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ON $\mathfrak{X}$-SATURATED FORMATIONS OF FINITE GROUPS

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ABSTRACT

In the paper, a Frattini-like subgroup associated with a class $\mathfrak{X}$ of simple groups is introduced and analysed. The corresponding $\mathfrak{X}$-saturated formations are exactly the $\mathfrak{X}$-local ones introduced by Förster. Our techniques are also very useful to highlight the properties and behaviour of ω-local formations. In fact, extensions and improvements of several results of Shemetkov are natural consequences of our study.

1 INTRODUCTION

In the sequel it is understood that all groups are finite.

Recall that a *formation* is a class of groups which is closed under taking epimorphic images and subdirect products. A formation $\mathfrak{F}$ is said to be *saturated* if the condition $G/\Phi(G) \in \mathfrak{F}$ always implies $G \in \mathfrak{F}$. Saturated formations play a very important role in the theory of classes of groups because they are exactly those formations $\mathfrak{F}$ for which every group has $\mathfrak{F}$-projectors. The family of local formations introduced by Gaschütz allows us to generate saturated formations of ever-increasing complexity. In fact, both families coincide. It was proved by Gaschütz and Lubeseder in the soluble universe, and by Schmid in the general one. Today these results are known as Gaschütz-Lubeseder-Schmid theorem (see [1; Section IV] for details).

In an unpublished manuscript, Baer has studied a different definition of local formation. It is more flexible than the original one because the simple components, rather than the primes dividing its order, are used to label a chief factor and its automorphism group. Therefore the actions on the insoluble chief factors can be independent of those on the abelian chief factors (see [1; Section IV] for details).

Baer-local formations are exactly the ones that are closed under extensions through the Frattini subgroup of the soluble radical, i. e., the solubly saturated ones. The presentation of Baer's theorem in [1; IV,4.17] includes a nice variation of the proof of Lubeseder's theorem in the soluble universe.

Baer-local formations were also considered by Shemetkov (see [2]). He calls them *composition formations*.

With the purpose of presenting a common extension of the Gaschütz-Lubeseder-Schmid and Baer theorems, Förster introduced the concept of $\mathfrak{X}$-local formation, where $\mathfrak{X}$ is a class of simple groups with a completeness property ([3]). If $\mathfrak{X} = \mathfrak{J}$, the class of all simple groups, $\mathfrak{X}$-local formations are exactly the local formations. When $\mathfrak{X} = \mathbb{P}$, the class of all abelian simple groups, the notion of $\mathfrak{X}$-local formation coincides with the concept of Baer-local formation. Förster also defined a Frattini-like subgroup $\Phi^*_{\mathfrak{X}}(G)$ for each group G, which enables him to introduce the concept of $\mathfrak{X}$-saturation. He proves that $\mathfrak{X}$-saturated formations are exactly the $\mathfrak{X}$-local ones. However, Förster's definition of $\mathfrak{X}$-saturation is not the natural one if our aim is to generalise the concepts of saturation and soluble saturation. Since $\mathrm{O}_{\mathfrak{J}}(G) = G$ and $\mathrm{O}_{\mathbb{P}}(G) = G_{\mathfrak{S}}$, we would expect the $\mathfrak{X}$-Frattini subgroup of a group G to be defined as $\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)$, where $\mathrm{O}_{\mathfrak{X}}(G)$ is the largest normal subgroup of G whose composition factors belong to $\mathfrak{X}$. In many cases $\Phi^*_{\mathfrak{X}}(G)$ does not coincide with $\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)$. Since $\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big) \leq \Phi^*_{\mathfrak{X}}(G)$ for every group G, we can deduce from Förster's theorem that every $\mathfrak{X}$-local formation $\mathfrak{F}$ fulfills the following property:

A group G belongs to $\mathfrak{F}$ if and only if $G/\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)$ belongs to $\mathfrak{F}$. (*)

Therefore from the very beginning the following question naturally arises:

Open question 1.1. *Let $\mathfrak{F}$ be a formation with the property* (*). *Is $\mathfrak{F}$ $\mathfrak{X}$-local?*

One of the purposes of the present paper is to draw near the solution of the above question. We introduce another $\mathfrak{X}$-Frattini subgroup which is smaller than Förster's one. We will prove that $\mathfrak{X}$-local formations are exactly the $\mathfrak{X}$-saturated ones defined using our $\mathfrak{X}$-Frattini subgroup.

The method used in the proof of this theorem can be applied to deduce that ω-saturated formations are ω-local. We will also get the main results of [4] as particular cases of ours.

For the basic definitions and results on the theory of formations, as well as for the standard notations, the reader is referred to the book of Doerk and Hawkes [1].

2 PRELIMINARIES

We begin with the concept of $\mathfrak{X}$-local formation due to Förster [3].

Denote by $\mathfrak{J}$ the class of all simple groups. For any subclass $\mathfrak{Y}$ of $\mathfrak{J}$, we put $\mathfrak{Y}' = \mathfrak{J} \setminus \mathfrak{Y}$. Denote by $\mathrm{E}\,\mathfrak{Y}$ the class of groups whose composition factors belong to $\mathfrak{Y}$. It is clear that $\mathrm{E}\,\mathfrak{Y}$ is a Fitting class, and so each group G has a largest normal $\mathrm{E}\,\mathfrak{Y}$-subgroup, the $\mathrm{E}\,\mathfrak{Y}$-radical $\mathrm{O}_{\mathfrak{Y}}(G)$. A chief factor of G which belongs to $\mathrm{E}\,\mathfrak{Y}$ is called a $\mathfrak{Y}$-chief factor, and if, moreover, p divides the order of a $\mathfrak{Y}$-chief factor H/K of G, we shall say that H/K is a $\mathfrak{Y}_p$-chief factor.

In the sequel it will be convenient to identify the prime p with the cyclic group C_p of order p.

