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# On finite groups generated by strongly cosubnormal subgroups

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**Abstract**

Two subgroups $A$ and $B$ of a group $G$ are cosubnormal if $A$ and $B$ are subnormal in their join $\langle A, B\rangle$ and are strongly cosubnormal if every subgroup of $A$ is cosubnormal with every subgroup of $B$. We find necessary and sufficient conditions for $A$ and $B$ to be strongly cosubnormal in $\langle A, B\rangle$ and, if $Z$ is the hypercentre of $G = \langle A, B\rangle$, we show that $A$ and $B$ are strongly cosubnormal if and only if $G/Z$ is the direct product of $AZ/Z$ and $BZ/Z$. We also show that projectors and residuals for certain formations can easily be constructed in such a group.

Two subgroups $A$ and $B$ of a group $G$ are $\mathfrak{N}$-connected if every cyclic subgroup of $A$ is cosubnormal with every cyclic subgroup of $B$. Though the concepts of strong cosubnormality and $\mathfrak{N}$-connectedness are clearly closely related, we give an example to show that they are not equivalent. We note however that if $G$ is the product of the $\mathfrak{N}$-connected subgroups $A$ and $B$, then $A$ and $B$ are strongly cosubnormal.

# 1 Introduction and statements of results

In the sequel it is understood that all groups are finite.

Following Wielandt [6], we say that two subgroups $A$ and $B$ of a group $G$ are *cosubnormal* in $G$ if $A$ and $B$ are subnormal subgroups of their join $\langle A, B\rangle$.

More recently, Knapp [5] introduces the notion of *strong cosubnormality*: two subgroups $A$ and $B$ of a group are called *strongly cosubnormal* if every subgroup of $A$ is cosubnormal with every subgroup of $B$. We write $A \operatorname{cs} B$ if $A$ and $B$ are cosubnormal and $A \operatorname{scs} B$ if $A$ and $B$ are strongly cosubnormal.

Notice that if $A$ and $B$ are $\mathfrak{N}$-connected, then every cyclic subgroup of $A$ is cosubnormal with every cyclic subgroup of $B$.

Knapp proves in [5] the following characterisation of strong cosubnormality in terms of the hypercentre:

**Theorem 1 ([5, Theorem 3.3]).** *Let $A$, $B$ be subgroups of a group $G$. Then the following are equivalent:*

1. *$A$ and $B$ are strongly cosubnormal.*
2. *$[A, B] \le Z_\infty(\langle A, B\rangle)$.*

Here $Z_\infty(G)$ denotes the hypercentre of a group $G$.

A natural sequel of Knapp's work would be the study of groups generated by strongly cosubnormal subgroups.

On the other hand, Carocca [3] introduces the concept of $\mathfrak{N}$-connected subgroups: two subgroups $A$ and $B$ of a group $G$ are $\mathfrak{N}$-*connected* when for every $a \in A$ and $b \in B$, the subgroup $\langle a, b\rangle$ is nilpotent ($\mathfrak{N}$ denotes the class of nilpotent groups).

It is very easy to show that if $A$ and $B$ are two strongly cosubnormal subgroups of a group $G$, then they are $\mathfrak{N}$-connected: if $a \in A$ and $b \in B$, then $\langle a\rangle$ and $\langle b\rangle$ are nilpotent subnormal subgroups of $\langle a, b\rangle$, and so $\langle a, b\rangle$ is nilpotent. However, $\mathfrak{N}$-connection and strong cosubnormality are not equivalent in general, as we will show in the Example at the end of Section 2.

We prove the following characterisation theorem:

**Theorem 2.** *Let $A$ and $B$ be two subgroups of $G$ such that $G = \langle A, B\rangle$ and let $Z = Z_\infty(G)$. The following statements are equivalent:*

1. *$A \operatorname{scs} B$.*
2. *$A \operatorname{cs} B$ and $A$ and $B$ are $\mathfrak{N}$-connected.*
3. *$A \operatorname{cs} B$ and if $p$ and $q$ are two different primes, $x$ is a $p$-element of $A$ and $y$ is a $q$-element of $B$, then $[x, y] = 1$.*
4. *$[A, B] \leq Z$.*

We observe from that cosubnormality and $\mathfrak{N}$-connection are closely related concepts. In the important case of products, they are indeed equivalent.

