NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1250

SUBSONIC GAS FLOW PAST A WING PROFILE

By S. A. Christianovich and I. M. Yuriev

Translation

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SUBSONIC GAS FLOW PAST A WING PROFILE *

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The use of the linearized equations of Chaplygin to calculate the subsonic flow of a gas permits solving the problem of the flow about a wing profile for absence and presence of circulation (reference 1).

The solution is obtained in a practical convenient form that permits finding all the required magnitudes for the gas flow (lift, lift moment, velocity distribution over the profile, and critical Mach number). This solution is not expressed in simple closed form; for a certain simplifying assumption, however, the equations of Chaplygin can be reduced to equations with constant coefficients, and solutions are obtained by using only the mathematical apparatus of the theory of functions of a complex variable.

The method for simplifying the equations was pointed out by Chaplygin himself (reference 2). Tsien (reference 3) applied similar equations to the solution of the flow problem and obtained a solution for the case of the absence of circulation. He did not succeed in obtaining the flow about the profile in the presence of circulation.

1. EQUATIONS OF MOTION

The equations of the two-dimensional adiabatic irrotational flow of a gas are of the form

$$\begin{gathered}\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \\ \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \qquad (1.1) \\ \frac{w^2}{2} + \frac{a^2}{\chi - 1} = \frac{a_*^2(\chi+1)}{2(\chi-1)}\end{gathered}$$

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where w is the magnitude of the velocity, ρ the density, a the velocity of sound, a_* the critical velocity, and u and v the projections of the velocity vector on the x- and y-axes.

From these equations it follows that the expressions

$$d\varphi = \frac{u}{a_*}\,dx + \frac{v}{a_*}\,dy$$
$$d\psi = -\frac{\rho}{\rho_0}\frac{v}{a_*}\,dx + \frac{\rho}{\rho_0}\frac{u}{a_*}\,dy \qquad (1.2)$$

are total differentials; ρ_0 is the density of the gas at $w=0$.

By solving these equations for dx and dy, there is obtained

$$dx = \frac{\cos\vartheta}{\lambda}\,d\varphi - \frac{\rho_0}{\rho}\frac{\sin\vartheta}{\lambda}\,d\psi$$
$$dy = \frac{\sin\vartheta}{\lambda}\,d\varphi + \frac{\rho_0}{\rho}\frac{\cos\vartheta}{\lambda}\,d\psi \qquad (1.3)$$

where $\lambda = w/a_*$.

From the conditions expressing the fact that the right sides of equations (1.3) are total differentials

$$\frac{\partial}{\partial\psi}\frac{\sin\vartheta}{\lambda} = \frac{\partial}{\partial\varphi}\left(\frac{\rho_0}{\rho}\frac{\cos\vartheta}{\lambda}\right)$$
$$\frac{\partial}{\partial\psi}\frac{\cos\vartheta}{\lambda} = -\frac{\partial}{\partial\varphi}\left(\frac{\rho_0}{\rho}\frac{\sin\vartheta}{\lambda}\right)$$

the equations of Chaplygin are obtained

$$\frac{\partial\vartheta}{\partial\varphi} = \frac{1}{\lambda}\left(1 - \frac{\chi-1}{\chi+1}\lambda^2\right)^{\frac{1}{\chi-1}}\frac{\partial\lambda}{\partial\varphi}$$

$$\frac{\partial\vartheta}{\partial\varphi} = -\frac{1-\lambda^2}{\lambda}\left(1 - \frac{\chi-1}{\chi+1}\lambda^2\right)^{\frac{\chi}{1-\chi}}\frac{\partial\lambda}{\partial\varphi}$$

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By introducing in place of the variable λ the variable s defined by the equation (reference 1)

$$ds = \sqrt{\frac{1-\lambda^2}{1-\lambda^2(\chi-1)/(\chi+1)}}\,\frac{d\lambda}{\lambda} \tag{1.4}$$

the equation of Chaplygin is transformed to the form

$$\begin{aligned} \frac{\partial s}{\partial \varphi} &= \sqrt{K}\,\frac{\partial \vartheta}{\partial \varphi} \\ \frac{\partial \vartheta}{\partial \varphi} &= -\sqrt{K}\,\frac{\partial s}{\partial \varphi} \end{aligned} \tag{1.5}$$

where

$$\sqrt{K} = \sqrt{(1-\lambda^2)\left(1-\frac{\chi-1}{\chi+1}\lambda^2\right)^{\frac{1+\chi}{1-\chi}}} = \frac{\rho_0}{\rho}\sqrt{1-M^2}$$

The Mach number is $M = w/a$. The constant of integration in equation (1.4) is chosen with the condition that $e^s/\lambda = 1$ for $\lambda \to 0$.

The function e^s is denoted in that which follows by $\tilde{\lambda}$. Tables of the values of $\tilde{\lambda}$ and $\sqrt{K}$ as functions of λ are given in reference 1.

If in equations (1.5) μ and ν are taken as the unkown variables connected with s and ϑ by the relation

$$s - i\vartheta = f\,(\mu + i\nu)$$

where $f(\mu + i\nu)$ is an analytic function of $\mu + i\nu$, equations (1.5) can be replaced by the system of equations

$$\begin{aligned} \frac{\partial s}{\partial \mu} &= -\frac{\partial \vartheta}{\partial \nu} \\ -\frac{\partial s}{\partial \nu} &= \frac{\partial \vartheta}{\partial \mu} \end{aligned} \tag{1.6}$$

and

$$\frac{\partial\varphi}{\partial\mu} = \sqrt{K}\,\frac{\partial\psi}{\partial\nu}$$

$$\frac{\partial\varphi}{\partial\nu} = -\sqrt{K}\,\frac{\partial\psi}{\partial\mu} \tag{1.7}$$

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The coordinates x and y are determined from equations (1.3). The functions satisfying equations (1.6) and (1.7) convert the right sides of equations (1.3) into total differentials.

2. APPROXIMATE EQUATIONS

It is assumed as an approximation in equations (1.5) that $\sqrt{K} \approx \sqrt{K_\infty}$ where $\sqrt{K_\infty}$ is the value of $\sqrt{K}$ corresponding to $\lambda = \lambda_\infty$.

Let

$$\sqrt{K_\infty}\,\psi = \overline{\Psi}$$

Equations (1.7) then assume the form

$$\frac{\partial\varphi}{\partial\mu} = \frac{\partial\overline{\Psi}}{\partial\nu}$$

$$\frac{\partial\varphi}{\partial\nu} = -\frac{\partial\overline{\Psi}}{\partial\mu} \tag{2.1}$$

Because of this simplification of equations (1.7), the right sides of equations (1.3) will no longer be total differentials after the solutions of the system of equations (1.6) and (2.1) are substituted in them. It is necessary to make changes in equations (1.3) to compensate for the simplification of equations (1.7).

