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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM

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EQUATIONS FOR ADIABATIC BUT ROTATIONAL STEADY
GAS FLOWS WITHOUT FRICTION

By Manfred Schäfer

Translation

“Gleichungen für Adiabatische, aber Wirbelbehaftete
Stationäre Gasströmungen ohne Reibung.”
Lehrstuhl für Technische Mechanik an der Technischen
Hochschule Dresden, Archiv. Nr. 44/1



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EQUATIONS FOR ADIABATIC BUT ROTATIONAL STEADY

GAS FLOWS WITHOUT FRICTION*

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ASSUMPTIONS

1. The flowing gases are assumed to have uniform energy distribution. ("Isoenergetic gas flows," that is, the energy equation is valid with the same constants for the entire flow.) This is correct, for example, for gas flows issuing from a region of constant pressure, density, temperature, and velocity. This property is not destroyed by compression shocks because of the universal validity of the energy law.

2. The gas behaves adiabatically, not during the compression shock itself but both before and after the shock. However, the adiabatic equation ($p/\rho^k = C$) is not valid for the entire gas flow with the same constant C but rather with an appropriate individual constant for each portion of the gas. For steady flows, this means that the constant C of the adiabatic equation is a function of the stream function. Consequently, a gas that has been flowing "isentropically", that is, with the same constant C of the adiabatic equation throughout (for example, in origination from a region of constant density, temperature, and velocity) no longer remains isentropic after a compression shock if the compression shock is not extremely simple (wedge shaped in a two-dimensional flow or cone shaped in a rotationally symmetrical flow).

The solution of nonisentropic flows is therefore an urgent necessity.

For the computations, the physical system of measurements (mass, length, time, degrees Kelvin (absolute temperature), and calorie) will be used. This system was chosen in order to avoid the frequent occurrence of g . In the final equations, the gas condition no longer explicitly appears but instead only properties of the velocity field based on the "limit velocity" V_m (corresponding to outflow into a vacuum) appear.

*"Gleichungen für Adiabatische, aber Wirbelbehaftete Stationäre Gasströmungen ohne Reibung." Lehrstuhl für Technische Mechanik an der Technischen Hochschule Dresden, Archiv. Nr. 44/1.

SYMBOLS

p	pressure
ρ	density
T	absolute temperature
\bar{v}	velocity vector [NACA comment: Changed from the German script \vec{v} for simplicity.] with velocity components u, v, and w in a cartesian x, y, z coordinate system
q	magnitude of velocity
ψ	stream function
V_m	limit velocity corresponding to outflow into a vacuum
a	local sonic velocity

BASIC EQUATIONS

Motion equations:

$$\left\{ \begin{array}{l} \rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = - \frac{\partial p}{\partial x} \\ \rho \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = - \frac{\partial p}{\partial y} \\ \rho \left(u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = - \frac{\partial p}{\partial z} \end{array} \right. \quad (1)$$

or

$$\rho \bar{v} \cdot \nabla \bar{v} = - \text{grad } p$$

Continuity equation:

$$\rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} = 0 \quad (2)$$

or

$$\text{div} (\rho \bar{v}) = 0$$

Along each stream line $\psi = \text{constant}$, the following adiabatic equation applies:

Adiabatic equation:

$$\frac{p}{\rho^\kappa} = f(\psi) \quad (3)$$

The flow line has direction cosines that are related by the proportion $u:v:w$; hence

$$\frac{\partial \psi}{\partial x} u + \frac{\partial \psi}{\partial y} v + \frac{\partial \psi}{\partial z} w = \text{grad } \psi \cdot \bar{v} = 0 \quad (4)$$

However, the stream function is not completely determined by this property. The numbering of the stream lines will be on the basis of the mass flowing through, in the cases of two-dimensional and of rotationally symmetrical flows.

The following gas equation is understood to apply:

Gas equation:

$$\frac{p}{\rho} = RT \quad (5)$$

The quantity $\frac{p}{\rho^\kappa}$ or, correspondingly, $f(\psi)$ is related in a readily apparent manner to the potential temperature used by meteorologists as well as to the entropy, namely:

The entropy per unit mass is

$$s = \frac{1}{J} \frac{R}{\kappa - 1} \ln \left(\frac{p}{\rho^\kappa} \right)$$

J, the mechanical heat equivalent, equals

$$4.18 \times 10^7 \frac{\text{gram cm}^2 \text{ sec}^{-2}}{\text{cal}}$$

The "potential temperature" θ is that temperature which a gas would reach if it were adiabatically brought to a suitably selected normal pressure p_n .

In accordance with the adiabatic equation

$$\frac{p}{\rho^\kappa} = \frac{p_n}{\rho_n^\kappa} = \frac{p_n^\kappa}{\rho_n^\kappa} p_n^{1-\kappa}$$

In accordance with the gas equation

$$\frac{p}{\rho^\kappa} = R^\kappa \theta^\kappa p_n^{1-\kappa}$$

Consequently

$$\theta^\kappa = \frac{p_n^{\kappa-1} p}{R^\kappa \rho^\kappa}$$

Instead of the potential temperature, F. Frankl introduces a "zero density" ρ_0 . (See discussion of the literature at end of report.)

