NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1310

CORRECTION FACTORS FOR WIND TUNNELS OF ELLIPTIC SECTION WITH PARTLY OPEN AND PARTLY CLOSED TEST SECTION

By F. Riegels

Translation of "Korrekturfaktoren für Windkanäle elliptischen Querschnitts mit teilweise offener und teilweise geschlossener Mess-strecke." Luftfahrtforschung Bd. 16, Lfg. 1, 1939

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SUMMARY

A wind tunnel of elliptic section with partly open and partly closed test section contains a wing with rectangular lift distribution. The additional flow caused by the wall interference is determined by conformal representation.

The correction factor for an elliptic jet of $1:\sqrt{2}$ axial ratio is plotted for several span-channel width ratios and several included angles, that is, the angles which the lines connecting the end points of the solid part of the tunnel boundary form with the center of the ellipse. (Compare figs. 1 and 5.) As on the circular jet (references 5 and 6), it is found that the correction for angle of attack and drag becomes zero at a certain included angle. This angle varies with the wing span.

The theory is so applied that it can be utilized also for elliptic tunnels with different axial ratio. An extension to include wings suspended over the median plane of the tunnel is likewise easily possible.

I. INTRODUCTION

The finite boundary of jets causes additional velocities in the flow, especially at the wing. As a consequence, the angle of attack and the drag coefficient measured in the jet at a given lift must be corrected. Calculations dealing with the effect of the jet boundaries are numerous. Open and closed jets have been explored and also jets

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whose boundaries consist partly of solid walls and partly of free jet boundaries (references 1 to 6).

The last arrangements have the advantage that they make it easy to conduct a greater number of tests and also permit choosing a boundary in such a way that the mean additional velocity at the wing, and hence the correction factor, becomes exactly zero for the measurements. If the wing model is suspended with the suction side downward, jets with fixed boundaries produce additional downward velocities, those with free jet boundaries, upward velocities. Therefore, it can be expected that, with suitable distribution of fixed and free boundaries over the section, no additional velocity is produced at all.

The present article deals with the effect of a partly open and partly closed jet of elliptic section.

2. ELLIPTIC JET WITH ONE SOLID WALL (FIG. 1)

With the assumption of small additional velocities whose squares are negligible, and a nondeformed jet boundary, the determination of the additional flow can be reduced to the consideration of the flow condition in a section infinitely far downstream, where an additional velocity exists which is twice as great as that at the wing (reference 1).

Suppose that the jet has the elliptic section with the axes $2a'$ and $2b'$ shown in figure 1; the wing of span $b = 2s$ and rectangular lift distribution is mounted in the center of the jet section. The chosen system of coordinates (x,y) has its origin in the center of the ellipse, so that the shed vortices of the wing push through the section at the points $x = \pm s$.

Now, the problem is to define an additional flow in such a way that the boundary conditions are satisfied. They are: for the solid part, disappearance of the normal velocity component; for the free jet boundary, constant pressure, which for the assumed smallness of the velocity is identical with the requirement that the tangential velocity component shall disappear (reference 1). But, as such an additional flow is difficult to define in the z-plane, it is attempted to find a plane by conformal representation in which the potential of the required additional flow is easily obtainable. Now, reference 5 cites a report by K. Kondo which treats the corresponding problem for circular jet. The mapping function

$$z'' = c \tan \zeta$$

is used which maps the inside of a circle in the z"-plane on the inside of a strip in the ζ-plane in such a way that a part of the circumference is changed in the one, the remaining part in the other of the straight lines bounding the strip (fig. 3). But in the plane of the strip, an additional potential that satisfies the boundary conditions is easily indicated and, if it succeeds in mapping the inside of the ellipse on the inside of a circle, the aforementioned representation is fundamentally accomplished.

The procedure is developed step by step. With the aid of the function

$$z' = \sqrt{k}\ \mathrm{sn}\left(\frac{2K}{\pi} \arcsin \frac{z}{e}\right) \tag{1}$$

the inside of the ellipse in the z-plane is mapped on the unit circle in the z'-plane (fig. 2), where $2K$ is the half real period, K the modulus of the Jakobian elliptic function $\mathrm{sn}\ Z$ and

$$e = \sqrt{a'^2 - b'^2}$$

half the distance of the foci of the ellipse.

As to the theory of this mapping function, the reader is referred to the report by de Haller (reference 7) and the "Schwarzschen Abhandlungen" (reference 8).

Next, the plane z' is rotated about $\pi/2$ and followed by a translation

$$z'' = \sqrt{1 - c^2} - iz' \tag{2}$$

as a result of which the axis of the ordinates of the new z"-plane exactly separates the fixed part of the circumference from the free part. (Compare figs. 2 and 3.) Both parts meet in the points $z'' = \pm ic$. The subsequent transformation

$$z'' = c \tan \zeta \tag{3}$$

projects these two points to infinity, where the arcs of the circle change into straight lines of distance $\pi/2$ (fig. 3). Combining these transformations produces finally the mapping function of the z-plane on the ζ-plane

$$\zeta = \text{arc tan}\,\frac{1}{c}\left[\sqrt{1 - c^2} - i\sqrt{k}\ \text{sn}\left(\frac{2K}{\pi}\ \text{arc sin}\,\frac{z}{e}\right)\right] \tag{4}$$

Since the derivative of this function is used later on, it is given here

$$\frac{d\zeta}{dz} = \frac{-i\frac{2K}{\pi}\sqrt{k}\ \text{ccn}Z\ \text{dn}\ Z}{\sqrt{e^2 - z^2}\left(1 - k\text{sn}^2 Z + 2i\sqrt{k\left(1 - c^2\right)}\,\text{sn}Z\right)} \tag{5}$$

where, for abbreviation, $\frac{2K}{\pi}\ \text{arc sin}\,\frac{z}{e} = Z$ is introduced.