Throughout this paper, $\mathfrak{X}$ denotes a fixed class of simple groups satisfying

$$\begin{aligned}\pi(\mathfrak{X}) &:= \{p \in \mathbb{P} \mid \text{there exists } G \in \mathfrak{X} \text{ such that } p \text{ divides } |G|\} \\ &= \{p \in \mathbb{P} \mid C_p \in \mathfrak{X}\} =: \operatorname{char}\mathfrak{X}.\end{aligned}$$

Definition 2.1 (Förster)**.** An *$\mathfrak{X}$-formation function* f associates to each $X \in \operatorname{char}(\mathfrak{X}) \cup \mathfrak{X}'$ a formation $f(X)$ (possibly empty). If f is an $\mathfrak{X}$-formation function, then the *$\mathfrak{X}$-local formation* $\mathrm{LF}_{\mathfrak{X}}(f)$ defined by f is the class of all groups G satisfying the following two conditions:

1. if H/K is an $\mathfrak{X}_p$-chief factor of G, then $G/\mathrm{C}_G(H/K) \in f(p)$, and
2. if G/K is a monolithic quotient of G such that the composition factor of its socle $\mathrm{Soc}(G/K)$ is isomorphic to $E \in \mathfrak{X}'$, then $G/K \in f(E)$.

A formation $\mathfrak{F}$ is said to be $\mathfrak{X}$-*local* if there exists an $\mathfrak{X}$-formation function f such that $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$.

Förster also defined an $\mathfrak{X}$-Frattini subgroup $\Phi^*_{\mathfrak{X}}(G)$ of a group G:

Definition 2.2 (Förster)**.** Let G be a group.

- For a prime p, we define $\Phi^p_{\mathfrak{X}}(G)$:
  - If $\mathrm{O}_{p'}(G) = 1$,
  $$\Phi^p_{\mathfrak{X}}(G) := \begin{cases} \Phi(G) & \text{if } \mathrm{Soc}\big(G/\Phi(G)\big) \text{ and } \Phi(G) \text{ belong to } \mathrm{E}\,\mathfrak{X}, \\ \Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big) & \text{otherwise.} \end{cases}$$
  - In general, $\Phi^p_{\mathfrak{X}}(G)$ is the subgroup of G such that
  $$\Phi^p_{\mathfrak{X}}(G)/\,\mathrm{O}_{p'}(G) = \Phi^p_{\mathfrak{X}}\big(G/\,\mathrm{O}_{p'}(G)\big).$$
- Finally
$$\Phi^*_{\mathfrak{X}}(G) := \mathrm{O}_{\mathfrak{X}}(G) \cap \left(\bigcap_{p \in \mathrm{char}(\mathfrak{X})} \Phi^p_{\mathfrak{X}}(G) \right).$$

As it is announced, we introduce another $\mathfrak{X}$-Frattini subgroup $\Phi_{\mathfrak{X}}(G)$ of a group G.

Definition 2.3. 1. Let p be a prime number. We say that a group G belongs to the class $\mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2)$ provided there exists an elementary abelian normal p-subgroup N of G such that

   (a) $N \leq \Phi(G)$ and G/N is a primitive group with a unique non-abelian minimal normal subgroup, i. e., G/N is a primitive group of type 2,

   (b) $\mathrm{Soc}(G/N) \in \mathrm{E}\,\mathfrak{X} \setminus \mathfrak{E}_{p'}$, and

   (c) $\mathrm{C}^h_G(N) \leq N$, where
   $$\mathrm{C}^h_G(N) := \bigcap \{\mathrm{C}_G(H/K) \mid H/K \text{ is a chief factor of } G \text{ below } N\}.$$

2. The $\mathfrak{X}$-Frattini subgroup of a group G is the subgroup $\Phi_{\mathfrak{X}}(G)$ defined as
$$\Phi_{\mathfrak{X}}(G) := \begin{cases} \Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big) & \text{if } G \notin \mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2) \text{ for all } p \in \mathrm{char}\,\mathfrak{X}, \\ \Phi(G) & \text{otherwise.} \end{cases}$$

Note that in the case $\mathfrak{X} = \mathfrak{J}$, the class of all simple groups, $\Phi_{\mathfrak{X}}(G) = \Phi(G)$ for every group G, and in the case $\mathfrak{X} = \mathbb{P}$, the class of all abelian simple groups, $\Phi_{\mathfrak{X}}(G) = \Phi(G_{\mathfrak{S}})$ for every group G, where $G_{\mathfrak{S}}$ is the soluble radical of G.

It can be proved that $\Phi_{\mathfrak{X}}(G) \leq \Phi^*_{\mathfrak{X}}(G)$ for every group G, but the equality is not true in general. In fact, for every class of simple groups $\mathfrak{X} \neq \mathfrak{J}$, there exists a group G such that $\Phi_{\mathfrak{X}}(G) < \Phi^*_{\mathfrak{X}}(G)$, as it is shown in the following example.

Example 2.4. If $\mathfrak{X} \neq \mathfrak{J}$, it follows that there exists a non-abelian simple group E and a prime $p \in \pi(E)$ such that $E \in \mathfrak{X}'$ and $p \in \operatorname{char}(\mathfrak{X})$. Let T be the group algebra $\mathrm{GF}(p)E$. Consider $G = [T]E$, the corresponding semidirect product. It is rather clear that $\Phi(G) = \operatorname{Rad}(T)$.

We have that $\Phi^p_{\mathbb{P}}(G) = \Phi(G)$, since $\mathrm{O}_{p'}(G) = 1$ and the groups $\Phi(G)$ and $\operatorname{Soc}\big(G/\Phi(G)\big)$ belong to $\mathrm{E}\,\mathfrak{X}$. If $q \neq p$, then $\Phi(G) \leq T \leq \mathrm{O}_{q'}(G) \leq \Phi^q_{\mathbb{P}}(G)$. Now it follows that $\Phi^*_{\mathbb{P}}(G) = \Phi(G)$. Since $p \in \pi(E)$, we have by Maschke's theorem that $\operatorname{Rad}(T) \neq 1$. Therefore $\Phi^*_{\mathfrak{X}}(G) \neq 1$. Clearly, $G \notin \mathrm{A}_{\mathfrak{X}_q}(\mathfrak{P}_2)$ for all $q \in \operatorname{char}(\mathfrak{X})$. Hence, $\Phi_{\mathfrak{X}}(G) = \Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big) = \Phi(T) = 1$.