**Theorem 3.** *If a group $G$ is the $\mathfrak{N}$-connected product of its subgroups $A$ and $B$, then $A$ and $B$ are strongly cosubnormal.*

Our next result describes the groups generated by strongly cosubnormal subgroups.

**Theorem 4.** *Let $G = \langle A, B\rangle$ and $Z = Z_\infty(G)$. Then the following statements are equivalent:*

1. *$A \operatorname{scs} B$.*
2. *$G/Z = AZ/Z \times BZ/Z$.*

In [1], Ballester-Bolinches and Pedraza-Aguilera proved that soluble $\mathfrak{N}$-connected products behave well with respect to saturated formations containing $\mathfrak{N}$. Following this idea, we study the behaviour of strongly cosubnormal subgroups in the finite (not necessarily soluble) universe with respect to formations.

Recall that a *formation* $\mathfrak{F}$ is a class of groups which is closed under taking epimorphic images and subdirect products. Every group $G$ has a smallest normal subgroup $G^{\mathfrak{F}}$ (called the $\mathfrak{F}$-residual of $G$) such that $G/G^{\mathfrak{F}} \in \mathfrak{F}$ (see [4, II.2] for details). If $\mathfrak{X}$ is a class of groups, a subgroup $E$ of $G$ is an $\mathfrak{X}$-*projector* of $G$ if $EN/N$ is $\mathfrak{X}$-maximal in $G/N$ for all normal subgroups $N$ of $G$. If $\mathfrak{F}$ is a formation, then every group $G$ has $\mathfrak{F}$-projectors if and only if $\mathfrak{F}$ is saturated, that is, if $G/\Phi(G) \in \mathfrak{F}$, then $G \in \mathfrak{F}$ (see [4, Chapter 4] for further details). Note that $\mathfrak{N}$ is a saturated formation.

The following results show that finite (not necessarily soluble) groups generated by strongly cosubnormal subgroups behave well with respect to (not necessarily saturated) formations containing $\mathfrak{N}$.

**Theorem 5.** *Let $\mathfrak{F}$ be a formation containing $\mathfrak{N}$ such that either $\mathfrak{F}$ is saturated, or $\mathfrak{F}$ is contained in the class of soluble groups. Suppose that $G = \langle A, B\rangle$ and $A \operatorname{scs} B$. Then $G^{\mathfrak{F}} = \langle A^{\mathfrak{F}}, B^{\mathfrak{F}}\rangle$.*

**Theorem 6.** *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{N}$. Suppose that $G = \langle A, B\rangle$ with $A \operatorname{scs} B$. Let $A_1$ be an $\mathfrak{F}$-projector of $A$ and let $B_1$ be an $\mathfrak{F}$-projector of $B$. Then $\langle A_1, B_1\rangle$ is an $\mathfrak{F}$-projector of $G$. Moreover, $A$ permutes with $B$ if and only if $A_1$ permutes with $B_1$.*

# 2 Proofs of the results

We begin with the following Lemma, whose proof is already contained in Knapp's paper.

**Lemma 1.** *Suppose that $A$ and $B$ are subgroups of a group $G$ such that the following conditions hold:*

1. *$G = \langle A, B\rangle$ and*
2. *if $p$ and $q$ are two different primes, $x$ is a $p$-element of $A$ and $y$ is a $q$-element of $B$, then $[x, y] = 1$.*

*Then:*

1. *if $p$ is a prime, then $O^{p'}(B) \le C_G\big(O^p(A)\big)$ and $O^{p'}(A) \le C_G\big(O^p(B)\big)$ and*
2. *$B^A \le C_G(A^{\mathfrak{N}})$ and $A^B \le C_G(B^{\mathfrak{N}})$.*

*In particular, $A^{\mathfrak{N}}$ and $B^{\mathfrak{N}}$ are normal subgroups of $G$.*

*Proof.* Let $p$ and $q$ be two different prime numbers. Let $A_p$ be a Sylow $p$-subgroup of $A$ and let $B_q$ be a Sylow $q$-subgroup of $B$. Then $[A_p, B_q] = 1$ by hypothesis.

