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On self-normalising subgroups of finite groups

A. Ballester-Bolinches* R. Esteban-Romero† Yangming Li‡

Abstract

The aim of this paper is to characterise the classes of groups in which every subnormal subgroup is normal, permutable, or S-permutable by the embedding of the subgroups (respectively, subgroups of prime power order) in their normal, permutable, or S-permutable closure, respectively.

Keywords: finite group, permutability, Sylow permutability, permutable closure, subnormal closure, PST-group, PT-group, T-group.

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Dedicated to Derek J. S. Robinson on the occasion of his seventieth birthday.

1 Introduction and statement of results

In this paper, we consider only finite groups.

A subgroup H of a group G is said to be *permutable* in G if H permutes with every subgroup of G. A group G is said to be a PT*-group* (respectively, T*-group*) if permutability (respectively, normality) is a transitive relation in G. By a result of Ore [13], PT-groups are exactly those groups where all subnormal subgroups are permutable. PST-groups are also defined via a transitivity property, namely with respect to S-permutability ([11]): a subgroup of a group G is called *S-permutable* if it permutes with all the Sylow

*Departament d'Àlgebra, Universitat de València, Dr. Moliner, 50, E-46100 Burjassot, València, Spain, email: Adolfo.Ballester@uv.es

†Departament de Matemàtica Aplicada-IUMPA, Universitat Politècnica de València, Camí de Vera, s/n, E-46022 València, Spain, email: resteban@mat.upv.es

‡Department of Mathematics, Guangdong College of Education, Guangzhou, 510310, People's Republic of China, email: liyangming@gdei.edu.cn

subgroups of G. By a result of Kegel ([11, Satz 1]), every S-permutable subgroup is subnormal and hence PST-groups are exactly those groups in which all subnormal subgroups are S-permutable.

Note that the class of T-groups is a proper subclass of the class of PT-groups, which in turns forms a proper subclass of the class of PST-groups. These classes have been studied in detail, with many characterisations available (see [1, 2, 3, 4, 5, 6, 7, 14, 15]).

The basic structure of soluble T-, PT-, and PST-groups were established by Gaschütz, Zacher, and Agrawal, respectively, and are presented in the following theorem.

Theorem 1. *Let L be the nilpotent residual of a group G. Then:*

1. *(Agrawal, [1]) G is a soluble PST-group if and only if L is an abelian Hall subgroup of odd order of G on which G acts by conjugation as a group of power automorphisms.*
2. *(Zacher, [15]) G is a soluble PT-group if and only if G is a soluble PST-group with modular Sylow subgroups.*
3. *(Gaschütz, [10]) G is a soluble T-group if and only if G is a soluble PST-group with Dedekind Sylow subgroups.*

The results of the present paper spring from a recent characterisation of soluble T-groups due to the third author. There he proves that a group is a soluble T-group if and only if every subgroup is self-normalising in its normal closure ([12, Theorem 3.1]). It is natural to wonder whether or not it is possible to get similar characterisations of soluble PT- and PST-groups by using permutable and S-permutable versions of the normal closure. This is the goal of the present paper.

Since the intersection of S-permutable subgroups of a group G is again a S-permutable subgroup of G, it seems reasonable to replace the normal closure of a subgroup H by the intersection of all S-permutable subgroups of G containing H in the PST-case. Unfortunately, the intersection of permutable subgroups of G is not permutable in general. Therefore the intersection of all permutable subgroups of G containing a given subgroup H is not the smallest permutable subgroup of G containing H in general. This is the main reason why the proofs concerning PT-groups are much more involved than the corresponding ones for T- and PST-groups. Despite this fact, the above subgroup will play a central role in our approach.

Definition 2. Let H be a subgroup of a group G.

1. The permutable closure $\mathrm{A}_G(H)$ of H in G is the intersection of all permutable subgroups of G containing H.

2. The S-permutable closure $\mathrm{B}_G(H)$ of H in G is the intersection of all S-permutable subgroups of G containing H.