The boundary conditions, disappearance of the normal component on the fixed part and disappearance of the tangential velocity on the free part of the circumference of the ellipse, can be satisfied now in the ζ-plane by repeated reflection of the original vortex doublet at the boundaries. The result is the vortex system represented in figure 4. The potential of this flow is readily defined, since the vortex rows can be added up in horizontal direction (reference 5):

$$F_o = \frac{\Gamma i}{2\pi}\log\frac{\tan\frac{\zeta - \xi' - i\eta'}{2}\tan\frac{\zeta + 2\xi_o + \xi' + i\eta'}{2}}{\tan\frac{\zeta - \xi' + i\eta'}{2}\tan\frac{\zeta + 2\xi_o + \xi' - i\eta'}{2}} \tag{6}$$

with $\xi' + i\eta'$ and $\xi' - i\eta'$ representing the points corresponding to the vortex points $z = \mp s$ in the ζ-plane. The potential of the additional flow is obtained by subtracting the potential of the original flow in the z-plane from the potential F_o

$$F = \frac{\Gamma i}{2\pi}\log\frac{\tan\frac{\zeta - \xi' - i\eta'}{2}\tan\frac{\zeta + 2\xi_o + \xi' + i\eta'}{2}}{\tan\frac{\zeta - \xi' + i\eta'}{2}\tan\frac{\zeta + 2\xi_o + \xi' - i\eta'}{2}} - \frac{\Gamma i}{2\pi}\log\frac{z + s}{z - s} \tag{7}$$

Posting $F = \Phi + i\psi$, the mean additional downward velocity over the span in the z-plane is

$$\overline{w} = \frac{\psi(-s) - \psi(s)}{b} \tag{8}$$

Hence the additional downwash at the wing is

$$\overline{w} = \frac{1}{2}\frac{\Gamma}{4\pi s} \log \left[s^2 \left|\frac{d\zeta}{dz}\right|^2_{(z = s)} \frac{\tan (\xi_o + \xi' + i\eta') \tan (\xi_o + \xi' - i\eta')}{\tan^2 (\xi_o + \xi') \tanh^2 \eta'} \right] \tag{9}$$

The mean downwash w serves to determine the angle-of-attack correction

$$\Delta\alpha = \frac{\overline{w}}{V}$$

Putting, as usual

$$\Delta\alpha = \frac{c_a}{8}\frac{F}{F_o}\delta \tag{10}$$

where δ is the correction factor, c_a the lift coefficient, F the wing area, F_o the cross-sectional area of the jet, and

$$\Gamma = \frac{c_a V t}{2} \quad (t = \text{wing chord})$$

the circulation, the correction factor is

$$\delta = \frac{a'b'}{4s^2}\left\{\log s^2 \left|\frac{d\zeta}{dz}\right|^2_{(z = s)} + \log \frac{\tan^2 (\xi_o + \xi') + \tanh^2\eta' + \tan^2 (\xi_o + \xi') \tanh^4\eta' + \tan^4 (\xi_o + \xi') \tanh^2\eta'}{\tan^2 (\xi_o + \xi') \tanh^2\eta' \left[1 + \tan^2 (\xi_o + \xi') \tanh^2\eta'\right]^2}\right\} \tag{11}$$

For the calculation, the following relations are added.

For points of the major axis, the mapping function (equation (1)) changes to

$$x' = \sqrt{k}\ \mathrm{sn}\left(\frac{2K}{\pi} \text{ arc sin } \frac{x}{e};\ k\right) \tag{12}$$

for $0 < x < e$, and

$$x' = \frac{\sqrt{k}}{\mathrm{dn}\left(\frac{2K}{\pi} \text{ arc cosh } \frac{x}{e};\ k'\right)} \tag{13}$$

for $e < x < a'$

with k' denoting the modulus $k' = \sqrt{1 - k^2}$ complementary to k. The points $z = \pm s$ in which the vortices in the z-plane lie change equations (12) and (13) to $z' = \pm s_1$; in the ζ-plane, the vortices lie then at the points

$$\xi' = \text{arc tan } \frac{1 - 2c^2 + s_1^2 + \sqrt{1 + s_1^2(2 - 4c^2 + s_1^2)}}{2c\sqrt{1 - c^2}} \tag{14}$$

$$\eta' = \text{arc tanh } \frac{1 + s_1^2 + \sqrt{1 + s_1^2(2 - 4c^2 + s_1^2)}}{2cs_1} \tag{15}$$

For the extreme case of a wing with zero span, formula (11) becomes

$$\delta = \frac{a'b'}{4(a'^2 - b'^2)} \frac{4K^2 k}{\pi^2} \left[- 4 \sqrt{1 - c^2} + 2c^2 + C\right] \tag{16}$$

where

$$C = \frac{1}{6} \frac{1}{1 - \frac{b'^2}{a'^2}} \left[\frac{4K^2}{\pi^2}(1 + k^2) - 1\right] + \frac{1}{k}\left(\frac{\pi^2}{4K^2} - 1\right) - k$$

is a constant solely dependent on the axial ratio of the elliptic tunnel section.