Since $B_q \leq C_G(A_p)$ for every $q \neq p$, we have that $A_p \leq C_G\big(O^p(B)\big)$. Analogously, $B_p \leq C_G\big(O^p(A)\big)$. This proves the first claim.

Since $A^{\mathfrak{N}} = \bigcap_{p \text{ prime}} O^p(A)$, we obtain that $B_p \leq C_G(A^{\mathfrak{N}})$ for all primes $p$, and hence $B \leq C_G(A^{\mathfrak{N}})$. Bearing in mind that $A^{\mathfrak{N}}$ is a normal subgroup of $A$, we get $B^A \leq C_G(A^{\mathfrak{N}})$. Analogously, we have that $A^B \leq C_G(B^{\mathfrak{N}})$. $\square$

*Proof of Theorem 2.* 1 implies 2 has been already noted in the introduction, whereas 4 implies 1 is just one of the implications of Knapp's result.

2 implies 3. Let $p$ and $q$ be two different prime numbers. Let $x$ be a $p$-element of $A$ and let $y$ be a $q$-element of $B$. Since $\langle x, y\rangle$ is nilpotent, it follows that $[x, y] = 1$.

3 implies 4. We argue by induction on $|G|$. We have that $[A, B]$ is a normal subgroup of $\langle A, B\rangle = G$. Suppose that $[A, B] \neq 1$, and let $N$ be a minimal normal subgroup of $G$ contained in $[A, B]$. If $N \cap G^{\mathfrak{N}} = 1$, then $N$ is central in $G$. Hence, by induction, $[A, B]/N \leq Z_\infty(G/N)$, which is equal to $Z/N$ because $N$ is central in $G$. Consequently $[A, B]$ is contained in $Z$ and the theorem is proved. Therefore we may assume that every minimal normal subgroup of $G$ contained in $[A, B]$ is also contained in $G^{\mathfrak{N}}$.

Since $[A, B]$ centralises $A^{\mathfrak{N}}$ and $B^{\mathfrak{N}}$ by Lemma 1, it follows that $[A, B]$ centralises $\langle A^{\mathfrak{N}}, B^{\mathfrak{N}}\rangle$, which is equal to $G^{\mathfrak{N}}$ by [5, Theorem W]. This implies that $N$ is central in $[A, B]$. Now $[A, B]/N \leq Z_\infty(G/N)$ by induction. Hence $[A, B]/N$ is nilpotent and so is $[A, B]$.

Suppose that there exists a minimal normal subgroup $C$ of $G$, $C \neq N$, and $C \leq [A, B]$. Then, by induction, $CN/N \leq Z_\infty(G/N)$. Thus $C$ is central in $G$. We can argue as in the previous case to conclude $[A, B] \leq Z$. Consequently, $[A, B]$ contains a unique minimal normal subgroup of $G$. Since $[A, B]$ is nilpotent, we have that $[A, B]$ is a $p$-group for some prime $p$.

Assume that there exists a minimal normal subgroup $N_1$ of $G$, $N_1 \neq N$. By induction, $[A, B]N_1/N_1 \leq Z_\infty(G/N_1)$, and so $NN_1/N_1$ is centralised by every $p'$-subgroup of $G/N_1$. In particular, $[N, O^p(A)] \leq N_1$ and $[N, O^p(B)] \leq N_1$. Since $[N, O^p(A)]$ and $[N, O^p(B)]$ are both contained in $N$, it follows that $[N, O^p(A)] = [N, O^p(B)] = 1$. This means that $N \leq C_G\big(\langle O^p(A), O^p(B)\rangle\big) = C_G\big(O^p(G)\big)$, because $O^p(G) = \langle O^p(A), O^p(B)\rangle$ ([5, Theorem W]). This implies that $N \leq Z$. Since $[A, B]/N \leq Z_\infty(G/N)$ and $Z_\infty(G/N) = Z/N$, we have that $[A, B] \leq Z$ and so $[A, B] \leq Z$.

