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On a graph related to permutability in finite groups

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Abstract

For a finite group G we define the graph $\Gamma(G)$ to be the graph whose vertices are the conjugacy classes of cyclic subgroups of G and two conjugacy classes $\mathcal{A}$, $\mathcal{B}$ are joined by an edge if for some $A \in \mathcal{A}$, $B \in \mathcal{B}$ A and B permute. We characterise those groups G for which $\Gamma(G)$ is complete.
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1 Introduction

There are many ways in which a graph has been associated with a finite group. Herzog, Longobardi and Maj [8] have defined a graph whose vertices are the conjugacy classes of a group, with two vertices joined by and edge if an element of one vertex commutes with some element of the other vertex. This is a generalisation of the commuting graph of a group, which has the elements of a group as vertices, joined by an edge if they commute (see [9]). In this paper we will consider a generalisation of the graph of Herzog, Longobardi and Maj. For a finite group G we define the graph $\Gamma(G)$ to be the graph whose vertices are the conjugacy classes of cyclic subgroups of G and two conjugacy classes $\mathcal{A}$, $\mathcal{B}$ are joined by an edge if for some $A \in \mathcal{A}$, $B \in \mathcal{B}$ A and B permute. We characterise those groups G for which $\Gamma(G)$ is complete.

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Recall that a subgroup H of a group G is said to be permutable in G if HL is a subgroup of G for every subgroup L of G. Permutability, like normality, is not a transitive relation in general. We say that a group G is a PT-group if the permutability is transitive in G, that is, if H is permutable in K and K is permutable in G, then H is permutable in G. According to a classical result of Ore [10] permutable subgroups of finite groups are subnormal. Hence a finite group is a PT-group if and only if every subnormal subgroup is permutable.

We prove:

Theorem 1. *A finite group G is a soluble* PT*-group if and only if the graph $\Gamma(G)$ is complete.*

2 Proof of Theorem 1

Suppose that G is a soluble PT-group. By a result of Zacher [12] $G = AH$ where A is an abelian normal subgroup of G, H is a nilpotent modular subgroup of G, A and H have coprime orders and every subgroup of A is normal in G. If X and Y are two cyclic subgroups of G, we can write $X = X_0X_1$ and $Y = Y_0Y_1$, where X_0 and Y_0 are subgroups of A and X_1 and Y_1 have orders dividing $|H|$. Since H is a Hall subgroup of G we can, by replacing X and Y by conjugates if necessary, assume that X_1 and Y_1 are subgroups of H. Since every subgroup of A is normal in G we have $Y_0X_1 = X_1Y_0$ and since H is modular we have X_1Y_1 is a subgroup of G. It now follows that $X_0X_1Y_0Y_1 = Y_0Y_1X_0X_1$ is a subgroup, that is, X and Y permute.

In the other direction, we argue by induction on the order of G. We begin by showing that G is a soluble PT-group if G has at least two minimal normal subgroups. If M and N are two minimal normal subgroups of G, then G/M and G/M clearly satisfy the hypothesis of the theorem. Hence G/M and G/N are soluble PT-groups. It follows that NM/M is a minimal normal subgroup of a soluble PT-group, and so is cyclic of prime order because every soluble PT-group is supersoluble. Similarly MN/N is cyclic of prime order and hence M and N have prime orders, p and q say (and G is soluble). Then, for any prime $r \neq p$, all r-chief factors of G/M are G-isomorphic. Further, by Zacher's Theorem [12], Sylow r-subgroups of G/M are abelian if r-chief factors are noncentral and modular if all r-chief factors are central. If $p \neq q$ by considering G/N we have all p-chief factors G-isomorphic and Sylow p-subgroups abelian if p-chief factors are noncentral and modular if p-chief factors are central. In this case G is a PT-group by [3, Corollary 3] and [4, Theorem 2] (note that G is supersoluble).