Then

$$\frac{d\lambda}{ds} = \lambda\,\frac{\rho_0}{\rho}\,\frac{1}{\sqrt{K}} \approx \lambda\frac{\rho_0}{\rho}\,\frac{1}{\sqrt{K_\infty}}$$

$$\frac{d}{ds}\left(\frac{1}{\lambda}\right) \approx -\frac{1}{\lambda}\,\frac{\rho_0}{\rho}\,\frac{1}{\sqrt{K_\infty}}$$

Moreover,

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$$\frac{d}{ds}\left(\frac{\rho_0}{\rho}\,\frac{1}{\lambda}\right) = -\frac{\sqrt{K}}{\lambda} \approx -\frac{\sqrt{K_\infty}}{\lambda}$$

$$\frac{d^2}{ds^2}\left(\frac{1}{\lambda}\right) = \frac{1}{\lambda}$$

Whence

$$\frac{1}{\lambda} \approx C_1 e^{s} + C_2 e^{-s}$$

$$\frac{\rho_0}{\rho}\,\frac{1}{\lambda} \approx \left(C_2 e^{-s} - C_1 e^{s}\right)\sqrt{K_\infty} \tag{2.2}$$

The values of the constants are chosen from the condition requiring the exact satisfying of equations (2.2) for $\lambda = \lambda_\infty$.

$$C_1 = \frac{\tilde{\lambda}_\infty}{2\lambda_\infty}\,\frac{1}{\tilde{\lambda}_\infty^2}\left(1 - \frac{1}{\sqrt{1-M_\infty^2}}\right)$$

$$C_2 = \frac{\tilde{\lambda}_\infty}{2\lambda_\infty}\left(1 + \frac{1}{\sqrt{1-M_\infty^2}}\right) \tag{2.3}$$

Substituting in equations (1.3) the corresponding expressions from equations (2.2), the following equation is obtained:

$$dx + i dy = C_1 e^{s+i\vartheta}\left(d\varphi - i d\overline{\psi}\right) - C_2 e^{-s+i\vartheta}\left(d\varphi + i d\overline{\psi}\right) \tag{2.4}$$

The right side of equation (2.4) by virtue of equations (1.6) and (2.1) is a total differential.

Consider the Jacobian $D(x,y)/D(\varphi,\overline{\psi})$: Making use of equation (2.4), it is found that

$$\frac{D(x,y)}{D(\varphi,\psi)} = C_2^2 e^{-2s} - C_1^2 e^{2s}$$

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In order that the determinant should not be zero, it is necessary that the following inequality be satisfied:

$$\frac{\tilde{\lambda}}{\lambda_\infty} < \sqrt{\frac{m+1}{m-1}} \qquad \left(m = \frac{1}{1-M_\infty^2}\right)$$

This condition is a consequence of the modification made in the equations of motion; however, it is not essential. This inequality is generally satisfied if the fundamental inequality $\bar{\lambda} < \bar{\lambda}_{cr}$ is satisfied where $\bar{\lambda}_{cr} = 0.7577 \ldots$ corresponds to $\lambda = 1$.

3. FORMULATION OF THE PROBLEM

In the plane $z = \mu + i\nu$, a closed contour about which is a flow of an incompressible fluid is considered. The direction of the flow will be taken to coincide with the direction of the x-axis and the velocity at infinity λ_∞. Let $f = f(z)$ be the complex potential corresponding to this flow and set

$$s - i\vartheta = \log \frac{df}{dz} \tag{3.1}$$

The solution of equations (1.6) as determined by equations (3.1) is to be treated as the solution of the problem of the flow about a profile in the μ,ν-plane of a fictitious stream of an incompressible fluid. The magnitude of the velocity at each point of this flow is equal to $\tilde{\lambda}$ and the angle of inclination of the velocity vector to the x-axis is equal to ϑ; thus

$$\frac{df}{dz} = \tilde{\lambda} e^{-i\vartheta}$$

The complex potential may be represented in the form

$$\left.\begin{aligned} f &= \tilde{\lambda}_\infty\left(\zeta + \frac{1}{\zeta} + \frac{\tilde{\gamma}}{2\pi i}\log \zeta\right) \\ z &= \chi(\zeta) \end{aligned}\right\} \tag{3.2}$$

where $z = \chi(\xi)$ is a function mapping the outer region with respect to a circle of unit radius in the ξ-plane on a region outside the contour considered. This function satisfies the conditions

$$\chi(\infty) = \infty$$

$$\left(\frac{d\chi}{d\xi}\right)_{\xi = \infty} = 1$$

The last condition determines the scale for the contour in the plane $\mu + i\nu$.

The expansion of $z = \chi(\xi)$ in the neighborhood of the point at infinity has the form

$$z = \xi + a_0 + \frac{a_1}{\xi} + \frac{a_2}{\xi^2} + \ldots \tag{3.3}$$

Hence in the neighborhood of the point at infinity

$$\frac{df}{dz} = \tilde{\lambda}_\infty \left[1 + \frac{\tilde{\gamma}}{2\pi i}\frac{1}{\xi} + \frac{a_1 - 1}{\xi^2} + \ldots\right] \tag{3.4}$$

The circulation of the velocity $\tilde{\Gamma}$ in the flow of an incompressible fluid is equal to $\tilde{\Gamma} = \tilde{\gamma}\tilde{\lambda}_\infty$.

In the ξ-plane, the velocity potential φ and the stream function $\bar{\psi}$ will be considered also as the velocity potential and the stream function of an incompressible fluid about a circular cylinder.

The complex potential $F = \varphi + i\bar{\psi}$ is introduced and represented in the form

$$F = \tilde{\lambda}_\infty \left(\xi + \frac{1}{\xi} + \frac{\gamma}{2\pi i} \log \xi\right) \tag{3.5}$$

where γ is as yet an arbitrary parameter determining the circulation of this flow.

In the solution of the exact equations it is necessary, in place of the function (3.5), to find the exact corresponding solution of the linear system (1.7). In reference 1, this solution is constructed by the method of successive approximations.

Equation (2.4), in the notation assumed, may be written in the form

$$dx + idy = C_2 \frac{1}{df/dz}\, dF + C_1 \overline{\frac{df}{dz}}\, \overline{dF} \qquad (3.6)$$

By considering all magnitudes as functions of the variable ξ, the following expression is obtained:

$$dx + idy = C_2 \frac{dF/d\xi}{df/dz}\, d\xi + C_1 \overline{\frac{dF}{d\xi}}\, \overline{\frac{df}{dz}}\, \overline{d\xi} \qquad (3.7)$$

or transforming it again

$$dx + idy = C_2 \frac{dF/d\xi}{df/d\xi}\, dz + C_1 \overline{\frac{dF}{d\xi}}\, \overline{\frac{df}{d\xi}} \left(\frac{d\xi}{dz}\right)^2 \overline{dz} \qquad (3.8)$$

Thus, knowing the function $z = \chi(\xi)$, it is possible to make use of equation (3.7) or (3.8) to determine x and y with the aid of quadratures.