Definition 2.5. A formation $\mathfrak{F}$ is said to be $\mathfrak{X}$-*saturated* whenever the condition $G/\Phi_{\mathfrak{X}}(G) \in \mathfrak{F}$ always implies $G \in \mathfrak{F}$.

There exist groups G for which $\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)$ is a proper subgroup of $\Phi_{\mathfrak{X}}(G)$, as the following example, suggested by John Cossey, shows:

Example 2.6. Let X be the maximal Frattini extension of the alternating group A_5 of degree 5 corresponding to the prime $p = 5$. Then X has an elementary abelian normal subgroup M of order 5^3 contained in $\Phi(X)$ and X/M is isomorphic to A_5 (see [1; Appendix β]). Let T be a non-abelian group of order 55, and consider $Y = X \wr T$. Let G be a subdirect product of Y and a cyclic group of order 25 with amalgamated factor group isomorphic to C_5. Consider the class $\mathfrak{X} = (A_5, C_2, C_3, C_5)$. Then G belongs to $\mathrm{A}_{\mathfrak{X}_5}(\mathfrak{P}_2)$, and so $\Phi_{\mathfrak{X}}(G) = \Phi(G)$, which has order $(5^3)^{55} \cdot 5$. On the other hand, $\Phi\big(\mathrm{O}_{\mathfrak{X}}(G)\big)$ has order $(5^3)^{55}$.

3 THE $\mathfrak{X}$-SATURATED FORMATIONS ARE $\mathfrak{X}$-LOCAL

The main goal in this section is to prove:

Theorem A. *The $\mathfrak{X}$-local formations are exactly the $\mathfrak{X}$-saturated formations.*

The following series of lemmas on $\mathfrak{X}$-saturated formations is needed to prove Theorem A.

Lemma 3.1. *Let $\mathfrak{F}$ be an $\mathfrak{X}$-saturated formation, X a group, and p a prime in* $\operatorname{char}(\mathfrak{X})$. *If there exists a faithful X-module M over* $\mathrm{GF}(p)$ *such that* $[M]X \in \mathfrak{F}$*, then* $[N]X \in \mathfrak{F}$ *for every irreducible* $\mathrm{GF}(p)X$*-module* N.

Proof. We can argue as in [1; IV,4.1], bearing in mind that the Hartley group used in the proof is a p-group and hence it belongs to $\mathrm{E}\,\mathfrak{X}$. □

Lemma 3.2. *Let $\mathfrak{F}$ be an $\mathfrak{X}$-saturated formation, G a group and let p be a prime in* $\operatorname{char}\mathfrak{X}$. *If $C_p \in \mathfrak{F}$ and N is a normal elementary abelian p-subgroup of G such that and $[N](G/N) \in \mathfrak{F}$, then $G \in \mathfrak{F}$.*

Proof. Analogous to [1; IV,4.15], noting that the Hartley group is a p-group as in the previous lemma. □

Lemma 3.3. *Let $\mathfrak{F}$ be an $\mathfrak{X}$-saturated formation and p a prime in* $\operatorname{char}\mathfrak{X}$. *If* $X \in \mathrm{R}_0\big(G/\mathrm{C}_G(H/K) \mid G \in \mathfrak{F}$ *and H/K is an $\mathfrak{X}_p$-chief factor of* $G\big)$, *then $[N]X \in \mathfrak{F}$ for every irreducible* $\mathrm{GF}(p)X$*-module* N.

Proof. By Lemma 3.1, it is enough to find a faithful X-module M over $\mathrm{GF}(p)$ such that $[M]X \in \mathfrak{F}$.

Since $X \in \mathrm{R}_0\big(G/\mathrm{C}_G(H/K) \mid G \in \mathfrak{F}$ and H/K is an $\mathfrak{X}_p$-chief factor of $G\big)$, there exist a natural number n and normal subgroups X_i of X, for $i = 1, 2, \ldots, n$, such that $\bigcap_{i=1}^{n} X_i = 1$ and $X/X_i \cong G_i/\mathrm{C}_{G_i}(H_i/K_i)$, where $G_i \in \mathfrak{F}$ and H_i/K_i is an $\mathfrak{X}_p$-chief factor of G_i.

Assume that H_i/K_i is non-abelian for some $i = 1, 2, \ldots, n$. Then $G := G_i/\mathrm{C}_{G_i}(H_i/K_i)$ is a primitive group of type 2. Consider the maximal Frattini extension E of G corresponding to the prime p. Then E has a elementary abelian normal p-subgroup $\mathrm{A}_p(G)$, the Frattini p-module of G, such that $E/\mathrm{A}_p(G) \cong G$. $\mathrm{A}_p(G)$ can be regarded as a $\mathrm{GF}(p)G$-module and so viewed we have that $\operatorname{Ker}\Big(G \text{ on } \operatorname{Soc}\big(\mathrm{A}_p(G)\big)\Big) = \mathrm{O}_{p',p}(G)$ (cf. [1; Appendix β]). In this case $\mathrm{O}_{p',p}(G) = 1$ and, therefore, there exists an irreducible $\mathrm{GF}(p)G$-submodule of $\mathrm{A}_p(G)$, say T, such that $\mathrm{C}_G(T) = 1$.

Note that $E \in \mathrm{A}_{\mathfrak{X}_p}(\mathfrak{P}_2)$. This means that $\Phi_{\mathfrak{X}}(E) = \Phi(E) = \mathrm{A}_p(G)$ and, therefore, $E/\Phi_{\mathfrak{X}}(E) \cong G \in \mathfrak{F}$. Since $\mathfrak{F}$ is $\mathfrak{X}$-saturated, it follows that $E \in \mathfrak{F}$. We have that T is an abelian $\mathfrak{X}_p$-chief factor of E such that $E/\mathrm{C}_E(T) \cong G$ and $E \in \mathfrak{F}$. Hence we can assume that H_i/K_i is abelian for all i.

By [1; IV,1.5], it follows that $[H_i/K_i]\big(G_i/\mathrm{C}_{G_i}(H_i/K_i)\big) \in \mathfrak{F}$. Consequently, $[H_i/K_i](X/X_i) \in \mathfrak{F}$.