The energy equation per unit mass flow is, as proved in thermodynamics,

$$JE_1 + \frac{p}{\rho} + \frac{1}{2} (u^2 + v^2 + w^2) = \text{constant}$$

where, according to the introductory remarks, the constant is supposed to be valid for the entire gas. The internal energy in thermal units is E_1 ; $JE_1 + \frac{p}{\rho}$ is the heat content i per unit mass, expressed in mechanical units.

In the ideal gases

$$E_i = c_v T$$

$$c_v = \frac{1}{J} \frac{R}{\kappa - 1}$$

$$\begin{aligned} i &= J E_i + \frac{p}{\rho} = \frac{RT}{\kappa - 1} + \frac{p}{\rho} \\ &= \frac{p}{\rho} \left(\frac{1}{\kappa - 1} + 1 \right) = \frac{p}{\rho} \frac{\kappa}{\kappa - 1} \end{aligned}$$

Hence the energy equation becomes

$$\frac{p}{\rho} \left(\frac{\kappa}{\kappa - 1} \right) + \frac{1}{2} (u^2 + v^2 + w^2) = \text{constant}$$

In terms of the maximum possible velocity V_m , which is the "limit velocity," $\frac{V_m^2}{2}$ may be written for the constant; for with outflow into a vacuum ($p/\rho = 0$), according to the previous equation

$$\frac{1}{2} (u^2 + v^2 + w^2) = \frac{V_m^2}{2}$$

Thus the final form of the energy equation is obtained if one more quantity, the magnitude of the velocity q is introduced.

$$\frac{p}{\rho} \frac{\kappa}{\kappa - 1} + \frac{q^2}{2} = \frac{V_m^2}{2} \tag{6}$$

n.b. For adiabatic flows, the following is true along a flow line:

$$\int \frac{dp}{\rho} = f(\psi) \kappa \int \rho^{\kappa - 2} d\rho = \frac{\kappa}{\kappa - 1} f(\psi) \rho^{\kappa - 1} = \frac{\kappa}{\kappa - 1} \frac{p}{\rho}$$

and this equation agrees with the first term of equation (6).

Because $\int \frac{dp}{\rho}$ occurs in the integration of the motion equation (1),

$\bar{v} \cdot \nabla \bar{v} = -\frac{\text{grad } p}{\rho}$, it is evident that the energy equation (6) may be derived by this method also, at least for adiabatic flows. Compared with this, however, the thermodynamic derivation used previously has the advantage of being valid for the entire flow independently of compression shocks.

By the use of equation (6), the following equation is obtained for local sound velocity a :

$$a^2 = \frac{dp}{d\rho} = \kappa f(\psi) \rho^{\kappa-1} = \kappa \frac{p}{\rho} = \frac{\kappa-1}{2} (V_m^2 - q^2) \quad (7)$$

DIFFERENTIAL EQUATIONS FOR GAS-VELOCITY FIELD ALONE

From equations (1), (2), (3), and (6) it is now intended to derive equations that contain only velocities; that is, p and ρ are to be eliminated.

In order to eliminate ρ from equation (2), $\text{div}(\rho \bar{v}) = 0$, the following equation is derived from equations (3) and (6):

$$f(\psi) \rho^{\kappa-1} \frac{\kappa}{\kappa-1} = \frac{V_m^2 - q^2}{2}$$

therefore

$$\rho = \left[\frac{\kappa-1}{2\kappa} \frac{V_m^2 - q^2}{f(\psi)} \right]^{\frac{1}{\kappa-1}}$$

and

$$\text{div}(\rho \bar{v}) = \rho \text{div } \bar{v} + \bar{v} \cdot \text{grad } \rho = 0 \quad (2a)$$

This equation may be replaced by the equivalent equation

$$\text{div } \bar{v} + \bar{v} \frac{\text{grad } \rho}{\rho} = \text{div } \bar{v} + \bar{v} \text{grad } \ln \rho = 0$$

Now with the new expression for ρ [NACA comment: The second equation in this section.]

$$\text{grad ln } \rho = \text{grad ln } [V_m^2 - q^2]^{\frac{1}{\kappa-1}} - \text{grad ln } f(\psi)^{\frac{1}{\kappa-1}}$$

Then it follows that

$$\bar{v} \cdot \text{grad ln } \rho = \bar{v} \cdot \text{grad ln } [V_m^2 - q^2]^{\frac{1}{\kappa-1}}$$

because

$$\bar{v} \cdot \text{grad ln } f(\psi)^{\frac{1}{\kappa-1}} = \frac{1}{f(\psi)} \frac{df(\psi)}{d\psi} \frac{1}{\kappa-1} \left\{ u \frac{\partial \psi}{\partial x} + v \frac{\partial \psi}{\partial y} + w \frac{\partial \psi}{\partial z} \right\}$$

and according to equation (4) becomes equal to 0. Therefore

$$\begin{aligned} \text{div } \bar{v} + \bar{v} \cdot \text{grad ln } \rho &= \text{div } \bar{v} + \bar{v} \cdot \text{grad ln } [V_m^2 - q^2]^{\frac{1}{\kappa-1}} \\ &= \text{div } \bar{v} + \frac{\bar{v} \cdot \text{grad } [V_m^2 - q^2]^{\frac{1}{\kappa-1}}}{[V_m^2 - q^2]^{\frac{1}{\kappa-1}}} = 0 \end{aligned}$$

For this can finally be written

$$[V_m^2 - q^2]^{\frac{1}{\kappa-1}} \text{div } \bar{v} + \bar{v} \cdot \text{grad } [V_m^2 - q^2]^{\frac{1}{\kappa-1}} = 0$$

or:

$$\text{div} \left(\bar{v} [V_m^2 - q^2]^{\frac{1}{\kappa-1}} \right) = 0 \quad (8)$$

This equation is already found in the work of L. Crocco, 1937. (See discussion of literature at end of report.)