Consequently we may assume that $G$ has a unique minimal normal subgroup, $N$ say, and $N \leq [A, B]$. Note that $A^B = A[A, B]$ is a normal subgroup of $G$ and $O^p(A^B) = O^p(A)$ because $[A, B]$ is a $p$-group. Analogously $O^p(B^A) = O^p(B)$. In particular, $O^p(A)$ and $O^p(B)$ are normal in $G$. Suppose that $O^p(A) \neq 1$. Then $N \leq O^p(A)$ and so $O^p(B) \leq C_G(N)$ by Lemma 1. If $O^p(B) \neq 1$, we also have $O^p(A) \leq C_G(N)$. This means that $O^p(G) \leq C_G(N)$

and $N \leq Z$.

Therefore we may suppose that $O^p(B) = 1$ and $B$ is a $p$-group. Then $N \leq B^A$ and $B^A \leq C_G\big(O^p(A)\big)$ by Lemma 1. Since $O^p(A) = O^p(G)$, it follows that $N \leq C_G\big(O^p(G)\big)$ and then $N \leq Z$. Arguing as above, we have that $[A, B] \leq Z$ and the theorem is proved. $\square$

*Proof of Theorem 3.* By Theorem 2, we need only prove that $A \operatorname{cs} B$ provided that $A$ and $B$ are $\mathfrak{N}$-connected and $G = AB$. Assume that this is not true and let $G$ be a counterexample of minimal order. Note that the hypotheses of Lemma 1 hold for $\mathfrak{N}$-connected subgroups. Consequently, $A^{\mathfrak{N}}$ and $B^{\mathfrak{N}}$ are normal subgroups of $G$. Suppose that $A$ is not subnormal in $G$. It is clear that $G/B^{\mathfrak{N}}$ is the $\mathfrak{N}$-connected product of $AB^{\mathfrak{N}}/B^{\mathfrak{N}}$ and $B/B^{\mathfrak{N}}$. Hence, if $B^{\mathfrak{N}} \neq 1$, we have that $AB^{\mathfrak{N}}$ is subnormal in $G$ by the minimality of $G$. Since $A \leq C_G(B^{\mathfrak{N}})$ by Lemma 1, it follows that $A$ is normal in $AB^{\mathfrak{N}}$. Therefore $A$ is subnormal in $G$, a contradiction. Consequently, $B$ is nilpotent. If $A^{\mathfrak{N}} \neq 1$, we have that $A/A^{\mathfrak{N}}$ is subnormal in $G/A^{\mathfrak{N}}$ by the minimal choice of $G$. Hence $A$ is subnormal in $G$, a contradiction.

Therefore $A$ and $B$ are nilpotent. By [3], $G$ is nilpotent, a contradiction. $\square$

*Proof of Theorem 4.* 1 implies 2. Suppose that $A \operatorname{scs} B$. Since $A \cap B \operatorname{scs} B_1$ for every $B_1 \leq B$, we have that $A \cap B \leq Z_\infty(B)$ by [5, Theorem 2.6]. Since $A_1 \operatorname{scs} A \cap B$ for every $A_1 \leq A$, we have that $A \cap B \leq Z_\infty(A)$ by [5, Theorem 2.6]. Consequently $A \cap B \leq Z_\infty(A) \cap Z_\infty(B)$, which is contained in $Z$ by [5, Proposition 3.2]. On the other hand, $[AZ/Z, BZ/Z] \leq [A, B]Z/Z = 1$, by Theorem 2, whence $G/Z = AZ/Z \times BZ/Z$.

2 implies 1. Suppose that $G/Z = AZ/Z \times BZ/Z$. Let $A_1$ be a subgroup of $A$ and let $B_1$ be a subgroup of $B$. Since $A_1$ is subnormal in $A_1Z$ and $A_1Z/Z$ is centralised by $B_1Z/Z$, it follows that $A_1$ is subnormal in $T = \langle A_1Z, B_1Z \rangle$. Analogously, $B_1$ is subnormal in $T$. Hence $A_1 \operatorname{cs} B_1$, as desired. $\square$

The proofs of Theorem 5 and 6 depend on the following Lemmas:

**Lemma 2.** *Let $\mathfrak{F}$ be a formation containing $\mathfrak{N}$. Suppose that $G = \langle A, B \rangle$ and $A \operatorname{scs} B$. If $A$ and $B$ belong to $\mathfrak{F}$, then $G \in \mathfrak{F}$.*

