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Sylow permutable subnormal subgroups of finite groups*

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Dedicated to John Cossey on the occasion of his sixtieth birthday

Abstract

An extension of the well-known Frobenius' criterion of p-nilpotence in groups with modular Sylow p-subgroups is proved in the paper. This result is useful to get information about the classes of groups in which every subnormal subgroup is permutable and Sylow permutable.

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1 Introduction and statements of results

Throughout the paper, the word *group* means finite group.

A celebrated theorem of Frobenius ([9, Satz IV.5.8]) asserts that if p is a prime and G is a group such that $N_G(H)$ is p-nilpotent for every p-subgroup H of G, then G is p-nilpotent.

Our first main result can be considered as an extension of Frobenius' theorem in groups with modular Sylow p-subgroups.

Theorem 1. *Let p be a prime and let G be a group with a modular Sylow p-subgroup P. Then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent.*

This result turns out to be useful to study the classes of PST-groups and PT-groups.

Recall that a subgroup H of a group G is said to be *S-permutable* (or *S-quasinormal*, or *π-quasinormal*) in G if $HP = PH$ for all Sylow subgroups P of G. It is clear that S-permutability is weaker than permutability and normality. According to a theorem of Kegel [10, Satz 1], every S-permutable subgroup is subnormal. S-permutability, like normality and permutability, is not a transitive relation.

We say that a group G is a *PST-group* if S-permutability is transitive in G, that is, if A is an S-permutable subgroup of B and B is an S-permutable subgroup of G, then A is S-permutable in G. Applying Kegel's theorem, PST-groups are exactly the groups in which every subnormal subgroup is S-permutable. This class contains the class of all groups in which normality is transitive (*T-groups*) and the class of all groups in which permutability is transitive (*PT-groups*). The last two classes have been widely studied ([1, 4, 6, 7, 10, 11, 15]).

The structure of soluble PST-groups was obtained by Agrawal in [1]. It is proved there that a group G is a soluble PST-group if and only if G has an abelian normal Hall subgroup of odd order N such that G/N is nilpotent and the elements of G induce power automorphisms in N. In that result, if we force G/N to be a Dedekind group, we find Gaschütz's characterisation of soluble T-groups ([7]), and if we impose that G/N is a nilpotent modular group, then we obtain Zacher's characterisation of soluble PT-groups ([15]).

The above results show that, in the soluble universe, the difference between these three classes is simply the Sylow structure. Our second result supports that claim and provides a unified viewpoint for the classes of PST, PT and T-groups in the general finite case.

Theorem 2. *Let G be a group.*

1. *Suppose that p is a prime number and that H is an S-permutable p-subgroup of G. If the Sylow p-subgroups of G are modular (respectively, Dedekind), then H is permutable (respectively, normal) in G.*

2. *Assume that H is an S-permutable subgroup of G. If the Sylow subgroups of G are modular (respectively, Dedekind), then H is permutable (respectively, normal) in G.*

Taking this result into account, it seems natural to look for characterisations of the above classes in terms of the Sylow structure. This was done by Robinson ([11]) for the class of T-groups and by Beidleman, Brewster and Robinson ([4]) for the class of PT-groups.

One of the purposes of this paper is to provide necessary and sufficient conditions on the Sylow structure for a group to be a soluble PST-group.

As in the PT and T-cases, the procedure of defining local versions in order to simplify the study of the global properties has revealed itself as considerably useful.

Since our approach depends heavily on a previous analysis of the classes of PT-groups and T-groups, the following definition needs to be stated.

Definition 1. Let G be a group and p a prime. We say that G:

1. Enjoys property $\mathcal{C}_p$ (see [11]) if each subgroup of a Sylow p-subgroup P of G is normal in the normaliser $N_G(P)$.

2. Satisfies property $\mathcal{X}_p$ (as in [4]) if each subgroup of a Sylow p-subgroup P of G is permutable in the normaliser $N_G(P)$.

Robinson ([11]) proved that a group G is a soluble T-group if and only if G satisfies property $\mathcal{C}_p$ for all primes p and, thirty-one years later, Beidleman, Brewster and Robinson proved that G is a soluble PT-group if and only if G satisfies property $\mathcal{X}_p$ for all primes p.