From the motion equation

$$\rho \bar{v} \cdot \nabla \bar{v} = -\text{grad } p$$

p and ρ are also eliminated.

From previous reasoning

$$\rho = \left[\frac{\kappa-1}{2\kappa} \frac{V_m^2 - q^2}{f(\psi)} \right]^{\frac{1}{\kappa-1}}$$

According to equation (3)

$$p = f(\psi) \rho^\kappa = f(\psi) \left[\frac{\kappa-1}{2\kappa} (V_m^2 - q^2) \right]^{\frac{\kappa}{\kappa-1}}$$

Therefore $\frac{\text{grad } p}{\rho}$ is expressed as:

$$\begin{aligned} \text{grad } p &= \text{grad} \left\{ f(\psi) \left[\frac{\kappa-1}{2\kappa} (V_m^2 - q^2) \right]^{\frac{\kappa}{\kappa-1}} \right\} \\ &= -\frac{1}{\kappa-1} f(\psi)^{-\frac{1}{\kappa-1}-1} \frac{df(\psi)}{d\psi} \left[\frac{\kappa-1}{2\kappa} (V_m^2 - q^2) \right]^{\frac{\kappa}{\kappa-1}} \text{grad } \psi \\ &\quad - f(\psi)^{-\frac{1}{\kappa-1}} \frac{\kappa}{\kappa-1} \left[\frac{\kappa-1}{2\kappa} (V_m^2 - q^2) \right]^{\frac{\kappa}{\kappa-1}-1} \frac{\kappa-1}{2\kappa} \text{grad } q^2 \\ &= -\frac{1}{\kappa-1} f(\psi)^{-\frac{\kappa}{\kappa-1}} \frac{df(\psi)}{d\psi} \left[\frac{\kappa-1}{2\kappa} (V_m^2 - q^2) \right]^{\frac{\kappa}{\kappa-1}} \text{grad } \psi \\ &\quad - f(\psi)^{-\frac{1}{\kappa-1}} \left[\frac{\kappa-1}{2\kappa} (V_m^2 - q^2) \right]^{\frac{1}{\kappa-1}} \text{grad } \frac{q^2}{2} \end{aligned}$$

$$\begin{aligned}
 - \frac{\text{grad } p}{\rho} &= \frac{1}{\kappa-1} f(\psi)^{-\frac{\kappa}{\kappa-1} + \frac{1}{\kappa-1}} \frac{df(\psi)}{d\psi} \left[\frac{\kappa-1}{2\kappa} (v_m^2 - q^2) \right]^{\frac{\kappa}{\kappa-1} - \frac{1}{\kappa-1}} \text{grad } \psi \\
 + f(\psi)^{-\frac{1}{\kappa-1} + \frac{1}{\kappa-1}} &\left[\frac{\kappa-1}{2\kappa} (v_m^2 - q^2) \right]^{\frac{1}{\kappa-1} - \frac{1}{\kappa-1}} \text{grad } \frac{q^2}{2} \\
 &= \frac{1}{\kappa-1} f(\psi)^{-1} \frac{df(\psi)}{d\psi} \left[\frac{\kappa-1}{2\kappa} (v_m^2 - q^2) \right] \text{grad } \psi + \text{grad } \frac{q^2}{2}
 \end{aligned}$$

Consequently, after taking into account also

$$\bar{v} \cdot \nabla \bar{v} = \text{grad } \frac{q^2}{2} + \text{rotation } \bar{v} \times \bar{v}$$

the following is obtained as the motion equation:

$$\text{rotation } \bar{v} \times \bar{v} = \frac{1}{2\kappa} (v_m^2 - q^2) \frac{d \ln f(\psi)}{d\psi} \text{grad } \psi \tag{9}$$

This equation appears to be new. It contains the expression of the Bjerknes vortex theorem for the flows in question.

Because

$$\frac{d \ln f(\psi)}{d\psi} \text{grad } \psi = \text{grad } \ln f(\psi)$$

L. Crocco's (1937) equation number (5) may also be immediately derived from this equation

$$\text{rotation } \left(\frac{\text{rotation } \bar{v} \times \bar{v}}{v_m^2 - q^2} \right) = 0$$

however, use of this equation is less informative than equation (9).

INTRODUCTION OF DIMENSIONLESS MAGNITUDES

If the procedure of other authors is followed (Castagna: Atti della R. Acc. delle Science di Torino, vol. 70) in basing the velocities on V_m , which appears to be the most useful procedure, the following system of equations is finally obtained

$$\text{rotation } \bar{v} \times \bar{v} = \frac{1}{2\kappa} (1 - q^2) \text{ grad } \ln f(\psi) \quad (\text{I})$$

$$\text{div} \left(\bar{v} [1 - q^2]^{\frac{1}{\kappa-1}} \right) = 0 \quad (\text{II})$$

[NACA comment: Compare equation (8).] and equation (4) as the third

$$\bar{v} \cdot \text{grad } \psi = 0 \quad (\text{III})$$

These equations furnish five scalar equations for the four unknown quantities u , v , w , and ψ . This apparent extra relation is caused by the fact that use has already been made of an integral of the gas-dynamic equations, namely the energy integral, equation (6). For this reason, the apparent extra relation introduces no contradictions and proves to be no impediment in practice.