The conditions the functions $z = \chi(\xi)$ and equation (3.5) must satisfy in order that the obtained solution correspond to a physically possible flow are considered.

In order that the solution have a physical sense, it is necessary that the region in the x,y-plane corresponding to this solution be a single-sheet region.

A necessary condition for this condition is that the integral of $dx + idy$ should be zero over any closed contour in the region $|\xi| > 1$.

Because the expressions on the right of equation (3.7) are the sum of analytic functions of ξ and $\overline{\xi}$ having no singularities in the region $|\xi| > 1$, the corresponding integrals do not depend on the shape of the path of integration and are equal to zero in going around any closed contour not including the circle $|\xi| = 1$.

The values of the integrals along any closed contour, including the circle $|\xi| = 1$, are equal; consider the integral of $dx + idy$ along a circle of large radius with center at the origin of coordinates.

In the neighborhood of the point at infinity

$$\frac{1}{df/dz} = \frac{1}{\lambda_\infty}\left[1 - \frac{\tilde{\gamma}}{2\pi i}\frac{1}{\xi} - \left(\frac{\tilde{\gamma}^2}{4\pi^2} + a_1 - 1\right)\frac{1}{\xi^2} + \ldots\right]$$

whence

$$\frac{dF/d\xi}{df/dz} = 1\,\frac{\gamma - \tilde{\gamma}}{2\pi i}\frac{1}{\xi} + \left(\frac{\gamma\tilde{\gamma} - \tilde{\gamma}^2}{2\pi^2} - a_1\right)\frac{1}{\xi^2} + \ldots$$

Moreover

$$\overline{\frac{df}{dz}}\,\overline{\frac{dF}{d\xi}} = \tilde{\lambda}_\infty^{\,2}\left[1 - \frac{\gamma + \tilde{\gamma}}{2\pi i}\frac{1}{\xi} - \left(\frac{\gamma\tilde{\gamma}}{4\pi^2} - \bar{a}_1 + 2\right)\frac{1}{\xi^2} + \ldots\right]$$

By substituting in equation (3.7), integrating over the circle $|\xi| = R$, and passing to the limit as $R \to \infty$

$$\int dx + idy = C_2\,(\gamma - \tilde{\gamma}) + C_1\tilde{\lambda}_\infty^{\,2}(\gamma + \tilde{\gamma})$$

Replacing C_1 and C_2 by the corresponding values from equations (2.3) gives

$$\int dx + idy = \frac{\tilde{\lambda}_\infty}{\tilde{\lambda}_\infty}\left(\gamma - \frac{\tilde{\gamma}}{\sqrt{1 - M_\infty^2}}\right)$$

Thus the condition that to any closed contour in the region $|\xi| > 1$ should correspond a closed contour in the x,y-plane consists in the fact that the previously undetermined value of γ in the function (3.5) must be taken equal to

$$\gamma = \frac{\tilde{\gamma}}{\sqrt{1 - M_\infty^2}} \qquad (3.9)$$

This value of γ is proportional to the potential in the passage around the closed contour in the x,y-plane and therefore is proportional to the circulation of the actual flow.

When equation (3.7) in the neighborhood of the point at infinity is considered, it is found that the ellipse

$$x - x_0 = \frac{\tilde{\lambda}_\infty}{\lambda_\infty} p + O\left(\frac{1}{R}\right)$$

$$y - y_0 = \frac{\tilde{\lambda}_\infty}{\lambda_\infty\sqrt{1 - M_\infty^2}} q + O\left(\frac{1}{R}\right) \tag{3.10}$$

corresponds to the circle of radius R in the plane $\xi = p + iq$.

Thus the neighborhood of the point at infinity in the ξ-plane is uniquely mapped on the region of the point at infinity in the x,y-plane.

4. INVESTIGATION OF CONTOUR IN X,Y-PLANE

Equation (3.7) is considered near the contour $|\xi| = 1$. The derivative of the function (3.2) becomes zero on the circle $|\xi| = 1$ at the two points

$$\xi_{01} = e^{i\vartheta_0}$$

$$\xi_{02} = e^{i(\pi - \theta_0)} \qquad \left(\sin\theta_0 = \frac{\tilde{\gamma}}{4\pi}\right) \tag{4.1}$$

Then

$$\frac{df}{d\xi} = \tilde{\lambda}_\infty \frac{(\xi - \xi_{01})(\xi - \xi_{02})}{\xi^2} \tag{4.2}$$

In an analogous manner, the derivative of the function (3.5) where γ is determined by equation (3.9) can be represented in the form

$$\frac{dF}{d\xi} = \tilde{\lambda}_\infty \frac{(\xi - \xi_{*1})(\xi - \xi_{*2})}{\xi^2} \tag{4.3}$$

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where

$$\xi_{*1} = e^{i\vartheta_*}$$

$$\xi_{*2} = e^{i(\pi-\theta_*)} \qquad \left(\sin\theta_* = \frac{\tilde{\gamma}}{4\pi\sqrt{1 - M_\infty^2}}\right) \tag{4.4}$$

Equation (3.7) will therefore be

$$dx + idy = C_2\frac{(\xi-\xi_{*1})(\xi-\xi_{*2})}{(\xi-\xi_{02})(\xi-\xi_{02})}\chi'(\xi)\,d\xi + C_1\tilde{\lambda}_\infty^2\frac{(\bar{\xi}-\bar{\xi}_{01})(\bar{\xi}-\bar{\xi}_{02})}{-\xi^4}\,(\bar{\xi}-\bar{\xi}_{*1})(\bar{\xi}-\bar{\xi}_{02})\,\frac{d\bar{\xi}}{\lambda'(\xi)} \tag{4.5}$$

In the absence of circulation, for example, for $\widetilde{\gamma} = \gamma = 0$,

$$dx + idy = C_2\chi'(\xi)\,d\xi + C_1\tilde{\lambda}_\infty^2\frac{(\bar{\xi}^2 - 1)^2}{\xi^4}\,\frac{d\bar{\xi}}{\overline{\chi'(\xi)}} \tag{4.6}$$

It is shown by equation (3.7) or (4.5) that the points of the contour corresponding to the zeros of the functions df/dz and $dF/d\xi$ will, generally speaking, be singular points of the contour. The presence of these singularities is a consequence of the modification of equations (1.7).