3. ELLIPTIC JET WITH TWO SOLID WALLS (FIG. 5)

In this case, it is appropriate to map the inside of the elliptic section on the inside of a rectangle (fig. 5). The fixed walls correspond then to two opposite sides of the rectangle and the free jet boundaries to the other two sides. The first step is the same as before, namely, the inside of the ellipse is mapped on the inside of the unit circle with the aid of the mapping function

$$z' = \sqrt{k}\, \mathrm{sn}\left(\frac{2K}{\pi} \text{arc sin } \frac{z}{e};\ k\right) \tag{17}$$

given by equation (1). The inside of the unit circle becomes the inside of a rectangle by means of the function

$$\mathrm{sn}\left(\zeta;\ \cos\frac{\Theta}{2}\right) = \frac{2z'}{1 + z'^2} \tag{18}$$

if ζ represents the coordinate of the plane of the rectangle and $\cos\frac{\Theta}{2}$ is the modulus of the elliptic function $\mathrm{sn}\,\zeta$ (compare reference 6, p. 170), where Θ is an angle specifying the tunnel section opening ratio in the plane of the circle. (Compare fig. 5.) The connection between the ζ-plane and the original z-plane is therefore given by the mapping function

$$\mathrm{sn}\left(\zeta;\ \cos\frac{\Theta}{2}\right) = \frac{2\sqrt{k}\,\mathrm{sn}\,(Z;\ k)}{1 + k\,\mathrm{sn}^2(Z;\ k)} \tag{19}$$

where, for abbreviation, $\frac{2K}{\pi}$ arc sin $\frac{z}{e} = Z$, as before. The modulus k is again dependent on the axial ratio of the elliptic section. The square of the derivative of this function is given by

$$\left(\frac{d\zeta}{dz}\right)^2 = \frac{\frac{16K^2}{\pi^2} k\,\mathrm{cn}^2 Z\ \mathrm{dn}^2\, Z}{(e^2 - z^2)(1 - 2k \cos\Theta\ \mathrm{sn}^2 Z + k^2 \mathrm{sn}^4 Z)} \tag{20}$$

The conditions at the jet boundaries are satisfied by repeated reflection of the vortex doublet on the sides of the rectangle, while the sense of rotation of the vortices is inverse for reflection at the sides that correspond to the fixed walls and remains the same for reflection on the sides of the rectangle that correspond to the free jet boundaries. The potential of such a system of vortices is given by (compare reference 6)

$$F_o = \frac{\Gamma i}{2\pi}\left\{-\log\frac{\operatorname{sn}\frac{\zeta - \xi'}{2}\operatorname{cn}\frac{\zeta + \xi'}{2}}{\operatorname{dn}\frac{\zeta + \xi'}{2}} + \log\frac{\operatorname{sn}\frac{\zeta + \xi'}{2}\operatorname{cn}\frac{\zeta - \xi'}{2}}{\operatorname{dn}\frac{\zeta - \xi'}{2}}\right\} \tag{21}$$

where $\mp\xi'$ is the location of the original vortex in the ζ-plane (in the z-plane, the wing is in the center of the jet, hence $z = \mp s$). Subtracting from this expression the potential of the original vortex in the z-plane

$$F_1 = \frac{\Gamma i}{2\pi}\log\frac{z + s}{z - s} \tag{22}$$

leaves the potential of the looked-for additional flow

$$F = F_o - F_1 \tag{23}$$

Since $F = \Phi + i\Psi$, the stream function can be deduced, and so yields the mean downwash over the span of a wing suspended in the center of the section

$$\bar{w} = \frac{1}{2}\,\frac{\Psi(-s) - \Psi(s)}{2s}$$

$$\bar{w} = \frac{\Gamma}{4\pi s}\log\left|\left(\frac{d\zeta}{dz}\right)_{z = s}\quad s\,\frac{\mathrm{cn}\xi'}{\mathrm{sn}\xi'\mathrm{dn}\xi'}\right| \tag{24}$$

With

$$\Gamma = \frac{c_a Vt}{2}$$

and from the relation for the additional angle of attack

$$\Delta\alpha = \frac{\bar{w}}{V} = \frac{c_a}{8}\,\frac{F}{F_o}\,\delta \tag{25}$$

the correction factor δ follows as

$$\delta = \frac{a'b'}{2s^2}\log\left|\frac{\frac{2K}{\pi}\mathrm{scn}Z\ \mathrm{dn}\ Z(1 - k^2\mathrm{sn}^4Z)}{\sqrt{e^2} - s^2\mathrm{sn}Z(1 - 2k\cos\Theta\mathrm{sn}^2Z + k^2\mathrm{sn}^4Z)}\right| \quad . . . (26)$$

where k is the modulus of the present elliptic function. For the extreme case of a wing with disappearing span, this equation simplifies to

$$\delta = \frac{a'b'}{6(a'^2 - b'^2)}\left\{1 - \frac{4K^2}{\pi^2}(1 - 6k\cos\Theta + k^2)\right\} \tag{27}$$

4. RESULT

For an elliptic tunnel of $1:\sqrt{2}$ axial ratio, the correction factors δ were computed by equations (11) or (16) and (26) or (27) for wings with rectangular lift distribution for the span-major ellipse axis ratios $\frac{b}{2a'} = 0, 0.2, 0.4, 0.6,$ and 0.8 and plotted in figures 6 and 7 against an angle φ or ϑ which is decisive for the ratio of fixed wall and free jet boundary. It is noted that the correction factors for $\varphi = 0$ or $\vartheta = \pi$ and $\varphi = 2\pi$ or $\vartheta = 0$, that is, for completely open and completely closed jets, assume the values given by Sanuki and Tani (reference 9).