Applying [11], $\mathrm{B}_G(H)$ is S-permutable in G. However, $\mathrm{A}_G(H)$ is not permutable in general, as the following example shows:

Example 3. Suppose that p is a prime and $m > 1$ is a natural number. Let $G = \langle a, b, c \mid a^{p^m} = b^p = c^p = 1, b^a = bc, c^a = c^b = c\rangle$. The subgroups $H_1 = \langle b, c\rangle$ and $H_2 = \langle b, ca^{p^{m-1}}\rangle$ are permutable in G, but the intersection $H = H_1 \cap H_2$ is not permutable in G. For the subgroup H, the permutable closure $\mathrm{A}_G(H) = H$ is not permutable in G.

However we have:

Theorem 4. *Assume that G is a group such that every subgroup is self-normalising in its permutable closure. Then $\mathrm{A}_G(H)$ is a permutable subgroup of G for each subgroup H of G.*

Theorem 4 is a consequence of a stronger result:

Theorem 5. *Let p be a prime. If every p-subgroup of a group G is self-normalising in its permutable closure, then $\mathrm{A}_G(H)$ is a permutable subgroup of G for every p-subgroup H of G.*

These results justify the study of the class of groups in which every subgroup is self-normalising in its permutable closure. It turns out that this class is the class of all soluble PT-groups.

If we fix a prime p, the class of groups for which every p-subgroup is self-normalising in its permutable closure is a subclass of a class which can be considered as a local version of the class of all soluble PT-groups.

Definition 6 ([5])**.** A group G *satisfies* $\mathcal{X}_p$ if and only if each subgroup of a Sylow p-subgroup P of G is permutable in the normaliser $\mathrm{N}_G(P)$.

Theorem 7. *If p is a prime and G is a group in which every p-subgroup is self-normalising in its permutable closure, then G satisfies $\mathcal{X}_p$.*

Theorem 8. *For a group G, the following statements are equivalent:*

1. *G is a soluble PT-group.*

2. *H is abnormal in $\mathrm{A}_G(H)$ for every subgroup H of G.*

3. $\mathrm{N}_G(H) \cap \mathrm{A}_G(H) = H$ *for every subgroup* H *of* G.

4. *For every prime* p *and every* p*-subgroup* H *of* G, H *is abnormal in* $\mathrm{A}_G(H)$.

5. *For every prime* p *and every* p*-subgroup* H *of* G, *we have* $\mathrm{N}_G(H) \cap \mathrm{A}_G(H) = H$.

Theorem 7 follows from the local strategy we use in the PST-case. In fact, this local point of view leads to the local defining property of the class of soluble PST-groups.

Recall that if p is a prime, a group G satisfies property $\mathcal{Y}_p$ if for each pair of p-subgroups H and K of G such that H is contained in K, then H is S-permutable in $\mathrm{N}_G(K)$ ([4]). A group G satisfies $\mathcal{C}_p$ if every subgroup of a Sylow p-subgroup P of G is normal in $\mathrm{N}_G(P)$ ([14]).

A group G is a soluble PST-group if and only if G satisfies $\mathcal{Y}_p$ for all primes ([4, Theorem 4]). Similar results hold for soluble PT-groups and property $\mathcal{X}_p$ ([5]), and soluble T-groups and property $\mathcal{C}_p$ ([14]). These results are consequences of the following:

Theorem 9 ([4, Theorem 3])**.** *Let* p *be a prime. A group* G *satisfies* $\mathcal{X}_p$ *(respectively,* $\mathcal{C}_p$*) if and only if* G *satisfies* $\mathcal{Y}_p$ *and the Sylow* p*-subgroups of* G *are modular (respectively, Dedekind).*

If p is a prime and every p-subgroup is self-normalising in its permutable closure, then Sylow subgroups are modular. Hence Theorem 7, by virtue of Theorem 9, is the permutable local version of the following:

Theorem 10. *If* p *is a prime and* G *is a group such that every* p*-subgroup of* G *is self-normalising in its S-permutable closure, then* G *satisfies* $\mathcal{Y}_p$.

The converse of the above result does not hold.

Example 11. Suppose that p and q are two primes such that q divides $p-1$. Let E be an extraspecial group of order p^3 and exponent p. Let i and j be two numbers such that $ij \equiv 1 \pmod{p-1}$ and i and j have order q modulo p. Let $\{x, y\}$ be a generating system for E and let z be an automorphism of order q of E given by $x^z = x^i$, $y^z = y^j$. Let $G = [E]\langle z\rangle$ be the corresponding semidirect product, then the S-permutable closure of $Z = \langle z\rangle$ is G, but $\mathrm{N}_G(Z) = \langle z, [x, y]\rangle$. However, the group G satisfies $\mathcal{Y}_q$ because it is q-nilpotent.