*Proof.* Suppose that the theorem is false. Let $G = \langle A, B \rangle$ be a counterexample with $|A| + |B|$ minimal. We can assume without loss of generality that $A$ is not nilpotent. Then we can write $A = A^{\mathfrak{N}}C$, where $C$ is an $\mathfrak{N}$-projector of $A$. On the other hand, $A^{\mathfrak{N}}$ is a normal subgroup of $G$ by Lemma 1 and Theorem 2 and $B \leq C_G(A^{\mathfrak{N}})$. This implies that $D = B^{\langle B,C \rangle} \leq C_G(A^{\mathfrak{N}})$. By [2, Lemma 1], bearing in mind that $G = A^{\mathfrak{N}}\langle C, B \rangle$, there exists an epimorphism $\theta \colon X = [A^{\mathfrak{N}}]\langle C, B \rangle \longrightarrow G$. Let us prove that $X \in \mathfrak{F}$. We have that

$X/A^{\mathfrak{N}} \in \mathfrak{F}$, because $\langle C, B\rangle \in \mathfrak{F}$ by minimality of $G$. Now $D$ is a normal subgroup of $X$, because $D$ is centralised by $A^{\mathfrak{N}}$. Moreover

$$X/D \cong [A^{\mathfrak{N}}](CD/D) \cong [A^{\mathfrak{N}}](C/D \cap C).$$

We see that $Y = [A^{\mathfrak{N}}]C \in \mathfrak{F}$. By [2, Lemma 1], there exists an epimorphism $\alpha\colon Y \longrightarrow A^{\mathfrak{N}}C = A$ such that $\operatorname{Ker}\alpha \cap A^{\mathfrak{N}} = 1$. Now, $Y/\operatorname{Ker}\alpha \in \mathfrak{F}$ and $Y/A^{\mathfrak{N}} \in \mathfrak{F}$. Since $\mathfrak{F}$ is a formation, it follows that $Y \in \mathfrak{F}$. It is clear that $X/D$ is isomorphic to a quotient of $Y$. Therefore $X/D \in \mathfrak{F}$. Since $\mathfrak{F}$ is a formation, we have that $X/A^{\mathfrak{N}} \cap D = X \in \mathfrak{F}$. This implies that $G \in \mathfrak{F}$, because $G$ is an epimorphic image of $X$. □

**Lemma 3.** *Let $\mathfrak{F}$ be a formation containing $\mathfrak{N}$. Assume that either $\mathfrak{F}$ is saturated or $\mathfrak{F}$ consists only of soluble groups. If $A$ and $B$ are strongly co-subnormal subgroups of $G$, $G = \langle A, B\rangle$ and $G$ belongs to $\mathfrak{F}$, then $A$ and $B$ belong to $\mathfrak{F}$.*

*Proof.* Assume that $\mathfrak{F}$ is a saturated formation. Let $G$ be a counterexample of minimal order to the theorem. If $Z = Z_\infty(G) = 1$, then $A \cap B = 1$ by Lemma 4 and $G = A \times B$. In particular, $A$ and $B$ belong to $\mathfrak{F}$. Hence $Z \neq 1$. Let $N$ be a minimal normal subgroup of $G$. Since $G/N$ satisfies the hypotheses of the theorem, it follows that $AN/N \in \mathfrak{F}$ and $BN/N \in \mathfrak{F}$. In particular, $A/A \cap N$ and $B/B \cap N$ belong to $\mathfrak{F}$. If $G$ has more than one minimal normal subgroup, we have that $A$ and $B$ belong to $\mathfrak{F}$. Hence $G$ has a unique minimal normal subgroup. Thus $N \leq Z$, whence $N \leq Z(G)$. In particular, $A \cap N \leq Z(A)$ and $B \cap N \leq Z(B)$. This implies that $A$ and $B$ belong to $\mathfrak{F}$, as desired.

Assume now that $\mathfrak{F}$ is a formation of soluble groups. Let $G = \langle A, B\rangle$ be a minimal counterexample with $|A| + |B|$ minimal. If, for example, $B$ is nilpotent, then $G = AF(G)$. By Bryant, Bryce and Hartley's Theorem ([4, IV.1.14]), it follows that $A \in \mathfrak{F}$.