The diagrams indicate the distance which the solid part of the jet boundary must reach to produce zero correction. The amount of this favorable coverage varies with the span of the wing, as seen from figure 8 where the angles for which $\delta = 0$ are plotted against the span-tunnel width ratio.

Clearer than the plotting of the angles is the representation of the ratio d/b', where d is the distance between the major axis and the point at which solid and free boundary meet (fig. 9).

In general, the span of a model wing for a tunnel of $1:\sqrt{2}$ axial ratio amounts to about $b = 0.7 \times 2a'$. So in this case with the use of one solid wall the covering would have to extend to $\frac{d}{b'} = 0.385$ to produce zero correction. By choosing a partial covering of the jet boundaries with two solid walls, the correction would disappear for $\frac{d}{b'} = 0.64$, so that the last arrangement seems more promising in many respects as far as experimental technique is concerned, but naturally calls for more elaborate test preparations.

Translated by J. Vanier
National Advisory Committee
for Aeronautics

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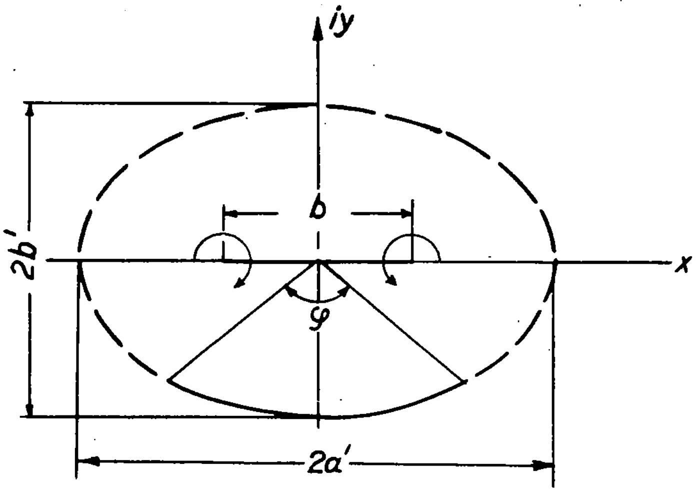

Figure 1.- The elliptic jet with one solid wall.

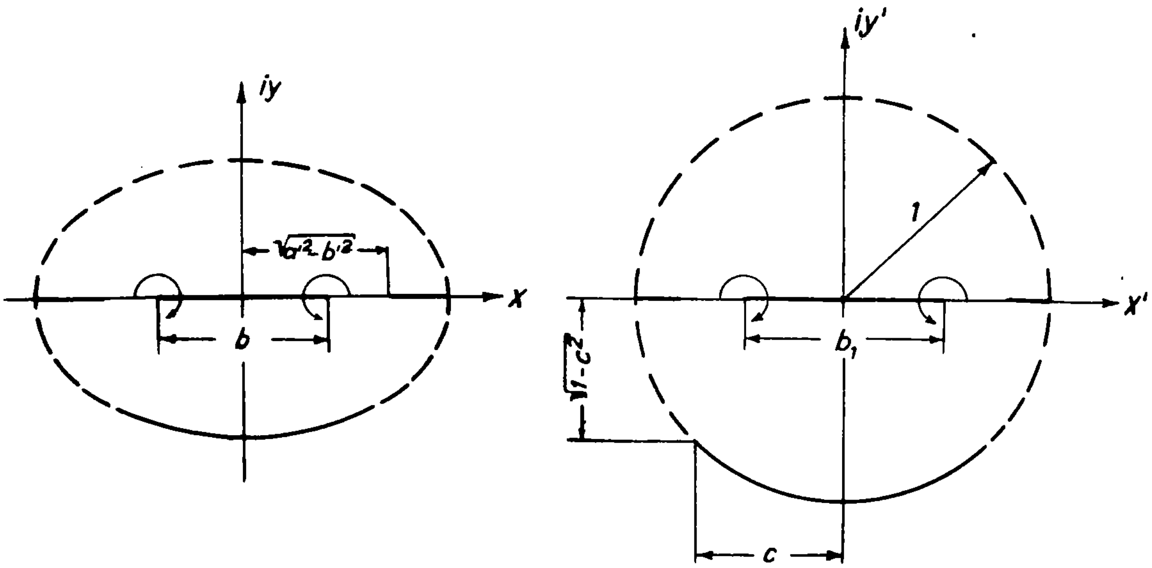

Figure 2.- Mapping of z-plane on the z'-plane.

