup of G, then $P \leq H$ or $Q \leq H$.*

Let G be a group. G is called a T-(resp. PT-)group if $H \trianglelefteq K \trianglelefteq G$ (resp. H is permutable in K and K is permutable in G) then $H \triangleleft G$ (resp. H is permutable in G). By Kegel's result G is a PT-group if and only if every subnormal subgroup of G is permutable. Many results about T- and PT-groups can be found in Chapter 2 of [2]. G is called a T_0-group if $G/\phi(G)$ is a T-group where $\phi(G)$ is the Frattini subgroup of G. T_0-groups have been studied in [4, 12, 14]. Several of the results on T_0-groups given in [4, 12] are contained in the next three lemmas and are needed in the proof of Theorem A.

A group G is called a PT_0-(resp. PST_0-)group provided that $G/\phi(G)$ is a PT-(resp. PST-)group. For solvable groups we have

Lemma 2.2 ([12])**.** $\mathcal{S}(T_0) = \mathcal{S}(PT_0) = \mathcal{S}(PST_0)$.

Lemma 2.3 ([4])**.** *Let G be a group. G is a solvable PST-group if and only if every subgroup of G is a T_0-group.*

3. Proofs of the Theorems.

Proof of Theorem A. Let G be a solvable PST-group and let L be the nilpotent residual of G. By Theorem 1.3 L is a normal abelian Hall subgroup of G on which G acts by conjugation as a group of power automorphisms. Let X be a subgroup of L. Since $X \lhd G$, $G = L\,\mathrm{N}_G(X)$. Let D be a system normalizer of G. By Theorem 9.2.7, p. 264 of [13] $G = [L]D$, the semidirect product of L by D. It follows by Theorem 1.2 that every primitive subgroup of G has prime power index and hence G is an $\mathfrak{X}$-group. Since every subgroup of G is a solvable PST-group, every subgroup of G is an $\mathfrak{X}$-group.

Conversely, assume that every subgroup of G is an $\mathfrak{X}$-group. We are to show that G is a solvable PST-group. Let H be a subgroup of G. Because of Theorem 1.1 $H/\phi(H)$ is a solvable PST-group and hence H is a solvable PST_0-group. By Lemma 2.2 H is a T_0-group. It follows that every subgroup of G is a solvable T_0-group and by Lemma 2.3 G is a solvable PST-group.

This completes the proof. □

Proof of Theorem B. Let G be a solvable PST-group. Using the proof of the first part of Theorem A and Theorem 1.6 we see that every subgroup of G is a $\mathcal{Y}$-group and (1) implies (2). Since $\mathcal{Y} \subseteq \mathfrak{X}$, (2) implies (3). By Theorem A we see that (3) implies (1). □

Proof of Theorem C. By Theorem B, $\mathcal{S}\mathfrak{X} = \mathcal{S}\mathcal{Y} = \mathcal{S}(\mathrm{PST}) = \mathcal{S}\mathrm{PST}$. Note by Theorem 1.1 $\mathcal{S}(\mathrm{T}_0) = \mathcal{S}\mathrm{T}_0 = \mathcal{S}\mathfrak{X}$. Finally, it follows that $\mathcal{S}(\mathrm{T}_0) = \mathcal{S}(\mathrm{PST}_0) = \mathcal{S}(\mathrm{PT}_0)$ by Lemma 2.2. Hence Theorem C holds. □

4. Examples.

Example 1. Let $P = \langle x, y \mid x^5 = y^5 = [x, y]^5 = 1\rangle$ be an extra-special group of order 125 of exponent 5. Let $z = [x, y]$ and note $Z(P) = \Phi(P) = \langle z\rangle$. P has an automorphism a of order 4 given by $x^a = x^2$, $y^a = y^2$ and $z^a = z^4 = z^{-1}$. Put $G = [P]\langle a\rangle$ and note $Z(G) = 1$, $\Phi(G) = \langle z\rangle$ and $G/\Phi(G)$ is a T-group. Thus G is a solvable T_0-group. Let $H = \langle y, z, a\rangle$ and notice $\Phi(H) = 1$. H is not a T-group since the nilpotent residual L of H is $\langle y, z\rangle$ and a does not act on L as a power automorphism. Thus H is not a T_0-group and hence not a solvable PST-group. By Theorem 1.1 G is an $\mathfrak{X}$-group and H is not an $\mathfrak{X}$-group.

Example 2. Let $P = \langle x, y \mid x^3 = y^3 = [x, y]^3 = 1\rangle$ be an extra-special group of order 3^3 and exponent 3. P has an automorphism b of order 2 given by $x^b = x^{-1}$, $y^b = y^{-1}$ and $[x, y]^b = 1$. Let $G = [P]\langle b\rangle$ and note $Z(G) = Z(P) = \langle [x, y]\rangle = \phi(G)$. $G/\phi(G)$ is a T-group and hence G is a T_0-group. By Lemma 2.3 G has a subgroup which is not a T_0-group and hence not a solvable PST-group. Note G is an $\mathfrak{X}$-group.

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Departament d'Àlgebra, Universitat de València, Dr. Moliner 50, 46100 Burjassot, València, Spain

E-mail address: adolfo.ballester@uv.es

Department of Mathematics, University of Kentucky, Lexington, KY 40506-0027, U.S.A.

E-mail address: clark@ms.uky.edu

Institut Universitari de Matemàtica Pura i Aplicada, Universitat Politècnica de València, Camí de Vera s/n, 46022 València, Spain

Current address: Departament d'Àlgebra, Universitat de València, Dr. Moliner 50, 46100 Burjassot, València, Spain

E-mail address: ramon.esteban@uv.es