These results would follow easily if one could prove that $\mathcal{C}_p$ and $\mathcal{X}_p$ are subgroup-closed. The subgroup-closed character of $\mathcal{C}_p$ follows from the abnormality of the Sylow normalisers. Nevertheless, in the Beidleman, Brewster and Robinson approach, the subgroup-closed character of $\mathcal{X}_p$ follows after an intensive study of the property $\mathcal{X}_p$ and its consequences for the group structure (see [4, Corollary 3]). In the following, we show that the subgroup-closed character of the property $\mathcal{X}_p$ follows as a natural consequence of Theorem 1 and a new property called $\mathcal{Y}_p$, which can be considered as the "PST-version" of the properties $\mathcal{C}_p$ and $\mathcal{X}_p$.

Definition 2. Let p be a prime number. A group G is said to be a *$\mathcal{Y}_p$-group* when for all p-subgroups H and S of G such that $H \leq S$, H is S-permutable in $N_G(S)$.

The above property can be compared to property $\mathcal{S}_p$ introduced by Beidleman and Heineken in [5].

Theorem 3. *A group G satisfies $\mathcal{X}_p$ (respectively, $\mathcal{C}_p$) if and only if G satisfies $\mathcal{Y}_p$ and the Sylow p-subgroups of G are modular (respectively, Dedekind).*

Since $\mathcal{Y}_p$ is subgroup-closed, this result has the virtue of showing that the subgroup-closed character of $\mathcal{X}_p$ depends exclusively on the modularity of the Sylow p-subgroups. It also shows that, in order to get a global characterisation of the soluble PST-groups, it is necessary to impose the subgroup-closed character in the definition $\mathcal{Y}_p$, as in the PST-case there are no restrictions on the Sylow p-subgroups.

Assume that G is a solubpe PST-group. If H and S are p-subgroups of G such that $H \leq S$, then H is subnormal in $N_G(S)$. Now, by Agrawal's Theorem, $N_G(S)$ is a PST-group. Therefore H is S-permutable in $N_G(S)$. Consequently every soluble PST-group has property $\mathcal{Y}_p$. Our next result confirms that the converse is also true.

Theorem 4. *A group G is a soluble PST-group if and only if G satisfies $\mathcal{Y}_p$ for all primes p.*

Note that Theorem A of [4] is a consequence of Theorems 3 and 4.

One of the main results of [4] is that a group G satisfies $\mathcal{X}_p$ if and only if G has modular Sylow p-subgroups and either G is p-nilpotent or a Sylow p-subgroup P of G is abelian and G satisfies $\mathcal{C}_p$.

This result is a consequence of Theorem 3 and the following:

Theorem 5. *A group G is a $\mathcal{Y}_p$-group if and only if G is either p-nilpotent, or G has abelian Sylow p-subgroups and G satisfies $\mathcal{C}_p$.*

Theorem C of [4] follows from Theorem 3 and

Corollary 1. *If p is the smallest prime divisor of the order of G, then G is a $\mathcal{Y}_p$-group if and only if G is p-nilpotent.*

Theorem 5 has revealed itself to be useful to prove some interesting results on PST-groups. For instance, it is proved in [5, Theorem H] that a soluble group G is a PST-group if and only if every subnormal subgroup permutes with every Carter subgroup of G and the subnormal subgroups are hypercentrally embedded in G. As an application of Theorem 5, we prove in [3, Corollary 2] that the permutability with the Carter subgroups can be removed.

Theorem 6 ([3])**.** *A soluble group G is a PST-group if and only if every subnormal subgroup of G is hypercentrally embedded in G.*

Another application of Theorem 5 is the following structure theorem for p-soluble groups with property $\mathcal{Y}_p$.

Theorem 7 ([3])**.** *A p-soluble group has property $\mathcal{Y}_p$ if and only if*

1. *either G is p-nilpotent, or*
2. *$G(p)/O_{p'}\big(G(p)\big)$ is an abelian normal Sylow p-subgroup of $G/O_{p'}\big(G(p)\big)$ such that the elements of $G/O_{p'}\big(G(p)\big)$ induce power automorphisms in $G(p)/O_{p'}\big(G(p)\big)$.*

Here, $G(p)$ denotes the p-nilpotent residual of G, that is, the smallest normal subgroup of G such that $G/G(p)$ is p-nilpotent.

The paper is organised as follows. In Section 2 we study property $\mathcal{Y}_p$ and its relation with the properties $\mathcal{C}_p$ and $\mathcal{X}_p$. The local approach to the class of soluble PST-groups developed in [2] plays an important role. The proofs of the main results appear in Section 3. Finally, we give some non-soluble examples of groups with property $\mathcal{Y}_p$ and a remark to show that any hope of creating a similar landscape out of the soluble universe which leads to a characterisation of PST, PT and T-groups is soon dispelled.