For brevity, equation (7) is also used in dimensionless form

$$a^2 = \frac{\kappa-1}{2} (1 - q^2)$$

APPLICATION OF EQUATIONS (I), (II), AND (III) TO

TWO-DIMENSIONAL AND ROTATIONALLY

SYMMETRICAL FLOWS

1. Two-Dimensional Flows

Equations (II) and (III) are satisfied by the expressions

$$\begin{cases} u (1 - q^2)^{\frac{1}{\kappa-1}} = \psi_y \\ v (1 - q^2)^{\frac{1}{\kappa-1}} = -\psi_x \end{cases} \quad (10)$$

The vector equation (I) yields the following two equations

$$-\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \underline{y} = \frac{1}{2\kappa} (1 - q^2) \frac{d \ln f(\psi)}{d\psi} \frac{\partial \psi}{\partial x}$$

and

$$\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \underline{u} = \frac{1}{2\kappa} (1 - q^2) \frac{d \ln f(\psi)}{d\psi} \frac{\partial \psi}{\partial y}$$

By expression of the underlined velocities in terms of ψ , the following single equation is obtained

$$\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2\kappa} (1 - q^2)^{\frac{\kappa}{\kappa-1}} \frac{d \ln f(\psi)}{d\psi} \quad (11)$$

and the apparent extra relation no longer exists.

This equation agrees with Crocco's equation, equation (10), which that author could reach only by way of integration processes, for the entropy in mechanical units is

$$S = \frac{R}{\kappa-1} \ln \frac{p}{\rho^\kappa} = \frac{R}{\kappa-1} \ln f(\psi)$$

Equation (11) now must be transformed. In accordance with equation (10)

$$\frac{\partial v}{\partial x} = -\psi_{xx} (1 - q^2)^{-\frac{1}{\kappa-1}} - \frac{\psi_x}{\kappa-1} (1 - q^2)^{-\frac{1}{\kappa-1}-1} \frac{\partial q^2}{\partial x}$$

By use of equation (7), from the previous equation the following is obtained:

$$-(1 - q^2)^{\frac{1}{\kappa-1}} \frac{\partial v}{\partial x} = \psi_{xx} + \frac{\psi_x}{2a^2} \frac{\partial q^2}{\partial x} = \psi_{xx} - \frac{v (1 - q^2)^{\frac{1}{\kappa-1}}}{2a^2} \frac{\partial q^2}{\partial x}$$

Correspondingly, it is true that

$$\frac{\partial u}{\partial y} = \psi_{yy} (1 - q^2)^{-\frac{1}{\kappa-1}} - \frac{\psi_y}{\kappa-1} (1 - q^2)^{-\frac{1}{\kappa-1} - 1} \frac{\partial q^2}{\partial y}$$

and

$$(1 - q^2)^{\frac{1}{\kappa-1}} \frac{\partial u}{\partial y} - \psi_{yy} + \frac{\psi_y}{2a^2} \frac{\partial q^2}{\partial y} = \psi_{yy} + \frac{u (1 - q^2)^{\frac{1}{\kappa-1}}}{2a^2} \frac{\partial q^2}{\partial y}$$

In order to transform $\frac{\partial q^2}{\partial x}$ and $\frac{\partial q^2}{\partial y}$, the starting point is taken as either

$$q^2 = u^2 + v^2 = (1 - q^2)^{-\frac{2}{\kappa-1}} (\psi_x^2 + \psi_y^2)$$

or

$$q^2 (1 - q^2)^{\frac{2}{\kappa-1}} = \psi_x^2 + \psi_y^2$$

When differentiated with respect to x , this yields

$$\begin{aligned} & \left\{ (1 - q^2)^{\frac{2}{\kappa-1}} - \frac{2q^2}{\kappa-1} (1 - q^2)^{\frac{2}{\kappa-1} - 1} \right\} \frac{\partial q^2}{\partial x} = (1 - q^2)^{\frac{2}{\kappa-1}} \left\{ 1 - \frac{2}{\kappa-1} \frac{q^2}{1 - q^2} \right\} \frac{\partial q^2}{\partial x} \\ & = (1 - q^2)^{\frac{2}{\kappa-1}} \left\{ 1 - \frac{q^2}{a^2} \right\} \frac{\partial q^2}{\partial x} = 2 \left\{ \psi_x \psi_{xx} + \psi_y \psi_{xy} \right\} \\ & = 2(1 - q^2)^{\frac{1}{\kappa-1}} \left\{ -v \psi_{xx} + u \psi_{xy} \right\} \end{aligned}$$

Consequently

$$\underline{(1 - q^2)^{\frac{1}{\kappa-1}} \left\{ 1 - \frac{q^2}{a^2} \right\} \frac{\partial q^2}{\partial x} = 2 \left\{ -v \psi_{xx} + u \psi_{xy} \right\}}$$