In considering in greater detail what takes place in the neighborhood of these points, let df/dz in the neighborhood of $\xi = \xi_0$ have an expansion in the form

$$\frac{df}{dz} = (\xi - \xi_0)^k\left[l_0 + l_1(\xi - \xi_0) + \ldots\right]$$

where $0 \leq k \leq 1$. To a cusp of the profile in the z-plane (for example, the tail point of the profile with zero angle) corresponds the value $k = 0$, and to a regular point of the contour corresponds the value $k = 1$. In the neighborhood of the point $\xi = \xi_0$ (from equation (3.7))

$$dx + idy = \frac{A}{(\xi - \xi_0)^k}\,d\xi + \ldots$$

where A is a certain coefficient; thus,

for $k<1$

$$x - x_0 + i (y - y_0) = \frac{A}{1 - k} (\xi - \xi_0)^{1-k} + \ldots$$

for $k = 1$

$$x - x_0 + i (y - y_0) = A \log (\xi - \xi_0) + \ldots$$

It follows that an angular point of the contour in the x,y-plane will correspond to the point $\xi = \xi_0$; this angular point with angle $\pi(1-k)$ is turned concave to the flow; for $k = 1$ the vertex of the angle approaches infinity.

The point $\xi = \xi_*$ at which $dF/d\xi$ becomes zero is now considered. Let df/dz at this point also become zero and the expansion of this function into a series in the neighborhood of this point take the form

$$\frac{df}{dz} = (\xi - \xi_0)^{l} \left[m_0 + m_1 (\xi - \xi_0) + \ldots\right]$$

The point at which $l = 0$ is a regular point of the contour in the plane $z = \mu+i\nu$. The point at which $1 \geq l > 0$ is an angular point of the contour. At this point, the angle is concave to the region outside the contour.

In the neighborhood of the point $\xi = \xi_*$,

$$dx + idy = B (\xi - \xi_*)^{1-l} d\xi + \ldots$$

and therefore

$$x - x_0 + i (y - y_0) = \frac{B}{2 - l} (\xi - \xi_*)^{2-l} + \ldots$$

To this point in the x,y-plane also corresponds a cusp. For $l = 0$ this point is a cusp with its sharp point facing the flow; for $l > 0$ this point is angular with angle $\pi(2-l)$; for $l = 1$ it is an ordinary point of the contour.

5. FLOW ABOUT AN ARC OF A CIRCLE

Before proceeding to the consideration of the flow about a contour of arbitrary shape, the example of the flow about a contour corresponding to an arc of a circle in the plane $z = \mu + i\nu$ should be considered. The function

$$z = \xi + ki + \frac{1 - k^2}{\xi + ki}$$

where $k < 1$ maps the region outside the "cut" along the arc of the circle shown in figure 1 into the region outside the circle $|\xi| = 1$.

The derivative

$$\frac{dz}{d\xi} = 1 - \frac{1 - k^2}{(\xi + ki)^2}$$

becomes zero at the points

$$\xi_{01} = - ki - \sqrt{1 - k^2}$$

$$\xi_{02} = - ki + \sqrt{1 - k^2}$$

corresponding to the ends of the arc. The angle θ_0 is therefore determined by the equation

$$\sin \theta_0 = - k$$

Then

$$\frac{dz}{d\xi} \quad \frac{(\xi - \xi_{01})(\xi - \xi_{02})}{(\xi + ki)^2}$$

Only the case of flow for which the velocities at the ends of the arc are finite can be considered. In this case,

$$\frac{df}{dz} = \tilde{\lambda}_\infty \frac{(\xi + ki)^2}{\xi^2}$$

$$\frac{\tilde{\gamma}}{2\pi i} = 2ki$$

Thus

$$\frac{\gamma}{2\pi i} \quad \frac{2ki}{\sqrt{1 - M_\infty^2}}$$

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$$\sin \theta_* = - \frac{k}{\sqrt{1 - M_\infty^2}}$$

The derivative of the function $F(\xi)$ with respect to ξ in this case will be

$$\frac{dF}{d\xi} = \tilde{\lambda}_\infty \frac{(\xi - \xi_{*1})(\xi - \xi_{*2})}{\xi^2}$$

where

$$\xi_{*1} = - \frac{ki}{\sqrt{1 - M_\infty^2}} - \frac{\sqrt{1 - k^2 - M_\infty^2}}{\sqrt{1 - M_\infty^2}}$$

$$\xi_{*2} = - \frac{ki}{\sqrt{1 - M_\infty^2}} + \frac{\sqrt{1 - k^2 - M_\infty^2}}{\sqrt{1 - M_\infty^2}}$$

Equation (3.7) assumes the following form:

$$dx + idy = C_2 \frac{(\xi - \xi_{*1})(\xi - \xi_{*2})}{(\xi + ki)^2} d\xi + C_1 \tilde{\lambda}_\infty^2 (\bar{\xi} - \bar{\xi}_{*1})(\bar{\xi} - \bar{\xi}_{*2}) \frac{(\bar{\xi} - ki)^2}{\bar{\xi}^4} d\bar{\xi}$$

By integrating this equation,

$$x + iy = C_2 \left\{ \xi + 2ki \left(\frac{1}{\sqrt{1 - M_\infty^2}} - 1 \right) \log (\xi + ki) + \right.$$

$$\left. \frac{1}{\xi + ki} \left[1 - k^2 \left(\frac{2}{\sqrt{1 - M_\infty^2}} - 1 \right) \right] \right\} + C_1 \tilde{\lambda}_\infty^2 \left\{ \xi + \frac{1}{\xi} - 2ki \left(\frac{1}{\sqrt{1 - M_\infty^2}} + 1 \right) \log \xi + \right. \qquad (5.1)$$

$$\left. \frac{k^2}{\xi} \left(\frac{4}{\sqrt{1 - M_\infty^2}} + 1 \right) - \frac{ki}{\xi^2} \left(1 + \frac{k^2}{\sqrt{1 - M_\infty^2}} \right) - \frac{k^2}{3\xi^3} \right\} + kiC$$

The constant of integration C is chosen so that for $M_\infty \to 0$ the initial arc of the circle is obtained. For $M_\infty \to 0$, equation (5.1) becomes

$$x + iy = \xi + kiC + \frac{1 - k^2}{\xi + ki}$$

It is assumed that $C = \tilde{\lambda}_\infty / \lambda_\infty$.

The ends of the contour obtained correspond to the points ξ_{*1} and ξ_{*2}, and the points ξ_{01} and ξ_{02} become ordinary points of the contour. The tangents at the angular points of the contour coincide at the upper and lower sides. With an accuracy up to k^2 for x and k^3 for y, the coordinates of the end points of the arc will be

$$x = \pm \left\{ 2\frac{\tilde{\lambda}_\infty}{\lambda_\infty} + \frac{\tilde{\lambda}_\infty}{\lambda_\infty} k^2 \left(-\frac{2 + M_\infty^2}{1 - M_\infty^2} + \frac{2}{3\sqrt{1 - M_\infty^2}} + \frac{1}{3} \right) \right\} + \ldots$$

$$y = \frac{\tilde{\lambda}_\infty}{\lambda_\infty} k^3 \left(\frac{1}{\sqrt{1 - M_\infty^2}} - 1 \right) + \ldots$$

The maximum thickness of the contour is equal to

$$y\left(\frac{\pi}{2}\right) - y\left(-\frac{\pi}{2}\right) = \frac{8}{3} \frac{\tilde{\lambda}_\infty}{\lambda_\infty} k^2 \left(\frac{1}{\sqrt{1 - M_\infty^2}} - 1 \right)$$

The arc corresponding to the case $k \approx 0.04$, $M_\infty = M_{cr} = 0.803$ is shown in figure 2.