Thus we suppose that all minimal normal subgroups have the same prime order p. If M and N are minimal normal subgroups and p divides $|G/MN|$, then both M and N are G-isomorphic to a (fixed) p-chief factor of G/MN and so are G-isomorphic. Thus all p-chief factors are G-isomorphic. Therefore G is supersoluble and all chief factors of the same order are G-isomorphic. By [3, Corollary 3] G is a group in which every subnormal subgroup permutes with all Sylow subgroups (G is a PST-group). If the p-chief factors are central, then G is a p-group with all proper quotients modular and so is itself modular, since by Theorem of Longobardi [7] such a group must have a unique minimal normal subgroup. Applying a result of Agrawal [2], G has an abelian Sylow p-subgroup and it then follows that G is a PT-group by [4, Theorem 2]. Assume now that MN is a Sylow p-subgroup of G. Let $M = \langle m \rangle$, $N = \langle n \rangle$. By hypothesis, given a p'-element $y \in G$, $\langle mn \rangle$ permutes with a conjugate $\langle y^g \rangle$ of $\langle y \rangle$. Hence $\langle mn \rangle \langle y^g \rangle \cap MN = \langle mn \rangle$ is normalised by y^g. Call $m^g = m^{a_1}$, $n^g = m^{a_2}$ and $m^y = m^{b_1}$, $n^y = n^{b_2}$ and $(mn)^{g^{-1}yg} = (mn)^c$. Hence $(mn)^{g^{-1}yg} = m^{b_1}n^{b_2} = (mn)^c$, which implies that $b_1 \equiv b_2 \equiv c \pmod p$. Consequently M and N are G-isomorphic. Since G has all Sylow subgroups modular, it follows that G is a PT-group by [3, Corollary 3] and [4, Theorem 2].

We now suppose that G has a unique minimal normal subgroup N. If N is not soluble, then $N = S_1 \times \cdots \times S_r$, where the S_i are isomorphic (nonabelian) simple groups. Let p and q be different primes dividing the order of S_1 and let x_1 and y_1 be elements of S_1 of orders p and q, respectively. For $2 \leq i \leq r$, let x_i, y_i be the images of x_1, y_1 under the isomorphism between S_1 and S_i. Then $\langle x_1 \cdots x_r \rangle$ permutes with a conjugate $\langle (y_1 \cdots y_r)^g \rangle$ of $\langle y_1 \cdots y_r \rangle$. The projection of $\langle x_1 \cdots x_r \rangle \langle (y_1 \cdots y_r)^g \rangle$ onto S_1 is then a subgroup of S_1 of order pq and so S_1 has subgroups of order pq for every pair of primes dividing its order. A result of Abe and Iiyori [1] shows that this is impossible. Consequently N is a p-group for some prime p.

If N is not contained in the Frattini subgroup of G, then G is a primitive soluble group and $G = NM$, where M is a maximal subgroup of G, $N \cap M = 1$ and N is self-centralising. Since M is isomorphic to the soluble PT-group G/N, M is the product of its nilpotent residual $F = M^{\mathfrak{N}}$, which is an abelian normal Hall subgroup of odd order, and a complement C which acts on F as power automorphisms ([12]). Let Q be a cyclic normal subgroup of M. Suppose that $QP \neq PQ$ for some cyclic subgroup P of N. We have $Q^gP = PQ^g$ for some $g \in G$ and we can assume that $g \in M$. Since Q is normal in M we have $Q^g = Q$, giving a contradiction. Thus P is normalised by Q since $P = N \cap PQ$. It now follows that every element of F acts as a power automorphism on N and hence F acts as a power automorphism group on N. Since power automorphisms are central in the power automorphism group of N ([5, Theorem 2.2.1]), F is central in M and so $F = 1$. Thus M is a nilpotent modular group. In particular M is a p'-group. Let Q be a non-abelian Sylow q-subgroup of M. By Iwasawa's Theorem ([11, Theorem 2.4.14] Q has an abelian normal subgroup Q_0 with cyclic supplement S with S acting as a power automorphism group on Q_0 or Q is Hamiltonian. In both cases every cyclic subgroup of Q_0 is normal in Q and hence in M. Let U be a cyclic subgroup of N and let R cyclic subgroup of Q_0. By hypothesis, there exits an element $a \in M$ such that RU^a is subgroup. Since $R^a = R$, we have that RU is also a subgroup and U is normalised by R. Further, there exists an element $m \in M$ such that S permutes with U^m and so S and R normalise U^m. This implies that Q normalises U and Q acts as power automorphisms on N. It now follows that M acts as power automorphisms on N and so N is a cyclic group of order p and M is cyclic of order dividing $p - 1$ and G is clearly a PT-group.