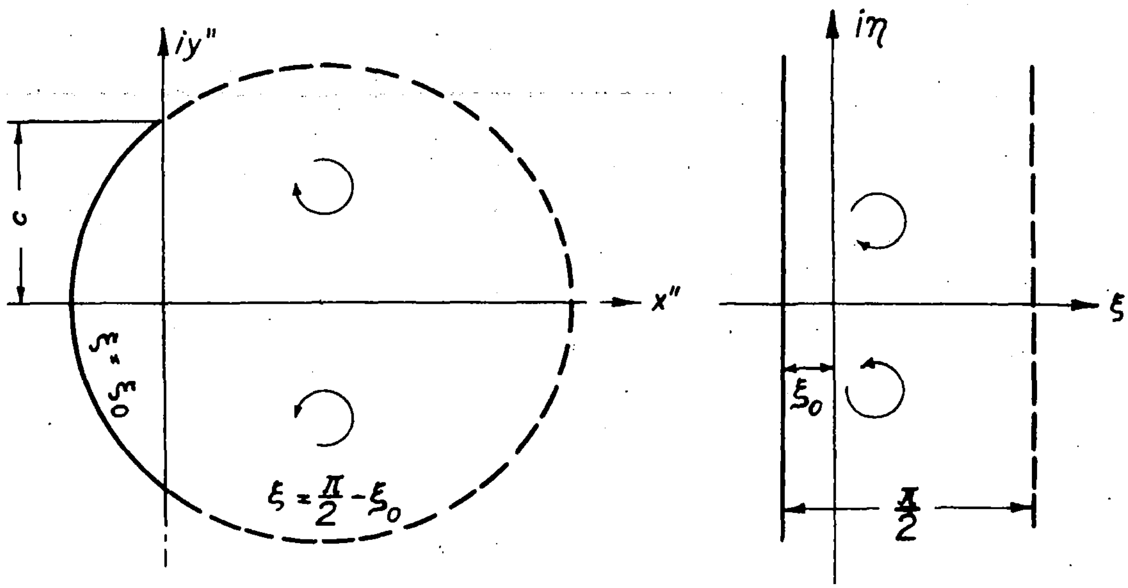

Figure 3.- Mapping of the z''-plane on the ζ-plane.

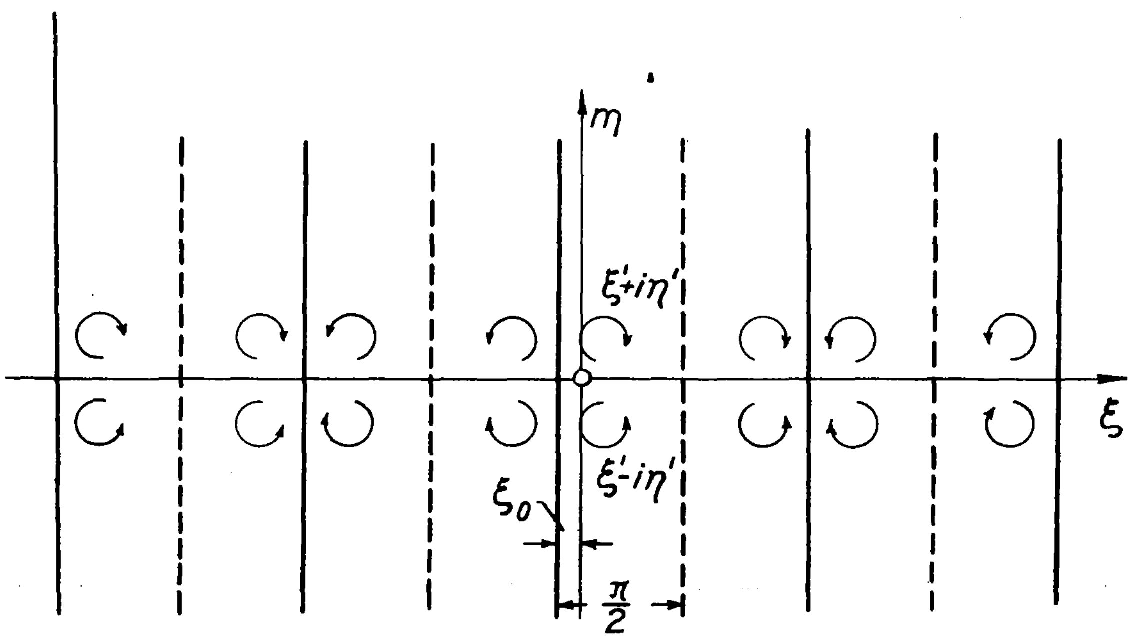

Figure 4.- The reflection system for compliance with the boundary conditions.