If in the expression for x there are retained only the magnitudes of the first order and in the expression for y only the magnitudes of the order of k, the profile will be an arc of a parabola

$$y = 2\frac{\tilde{\lambda}_\infty}{\lambda_\infty} k \left(1 - \frac{x^2}{4\tilde{\lambda}_\infty^2 / \lambda_\infty^2} \right)$$

similar to the initial arc but with an accuracy up to higher orders. In figure 2 this parabola is shown by a dashed line.

Still another example of an irrotational flow about a contour corresponding to an ellipse in the plane $z = \mu + i\nu$ shall be considered. The function

$$z = \xi + \frac{r^2}{\xi}$$

where $0 \leqslant r \leqslant 1$ maps the region $|\xi| > 1$ on the region outside the ellipse

$$\frac{\mu^2}{(1 + r^2)^2} + \frac{\nu^2}{)1 - r^2)^2} = 1$$

After integration equation (3.7) gives

$$x + iy = C_2 \left(\xi + \frac{r^2}{\xi}\right) + C_1 \tilde{\lambda}_\infty{}^2 \left[\frac{1 + r^2\ (r^2 - 2)}{2r^3} \log \overline{\frac{\xi - r}{\xi + r}} + \bar{\xi} + \frac{1}{r^2 \bar{\xi}}\right] \tag{5.2}$$

Because

$$\lim_{r \to 0} \left(\frac{1 + r^2\ (r^2 - 2)}{2r^3} \log \frac{\xi - r}{\xi + r} + \xi + \frac{1}{r^2 \xi}\right) = \xi + \frac{2}{\xi} - \frac{1}{3\xi^3}$$

equation (5.2) for $r = 0$ assumes the form

$$x + iy = C_2 \xi + C_1 \tilde{\lambda}_\infty{}^2 \left(\bar{\xi} + \frac{2}{\bar{\xi}} - \frac{1}{3\bar{\xi}^3}\right)$$

If $\epsilon = 1-r$ is denoted and in equation (5.2) magnitudes of the order ϵ^2 are neglected

$$x + iy = C_2 \left(\xi + \frac{1 - 2\epsilon}{\xi}\right) + C_1 \tilde{\lambda}_\infty{}^2 \left(\bar{\xi} + \frac{1 + 2\epsilon}{\bar{\xi}}\right)$$

The profile represented by this last equation is very close to the initial ellipse.

6. FLOW ABOUT AN ARBITRARY CONTOUR

In order to obtain in the x,y-plane a contour without angular points, it is necessary in the auxiliary plane of the variable $z = \mu + i\nu$ to be given a contour having angular points chosen in a special manner.

A certain auxiliary function is constructed that enables the obtainment of the necessary singularities on the contour in all cases considered.

In the plane of the complex variable ζ, a region outside the contour shown in figure 3 is considered. This contour consists of the circle D and the cut along the arc of the circle D' tangent to D at point c. The points a and b are the angular points of the contour. The region considered is mapped the region outside the unit circle (fig. 4) so that the points a, b, and c correspond to the given points A, B, and C on this circle and the derivative of the mapping function at the point of infinity is equal to unity. The radii of the circles D and D' and the length of the arc abc are then determined.

The required mapping function is determined by the following equations:

$$\eta = -i\frac{\xi + \xi_c}{\xi - \xi_c}$$

$$u = \eta + (\eta_* - \eta_0)\log(\eta_* - \eta)$$

$$\zeta = -i\frac{\eta_0 + i}{\eta_* + i}\frac{\xi_c}{u_\infty}\frac{u + u_\infty}{u - u_\infty}$$

where

$$u_\infty = -i + (\eta_* - \eta_0)\log(\eta_* + i)$$

where ξ_*, ξ_0, and ξ_c are the coordinates of the points A, B, and C in the plane ξ ; and η_* and η_0 are the coordinates of the corresponding points in the plane η.

For the derivative,

$$\frac{d\zeta}{d\xi} = \left(\frac{\xi_* - \xi_c}{\xi_0 - \xi_c}\right)^2 \frac{\xi - \xi_0}{\xi - \xi_*} \left[1 + \frac{(\xi_0 - \xi_*)(\xi - \xi_c)}{(\xi_* - \xi_c)(\xi_0 - \xi_c)} \log \frac{\xi - \xi_*}{\xi - \xi_c}\right]^{-2} \tag{6.1}$$

The function considered is denoted by $\zeta = g(\xi)$. As is seen from equation (6.1), this function satisfies the condition that $g'(\infty) = 1$.

The function $g(\xi)$, as follows from its geometric sense, is regular for $|\xi| > 1$ and on the circle $|\xi| = 1$ has two singular points $\xi = \xi_0$ and $\xi = \xi_*$.

At the point $\xi = \xi_c$, the derivative $d\zeta/d\xi$ is finite

The contour E in the μ,ν-plane is considered as having no angular points. Let the function $z = \chi(\xi)$ map the region outside the contour onto the region outside the circle $|\xi| = 1$. Let ξ_{01} and ξ_{02} be points on the circle $|\xi| = 1$ corresponding to the critical points for the flow considered about the contour, and ξ_{*1} and ξ_{*2} be the corresponding critical points for the flow about the circle determined by the function $F(\xi)$. By $g_1(\xi)$ and $g_2(\xi)$, the values of the function g corresponding to the points ξ_{01}, ξ_{*1} and ξ_{02}, ξ_{*2} are denoted. The values ξ_{c1} and ξ_{c2} may be chosen arbitrarily. Instead of the contour E in the μ,ν-plane, the contour E^* is considered, which bounds the area on which the region $|\xi| > 1$ is mapped by the function determined by the equation

$$\frac{d\chi^*}{d} = \chi'(\xi)\, g_1'(\xi)\, g_2'(\xi) \tag{6.2}$$

To this contour in the x,y-plane will correspond a contour without angular points.

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The functions g_1' and g_2' may be represented in the form

$$g_1' = \frac{\xi - \xi_{01}}{\xi - \xi_{*1}} h_1$$

$$g_2' = \frac{\xi - \xi_{02}}{\xi - \xi_{*2}} h_2$$

where h_1 and h_2 become zero at the points $\xi = \xi_{*1}$ and $\xi = \xi_{*2}$ respectively, as $[\log(\xi - \xi_*)]^{-2}$. Equation (4.5), which serves for computing the coordinates in the x,y-plane, then assumes the form

$$dx + idy = C_2 \chi'(\xi)\, h_1 h_2\, d\xi + \tilde{\lambda}_\infty^2\, C_1 \frac{(\bar{\xi} - \bar{\xi}_{*1})^2 (\bar{\xi} - \bar{\xi}_{*2})^2}{\bar{\xi}^4\, \bar{\chi}^{*\prime}(\bar{\xi})\, \bar{h}_1 \bar{h}_2} \qquad (6.3)$$

On integration, the singularities corresponding to h_1 and h_2 vanish.