2 The property $\mathcal{Y}_p$

In the sequel p will be a fixed prime.

Our first result confirms the subgroup-closed character of the property $\mathcal{C}_p$. This is a consequence of the abnormality of the normalisers of the Sylow subgroups.

Lemma 1. *$\mathcal{C}_p$ is inherited by subgroups.*

Proof. Assume that G has the property $\mathcal{C}_p$ and let B a subgroup of G. If C is a Sylow p-subgroup of B and D is contained in C, then D is normal in $N_G(P)$ for every Sylow p-subgroup P of G containing C. Therefore if $g \in N_G(C)$, then D is normal in $\langle N_G(P), N_G(P^{g^{-1}})\rangle$. Since $N_G(P)$ is abnormal, it follows that $g^{-1} \in \langle N_G(P), N_G(P^{g^{-1}})\rangle$ and so $g \in N_G(D)$. Therefore D is normal in $N_G(C)$ and B has property $\mathcal{C}_p$. $\square$

Bryce and Cossey ([6]) established local versions of some results of soluble T-groups. In particular, they characterised the soluble groups with the property $\mathcal{C}_p$ as the groups G in which every p'-perfect subnormal subgroup of G is normal in G.

Following Bryce and Cossey's approach, it is proved in [2] that the soluble groups with property $\mathcal{X}_p$ are those whose p'-perfect subnormal subgroups are permutable with the Hall p'-subgroups and the Sylow p-subgroups are modular (see [2, Theorems 6 and 7]). Then the following definition arose:

Definition 3 ([2])**.** We say that a group G is a PST_p-group if G is p-soluble and every p'-perfect subnormal subgroup is permutable with the Hall p'-subgroups of G.

According to [2, Theorem 8], a soluble group G is a PST-group if and only if G is a PST_p-group for all primes p.

We say that a group $G \in \mathcal{U}_p^*$ if it is p-soluble, and the p-chief factors of G are cyclic groups and are G-isomorphic when regarded as G-groups by conjugation.

In [2, Theorem 6] it is proved that a soluble group G belongs to PST_p if and only if $G \in \mathcal{U}_p^*$. The arguments used there still hold in the p-soluble universe. Therefore we have:

Theorem 8. $PST_p = \mathcal{U}_p^*$.

In [2, Lemma 2] it is proved that the class of the PST_p-groups is quotient-closed. Theorem 8 shows that this class is also subgroup-closed.

The characterisation of soluble PST-groups in terms of the Sylow structure follows from the following:

Theorem 9. *A p-soluble group is a PST_p-group if and only if it satisfies $\mathcal{Y}_p$.*

We need the following elementary lemma.

Lemma 2. *Let G be a group.*

1. *If G has property $\mathcal{Y}_p$ and A is a normal p-subgroup of G, then G/A has property $\mathcal{Y}_p$.*
2. *If G has property $\mathcal{Y}_p$ and N is a normal p'-subgroup of G, then G/N has property $\mathcal{Y}_p$.*

Proof. 1. This follows immediately from the definition.

2. Assume that G has property $\mathcal{Y}_p$ and let $H/N \leq S/N$ be p-subgroups of G/N. Then there exist Sylow p-subgroups H_1 and S_1 of H and S, respectively, such that H_1 is contained in S_1 and $H = H_1N$ and $S = S_1N$. Since G has $\mathcal{Y}_p$, it follows that H_1 is S-permutable in $N_G(S_1)$. Therefore $H/N = H_1N/N$ is S-permutable in $N_G(S_1)N/N = N_{G/N}(S/N)$. This implies that G/N has $\mathcal{Y}_p$. □

Proof of Theorem 9. Assume that G satisfies $\mathcal{Y}_p$. We prove that G is a PST_p-group by induction on $|G|$. Denote $O_{p'}(G)$ by A and suppose that $A \neq 1$. Let H be a p'-perfect subnormal subgroup of G and let B be a Hall p'-subgroup of G. Then $A \leq B$ and B/A is a Hall p'-subgroup of G/A. Since G/A is a PST_p-group, it follows that HA/A permutes with B/A. Consequently H permutes with B and hence G is a PST_p-group. Therefore we may assume that $A = O_{p'}(G) = 1$.