Now suppose that N is contained in the Frattini subgroup of G. If G is nilpotent, then G is a p-group since it has a unique minimal normal subgroup and G/N is an modular group. Assume that G is not modular. Let $M(p)$ denote the nonabelian group of order p^3 and exponent p for p odd and the dihedral group of order 8 for $p = 2$. By the Theorem of Longobardi [7] either G is the central product of a subgroup P isomorphic to $M(p)$ and another subgroup or G is isomorphic to

$$G_0 = \langle a, b, w : a^{p^n} = w^p = 1, a^b = a^{1+p^s}, b^{p^j} = a^{p^{n-s}}, a^w = a^{1+p^{n-1}}, b^w = b\rangle,$$

where $0 < s < n$, $s \geq 2$ if $p = 2$ and $j \geq n - s$. In the first case it is clear that if P is generated by a and b of order p no conjugate of a will commute with b and hence will not permute with b. Now consider G_0 and let $C = \langle a^\alpha b^\beta\rangle$ be a cyclic subgroup of $H = \langle a, b\rangle$. Then C permutes with $\langle w^g\rangle$ for some $g \in G_0$ and so, being of index p in $C\langle w^g\rangle$, is normalised by w^g. Since $[w, g] \in \langle a^{p^{n-1}}\rangle$, C is also normalised by w. Thus w acts as a power automorphism on H. If p is odd, H is regular ([6, III, Satz 11.4]) and so it acts as a universal power automorphism on H by [5, Theorem 5.3.1], a contradiction. Hence we suppose $p = 2$. Since w centralises b and b has order at least 4, w acts as universal power automorphism on H by a theorem of Napolitani [11, Theorem 2.3.24], again a contradiction. Thus G cannot be nilpotent.

If G is not nilpotent, then $E = G^{\mathfrak{N}} \neq 1$ and so $N \leq E$. Since G/N is a PT-group, G supersoluble, E is nilpotent and so it is a p-group. Furthermore E/N is abelian and complemented in G/N by a p'-subgroup B/N say which acts on E/N as power automorphisms. Then there exists a p'-subgroup D of G complementing E in G. If $C_D(E/N) \neq 1$ then $C_D(E/N)$ is a nontrivial normal subgroup of G, a contradiction. It follows that D is cyclic of order dividing $p-1$. We have that $N \leq \mathrm{Z}(E)$. Consequently $[a^p, b] = [a, b]^p = 1$ for every a, $b \in E$ by [6, III, Hilfssatz 1.3]. This implies that $E^p \leq \mathrm{Z}(E)$. Assume that E^p is not trivial. Hence E^p is cyclic because E^p is an abelian normal subgroup of G. Let $E^p = \langle a \rangle$, where a has order p^n. Suppose that N is a proper subgroup of E^p. Given $x \in G$, we have that $a^x = a^i$ and so all chief factors of G below P^p are G-isomorphic. Appying again [3, Corollary 3] and [4, Theorem 2], G is a PT-group. Assume now that $E^p = N$. Let x be an element of E not in N. If x has order greater than p, then $N \leq \langle x^p \rangle \leq E^p$ and $\langle x \rangle / N$ is a normal subgroup of G/N . Since all chief factors of G between N and E are G-isomorphic and the same happens with all chief factors of G below $\langle x \rangle$, we conclude again that all chief factors of G are G-isomorphic and G is a PT-group. Thus all elements of E are of order p. If E is abelian then E is cyclic and so $E = N \leq \Phi(G)$, a contradiction. Suppose that E is non-abelian and $x, y \in E$ do not commute, so that $N = \langle [x, y] \rangle$. Since $\langle x \rangle$ permutes with $\langle y^g \rangle$ for some $g \in G$, we have that $\langle x, y^g \rangle$ is of order p^2 and hence abelian. Since $\langle yN \rangle$ is a normal subgroup of G/N, we have that $\langle y^g \rangle \leq \langle y, N \rangle$ and it follows that $y^g = y^j n$ for some $n \in N$ and some integer j coprime with p. Then $[x, y^g] = [x, y^j n] = [x, y^j] = [x, y]^j \neq 1$, a contradiction. This completes the proof.

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