Hence we can assume that $A^{\mathfrak{N}} \neq 1$ and $B^{\mathfrak{N}} \neq 1$. Since $G$ is soluble, it follows that there exist a maximal subgroup $A_0$ of $A$ such that $AF(G) = A_0F(G)$ and a maximal subgroup $B_0$ of $B$ such that $BF(G) = B_0F(G)$. Note that $G = \langle A, B_0\rangle F(G) = \langle A_0, B\rangle F(G)$. From Bryant, Bryce and Hartley's Theorem ([4, IV.1.14]), we have that $\langle A, B_0\rangle$ and $\langle A_0, B\rangle$ belong to $\mathfrak{F}$. On the other hand, bearing in mind that $A \operatorname{scs} B_0$ and $A_0 \operatorname{scs} B$, the minimality of $|A| + |B|$ implies that $A \in \mathfrak{F}$ and $B \in \mathfrak{F}$, a contradiction. □

*Proof of Theorem 5.* Since $\mathfrak{N} \subseteq \mathfrak{F}$, we have that $G^{\mathfrak{F}} \leq G^{\mathfrak{N}}$, $A^{\mathfrak{F}} \leq A^{\mathfrak{N}}$ and $B^{\mathfrak{F}} \leq B^{\mathfrak{N}}$. Hence $B^A \leq C_G(A^{\mathfrak{N}})$ implies that $B \leq C_G(A^{\mathfrak{F}})$. Thus $A^{\mathfrak{F}}$ and, analogously, $B^{\mathfrak{F}}$ are normal subgroups of $G$. Since $G/G^{\mathfrak{F}} = \langle AG^{\mathfrak{F}}/G^{\mathfrak{F}}, BG^{\mathfrak{F}}/G^{\mathfrak{F}}\rangle$

belongs to $\mathfrak{F}$, we have that $AG^{\mathfrak{F}}/G^{\mathfrak{F}} \in \mathfrak{F}$ by Lemma 3. Hence $A/A \cap G^{\mathfrak{F}} \in \mathfrak{F}$. This implies that $A^{\mathfrak{F}} \leq A \cap G^{\mathfrak{F}}$. In particular, $A^{\mathfrak{F}} \leq G^{\mathfrak{F}}$. Analogously, $B^{\mathfrak{F}} \leq G^{\mathfrak{F}}$. This proves that $\langle A^{\mathfrak{F}}, B^{\mathfrak{F}}\rangle \leq G^{\mathfrak{F}}$.

We prove that $G^{\mathfrak{F}} = \langle A^{\mathfrak{F}}, B^{\mathfrak{F}}\rangle$ by induction on $|G|$. If $A^{\mathfrak{F}} = B^{\mathfrak{F}} = 1$, then $A$, $B \in \mathfrak{F}$ and, by Lemma 2 we have that $G = \langle A, B\rangle \in \mathfrak{F}$. Consequently we can assume that $N = A^{\mathfrak{F}} \neq 1$. Moreover, $N \leq G^{\mathfrak{F}}$. Hence $G^{\mathfrak{F}}/N = (G/N)^{\mathfrak{F}} = \langle (A/N)^{\mathfrak{F}}, (BN/N)^{\mathfrak{F}}\rangle \leq B^{\mathfrak{F}}N/N = \langle N, B^{\mathfrak{F}}\rangle/N$, because $A/N \operatorname{scs} BN/N$, $G/N = \langle A/N, BN/N\rangle$ and $(BN/N)^{\mathfrak{F}} \leq B^{\mathfrak{F}}N/N$. Consequently $G^{\mathfrak{F}} \leq \langle A^{\mathfrak{F}}, B^{\mathfrak{F}}\rangle$, and the proof is complete. □

*Proof of Theorem 6.* Assume that the theorem is false. Let $G$ be a counterexample of minimal order.