In the x,y-plane a closed contour without angular points is obtained. The actual construction of the contour E^* is unnecessary. For constructing the contour in the x,y-plane and computing the velocity field, only the derivative (6.2) is required.

The contour in the x,y-plane will approximate the initial contour in the plane $z = \mu + i\nu$ and will approach coincidence with the latter as $M_\infty \to 0$. The greatest difference in the contours will be near the critical points. By a rational choice of the magnitudes ξ_{c1} and ξ_{c2}, the difference can be made a minimum. The contour shown in figure 5 is that one corresponding to the case where the function $z = \chi(\xi)$ is $z = \xi$ where

$$M_\infty = 0.333$$

$$\tilde{\gamma} = C_y = 1$$

$$\theta_{c1} = 150^\circ$$

$$\theta_{c2} = 30^\circ$$

It is evident that the indicated method of constructing the function g is not the only one possible; it was discussed only from considerations of simplicity. It is not difficult to define a function g making possible the construction in the x,y-plane of a contour with any given angular point.

7. APPROXIMATE CONSTRUCTION OF CONTOUR

It was previously shown how to obtain in the x,y-plane a contour without angular points. The effect the angular points of the contour will have, if they exist, on the flow is now considered. For this purpose, in the ξ-plane two auxiliary circles are drawn with centers at the points ξ_{01} and ξ_{02} passing through the points ξ_{*1} and ξ_{*2}. The closed contour formed by the arcs of the circle $|\xi| = 1$ and the arcs of the auxiliary circles are mapped into a closed contour in the x,y-plane.

By making use of equations (3.2) and (3.5) and noting that $|\xi| = 1$, it is seen that

$$\frac{df}{d\xi} = 2i\tilde{\lambda}_\infty e^{-i\vartheta} (\sin\theta - \sin\theta_0)$$

$$\frac{d\xi}{dz} = \frac{d\xi}{dz} \exp\left[i\left(\theta - \vartheta - \frac{\pi}{2}\right)\right.$$

$$\frac{dF}{d\xi} = 2i\tilde{\lambda}_\infty e^{-i\vartheta} (\sin\theta - \sin\theta_*) \quad \frac{d\bar{z}}{dz} = e^{-2i\vartheta}$$

Then

$$dx+idy = \left\{C_2\frac{\sin\theta-\sin\theta_*}{\sin\theta-\sin\theta_0} + 4C_1\tilde{\lambda}_\infty^{\,2}\left|\frac{d\xi}{dz}\right|^2 (\sin\theta-\sin\theta_0)(\sin\theta-\sin\theta_*)\right\} dz \tag{7.1}$$

In particular in the absence of circulation

$$dx + idy = \left\{ C_2 + 4C_1 \tilde{\lambda}_\infty^2 \left| \frac{d\xi}{dz} \right|^2 \sin^2\theta \right\} dz$$

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The coefficient of dz is nearly constant for points removed from ξ_0 and ξ_*. The contour in the x,y-plane therefore approximates very closely the contour in the z-plane.

The radius of the previously mentioned auxiliary circles is equal to

$$\rho = 2 \sin \frac{\theta_0}{2} \left(\frac{1}{\sqrt{1 - M_\infty^2}} - 1 \right) \simeq \theta_0 \left(\frac{1}{\sqrt{1 - M_\infty^2}} - 1 \right)$$

On an auxiliary circle $\Delta\xi = \xi - \xi_0 = \rho e^{i\epsilon}$; therefore

$$d\Delta\xi = i\rho\, e^{i\epsilon}\, d\epsilon$$

By expanding $dz/d\xi$ in a series in the neighborhood of the point $\xi = \xi_0$, it can be seen that

$$\frac{dz}{d\xi} = a + b\Delta\xi + \ldots$$

where

$$a = \left| \frac{dz}{d\xi} \right|_{\xi=\xi_0} \exp\left[i \left(\vartheta_0 - \theta_0 + \frac{\pi}{2} \right) \right]$$

With an accuracy up to magnitudes containing ρ to the first degree

$$dx + idy = \left[\left(\frac{1}{\sqrt{1 - M_\infty^2}} - 1 \right) \operatorname{tg} \theta_0 e^{i\vartheta_0} + i\rho e^{i\epsilon} \right] C_2 a d\epsilon + O(\rho^2) \tag{7.2}$$

By making use of equations (7.1) and (7.2), in the x,y-plane the closed contour is constructed. The part of the contour corresponding to the auxiliary circles is not a streamline.

The details of the flow near the singular points may be seen in figure 6, which gives the mapping of the circle for $M_\infty = 0.333$ and $Cy = \tilde{\gamma} = 1$.

The solution obtained may be interpreted as the flow about a contour with suction of a small mass of gas through the surface of the contour near the critical points. It can be noted that the value of ρ is always very small; furthermore for very large values of the lift force θ_0 is small, for example, $\theta_0 \simeq 4.5^{\circ}$ for $C_y = 1$. For large values of C_y the value of M_∞ is small because M_∞ must not exceed M_{cr}; for small values of C_y the values of θ_* are small. 1258

Taking into account the fact that the singular points are located in the neighborhood of the critical points where the velocities are near zero and that ρ is in all cases of very small magnitude, the practical result is obtained that it is possible in many cases to neglect the effect of the singular points on the flow about the remaining part of the contour and to carry out its construction without the use of the function g.

8. THEOREM OF JOUKOWSKI, COMPUTATION OF THE MOMENT OF THE AERODYNAMIC FORCES

In order to compute the forces exerted by the flow on the airfoil, use is made of the well-known formulas obtained in the application of the law of conservation of momentum to the mass of fluid contained between the contour of the wing and a certain closed contour E which includes the wing contour.

$$\begin{aligned} P_x &= -\int_E p\,dy - \int_E \rho u\,(u\,dy - v\,dx) \\ P_y &= -\int_E p\,dx - \int_E \rho v\,(u\,dy - v\,dx) \end{aligned} \qquad (8.1)$$

where P_x and P_y are the projections on the x- and y-axes of the pressure forces of the flow on the wing. The integrals in equations (8.1) are taken in the direction for which the passage about the contour E is effected, that is, in the counterclockwise direction.