Consider now $W := [H_i/K_i]X$. We have that $W\big/(H_i/K_i) \in \mathfrak{F}$ and $W/X_i \in \mathfrak{F}$. Therefore $W \in \mathrm{R}_0\,\mathfrak{F} = \mathfrak{F}$.

Write $M := H_1/K_1 \times H_2/K_2 \times \cdots \times H_n/K_n$. We have that H_i/K_i is a faithful $G_i/\operatorname{C}_{G_i}(H_i/K_i)$-module over $GF(p)$. Since $G_i/\operatorname{C}_{G_i}(H_i/K_i) \cong X/X_i$, we obtain that H_i/K_i is a $GF(p)X$-module and $\operatorname{C}_X(H_i/K_i) = X_i$. Since $\bigcap_{i=1}^{n} X_i = 1$, it follows that M is a faithful X-module over $GF(p)$. Moreover, $[M]X \in \mathrm{R}_0\,\mathfrak{F} = \mathfrak{F}$, as desired. □

Theorem 3.4. *If $\mathfrak{F}$ is an $\mathfrak{X}$-saturated formation, then $\mathfrak{F}$ is $\mathfrak{X}$-local.*

Proof. Let f be the $\mathfrak{X}$-formation function defined as

$$f(X) = \begin{cases} \mathrm{Q}\,\mathrm{R}_0\bigl(G/\operatorname{C}_G(H/K) \mid G \in \mathfrak{F} \text{ and } H/K \text{ is an } \mathfrak{X}_p\text{-chief factor of } G\bigr) \\ \hfill \text{if } X = p \in \operatorname{char}(\mathfrak{X}), \\ \mathfrak{F} \hfill \text{if } X \in \mathfrak{X}'. \end{cases}$$

It is clear that $\mathfrak{F} \subseteq \mathrm{LF}_{\mathfrak{X}}(f)$. Suppose that $\mathfrak{F} \neq \mathrm{LF}_{\mathfrak{X}}(f)$ and take a minimal group in $\mathrm{LF}_{\mathfrak{X}}(f) \setminus \mathfrak{F}$. Clearly, G is a monolithic group and $N := \operatorname{Soc}(G)$ is an abelian $\mathfrak{X}$-chief factor of G. Let p be the prime dividing the order of N. Thus, $f(p) \neq \emptyset$ and we can take a group

$$Y \in \mathrm{R}_0\bigl(G/\operatorname{C}_G(H/K) \mid G \in \mathfrak{F} \text{ and } H/K \text{ is an } \mathfrak{X}_p\text{-chief factor of } G\bigr).$$

Consider the trivial $\mathrm{GF}(p)Y$-module V. By Lemma 3.3, we have that $[V]Y \cong V \times Y \in \mathfrak{F}$ and thus $C_p \cong V \in \mathfrak{F}$.

Now we distinguish two cases:

- If $\operatorname{C}_G(N) = N$, we have that $G/N \in f(p)$. Therefore there exist $X \in \mathrm{R}_0\bigl(G/\operatorname{C}_G(H/K) \mid G \in \mathfrak{F} \text{ and } H/K \text{ is an } \mathfrak{X}_p\text{-chief factor of } G\bigr)$ and $T \trianglelefteq X$ such that $G/N \cong X/T$.

  N can be regarded as an irreducible $\mathrm{GF}(p)G$-module and as an irreducible $\mathrm{GF}(p)X$-module. By Lemma 3.3, we have that $[N]X \in \mathfrak{F}$. This means that $[N](X/T)$ and $[N](G/N)$ also belong to $\mathfrak{F}$. We can now apply Lemma 3.2 to obtain that $G \in \mathfrak{F}$, a contradicction.

- Assume now that $N < \operatorname{C}_G(N)$. By [1; IV,1.5], the group $B := [N](G/N)$ belongs to $\mathrm{LF}_{\mathfrak{X}}(f)$.

  Consider $M := \operatorname{C}_B(N) \cap (G/N)$. Since $M \neq 1$, the minimality of G implies that $B/M \in \mathfrak{F}$. Moreover, since $B/N \in \mathfrak{F}$ and $M \cap N = 1$, we have that $B \in \mathrm{R}_0\,\mathfrak{F} = \mathfrak{F}$. By Lemma 3.2, we obtain that $G \in \mathfrak{F}$, a contradiction.

  We have proved that $\mathrm{LF}_{\mathfrak{X}}(f) \subseteq \mathfrak{F}$ and, thus, $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$. □

Förster's main result ([3]) and Theorem 3.4 can be combined to get the following theorem:

Theorem 3.5. *Let $\mathfrak{F}$ be a formation and $\mathfrak{X}$ a class of simple groups. The following statements are pairwise equivalent:*

1. *$\mathfrak{F}$ is $\mathfrak{X}$-local,*
2. *$G \in \mathfrak{F}$ whenever $G/\Phi^*_{\mathfrak{X}}(G) \in \mathfrak{F}$, and*
3. *$\mathfrak{F}$ is $\mathfrak{X}$-saturated.*

Now we recall the following concepts and notations given by Skiba and Shemetkov in [5]:

Definition 3.6. Let ω be a nonempty set of prime numbers.

An *ω-satellite* f associates to each element of $\omega \cup \{\omega'\}$ a formation (possibly empty).

We use the symbol $G_{\omega d}$ to denote the largest normal subgroup N of G such that $\omega \cap \pi(H/K) \neq \emptyset$ for every composition factor H/K of N (if $\omega \cap \pi\bigl(\mathrm{Soc}(G)\bigr) = \emptyset$, then we just put $G_{\omega d} = 1$).

If f is an ω-local satellite, then $\mathrm{LF}_\omega(f)$ denotes the class of groups satisfying the following two conditions:

1. If H/K is a chief factor of G and $p \in \pi(H/K) \cap \omega$, then $G/\mathrm{C}_G(H/K) \in f(p)$, and
2. $G/G_{\omega d} \in f(\omega')$.

A formation $\mathfrak{F}$ is said to be *ω-local* if there exists an ω-local satellite f such that $\mathfrak{F} = \mathrm{LF}_\omega(f)$. In this case, f is called an ω-local satellite of $\mathfrak{F}$.