Note that if H is a p-subgroup of G and $\mathrm{N}_G(H) \cap \langle H^G\rangle = H$, where $\langle H^G\rangle$ is the normal closure of H in G, we have that H is a Sylow p-subgroup of $\langle H^G \rangle$. Hence if the above condition holds for every p-subgroup of G, it follows that the Sylow p-subgroups of G are Dedekind groups. Therefore applying Theorems 9 and 10 we have:

Corollary 12. *Let p be a prime and let G be a group. If every p-subgroup of G is self-normalising in its normal closure, then G satisfies property $\mathcal{C}_p$.*

We are now in a position to give characterisations of groups in which every subgroup is self-normalising in its S-permutable closure: they turn out to be the soluble PST-groups.

Theorem 13. *Let G be a group. Any two of the following five statements are equivalent:*

1. *G is a soluble PST-group.*
2. *For every subgroup H of G, H is abnormal in $\mathrm{B}_G(H)$.*
3. *For every subgroup H of G, the equality $\mathrm{N}_G(H) \cap \mathrm{B}_G(H) = H$ holds.*
4. *If p is any prime and H is a p-subgroup of G, then H is abnormal in $\mathrm{B}_G(H)$.*
5. *If p is any prime and H is a p-subgroup of G, it follows that $\mathrm{N}_G(H) \cap \mathrm{B}_G(H) = H$.*

Combining Theorem 13 and Corollary 12, we have:

Corollary 14 ([12, Theorem 3.1])**.** *Any two of the following assertions about a group G are equivalent:*

1. *G is a soluble T-group.*
2. *H is abnormal in $\langle H^G \rangle$ for all subgroups H of G.*
3. *$\mathrm{N}_G(H) \cap \langle H^G \rangle = H$ for all subgroups H of G.*
4. *H is abnormal in $\langle H^G \rangle$ for every p-subgroup H of G and every prime p.*
5. *$\mathrm{N}_G(H) \cap \langle H^G \rangle = H$ for every p-subgroup H of G and every prime p.*

2 Proofs

We begin the section with a pair of lemmas, which will be used several times in subsequent proofs.

Lemma 15. *Let G be a group and let N be a normal subgroup of G. If H is a subgroup of G, then $\mathrm{A}_G(H)N/N$ and $\mathrm{B}_G(H)N/N$ are subgroups of $\mathrm{A}_{G/N}(HN/N)$ and $\mathrm{B}_{G/N}(HN/N)$, respectively. If N is contained in $\mathrm{A}_G(H)$, then $\mathrm{A}_G(H)/N = \mathrm{A}_{G/N}(HN/N)$. The same is true for $\mathrm{B}_G(H)$.*

Proof. The assertions made in the lemma follow from the fact that a subgroup X of G containing N is permutable (respectively, S-permutable) in G if and only if X/N is permutable (respectively, S-permutable) in G/N. □

Lemma 16. *Let H be a subgroup of a group G and let S be a subgroup of G containing H. Then $\mathrm{A}_S(H) \leq \mathrm{A}_G(H)$ and $\mathrm{B}_S(H) \leq \mathrm{B}_G(H)$.*

Proof. Note that if X is a permutable (respectively, S-permutable) subgroup of G containing H, then $X \cap S$ is a permutable (respectively, S-permutable) subgroup of S containing H. □

Corollary 17. *The classes of all groups in which every subgroup is self-normalising in its permutable (respectively, S-permutable) closure are closed under taking subgroups and factor groups.*