Correspondingly

$$\underline{(1 - q^2)^{\frac{1}{\kappa-1}} \left\{ 1 - \frac{q^2}{a^2} \right\} \frac{\partial q^2}{\partial y} = 2 \left\{ -v \psi_{xy} + u \psi_{yy} \right\}}$$

The four underlined equations may be summarized as follows:

$$\begin{aligned} -(1 - q^2)^{\frac{1}{\kappa-1}} \left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right] &= \psi_{xx} + \psi_{yy} + \frac{v^2 \psi_{xx} - 2uv \psi_{xy} + u^2 \psi_{yy}}{a^2 - q^2} \\ &= \frac{(a^2 - u^2) \psi_{xx} - 2uv \psi_{xy} + (a^2 - v^2) \psi_{yy}}{a^2 - q^2} \end{aligned}$$

Therefore equation (11) becomes

$$\begin{aligned} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} &= - \frac{(a^2 - u^2) \psi_{xx} - 2uv \psi_{xy} + (a^2 - v^2) \psi_{yy}}{(a^2 - q^2)(1 - q^2)^{\frac{1}{\kappa-1}}} \\ &= \frac{1}{2\kappa} (1 - q^2)^{\frac{\kappa}{\kappa-1}} \frac{d \ln f(\psi)}{d\psi} \end{aligned}$$

This further yields

$$\left(1 - \frac{u^2}{a^2} \right) \psi_{xx} - 2 \frac{uv}{a^2} \psi_{xy} + \left(1 - \frac{v^2}{a^2} \right) \psi_{yy} = - \frac{1}{2\kappa} \left(1 - \frac{q^2}{a^2} \right) (1 - q^2)^{\frac{\kappa+1}{\kappa-1}} \frac{d \ln f(\psi)}{d\psi}$$

and finally

$$\left(1 - \frac{u^2}{a^2}\right)\psi_{xx} - 2\frac{uv}{a^2}\psi_{xy} + \left(1 - \frac{v^2}{a^2}\right)\psi_{yy} = \frac{1}{2\kappa} \left(\frac{q^2}{a^2} - 1\right) \left(1 - q^2\right)^{\frac{\kappa+1}{\kappa-1}} \frac{d \ln f(\psi)}{d\psi} \quad (12)$$

This is the differential equation for the two-dimensional velocity field, with which also belong the defining equations, (10) and (7), for u , v , and a . All velocities are based on the limit velocity V_m .

In the familiar special case wherein $f(\psi)$ is constant throughout the whole flow

$$\left(1 - \frac{u^2}{a^2}\right)\psi_{xx} - 2\frac{uv}{a^2}\psi_{xy} + \left(1 - \frac{v^2}{a^2}\right)\psi_{yy} = 0 \quad (13)$$

Because according to equation (I) the flow is then free from rotation, and hence with the velocity potential φ

$$u = \frac{\partial \varphi}{\partial x}, \quad v = \frac{\partial \varphi}{\partial y} \quad (14)$$

the following equation is obtained from equation (II):

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \left(1 - q^2\right)^{\frac{1}{\kappa-1}} + \frac{1}{\kappa-1} \left(1 - q^2\right)^{\frac{1}{\kappa-1} - 1} \left[-2u^2 \frac{\partial u}{\partial x} - 2uv \frac{\partial v}{\partial x} - 2uv \frac{\partial u}{\partial y} - 2v^2 \frac{\partial v}{\partial y}\right] = 0$$

$$\varphi_{xx} + \varphi_{yy} - \frac{1}{a^2} \left[u^2 \varphi_{xx} + 2uv \varphi_{xy} + v^2 \varphi_{yy}\right] = 0$$

or

$$\left(1 - \frac{u^2}{a^2}\right)\varphi_{xx} - 2\frac{uv}{a^2}\varphi_{xy} + \left(1 - \frac{v^2}{a^2}\right)\varphi_{yy} = 0 \quad (15)$$

Equation (15) for φ is analogous in form to equation (13) for ψ ; however, it must be remembered in this connection that u and v are expressed altogether differently in terms of φ than in terms of ψ (compare equations (10) and (14)).

Holenbroek and Tschaplign (Hodographmethode der Gasdynamik) have done work involving the comparison of the two differential equations (13) and (15).

2. Rotationally Symmetrical Flows

The cylindrical coordinates r , ω , and z will now be used. The velocity components $u \parallel z$ (axial) and $v \parallel r$ (radial) are independent of ω .

Equation (II) becomes:

$$\frac{\partial}{\partial z} \left[u(1 - q^2)^{\frac{1}{\kappa-1}} \right] + \frac{1}{r} \frac{\partial}{\partial r} \left[rv(1 - q^2)^{\frac{1}{\kappa-1}} \right] = 0$$

where

$$q^2 = u^2 + v^2$$

This equation is satisfied by:

$$\begin{cases} ru(1 - q^2)^{\frac{1}{\kappa-1}} = \frac{\partial \psi}{\partial r} \\ rv(1 - q^2)^{\frac{1}{\kappa-1}} = - \frac{\partial \psi}{\partial z} \end{cases} \quad (16)$$

where $\psi(r, z)$ is the stream function, (compare Stokes's stream function for incompressible flows).