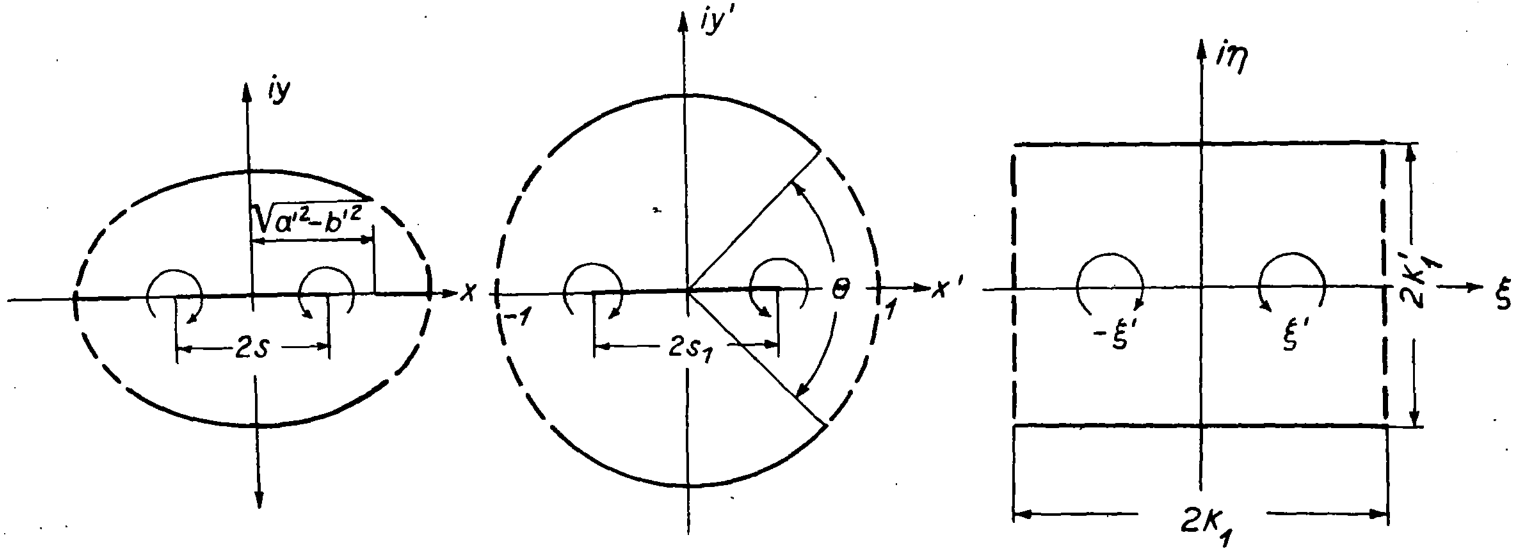

Figure 5.- Mapping the elliptic section bounded by two walls on a rectangle.

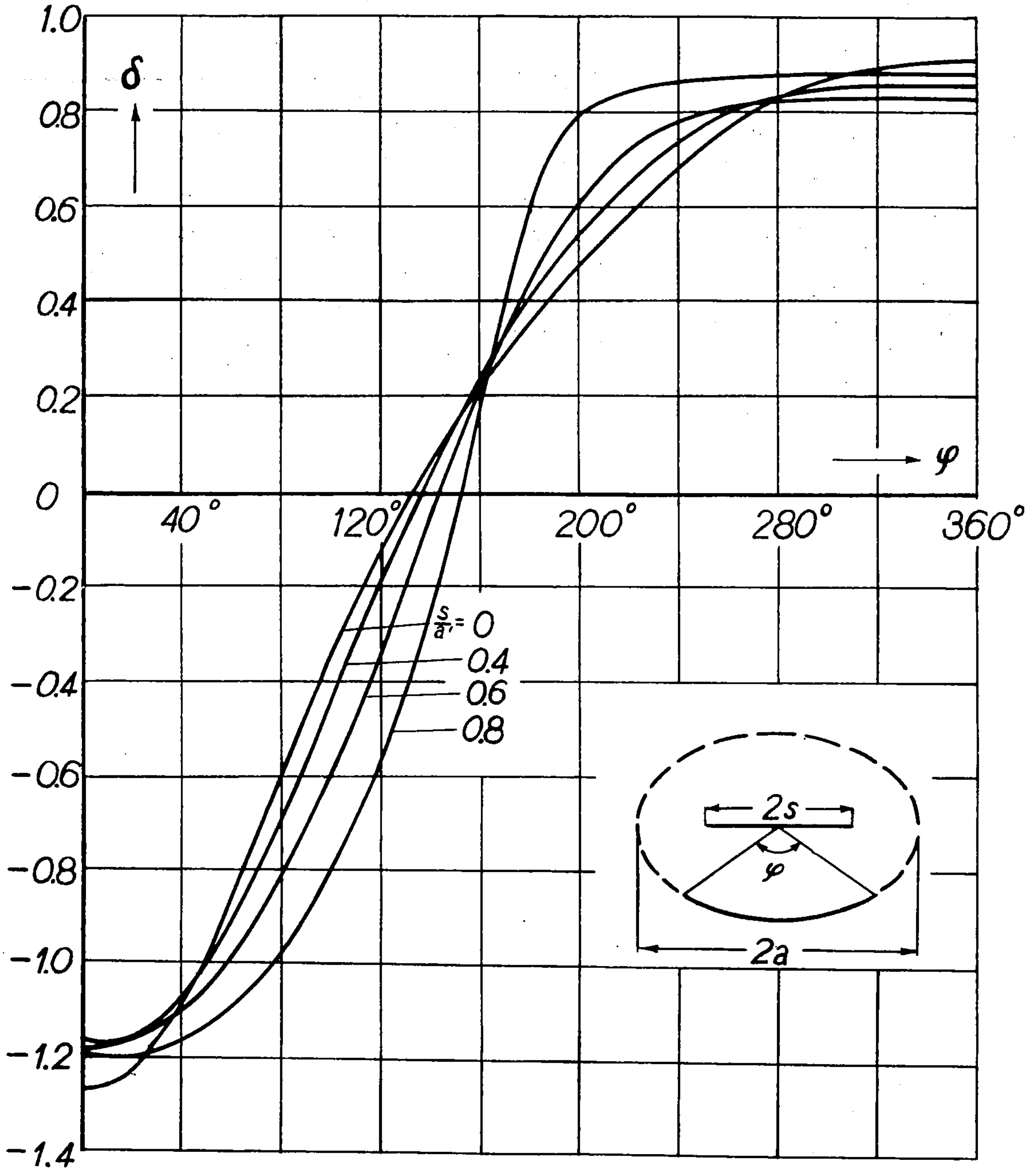

Figure 6.- The correction factor δ plotted against angle φ - one solid wall.