Let N be a minimal normal subgroup of G. Then N is a p-group because G is p-soluble. If N_0 is a subgroup of N, then N_0 is S-permutable in $N_G(N) = G$. This means that if Q is a Sylow q-subgroup of G for $q \neq p$, then N_0 is a Sylow p-subgroup of N_0Q and so Q normalises N_0.

Therefore $O^p(G)$ normalises every subgroup of N. Let P be a Sylow p-subgroup of G and let N_1 be a minimal normal subgroup of P contained in N. Then $PO^p(G) = G$ normalises N_1 and so $N_1 = N$. This means that N is cyclic of order p. By Lemma 2, we know that G/N has $\mathcal{Y}_p$. Therefore G/N is a PST_p-group by induction.

Applying Theorem 8, we have that G/N is a $\mathcal{U}_p^*$-group. In particular, G/N is p-supersoluble. Since N is cyclic, it follows that G is p-supersoluble. Then G has a normal Sylow p-subgroup P containing the derived subgroup G' by [2, Lemma 1]. Let H be a p'-perfect subnormal subgroup of G. Then $P \cap H$ is a normal Sylow p-subgroup of H and so $P \cap H = H$ since H is p'-perfect. Hence H is a p-group and $H \leq P$. Therefore H is S-permutable in $N_G(P) = G$. In particular, H permutes with the Hall p'-subgroups of G. Therefore G is a PST_p-group.

Conversely, suppose that G is a PST_p-group. Suppose that H and S are p-subgroups of G such that $H \leq S$. Then H is a subnormal subgroup of $N_G(S)$, H is p'-perfect and $N_G(S)$ is a PST_p-group because the class of PST_p-groups is subgroup-closed. Thus H permutes with every Hall p'-subgroup Q of $N_G(S)$ and $X = HQ$ is a subgroup of G. Then $H \leq O_p(X)$ and $O_p(X) = H\big(O_p(X) \cap Q\big) = H$. Therefore H is normalised by Q. Consequently, $O^p\big(N_G(S)\big)$ normalises H and G has $\mathcal{Y}_p$. □

Note that every p-nilpotent group is $\mathcal{U}_p^*$-group. Therefore by Theorem 8 and 9 we have:

Corollary 2. *If G is p-nilpotent, then G has $\mathcal{Y}_p$.*

Another relevant property of groups with $\mathcal{Y}_p$ is:

Lemma 3. *If G has $\mathcal{Y}_p$ and if P is a non-abelian Sylow p-subgroup of G, then $N_G(P)$ is p-nilpotent.*

Proof. Let H be a subgroup of P. If Q is a Sylow q-subgroup of $N_G(P)$ for a prime $p \neq q$, then HQ is a subgroup of G. This implies that H is a subnormal Sylow p-subgroup of HQ and then Q normalises H. Therefore every p'-element of $N_G(P)$ normalises every subgroup of P. Since P is non-abelian, we can apply [8, Hilfssatz 5] to conclude that every p'-element of $N_G(P)$ actually centralises P. Consequently, $N_G(P)$ is p-nilpotent. □

Our proof of Theorem 5 depends on the relation between $\mathcal{Y}_p$ and p-normality.

Recall that if p is a prime, a group G is said to be *p-normal* if it satisfies the following property:

> If P is a Sylow p-subgroup of G and $Z(P)$ is contained in P^g for some $g \in G$, then $Z(P) = Z(P^g)$.

This property is closely related to property $\mathcal{Y}_p$. In fact, we have:

Lemma 4. *If G satisfies $\mathcal{Y}_p$, then G is p-normal.*

Proof. Suppose that G satisfies $\mathcal{Y}_p$. Let P be a Sylow p-subgroup of G and let g be an element of G such that $Z = Z(P) \leq P^g$. Suppose that Z is not a normal subgroup of P^g. Then (see Burnside's Theorem, [9, Satz IV.5.1]) there exists an element $g \in G$ of order q^b for a prime $q \neq p$ such that $g \notin N_G(Z)$, $J = ZZ^g \cdots Z^{g^{q^b}-1}$ is a p-group and $g \in N_G(J) \setminus C_G(J)$. But g is a p'-element of $N_G(J)$ and G is a $\mathcal{Y}_p$-group. Consequently g induces a power automorphism on J. In particular, we get the contradiction $g \in N_G(Z)$.