The result is clear if $Z = Z_\infty(G) = 1$ by [4, III.6.3] and Theorem 4. Moreover, if $A^{\mathfrak{F}} = B^{\mathfrak{F}} = 1$, then we have that $A$, $B \in \mathfrak{F}$ and, by Theorem 2, we obtain that $\langle A, B\rangle = G$ is an $\mathfrak{F}$-projector of $G$. Therefore we can assume, without loss of generality, that $A^{\mathfrak{F}} \neq 1$. From Lemma 1, it follows that there exists a minimal normal subgroup $N$ of $G$ such that $N \leq A^{\mathfrak{F}}$. Let $A_1$ be an $\mathfrak{F}$-projector of $A$ and let $B_1$ be an $\mathfrak{F}$-projector of $B$. Then $\langle A_1, B_1\rangle N/N$ is an $\mathfrak{F}$-projector of $G/N$ by minimality of $G$. Let $X = \langle A_1, B_1\rangle N = \langle A_1N, B_1\rangle$. Since $A_1N \leq A$, we have that $A_1N \operatorname{scs} B$. Assume $X < G$. From [4, III.3.14] and [4, III.3.18], it follows that $A_1$ is an $\mathfrak{F}$-projector of $A_1N$. Hence, by minimality of $G$, we get that $\langle A_1, B_1\rangle$ is an $\mathfrak{F}$-projector of $X$ and, by [4, III.3.7], we obtain that $\langle A_1, B_1\rangle$ is an $\mathfrak{F}$-projector of $G$. Therefore $X = \langle A_1, B_1\rangle N = G$.

Now $\langle A_1, B_1\rangle \in \mathfrak{F}$ by Theorem 2. Therefore $G^{\mathfrak{F}} \leq N$ and, since $A^{\mathfrak{F}} \leq G^{\mathfrak{F}}$ by Theorem 5, we have that $N = G^{\mathfrak{F}}$. Assume that $N$ is abelian. Then $\langle A_1, B_1\rangle$ is a maximal subgroup of $G$. Hence $\langle A_1, B_1\rangle$ is an $\mathfrak{F}$-projector of $G$, a contradiction.

Now assume that $N$ is not abelian. Assume that $B^{\mathfrak{F}} \neq 1$. Then $N = B^{\mathfrak{F}} = A^{\mathfrak{F}} \leq A \cap B \leq Z_\infty(G)$ by Theorem 4. In particular, $N$ is abelian, a contradiction. Hence $B^{\mathfrak{F}} = 1$ and $B \in \mathfrak{F}$. Moreover $N$ is the unique minimal normal subgroup of $G$, because the argument above shows that if $T$ is a minimal normal subgroup of $G$, then $\langle A_1, B_1\rangle T = G$ and so $G^{\mathfrak{F}} \leq T$, whence $N = T$. Since $B \leq C_G(A^{\mathfrak{F}})$, we have that $B \leq C_G(N)$. If $C_G(N) \neq 1$, then there exists a minimal normal subgroup $T$ of $G$ contained in $C_G(N)$ and so $N \leq C_G(N)$, a contradiction, because $N$ is not abelian. Hence $C_G(N) = 1$ and so $B = 1$. In particular, $G = A$ and $A_1 = \langle A_1, B\rangle$ is an $\mathfrak{F}$-projector of $G$, a contradiction.

Assume now that $A_1$ and $B_1$ permute. We know that $G^{\mathfrak{F}} = A^{\mathfrak{F}}B^{\mathfrak{F}}$ by Theorem 5 and $A^{\mathfrak{F}}$ and $B^{\mathfrak{F}}$ are normal subgroups of $G$. On the other hand, $A = A^{\mathfrak{F}}A_1$ and $B = B^{\mathfrak{F}}B_1$. Consequently we have that $G = \langle A^{\mathfrak{F}}A_1, B^{\mathfrak{F}}B_1\rangle =$

$A^{\mathfrak{F}}\langle A_1, B_1\rangle B^{\mathfrak{F}} = (A^{\mathfrak{F}}A_1)(B^{\mathfrak{F}}B_1) = AB$. Hence $A$ and $B$ permute.