At the same time, equation (III) is also satisfied by this expression, therefore only the vector equation (I) remains to be dealt with. Equation (I) yields:

$$\left(\frac{\partial v}{\partial z} - \frac{\partial u}{\partial r}\right) \underline{u} = \frac{1}{2\kappa} (1 - q^2) \frac{d \ln f(\psi)}{d\psi} \frac{\partial \psi}{\partial r}$$

$$- \left(\frac{\partial v}{\partial z} - \frac{\partial u}{\partial r}\right) \underline{v} = \frac{1}{2\kappa} (1 - q^2) \frac{d \ln f(\psi)}{d\psi} \frac{\partial \psi}{\partial z}$$

By expressing u and v in terms of ψ , the following single equation is obtained:

$$\left(\frac{\partial v}{\partial z} - \frac{\partial u}{\partial r}\right) = \frac{r}{2\kappa} (1 - q^2)^{\frac{\kappa}{\kappa-1}} \frac{d \ln f(\psi)}{d\psi} \quad (17)$$

This equation is equivalent to L. Crocco's equation, equation (13), which, however, he could only reach by integration.

The necessity of suitably transforming this last differential equation, equation (17), remains.

From equation (16) is obtained

$$r \frac{\partial v}{\partial z} = -\psi_{zz} (1 - q^2)^{-\frac{1}{\kappa-1}} - \frac{\psi_z}{\kappa-1} (1 - q^2)^{-\frac{1}{\kappa-1}-1} \frac{\partial q^2}{\partial z}$$

and making use of equation (7)

$$\underline{-(1 - q^2)^{\frac{1}{\kappa-1}} r \frac{\partial v}{\partial z} = \psi_{zz} + \frac{\psi_z}{2a^2} \frac{\partial q^2}{\partial z} = \psi_{zz} - \frac{vr(1 - q^2)^{\frac{1}{\kappa-1}}}{2a^2} \frac{\partial q^2}{\partial z}}$$

By analogy

$$r \frac{\partial u}{\partial r} = \psi_{rr} (1 - q^2)^{-\frac{1}{\kappa-1}} + \frac{\psi_r}{\kappa-1} (1 - q^2)^{-\frac{1}{\kappa-1}-1} \frac{\partial q^2}{\partial r} - \frac{\psi_r}{r} (1 - q^2)^{-\frac{1}{\kappa-1}}$$

and

$$\underline{(1 - q^2)^{\frac{1}{\kappa-1}} r \frac{\partial u}{\partial r} = \psi_{rr} - \frac{\psi_r}{r} + \frac{\psi_r}{2a^2} \frac{\partial q^2}{\partial r} = \psi_{rr} - \frac{\psi_r}{r} + \frac{ur}{2a^2} (1 - q^2)^{\frac{1}{\kappa-1}} \frac{\partial q^2}{\partial r}}$$

In order to transform $\frac{\partial q^2}{\partial z}$ and $\frac{\partial q^2}{\partial r}$, the starting point is taken as

$$q^2 = u^2 + v^2 = \frac{(1 - q^2)^{-\frac{2}{\kappa-1}}}{r^2} (\psi_z^2 + \psi_r^2)$$

or

$$q^2 r^2 (1 - q^2)^{\frac{2}{\kappa-1}} = \psi_z^2 + \psi_r^2$$

By differentiation with respect to z

$$\begin{aligned} & \left\{ (1 - q^2)^{\frac{2}{\kappa-1}} r^2 - \frac{2q^2 r^2}{\kappa-1} (1 - q^2)^{\frac{2}{\kappa-1} - 1} \right\} \frac{\partial q^2}{\partial z} \\ &= (1 - q^2)^{\frac{2}{\kappa-1}} r^2 \left\{ 1 - \frac{2}{\kappa-1} \frac{q^2}{1-q^2} \right\} \frac{\partial q^2}{\partial z} \\ &= (1 - q^2)^{\frac{2}{\kappa-1}} r^2 \left\{ 1 - \frac{q^2}{a^2} \right\} \frac{\partial q^2}{\partial z} \\ &= 2 \left\{ \psi_z \psi_{zz} + \psi_r \psi_{rz} \right\} \\ &= 2 (1 - q^2)^{\frac{1}{\kappa-1}} r \left\{ -v \psi_{zz} + u \psi_{rz} \right\} \end{aligned}$$

From these equations it follows that

$$\left(1 - q^2\right)^{\frac{1}{\kappa-1}} r \left\{1 - \frac{q^2}{a^2}\right\} \frac{\partial q^2}{\partial z} = 2 \left\{-v\psi_{zz} + u\psi_{rz}\right\}$$

and finally

$$\frac{\left(1 - q^2\right)^{\frac{1}{\kappa-1}} r \frac{\partial q^2}{\partial z}}{2a^2} = \frac{-v\psi_{zz} + u\psi_{rz}}{a^2 - q^2}$$