Therefore $Z(P)$ is a normal subgroup of P^g. Then $Z(P^{g^{-1}}) = \big(Z(P)\big)^{g^{-1}}$ is a normal subgroup of P. By [9, Hilfssatz IV.5.2], since $Z(P)$ is a characteristic subgroup of P, we have that $Z(P) = Z(P^{g^{-1}})$ and $Z(P) = Z(P^g)$. That proves that G is p-normal. □

3 Proofs of the main results

The next result is the p-soluble version of Theorem 1.

Lemma 5. *Let p be a prime. Assume that G is a p-soluble group with modular Sylow p-subgroups. If P is a Sylow p-subgroup of G such that $N_G(P)$ is p-nilpotent, then G is p-nilpotent.*

Proof. Assume the result is false and let G be a counterexample of least order. Then for each non-trivial normal subgroup N of G, it follows that G/N is p-nilpotent. Therefore, since the class of p-nilpotent groups is a saturated formation, it follows that G has a unique minimal normal subgroup N such that N is an elementary abelian p-group, $C_G(N) = N$ and N is complemented in G by a core-free maximal subgroup M. It is clear that N is contained in P. Suppose that N is a proper subgroup of P. Then, since $G = NM$, we have that $P = N(P \cap M)$ and $P \cap M \neq 1$. Let $x \in P \cap M$ be an element of order p. If $n \in N$, then $\langle n, x\rangle = \langle n\rangle\langle x\rangle$ is an elementary abelian p-group because P is modular. Therefore $x \in C_G(n)$. This implies that $1 \neq P \cap M \cap C_G(N) = P \cap M \cap N$, a contradiction. Hence $P = N$ and so $G = N_G(P)$ is p-nilpotent, final contradiction. □

Proof of Theorem 1. Let G be a group with modular Sylow p-subgroups and p-nilpotent Sylow normalisers with least order subject to not being p-nilpotent. Let P be a Sylow p-subgroup of G. From Burnside's p-nilpotence criterion ([9, Hauptsatz IV.2.6]) we have that P is non-abelian. Assume that P is Dedekind. Then $p = 2$ and P is a direct product of a quaternion group and an elementary abelian 2-group. Hence, if $\Omega_1(P)$ is the subgroup generated by the involutions of P, it follows that $\Omega_1(P) \leq Z(P)$. Suppose that $C_G\big(Z(P)\big)$ is a proper subgroup of G. Then $C_G\big(Z(P)\big)$ inherits the hypotheses of the theorem. By minimality of G, it follows that $C_G\big(Z(P)\big)$ is p-nilpotent. The p-nilpotence of G follows now from [16, Theorem 1]. Suppose that $C_G\big(Z(P)\big) = G$. Then $1 \neq Z(P)$ is central in G. From the minimality of G, it follows that $G/Z(P)$ is p-nilpotent and so G is p-nilpotent, a contradiction.

Suppose now that P is not Dedekind. Then, applying [14, Exercise 4.4.1], we have that $N = O^p(G) \neq G$. It is clear that $N_G(P) \leq N_G(P \cap N)$ and $P \cap N$ is a modular Sylow p-subgroup of N. Suppose that $P \cap N = 1$. Then N is a normal Hall p'-subgroup of G because $G = NP$. This implies that G is p-nilpotent, a contradiction. Therefore $P \cap N \neq 1$. Suppose that $N_G(P \cap N) = G$. Then there exists a minimal normal subgroup A of G such that $A \leq P \cap N$. By minimality of G, it follows that G/A is p-nilpotent. In particular, G is p-soluble. Applying Lemma 5, we have that G is p-nilpotent, a contradiction. Consequently $N_G(P \cap N)$ is a proper subgroup of G and it inherits the properties of G. The minimal choice of G implies that $N_G(P \cap N)$ is p-nilpotent. Then $N_N(P \cap N)$ is also p-nilpotent and so N satisfies the hypotheses of the theorem. Since $N \neq G$, it follows that N is p-nilpotent. Hence G is p-nilpotent, a contradiction. □