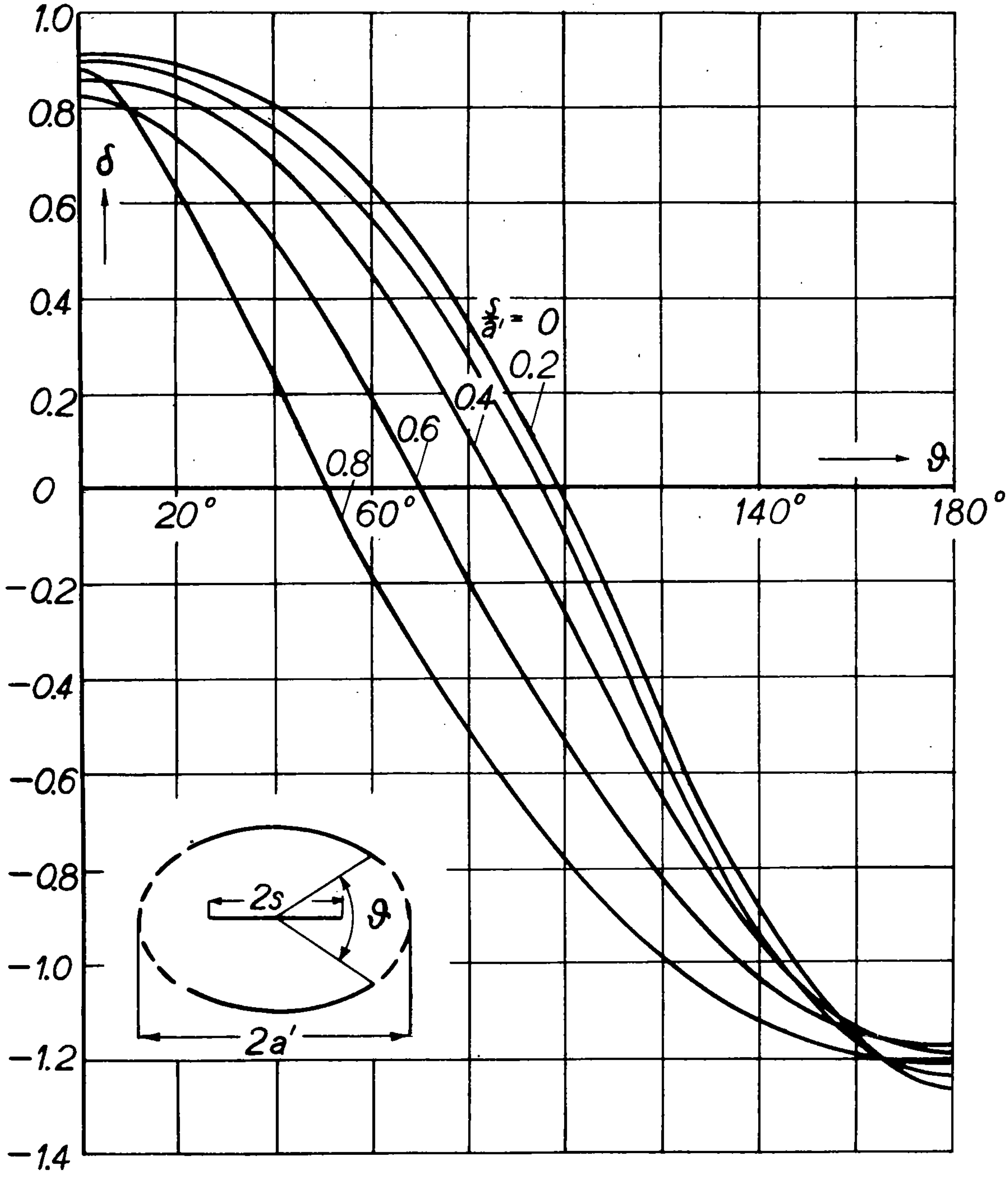

Figure 7.- The correction factor δ plotted against angle ϑ - two solid walls.