By applying the law of conservation of momentum for computing the aerodynamic moment of the wing profile,

$$M = \int_E p\,(x\,dx + y\,dy) - \int_E \rho\,(xv - yu)(u\,dy - v\,dx) \qquad (8.2)$$

By making use of equations (3.7) and (6.2), the integrals entering equations (8.1) and (8.2) may be transformed into integrals taken over the corresponding contour in the ξ-plane. The contour E is so chosen that to it in the ξ-plane will correspond a circle with center at the origin. All the magnitudes entering under the integral sign in equations (8.1) and (8.2) are expanded into series in the neighborhood of the point at infinity in the ξ-plane. From equation (3.7),

$$dx + i dy = C_2\left\{1 + \frac{\tilde{\gamma}}{2\pi i \xi}\left(\frac{1}{\sqrt{1 - M_\infty^2}} - 1\right) + \frac{1}{\xi^2}\left[-1 - a_1 + \frac{\tilde{\gamma}^2}{4\pi^2}\left(\frac{1}{\sqrt{1 - M_\infty^2}} - 1\right)\right] + \dots\right\} d\xi +$$

$$\tilde{\lambda}_\infty^2 C_1\left\{1 - \frac{\tilde{\gamma}}{2\pi i \tilde{\xi}}\left(\frac{1}{\sqrt{1 - M_\infty^2}} + 1\right) + \frac{1}{\tilde{\xi}^2}\left[-1 + \bar{a}_1 - \frac{\tilde{\gamma}^2}{4\pi^2}\frac{1}{\sqrt{1 - M_\infty^2}}\right] + \dots\right\} d\tilde{\xi}$$

On integrating, this equation becomes

$$x + iy = C_2\left\{\xi + \frac{\tilde{\gamma}}{2\pi i}\left(\frac{1}{\sqrt{1 - M_\infty^2}} - 1\right)\log \xi - \frac{1}{\xi}\left[-1 - a_1 + \frac{\tilde{\gamma}}{4\pi^2}\left(\frac{1}{\sqrt{1 - M_\infty^2}} - 1\right)\right] + \dots\right\} + \tilde{\lambda}_\infty^2 C_1\left\{\tilde{\xi} - \frac{\tilde{\gamma}}{2\pi i}\left(\frac{1}{\sqrt{1 - M_\infty^2}} + 1\right)\log \tilde{\xi} + \frac{1}{\tilde{\xi}}\left[-1 + \bar{a}_1 - \frac{\tilde{\gamma}}{4\pi^2}\frac{1}{\sqrt{1 - M_\infty^2}}\right] + \dots\right\} + a_0' + a_0'' i \qquad (a'')$$

where $a_0' + a_0'' i$ is the constant of integration and $a_1 = \beta_1 + i\gamma_1$ is the coefficient of ξ^{-2} in the expansion of df/dz in the neighborhood of the point $\xi = \infty$.

Expanding u, v, p, ρ, x, y, dx, and dy in powers of $1/r$ and integrating, the following expression is obtained:

$$\left.\begin{aligned} P_x &= 0 \\ P_y &= -\rho_\infty w_\infty \Gamma \qquad \left(w_\infty = a_* \lambda_\infty,\ \Gamma = \frac{\tilde{\Gamma}}{\sqrt{1 - M_\infty^2}}\right) \end{aligned}\right\} \qquad (8.3)$$

The theorem of Joukowski is thus valid also for the flow about a body of a gas with large subsonic velocities. This theorem was proven earlier in reference 1 by making use of the exact equations.

In a similar manner, the equation for the moment of the aerodynamic forces is

$$M = \frac{\rho_\infty w_\infty^2 \alpha_\infty^2}{\sqrt{1 - M_\infty^2}} \left(2\pi \gamma_1 - \frac{a_0' \tilde{\gamma}}{\alpha_\infty}\right) \qquad \left(\alpha_\infty = \frac{\tilde{\lambda}_\infty}{\lambda_\infty}\right) \tag{8.4}$$

The constant a_0' determines the position of the point relative to which the moment is computed. The moment for the corresponding profile in an incompressible flow is equal to

$$M = \rho_\infty \tilde{w}_\infty^2 \left(2\pi\gamma_1 - \tilde{\gamma}\beta_0\right) \qquad \left(\tilde{w}_\infty = a_* \tilde{\lambda}_\infty\right)$$

where β_0 is the real part of the free term in the expansion of the function $z = X^*(\xi)$ in the neighborhood of the point $\xi = \infty$. Let $a_0'/\alpha_\infty = \beta_0$. The chord of the deformed profile in the compressible flow will be denoted by b and the chord of the profile in the corresponding incompressible flow by b^o; in general the magnitudes referring to the incompressible flow will be denoted by a superscript circle.

The moment coefficient for an incompressible flow C_m^o is given by

$$C_m^o = \frac{2\pi\gamma_1 - \tilde{\gamma}\beta_0}{b^2}$$

The moment coefficient for a compressible flow C_m is given by

$$C_m = C_m^o \frac{\alpha_\infty^2}{\sqrt{1 - M_\infty^2}} \frac{b^{o2}}{b^2}$$

$C_y^{\circ} = \tilde{\gamma}/b^{\circ}$, and therefore

$$C_m^{\circ} = -\frac{\beta_0}{b^{\circ}} C_y^{\circ} + \frac{2\pi\gamma_1}{b^{02}}$$

The origin of coordinates is chosen at the nose of the profile. Then

$$\frac{dC_m^{\circ}}{dC_y^{\circ}} = -\frac{\beta_2}{b^{\circ}}$$

is a function of the focal distance from the nose of the profile and the expression for the moment C_m° may be given in the form

$$C_m^{\circ} = \frac{dC_m^{\circ}}{dC_y^{\circ}} C_y^{\circ} + C_m^{\circ}$$

where

$$C_m^{\circ} = \frac{2\pi\gamma_1}{b^{02}}$$

is the moment of the profile in the incompressible flow relative to the focal point.

The final expression for C_m can be given in the following form:

$$C_m = \left[\frac{dC_m^{\circ}}{dC_y^{\circ}} C_y^{\circ} + C_m^{\circ}\right] \frac{\alpha_{\infty}^{2}}{1 - M_{\infty}^{2}} \frac{b^{02}}{b^2} \tag{8.5}$$

Translated by S. Reiss
National Advisory Committee
for Aeronautics.

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3. Tsien, Hsue-Shen: Two-Dimensional Subsonic Flow of Compressible Fluids. Jour. Aero. Sci., vol. 6, no. 10, Aug. 1939, pp. 399-407.