Proof of Theorem 2. 1. Let A be a subgroup of G and denote $T = \langle A, H\rangle$. Since H is S-permutable in T, then H is a subnormal subgroup of T and H is contained in $O_p(T)$, which is contained in every Sylow p-subgroup P of T. Therefore $T = \langle H, A\rangle \leq \langle O_p(T), A\rangle = O_p(T)A \leq T$. Let A_q be a Sylow

q-subgroup of A for a prime $q \neq p$, and let G_q be a Sylow q-subgroup of G containing A_q. We have that A_q is a Sylow q-subgroup of T, and $A_q = G_q \cap T$ because $A_q \leq G_q \cap T$. Hence $HA_q = H(G_q \cap T) = HG_q \cap T$ is a subgroup of T. Moreover $O_p(T) \cap HA_q = H$. Therefore H is normalised by A_q. On the other hand, since P is modular (respectively, Dedekind), we have that H permutes with (respectively, is normalised by) a Sylow p-subgroup A_p of A. Therefore H permutes with (respectively, is normalised by) all Sylow subgroups of A. In particular, H permutes with A (respectively, H is normalised by A). This implies that H is a permutable (respectively, normal) subgroup of G.

2. Suppose that G is a counterexample of minimal order to the theorem. Then there exists an S-permutable subgroup H of G such that H is not permutable (respectively, normal) in G. We take H of minimal order. Let N be a minimal normal subgroup of G. Since HN/N is S-permutable in G/N, we have that HN/N is permutable (respectively, normal) in G/N. Assume that $\mathrm{Core}_G(H) = H_G \neq 1$. Then we may suppose that $N \leq H$ and then H is permutable (respectively, normal) in G, a contradiction. Therefore we have that $H_G = 1$. According to [13, Proposition A], we have that H is a nilpotent group. By [13, Proposition B], every Sylow subgroup of H is S-permutable in G. From the minimality of H, we can suppose that H is a p-group for some prime p; otherwise, if all Sylow subgroups of H are permutable (respectively, normal) in G, H would be permutable (respectively, normal) in G. We conclude then that H is a p-group for some prime p. By 1, we conclude that H is permutable (respectively, normal) in G. □ □

Proof of Theorem 3. Suppose that G satisfies $\mathcal{Y}_p$ and a Sylow p-subgroup P of G is modular. By Theorem 2, we have that every subgroup of P is permutable in $N_G(P)$.

Conversely, suppose that G satisfies $\mathcal{X}_p$. Then it is clear that every Sylow p-subgroup P of G is modular. Moreover, by [4, Lemma 2], every subgroup of P is normalised by the p'-elements of $N_G(P)$. Therefore, if P is abelian, every subgroup of P is normal in $N_G(P)$ and then G satisfies property $\mathcal{C}_p$. Since $\mathcal{C}_p$ is subgroup closed by Lemma 1, G satisfies $\mathcal{Y}_p$.

If P is non-abelian, then $N_G(P)$ is p-nilpotent by [4, Corollary 2]. By Theorem 1, we have that G itself is p-nilpotent. Let $H \leq S$ be p-subgroups of G. Then $N_G(S)$ is p-nilpotent. Therefore H is centralised by each p'-element of $N_G(S)$. This implies that H is S-permutable in $N_G(S)$. Consequently G has $\mathcal{Y}_p$. □

Proof of Theorem 4. Assume that G is a soluble PST-group. Then G is a p-soluble PST_p-group for all primes p by [2, Theorem 8]. By Theorem 9, it follows that G is a $\mathcal{Y}_p$-group for all primes p.

Conversely, suppose that G satisfies $\mathcal{Y}_p$ for all primes p. Then every subgroup of G has the same property. Therefore if G is a group with least order subject to not being a soluble PST-group, then every proper subgroup of G is a soluble PST-group. According to Agrawal's Theorem, every soluble PST-group is supersoluble. Therefore either G is supersoluble, or G is a minimal non-supersoluble group. In both cases, we have that G is soluble (the solubility of G follows from [9, Satz VI.9.6] in the second case). Since $\mathcal{Y}_p$ coincides with PST_p in the p-soluble universe by Theorem 9, it follows that G is a PST_p-group for all p. Then G is a PST-group by [2, Theorem 8]. □

Proof of Theorem 5. Suppose that G is p-nilpotent. Then G satisfies $\mathcal{Y}_p$ by Corollary 2. Assume now that G has abelian Sylow p-subgroups and that G satisfies $\mathcal{C}_p$. It follows from Theorem 3 that G satisfies $\mathcal{Y}_p$.