By differentiation with respect to r

$$\begin{aligned} & \left\{ \left(1 - q^2\right)^{\frac{2}{\kappa-1}} r^2 - 2 \frac{q^2 r^2}{\kappa-1} \left(1 - q^2\right)^{\frac{2}{\kappa-1} - 1} \right\} \frac{\partial q^2}{\partial r} + 2rq^2 \left(1 - q^2\right)^{\frac{2}{\kappa-1}} \\ &= \left(1 - q^2\right)^{\frac{2}{\kappa-1}} r^2 \left\{ \left(1 - \frac{q^2}{a^2}\right) \frac{\partial q^2}{\partial r} + \frac{2q^2}{r} \right\} = 2 \left(\psi_z \psi_{rz} + \psi_r \psi_{rr} \right) \\ &= 2 \left(1 - q^2\right)^{\frac{1}{\kappa-1}} r \left\{ -v\psi_{rz} + u\psi_{rr} \right\} \end{aligned}$$

$$\left(1 - q^2\right)^{\frac{1}{\kappa-1}} r \left(1 - \frac{q^2}{a^2}\right) \frac{\partial q^2}{\partial r} = 2 \left(-v\psi_{rz} + u\psi_{rr} \right) - 2 \left(1 - q^2\right)^{\frac{1}{\kappa-1}} q^2$$

$$\frac{\left(1 - q^2\right)^{\frac{1}{\kappa-1}} r \frac{\partial q^2}{\partial r}}{2a^2} = \frac{\left(-v\psi_{rz} + u\psi_{rr} \right) - \left(1 - q^2\right)^{\frac{1}{\kappa-1}} q^2}{a^2 - q^2}$$

From the four underlined equations it follows that

$$\begin{aligned}
 & - \left(1 - q^2\right)^{\frac{1}{\kappa-1}} r \left[\frac{\partial v}{\partial z} - \frac{\partial u}{\partial r} \right] = \psi_{zz} + \psi_{rr} - \frac{\psi_r}{r} \\
 & + \frac{v^2 \psi_{zz} - 2uv \psi_{rz} + u^2 \psi_{rr} - u(1 - q^2)^{\frac{1}{\kappa-1}} q^2}{a^2 - q^2} \\
 = & \frac{\left(\psi_{zz} + \psi_{rr} - \frac{\psi_r}{r}\right)(a^2 - q^2) + v^2 \psi_{zz} - 2uv \psi_{rz} + u^2 \psi_{rr} - \frac{\psi_r}{r} q^2}{a^2 - q^2} \\
 = & \frac{(a^2 - u^2) \psi_{zz} + (a^2 - v^2) \psi_{rr} - 2uv \psi_{rz} - \frac{\psi_r}{r} a^2}{a^2 - q^2}
 \end{aligned}$$

By use of this expression, differential equation (17) for ψ now becomes

$$\begin{aligned}
 & (a^2 - u^2) \psi_{zz} - 2uv \psi_{rz} + (a^2 - v^2) \psi_{rr} - \frac{\psi_r}{r} a^2 \\
 & = - \frac{r^2}{2\kappa} (a^2 - q^2) (1 - q^2)^{\frac{\kappa}{\kappa-1} + \frac{1}{\kappa-1}} \frac{d \ln f(\psi)}{d\psi}
 \end{aligned}$$

or finally

$$\begin{aligned}
 & \left(1 - \frac{u^2}{a^2}\right) \psi_{zz} - 2 \frac{uv}{a^2} \psi_{rz} + \left(1 - \frac{v^2}{a^2}\right) \psi_{rr} - \frac{\psi_r}{r} \\
 & = \frac{r^2}{2\kappa} \left(\frac{q^2}{a^2} - 1\right) (1 - q^2)^{\frac{\kappa+1}{\kappa-1}} \frac{d \ln f(\psi)}{d\psi} \quad (18)
 \end{aligned}$$

This equation is the differential equation for the rotationally symmetrical velocity field, with which belong also the defining equations (7) and (16) for u , v , and a . All velocities are based on the limit velocity V_m .

In the familiar special case in which $f(\psi)$ is constant throughout the whole flow,

$$\left(1 - \frac{u^2}{a^2}\right)\psi_{zz} - \frac{2uv}{a^2}\psi_{rz} + \left(1 - \frac{v^2}{a^2}\right)\psi_{rr} - \frac{\psi_r}{r} = 0 \quad (19)$$

Because, according to equation (I), the flow is then free of rotation and with the velocity potential defined as follows:

$$u = \frac{\partial\varphi}{\partial z} \quad v = \frac{\partial\varphi}{\partial r} \quad (20)$$

the following is obtained by insertion in equation (II) (the continuity equation):

$$\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} + \frac{v}{r}\right)\left(1 - q^2\right)^{\frac{1}{\kappa-1}} + \frac{1}{\kappa-1}\left(1 - q^2\right)^{\frac{1}{\kappa-1}-1} \left[-2u^2 \frac{\partial u}{\partial z} - 2uv \frac{\partial v}{\partial z} - 2uv \frac{\partial u}{\partial r} - 2v^2 \frac{\partial v}{\partial r}\right] = 0$$

$$\varphi_{zz} + \varphi_{rr} + \frac{1}{r}\varphi_r - \frac{1}{a^2}\left[u^2\varphi_{zz} + 2uv\varphi_{rz} + v^2\varphi_{rr}\right] = 0$$

or

$$\left(1 - \frac{u^2}{a^2}\right)\varphi_{zz} - \frac{2uv}{a^2}\varphi_{rz} + \left(1 - \frac{v^2}{a^2}\right)\varphi_{rr} + \frac{\varphi_r}{r} = 0 \quad (21)$$

Differential equation (21) for φ is the same, except for the sign of the last term, as that for ψ , equation (19). However, u and v are expressed altogether differently in terms of ψ than in terms of φ (compare equations (16) and (20)).