Definition 3.7. Let p be a prime number.

A formation $\mathfrak{F}$ is said to be *p-saturated* if $G \in \mathfrak{F}$ whenever $G/\mathrm{O}_p(G) \cap \Phi(G) \in \mathfrak{F}$.

If ω is a set of primes, a formation $\mathfrak{F}$ is said to be *ω-saturated* if it is p-saturated for every prime $p \in \omega$.

It is shown in [6] that ω-saturated formations are $\mathfrak{X}_\omega$-saturated, where $\mathfrak{X}_\omega$ is the class of all simple ω-groups. But, in general, the family of $\mathfrak{X}_\omega$-saturated formations does not coincide with the one of ω-saturated formations, as it is shown in the following example (see [6] for more details):

Example 3.8. Let us consider the formation $\mathfrak{F} = \mathrm{E}\,\mathfrak{Y}$, where $\mathfrak{Y} = (A_n \mid n \geq 5)$, i. e., the formation of all finite groups whose composition factors are isomorphic to groups in $\mathfrak{Y}$. $\mathfrak{F}$ is $\mathfrak{X}$-saturated for every class $\mathfrak{X}$ of abelian simple groups, but $\mathfrak{F}$ is not ω-saturated for any set ω of primes.

However, the methods used to prove Theorem 3.4 can be applied to prove the following result:

Theorem 3.9. *If $\mathfrak{F}$ is an ω-saturated formation, then $\mathfrak{F}$ is ω-local.*

This result has been proved by Skiba and Shemetkov in [5]. A standard argument confirms that the converse of Theorem 3.9 is also true (see [7]).

4 MORE ON $\mathfrak{X}$-SATURATED FORMATIONS

In the sequel we will tacitly assume the equivalence between 1 and 3 in Theorem 3.5. It will be used to get a lot of additional information about $\mathfrak{X}$-saturated formations. In particular, this allows us to deduce the main results of [4].

Throughout this section, $\mathfrak{X}$ is assumed to be a class of simple groups satisfying $\pi = \pi(\mathfrak{X}) = \mathrm{char}(\mathfrak{X})$.

Our next main result describes a full and integrated $\mathfrak{X}$-formation function defining an $\mathfrak{X}$-saturated formation. For saturated ones, this result was proved in [1; IV.3.7].

We begin with an elementary result.

Lemma 4.1. *Let $\mathfrak{F}$ be an $\mathfrak{X}$-local formation and let f be an $\mathfrak{X}$-formation function defining $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$. If N is a normal subgroup of G such that $N \in \mathrm{E}\,\mathfrak{X}$, $G/N \in \mathfrak{F}$, and $G/\mathrm{C}_G(N) \in f(p)$ for every $p \in \pi(N)$, then $G \in \mathfrak{F}$.*

Definition 4.2. Let $\mathfrak{F}$ be an $\mathfrak{X}$-saturated formation and let f be an $\mathfrak{X}$-formation function such that $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$.

We say that f is *full* if $\mathfrak{S}_p f(p) = f(p)$ for every $p \in \mathrm{char}(\mathfrak{X})$.

We say that the $\mathfrak{X}$-formation function f is *integrated* if $f(S) \subseteq \mathfrak{F}$ for every simple group $S \in \mathrm{char}(\mathfrak{X}) \cup \mathfrak{X}'$.

Definition 4.3. For a class of groups $\mathfrak{K}$ and a prime p, we define the formation:

$$\mathfrak{K}_{\mathfrak{X}_p} := \mathfrak{S}_p \,\mathrm{Q}\,\mathrm{R}_0\big(G/\mathrm{C}_G(H/K) \mid G \in \mathfrak{K} \text{ and } H/K \text{ is an } \mathfrak{X}_p\text{-chief factor of } G\big).$$

If $\mathfrak{K}$ is a class of groups, $\mathrm{form}_{\mathfrak{X}}(\mathfrak{K})$ denotes the smallest $\mathfrak{X}$-saturated formation containing $\mathfrak{K}$, i.e., the intersection of all $\mathfrak{X}$-saturated formations containing $\mathfrak{K}$. If $\mathfrak{X}$ is the class of all abelian simple groups, we write $\mathrm{bform}(\mathfrak{K})$ instead of $\mathrm{form}_{\mathfrak{X}}(\mathfrak{K})$.

The following lemma describes a full and integrated $\mathfrak{X}$-formation function defining $\mathrm{form}_{\mathfrak{X}}(\mathfrak{K})$.

Lemma 4.4. *1. Let $\mathfrak{K}$ be a class of groups. Then $\mathrm{form}_{\mathfrak{X}}(\mathfrak{K}) = \mathrm{LF}_{\mathfrak{X}}(f)$, where f is the following $\mathfrak{X}$-formation function:*

$$\begin{cases} f(p) = \mathfrak{K}_{\mathfrak{X}_p} & \text{if } p \in \mathrm{char}(\mathfrak{X}) \\ f(E) = \mathrm{Q}\,\mathrm{R}_0(\mathfrak{K}) & \text{if } E \in \mathfrak{X}' \end{cases}$$

Moreover, f is full and integrated.

2. *If $\mathfrak{F}$ is an $\mathfrak{X}$-local formation, then $\mathfrak{F}_{\mathfrak{X}_p} = (G \mid C_p \wr G \in \mathfrak{F})$ for every $p \in \mathrm{char}(\mathfrak{X})$.*

3. *If $\mathfrak{F}$ is an $\mathfrak{X}$-local formation, then*

$$\begin{aligned} &\mathfrak{S}_p\,\mathrm{Q}\,\mathrm{R}_0\big(G/\mathrm{C}_G(H/K) \mid G \in \mathfrak{F} \text{ and } H/K \text{ is an } \mathfrak{X}_p\text{-chief factor of } G\big) \\ &\quad = \mathfrak{S}_p\,\mathrm{Q}\,\mathrm{R}_0\big(G/\mathrm{C}_G(H/K) \mid G \in \mathfrak{F} \text{ and } H/K \text{ is an abelian} \\ &\qquad\qquad p\text{-chief factor of } G\big). \end{aligned}$$

Proof. 1. Let g be an $\mathfrak{X}$-formation function such that $\mathrm{form}_{\mathfrak{X}}(\mathfrak{K}) = \mathrm{LF}_{\mathfrak{X}}(g)$. It is clear that $\mathfrak{K} \subseteq \mathrm{LF}_{\mathfrak{X}}(f)$ and, hence, $\mathrm{form}_{\mathfrak{X}}(\mathfrak{K}) \subseteq \mathrm{LF}_{\mathfrak{X}}(f)$.