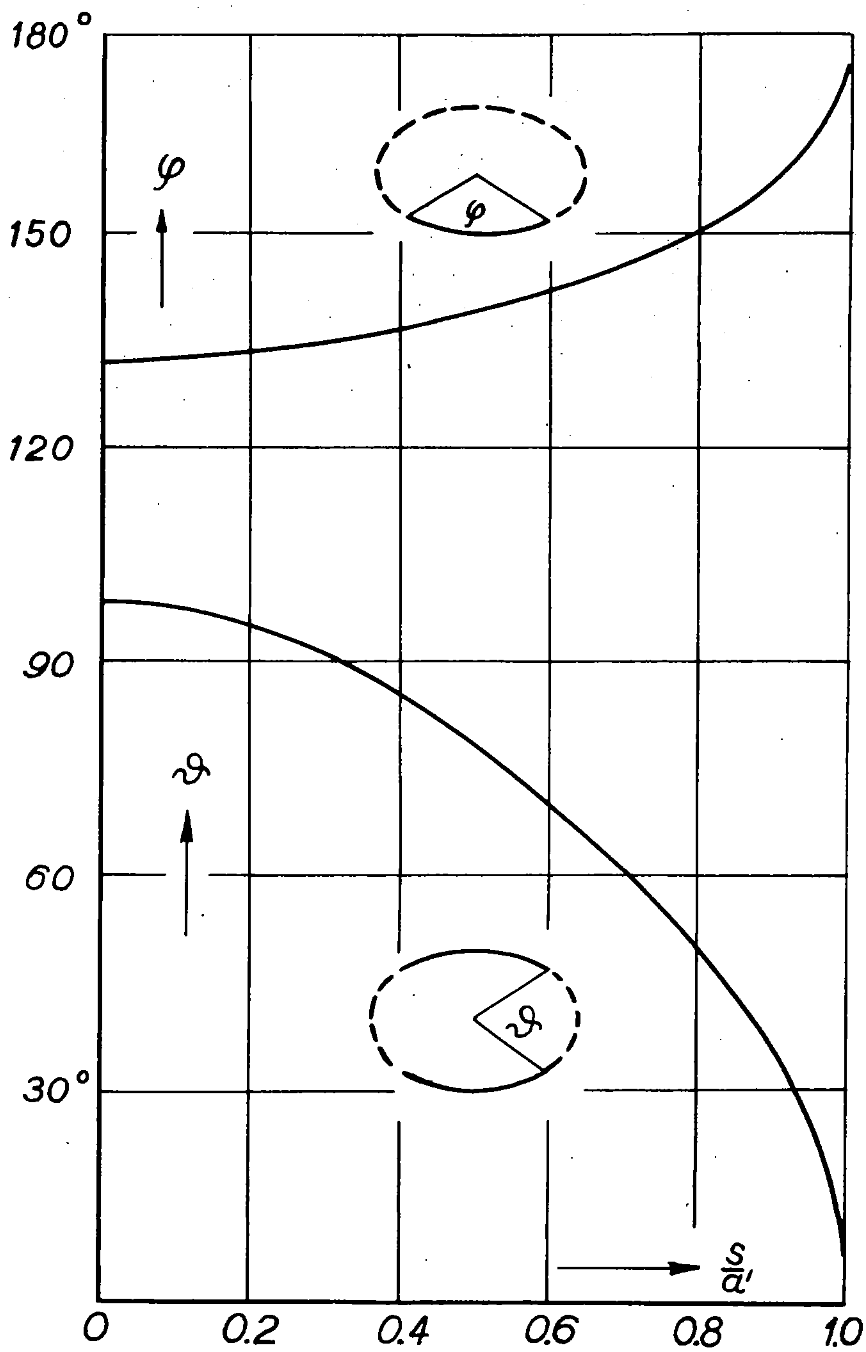

Figure 8.- The values of φ and ϑ for zero correction factor plotted against the span-tunnel width ratio.

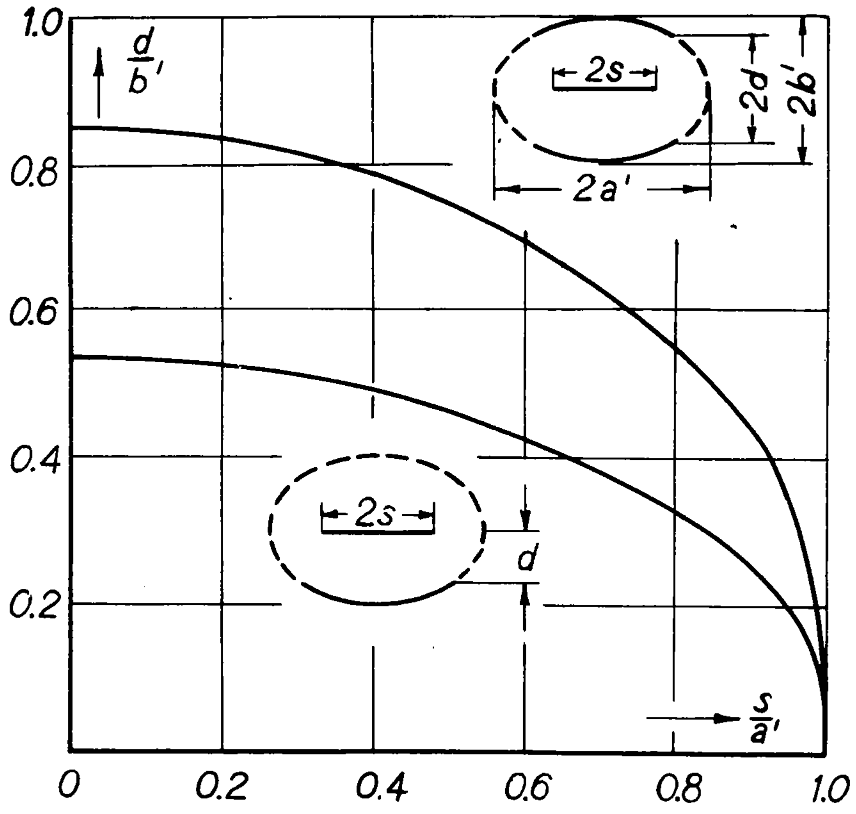

Figure 9.- $\frac{d}{b'}$ for $\delta = 0$.