THE LITERATURE

1. Frankl, F. and Aleksjeva, R.: Two Boundary-Value Problems Involving the Theory of Hyperbolic Differential Equations of the Second Order with Applications to Gas Flows of Supersonic Velocity (Russian). *Matematicheski Sbornik*, 1934, p. 483. There exists a German script translation of this work made at that time at my request by Mr. Bl. Dolaptschiew of Sofia.

In this report a differential equation occurs for ψ in the two-dimensional case of the general flow ($f(\psi) \neq \text{constant}$) that is copied from a paper by F. Frankl in the collection "Soviet Union Rockets" which is unavailable. The differential equation set forth by Mr. F. Frankl is as follows:

$$\left(1 - \frac{u^2}{a^2} - \frac{k-1}{k+1} \frac{v^2}{a^2}\right) \frac{\partial^2 \psi}{\partial x^2} - \frac{4}{k+1} \frac{uv}{a^2} \frac{\partial^2 \psi}{\partial x \partial y} + \left(1 - \frac{v^2}{a^2} - \frac{k-1}{k+1} \frac{u^2}{a^2}\right) \frac{\partial^2 \psi}{\partial y^2} + \frac{k+1}{2k} a^2 \left[\left(\frac{k-1}{k+1}\right)^2 \frac{w^4}{a^4} - 1 \right] \left(1 - \frac{k-1}{k+1} \frac{w^2}{a^2}\right)^{\frac{2}{k-1}} \rho_0 \frac{d\rho_0}{d\psi} = 0$$

In the report from which the quotation is taken, the only statement made regarding definitions is that the "zero density" ρ_0 is the density obtained by the adiabatic reduction of the velocity to zero. Aside from the unprofitable introduction of ρ_0 , F. Frankl's equation may be immediately obtained from equation (18) if the sonic velocity a is expressed in terms of the critical velocity or critical sonic velocity a^* ($u^2 + v^2 = a^{*2}$) in accordance with

$$a^2 = \frac{k+1}{2} a^{*2} - \frac{k-1}{2} (u^2 + v^2)$$

The meaning of the other symbols insofar as they differ from these used here is obviously as follows:

w^2 corresponds to q^2

k corresponds to κ .

2. Crocco, L.: Una nuova funzione di corrente per lo studio del moto rotazionale dei gas, Rend. della R. Accad. dei Lincei, Vol. XXIII, serie 6^a, 1^o sem. fasc. 2, 1936.

Translated as "Eine neue Stromfunktion für die Erforschung der Bewegung der Gase mit Rotation", ZAMM 17, p. 1, 1937.

Crocco's final equations (10'') and (13') agree with equations (12) and (18). Mention has previously been made of the differences in derivation.

NOTES

1. With regard to page 3 it should be noted that up to the present time a stream function ψ has been defined only in the two-dimensional and rotationally symmetrical flows, which cases are, to be sure, of principal interest. In the general three-dimensional case, $\psi = \text{constant}$ is to be considered as the equation applying to those flow surfaces (built up out of flow lines) on which the function $f(\psi)$ is constant.

The Crocco stream function is used chiefly as the stream function for two-dimensional and rotationally symmetrical flows. The numbering of the flow lines is, with the exception of the isentropic case (chapters III and IV), not done by Crocco on the basis of the mass flowing through, but corresponds instead to his form of the continuity equation (8). Reasons of expediency led, even in the case of the nonisentropic rotational flows (chapter VI), to reversion from the Crocco stream function to the customary stream function.

2. Regarding page 6 it is to be noted that the relation between the new vortex theorem as expressed in equations (9) and (I) and the Bjerknes vortex theorem has recently received proper clarification for the first time in a paper in the Wieselsberger-Gedächtnisheft, ZWB (W. Tollmien: Ein Wirbelsatz für stationäre isoenergetische Gasströmungen.) In the same paper the apparent excessive determination of the equation system (I), (II), and (III) is decisively explained by the fact that the vortex theorem (I), contrary to first appearances, yields only two scalar equations.

3. The relation between pressure, density, and velocity for isentropic flows that will be used frequently in the following lectures is appended here. When using dimensioned quantities, in accordance with the energy equation (6)

$$\frac{\kappa}{\kappa-1} \frac{p}{\rho} = \frac{V_m^2 - q^2}{2}$$

and in accordance with the adiabatic equation for isentropic flows

$$\frac{p}{\rho^\kappa} = c$$

From this, by the elimination of ρ , is obtained

$$p = \left[\frac{\kappa}{\kappa-1} \frac{V_m^2}{2c^{1/\kappa}} \right]^{\frac{\kappa}{\kappa-1}} \left(1 - \frac{q^2}{V_m^2} \right)^{\frac{\kappa}{\kappa-1}}$$

The first factor is equal to the reservoir pressure p_0 , as appears by setting q equal to zero. According to this

$$p = p_0 \left(1 - \frac{q^2}{V_m^2} \right)^{\frac{\kappa}{\kappa-1}}$$

From the adiabatic equation it then follows that

$$\rho = \rho_0 \left(1 - \frac{q^2}{V_m^2} \right)^{\frac{1}{\kappa-1}}$$

where ρ_0 is the reservoir density.

Translation by Edward S. Shafer,
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for Aeronautics.



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