Assume that $\mathrm{LF}_{\mathfrak{X}}(f) \setminus \mathrm{form}_{\mathfrak{X}}(\mathfrak{K}) \neq \emptyset$ and let $G \in \mathrm{LF}_{\mathfrak{X}}(f) \setminus \mathrm{form}_{\mathfrak{X}}(\mathfrak{K})$ be a group of minimal order. Then G has a unique minimal normal subgroup N such that $N \in \mathrm{E}\,\mathfrak{X}$ and $G/N \in \mathrm{LF}_{\mathfrak{X}}(g)$. Let p be a prime in $\pi(N)$. Since $G/\mathrm{C}_G(N) \in \mathfrak{K}_{\mathfrak{X}_p}$ and $\mathrm{O}_p\big(G/\mathrm{C}_G(N)\big) = 1$, it follows that $G/\mathrm{C}_G(N) \in \mathrm{Q}\,\mathrm{R}_0\big(G/\mathrm{C}_G(H/K) \mid G \in \mathfrak{K}$ and H/K is an $\mathfrak{X}_p$-chief factor of $G\big) \subseteq g(p)$. Now by Lemma 4.1, we have that $G \in \mathrm{LF}_{\mathfrak{X}}(g)$, contradicting the choice of G. Therefore $\mathrm{LF}_{\mathfrak{X}}(f) \subseteq \mathrm{form}_{\mathfrak{X}}(\mathfrak{K})$.

It is clear that f is full. We prove that f is integrated. Let $E \in \mathfrak{X}'$. Clearly, $f(E) \subseteq \mathrm{form}_{\mathfrak{X}}(\mathfrak{K})$. Let p be a prime in $\mathrm{char}(\mathfrak{X})$. Assume that $\mathfrak{K}_{\mathfrak{X}_p}$ is not contained in $\mathrm{form}_{\mathfrak{X}}(\mathfrak{K})$ and let $G \in \mathfrak{K}_{\mathfrak{X}_p} \setminus \mathrm{form}_{\mathfrak{X}}(\mathfrak{K})$ be a group of least order. It follows that G is monolithic and its socle N is a p-group. Since $G/\mathrm{C}_G(N) \in \mathfrak{K}_{\mathfrak{X}_p} = f(p)$, we can apply Lemma 4.1 to get the contradiction $G \in \mathrm{LF}_{\mathfrak{X}}(f)$. Hence $\mathfrak{K}_{\mathfrak{X}_p} \subseteq \mathrm{form}_{\mathfrak{X}}(\mathfrak{K})$.

2. Consider $\overline{\mathfrak{F}_p} := (G \mid C_p \wr G \in \mathfrak{F})$. Let G be a group in $\mathfrak{F}_{\mathfrak{X}_p}$. Then $C_p \wr G \in \mathfrak{S}_p \mathfrak{F}_{\mathfrak{X}_p} = \mathfrak{F}_{\mathfrak{X}_p} \subseteq \mathfrak{F}$ and, hence, $G \in \overline{\mathfrak{F}_p}$.

   Let $G \in \overline{\mathfrak{F}_p}$ and form the wreath product $W := C_p \wr G \in \mathfrak{F}$. Let B denote the base group of W and consider the normal subgroup of W:

$$A := \bigcap \{\mathrm{C}_W(H/K) \mid H \leq B \text{ and } H/K \text{ is a chief factor of } W\}.$$

   We prove that A is a p-group. Since A acts nilpotently on B and B is a p-group, we have that $A/\mathrm{C}_A(B)$ is a p-group. Since $\mathrm{C}_A(B) = \mathrm{C}_W(B) \cap A = B \cap A$ is a p-group, we can conclude that A is also a p-group. Now $W/A \in \mathfrak{F}_{\mathfrak{X}_p}$. This implies that $W \in \mathfrak{S}_p \mathfrak{F}_{\mathfrak{X}_p} = \mathfrak{F}_{\mathfrak{X}_p}$ and, therefore, $G \in \mathrm{Q}\mathfrak{F}_{\mathfrak{X}_p} = \mathfrak{F}_{\mathfrak{X}_p}$, as desired.

3. It is clear from 2 and the fact that $\mathfrak{X}$-local formations are (C_p)-local formations for all primes $p \in \mathrm{char}(\mathfrak{X})$. □

Lemma 4.5. *Let $\mathfrak{F}$ be a formation. Let $\{\mathfrak{X}_i \mid i \in I\}$ be a family of classes of simple groups such that $\pi(\mathfrak{X}_i) = \mathrm{char}(\mathfrak{X}_i)$. Set $\mathfrak{X} = \bigcup_{i \in I} \mathfrak{X}_i$. If $\mathfrak{F}$ is $\mathfrak{X}_i$-local for all i, then $\mathfrak{F}$ is $\mathfrak{X}$-local.*

Proof. First notice that $\pi(\mathfrak{X}) = \mathrm{char}(\mathfrak{X})$ and hence the statement makes sense.

For $i \in I$, we have that $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}_i}(f_i)$, where

$$\begin{cases} f(p) = (G \in \mathfrak{E} \mid C_p \wr G \in \mathfrak{F}) & \text{if } p \in \mathrm{char}(\mathfrak{X}_i) \\ f(E) = \mathfrak{F} & \text{if } E \in \mathfrak{X}_i' \end{cases}$$

by Lemma 4.4.