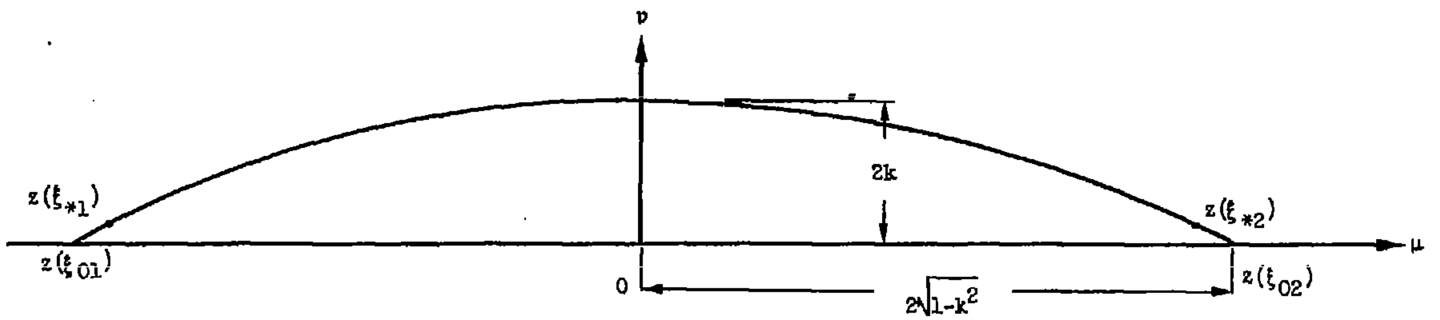

Figure 1.

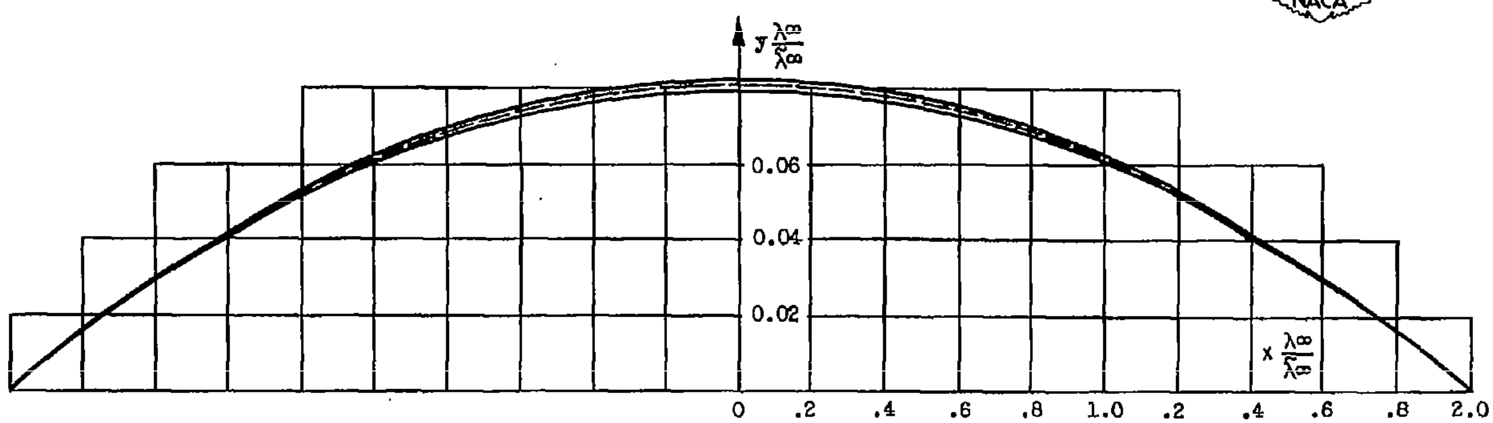

Figure 2.

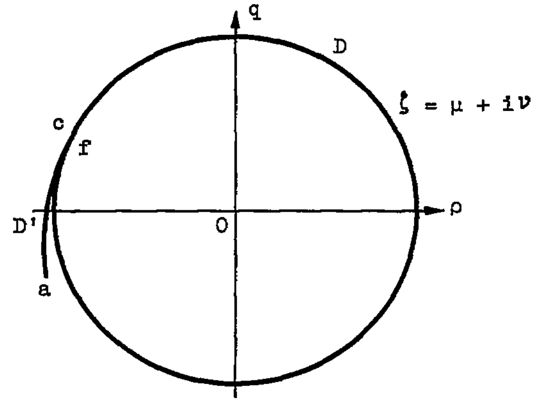

Figure 3.

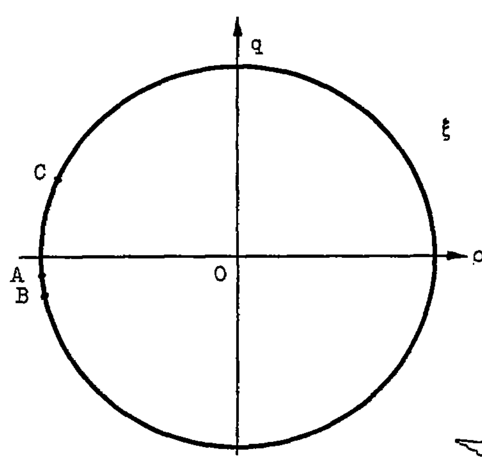

Figure 4.

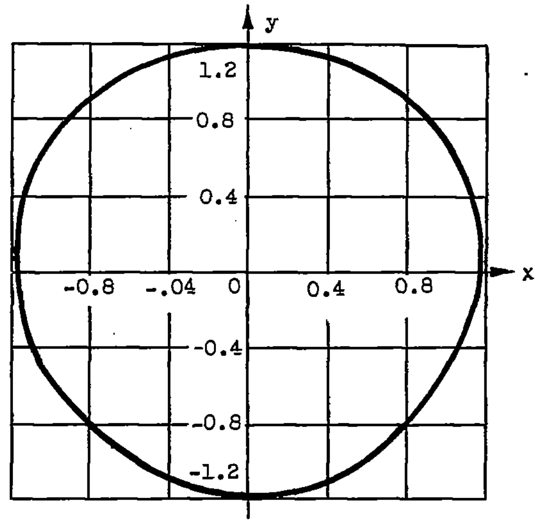

Figure 5

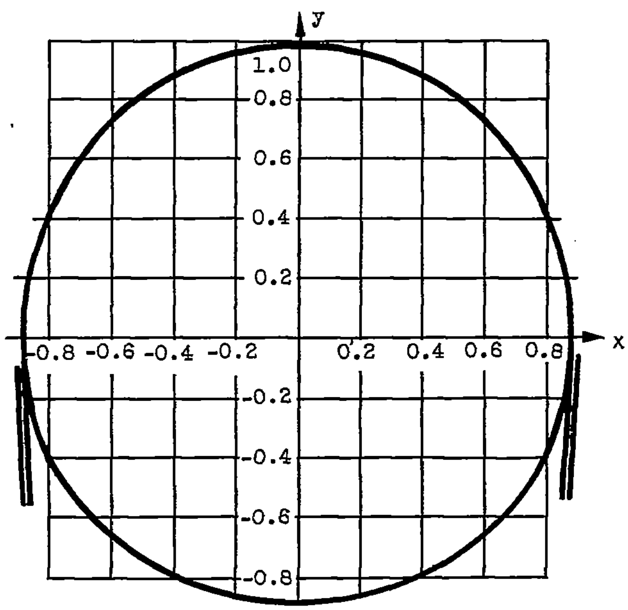

Figure 6

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