Assume that the converse is not true and let G be a counterexample of minimal order. If G had an abelian Sylow p-subgroup, then G would satisfy $\mathcal{C}_p$ by Theorem 3. Therefore G has a non-abelian Sylow p-subgroup P and G is not p-nilpotent. Suppose that $P_G = \mathrm{Core}_G(P) = 1$. Therefore $N_G\big(Z(P)\big)$ is a proper subgroup of G. Hence $N_G\big(Z(P)\big)$ is p-nilpotent by the minimal choice of G. Applying Lemma 4 and [12, Exercise 594], we have that G is p-nilpotent, a contradiction.

Consequently $P_G \neq 1$. Let N be a minimal normal subgroup of G contained in P. Since G has minimal order and G/N is a $\mathcal{Y}_p$-group, it follows that either G/N is p-nilpotent or P/N is abelian.

Suppose that P/N is abelian. Since P is non-abelian, then $N_G(P)$ is p-nilpotent by Lemma 3, and so $N_G(P)/N = P/N \times O_{p'}\big(N_G(P)\big)N/N$ and P/N lies in the center of $N_{G/N}(P/N)$. From Burnside's p-nilpotence criterion (see [9, Hauptsatz IV.2.6]), we have that G/N is p-nilpotent. But if G/N is p-nilpotent, bearing in mind that G is a $\mathcal{Y}_p$-group and hence a $\mathcal{U}_p^*$-group by Theorems 8 and 9, we have that $|N| = p$ and p divides $|G/N|$ (otherwise, G would have an abelian Sylow p-subgroup $N = P$). It follows that G is p-nilpotent, because G acts centrally on the chief p-factors of G/N and hence G must act centrally on N. This contradiction proves the theorem. □

Proof of Corollary 1. Suppose that G is a non-p-nilpotent $\mathcal{Y}_p$-group of minimal order. Since all proper subgroups of G satisfy $\mathcal{Y}_p$, from the minimality it follows that all the proper subgroups of G are p-nilpotent. From Itô's Theorem (see [9, Satz IV.5.4]), we have that G has a normal Sylow p-subgroup P. But from Lemma 3, we have that $G = N_G(P)$ is p-nilpotent, a contradiction. □

4 Examples and remarks

1. Property $\mathcal{Y}_p$ does not imply property PST_p in general. The alternating group $G = A_5$ of degree 5 has Sylow 3-subgroups of order 3 and Sylow 5-subgroups of order 5. Hence it satisfies $\mathcal{C}_3$ and $\mathcal{C}_5$. By Theorem 5, G satisfies $\mathcal{Y}_3$ and $\mathcal{Y}_5$. But G is not 3-soluble nor 5-soluble. Hence it is clear that G is not a PST_3-group nor a PST_5-group.

2. Theorem 5 indicates the way for constructing non-soluble examples of groups with property $\mathcal{Y}_p$ which are not p-nilpotent.

   Let A be an abelian p-group and let B be a p'-group of power automorphisms of A. Denote by $H = [A]B$, the corresponding semidirect product. Suppose that H is not p-nilpotent. If S is any non-abelian simple group such that p does not divide S, then the regular wreath product G of S by H is a non-soluble group with property $\mathcal{Y}_p$ which is not p-nilpotent.

3. One might think that the most natural candidate for the "PST-version" of properties $\mathcal{C}_p$ and $\mathcal{X}_p$ could be:

   > A group G satisfies $\mathcal{Y}_p^*$ if every subgroup of a Sylow p-subgroup P is S-permutable in $N_G(P)$.

   In $G = \Sigma_4$, the symmetric group of degree 4, the Sylow 2-subgroups are self-normalising, hence every subgroup of a Sylow 2-subgroup P of G is S-permutable in $N_G(P)$, and the subgroups of a Sylow 3-subgroup Q of G are S-permutable in $N_G(Q)$. Consequently G satisfies $\mathcal{Y}_p^*$ for every p, but G is not a PST-group, because the cyclic subgroups of the Klein 4-group are not permutable with the Sylow 3-subgroups of G.

   Note that the above example shows that $\mathcal{Y}_p^*$ is not subgroup-closed.

4. Any hope of creating a similar landscape outside of the soluble universe is soon dispelled. As soon as we have a local property $\mathcal{J}_p$ which is subgroup-closed and such that a finite group G is a PST-group if and only if G satisfies $\mathcal{J}_p$ for every prime p, then $\bigcap_{p\in\mathbb{P}} \mathcal{J}_p$ is contained in the class of soluble groups. Therefore a group satisfying $\mathcal{J}_p$ for all p should be soluble.

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