We will see that $\mathfrak{F} = \mathrm{LF}_{(\mathfrak{X})}(f)$, where f is the $\mathfrak{X}$-formation function defined as:

$$\begin{cases} f(p) = (G \in \mathfrak{E} \mid C_p \wr G \in \mathfrak{F}) & \text{if } p \in \mathrm{char}(\mathfrak{X}), \\ f(E) = \mathfrak{F} & \text{if } E \in \mathfrak{X}'. \end{cases}$$

It is clear that $\mathfrak{F} \subseteq \mathrm{LF}_{\mathfrak{X}}(f)$. Assume that the inclusion is proper and consider G of minimal order in $\mathrm{LF}_{\mathfrak{X}}(f) \setminus \mathfrak{F}$. As usual, G is a monolithic group whose socle N is a p-group for a prime $p \in \mathrm{char}(\mathfrak{X})$.

There exists $i \in I$ such that $p \in \mathrm{char}(\mathfrak{X}_i)$. Thus $N \in \mathrm{E}\mathfrak{X}_i$, $G/N \in \mathfrak{F} = \mathrm{LF}_{\mathfrak{X}_i}(f_i)$, and $G/\mathrm{C}_G(N) \in f(p) = f_i(p)$. By Lemma 4.1, we have that $G \in \mathrm{LF}_{\mathfrak{X}_i}(f_i)$, contradicting the choice of G. Therefore $\mathrm{LF}_{\mathfrak{X}}(f) \subseteq \mathfrak{F}$. □

Corollary 4.6. *A formation $\mathfrak{F}$ is (C_p)-saturated for all primes p if and only if $\mathfrak{F}$ is Baer-local.*

Theorem 4.7. *Let $\mathfrak{F}$ be a formation.*

1. *If $\mathrm{form}_{\mathfrak{X}}(\mathfrak{F}) \subseteq \mathfrak{E}_{\pi'}\mathfrak{F}$, then $\mathfrak{F}$ is $\mathfrak{X}$-saturated.*

2. *If $\mathfrak{X}$ is a class of abelian simple groups and $\mathfrak{F}$ is $\mathfrak{X}$-saturated, then $\mathfrak{N}_{\pi'}\mathfrak{F}$ is $\mathfrak{X}$-saturated.*

Proof. 1. Assume that $G/\Phi_{\mathfrak{X}}(G) \in \mathfrak{F}$. Since $G \in \mathrm{form}_{\mathfrak{X}}(\mathfrak{F}) \subseteq \mathfrak{E}_{\pi'}\mathfrak{F}$, there exists a normal π'-subgroup M of G such that $G/M \in \mathfrak{F}$. Since $\Phi_{\mathfrak{X}}(G)$ is a π-group, it follows that $M \cap \Phi_{\mathfrak{X}}(G) = 1$ and so $G \in \mathfrak{F}$. Hence $\mathfrak{F}$ is $\mathfrak{X}$-saturated.

2. Assume that $\mathfrak{N}_{\pi'}\mathfrak{F}$ is not $\mathfrak{X}$-saturated and let G be a group of minimal order satisfying $G/\Phi_{\mathfrak{X}}(G) \in \mathfrak{N}_{\pi'}\mathfrak{F}$ and $G \notin \mathfrak{N}_{\pi'}\mathfrak{F}$. If M is a normal subgroup of G, we have that $(G/M)/\Phi_{\mathfrak{X}}(G/M) \in \mathfrak{N}_{\pi'}\mathfrak{F}$, since $\Phi_{\mathfrak{X}}(G)M/M \leq \Phi_{\mathfrak{X}}(G/M)$. This means that G is a monolithic group. Since $N := \mathrm{Soc}(G) \leq \Phi_{\mathfrak{X}}(G)$, we have that N is a π-group.

   Let M be a normal subgroup of G such that $M/\Phi_{\mathfrak{X}}(G) \in \mathfrak{N}_{\pi'}$ and
   $$\big(G/\Phi_{\mathfrak{X}}(G)\big)/\big(M/\Phi_{\mathfrak{X}}(G)\big) \cong G/M \in \mathfrak{F}.$$
   Since M is nilpotent, we have that $M = \Phi_{\mathfrak{X}}(G) \times \overline{M}$, where $\overline{M}$ is a normal Hall π'-subgroup of M. Since $\mathrm{O}_{\pi'}(G) = 1$, it follows that $M = \Phi_{\mathfrak{X}}(G)$ and $G/\Phi_{\mathfrak{X}}(G) \in \mathfrak{F}$. Therefore $G \in \mathfrak{F} \subseteq \mathfrak{N}_{\pi'}\mathfrak{F}$, a contradiction. □

Corollary 4.8. *Assume that $\mathfrak{X}$ is a class of abelian simple groups and let $\mathfrak{F}$ be a formation. The following statements are pairwise equivalent:*

1. *$\mathfrak{F}$ is $\mathfrak{X}$-saturated,*

2. *$\mathfrak{N}_{\pi'}\mathfrak{F}$ is $\mathfrak{X}$-saturated, and*

3. *$\mathrm{form}_{\mathfrak{X}}(\mathfrak{F}) \subseteq \mathfrak{N}_{\pi'}\mathfrak{F}$*

Theorem 4.9. *Let $\mathfrak{F}$ be a formation and let σ be a set of primes. Denote by $\mathfrak{Y} := \mathfrak{J} \cap \mathfrak{E}_{\sigma}$, the class of all simple σ-groups. The formation $\mathfrak{N}_{\sigma}\mathfrak{F}$ is $\mathfrak{Y}$-saturated and it can be defined by the following $\mathfrak{Y}$-formation function:*

$$\begin{cases} f(p) = \mathfrak{F} & \text{if } p \in \sigma \\ f(E) = \mathfrak{N}_{\sigma}\mathfrak{F} & \text{if } E \in \mathfrak{Y}' \end{cases}$$

Proof. Clearly, $\mathfrak{N}_\sigma\mathfrak{F} \subseteq \mathrm{LF}_{\mathfrak{Y}}(f)$.