Proof of Theorem 5. Assume that every p-subgroup of G is self-normalising in its permutable closure. We prove that $\mathrm{A}_G(H)$ is permutable in G by induction on the order of G. If N is a minimal normal subgroup of G contained in $\mathrm{A}_G(H)$, then $\mathrm{A}_G(H)/N = \mathrm{A}_{G/N}(HN/N)$ by Lemma 15. Since the hypotheses of the theorem hold in G/N, we have that $\mathrm{A}_G(H)/N$ is permutable in G/N by induction. Hence $\mathrm{A}_G(H)$ is permutable in G, as required. Therefore we can suppose that $\mathrm{Core}_G\big(\mathrm{A}_G(H)\big) = 1$. On the other hand, applying [11, Satz 2], $\mathrm{A}_G(H)$ is S-permutable in G and so $\mathrm{A}_G(H)$ is nilpotent by a result of Deskins [8, Theorem 1]. Since H is self-normalising in $\mathrm{A}_G(H)$, it follows that $\mathrm{A}_G(H)$ is a p-group and so $H = \mathrm{A}_G(H)$ is S-permutable in G. By [11, Satz 1], H is subnormal in G. Let P be a Sylow p-subgroup of G. Then H is contained in P. If P were a proper subgroup of G, then H would be permutable in P because P inherits the hypotheses of the theorem. It would imply the permutability of H in G, as required. Hence we may assume that $P = G$. Then a minimal normal subgroup N of G is central. Since H is self-normalising in G, it follows that N must be contained in H, contradicting the fact that $\mathrm{Core}_G(H) = 1$. Therefore H is permutable in P and so in G. □

Proof of Theorem 10. Let H and K be p-subgroups of G such that H is contained in K and let P be a Sylow p-subgroup of G containing K. Write $T = \mathrm{N}_G(K)$. We must prove that H is S-permutable in T. Applying Lemma 16, H is self-normalising in $\mathrm{B}_T(K)$. If H is not S-permutable in T, then H is a proper subgroup of $\mathrm{B}_T(H)$. It implies that H is a proper subgroup of its normaliser in $\mathrm{B}_T(H)$ as H is a subnormal subgroup of T. This contradiction shows that H is S-permutable in $\mathrm{N}_G(K)$ and G has property $\mathcal{Y}_p$. □

Proof of Theorem 13. 1 implies 2. We suppose that the implication is not true and derive a contradiction. Let (G, H) be a counterexample with $|G|$ minimal. Assume that $A = \mathrm{B}_G(H)$ is a proper subgroup of G. Then, by Theorem 1, A is a soluble PST-group and so H is abnormal in $\mathrm{B}_A(H)$. Since $\mathrm{B}_A(H)$ is S-permutable in A and A is S-permutable in G, it follows that $\mathrm{B}_A(H)$ is S-permutable in G because G is a PST-group. Consequently $A = \mathrm{B}_A(H)$ and then H is abnormal in A. This contradiction shows that $A = G$. If $N := \mathrm{Core}_G(H) \neq 1$, we have that H/N is abnormal in G/N by the minimal choice of G. Hence H is abnormal in G. Thus we can assume that $N := \mathrm{Core}_G(H) = 1$. Applying Agrawal's theorem (Theorem 1), $G = LM$, where L is the nilpotent residual of G, $L \cap M = 1$, and L is an abelian normal Hall subgroup of G of odd order acted upon by conjugation as a group of power automorphisms by M. It implies that every subgroup of L is normal in G and so $L \cap H = 1$. Hence we can assume, without loss of generality, that H is contained in M. Since M is nilpotent, we have that LH is S-permutable in G. Thus $\mathrm{B}_G(H) = G = LH$ and $H = M$. Applying [9, IV, 5.18 and III, 4.6], H is a Carter subgroup of G. Applying [9, IV, Section 3 and 4.6, and I, 6.21], H is abnormal in G. This is the desired contradiction.

On the other hand, by virtue of [9, I, 6.21], every abnormal subgroup is self-normalising. Therefore 2 implies 3 and 4 implies 5. It is clear that 2 implies 4 and 3 implies 5.

To complete the proof we now show that 5 implies 1. Assume that every p-subgroup is self-normalising in its S-permutable closure for each prime p. By Theorem 10, G satisfies $\mathcal{Y}_p$ for each prime p. Applying [4, Theorem 4], we conclude that G is a soluble PST-group. □

Proof of Theorem 8. The same arguments to those used in the proof of Theorem 13 replacing Agrawal's result by Zacher's result (see Theorem 1) show that 1 implies 2. It is clear that 2 implies 3 and 4 implies 5. Obviously 2 implies 4 and 3 implies 5.

Now if every p-subgroup is self-normalising in its permutable closure for each prime p, then G satisfies $\mathcal{X}_p$ for each prime p by virtue of Theorem 7. Applying [5, Theorem A], G is a soluble PT-group. □

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