Assume that $\mathrm{LF}_{\mathfrak{Y}}(f) \setminus \mathfrak{N}_\sigma\mathfrak{F} \neq \emptyset$ and consider a group G of least order in $\mathrm{LF}_{\mathfrak{Y}}(f) \setminus \mathfrak{N}_\sigma\mathfrak{F}$. Then G is a monolithic group whose socle N is a p-group for a prime $p \in \sigma$. We have that $G/\mathrm{C}_G(N) \in f(p) = \mathfrak{F}$. Moreover, since $G/N \in \mathfrak{N}_\sigma\mathfrak{F}$, there exists a normal subgroup M of G such that $N \leq M$, $M/N \in \mathfrak{N}_\sigma$ and $G/M \in \mathfrak{F}$. We observe that M is a σ-group. If $N \leq \Phi(G)$, we would obtain that M is nilpotent and, therefore, $G \in \mathfrak{N}_\sigma\mathfrak{F}$. This could not be possible. Therefore N is complemented in G and $\mathrm{C}_G(N) = N$. In particular, G belongs to $\mathfrak{N}_\sigma\mathfrak{F}$, a contradiction. Consequently $\mathrm{LF}_{\mathfrak{Y}}(f) \subseteq \mathfrak{N}_\sigma\mathfrak{F}$, as desired. □

Corollary 4.10. *Let $\mathfrak{F}$ be a formation. If $\mathfrak{F}$ is $\mathfrak{X}$-saturated, then $\mathfrak{N}_{\pi'}\mathfrak{F}$ is $\overline{\mathfrak{X}}$-local, where $\overline{\mathfrak{X}} := (\mathfrak{X} \cap \mathbb{P}) \cup (\mathfrak{J} \cap \mathfrak{E}_{\pi'})$. In particular, $\mathfrak{N}_{\pi'}\mathfrak{F}$ is a Baer-local formation.*

Proof. Since $\mathfrak{F}$ is $\mathfrak{X}$-saturated, we have that $\mathfrak{F}$ is also $(\mathfrak{X} \cap \mathbb{P})$-local. By Theorem 4.7, it follows that $\mathfrak{N}_{\pi'}\mathfrak{F}$ is $(\mathfrak{X} \cap \mathbb{P})$-local, since $\mathrm{char}(\mathfrak{X} \cap \mathbb{P}) = \mathrm{char}(\mathfrak{X}) = \pi$. Moreover, by Theorem 4.9, we have that $\mathfrak{N}_{\pi'}\mathfrak{F}$ is $(\mathfrak{J} \cap \mathfrak{E}_{\pi'})$-local. Now, by Lemma 4.5, we obtain that $\mathfrak{N}_{\pi'}\mathfrak{F}$ is $\overline{\mathfrak{X}}$-local. Since $\mathrm{char}(\overline{\mathfrak{X}}) = \mathbb{P}$, we have that $\mathfrak{N}_{\pi'}\mathfrak{F}$ is, in particular, a Baer-local formation. □

Corollary 4.11. *Let $\mathfrak{F}$ be a formation, $\mathfrak{X}$ a class of abelian simple groups and $\pi := \mathrm{char}(\mathfrak{X})$. Then $\mathrm{bform}(\mathfrak{F}) \subseteq \mathfrak{N}_{\pi'}\mathfrak{F}$ if and only if $\mathfrak{F}$ is $\mathfrak{X}$-saturated.*

If we take $\mathfrak{X} = (C_p)$ in Corollary 4.11, we obtain Shemetkov's main result in [4] (Theorem 3.2).

Lemma 4.12. *Let ω be a set of primes and $\mathfrak{F}$ an ω-saturated formation. Then $\mathfrak{N}_{\omega'}\mathfrak{F}$ is also ω-saturated.*

Proof. Assume that the result is not true and consider a group G of minimal order satisfying $G/\mathrm{O}_\omega(G) \cap \Phi(G) \in \mathfrak{N}_{\omega'}\mathfrak{F}$ and $G \notin \mathfrak{N}_{\omega'}\mathfrak{F}$. Note that G is a monolithic group whose socle is an ω-group. There exists a normal subgroup N_1 of G such that $G/N_1 \in \mathfrak{F}$ and $N_1/\mathrm{O}_\omega(G) \cap \Phi(G) \in \mathfrak{N}_{\omega'}$. Since N_1 is nilpotent, we have that $N_1 = \big(\mathrm{O}_\omega(G) \cap \Phi(G)\big) \times N_2$, where N_2 is a normal Hall ω'-subgroup of N_1. Since $\mathrm{O}_{\omega'}(G) = 1$, it follows that $N_2 = 1$ and, hence, $G/\mathrm{O}_\omega(G) \cap \Phi(G) \in \mathfrak{F}$. This implies that $G \in \mathfrak{F} \subseteq \mathfrak{N}_{\omega'}\mathfrak{F}$. This contradiction completes the proof. □

Corollary 4.13. *Let ω be a set of primes and let $\mathfrak{F}$ be an ω-saturated formation. Then $\mathrm{bform}(\mathfrak{F})$ is ω-saturated.*

Proof. Assume that the result is false and consider a group G of least order such that $G/\operatorname{O}_\omega(G) \cap \Phi(G) \in \operatorname{bform}(\mathfrak{F})$ and $G \notin \operatorname{bform}(\mathfrak{F})$. We have that G is a monolithic group whose socle is an ω-group. We know that $\mathfrak{F}$ is $\mathfrak{X}_\omega$-saturated by [6; Section 3], where $\mathfrak{X}_\omega$ is the class of all simple ω-groups. By Corollary 4.10, $\mathfrak{N}_{\omega'}\mathfrak{F}$ is a Baer-local formation. Therefore $G/\operatorname{O}_\omega(G) \cap \Phi(G) \in \mathfrak{N}_{\omega'}\mathfrak{F}$, since $\operatorname{bform}(\mathfrak{F}) \subseteq \mathfrak{N}_{\omega'}\mathfrak{F}$. By Lemma 4.12, it follows that $\mathfrak{N}_{\omega'}\mathfrak{F}$ is ω-saturated and, hence, $G \in \mathfrak{N}_{\omega'}\mathfrak{F}$. Bearing in mind that $\operatorname{O}_{\omega'}(G) = 1$, we conclude $G \in \operatorname{bform}(\mathfrak{F})$, a contradiction. □

As a particular case of Corollary 4.13, taking $\omega = \{p\}$, we get a result proved by Shemetkov [4; Theorem 3.1].

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