Suppose now that the converse is false. Let $G$ be a counterexample of minimal order. We have that $G = AB$, but $A_1$ is an $\mathfrak{F}$-projector of $A$ and $B_1$ is an $\mathfrak{F}$-projector of $B$ such that $A_1$ and $B_1$ do not permute. We can assume that $Z_\infty(G) \neq 1$, because otherwise $G = A \times B$ and so $A_1$ would be centralised by $B_1$. Let $N$ be a minimal normal subgroup of $G$ contained in $Z_\infty(G)$. It is clear that $N \leq Z(G)$. We know that $X = \langle A_1, B_1\rangle$ is an $\mathfrak{F}$-projector of $G$. Since $XN/N \in \mathfrak{F}$ and $N \leq Z(G)$, we have that $XN \in \mathfrak{F}$. From the maximality of $X$, we conclude that $N \leq X$. From the minimality of $G$, we have that $A_1N/N$ and $B_1N/N$ permute. Hence $X = (A_1N)B_1$.

If $A$ and $B$ belong to $\mathfrak{F}$, we have that $A_1 = A$ and $B_1 = B$, a contradiction to the choice of $G$.

Suppose that $A$ does not belong to $\mathfrak{F}$. Since $A^{\mathfrak{F}}$ is a non-trivial normal subgroup of $G$, we can consider a minimal normal subgroup $T$ of $G$ contained in $A^{\mathfrak{F}}$. Assume that $Y = \langle A_1, B_1\rangle T$ is a proper subgroup of $G$. From the minimality of $G$, since $G/T = (A/T)(BT/T)$ and $A_1T/T$ is an $\mathfrak{F}$-projector of $A/T$ and $B_1T/T$ is an $\mathfrak{F}$-projector of $BT/T$, we have that $A_1T/T$ permutes with $B_1T/T$. This implies that $A_1T$ permutes with $B_1$. Since $Y = \langle A_1T, B_1\rangle$, $A_1T$ and $B_1$ are strongly cosubnormal in $Y$, $A_1$ is an $\mathfrak{F}$-projector of $A_1T$ by [4, III.3.14] and [4, III.3.18], and $B_1$ is an $\mathfrak{F}$-projector of $B_1$, the minimality of $G$ yields that $A_1$ permutes with $B_1$, a contradiction. Hence $\langle A_1, B_1\rangle T = G$. This implies that $G^{\mathfrak{F}} = T$, because if $G \in \mathfrak{F}$, we would have that $A_1 = A$ and $B_1 = B$ and $A_1$ and $B_1$ would permute.

Assume that $B^{\mathfrak{F}} \neq 1$. Since $B^{\mathfrak{F}} \leq G^{\mathfrak{F}} = T$, we have that $B^{\mathfrak{F}} = T$ and hence $T \leq A \cap B \leq Z_\infty(G)$ by Theorem 4. The above argument shows that $T \leq X$. Thus $X = (X \cap A)B$. But $G = XT$ and, since $T$ is abelian, we have that $X \cap T = 1$ by [4, IV.5.18]. Moreover, $X \cap A = X \cap A_1T = A_1(X \cap T) = A_1$. Consequently $X = A_1B = A_1B_1$ and $A_1$ permutes with $B_1$, final contradiction. □

*Example.* Let $X = \langle x\rangle$ be a cyclic group of order 8. Let $Y = \langle z, y\rangle$ be a direct product of two cyclic groups of order 2. The group $Y$ acts on $X$ via $x^y = x^{-1}$, $x^z = x^5$. Let $H$ be the corresponding semidirect product. The group $H$ has an irreducible and faithful module $V = \langle v_1, v_2, v_3, v_4\rangle$ over the field of 3 elements of dimension 4, given by

$$\begin{array}{lll}
v_1^x = v_3^2, & v_1^y = v_1v_2, & v_1^z = v_1, \\
v_2^x = v_3^2v_4, & v_2^y = v_2, & v_2^z = v_2, \\
v_3^x = v_1v_2, & v_3^y = v_3^2, & v_3^z = v_3^2, \\
v_4^x = v_2^2, & v_4^y = v_3^2v_4, & v_4^z = v_4^2.
\end{array}$$

Let us consider now the corresponding semidirect product $G = [V]H$. Let $w = (xy)^{v_1}$, $A = \langle w \rangle$ and $B = \langle y, z \rangle$. In the dihedral group $\langle x, y \rangle$, we have that $xy$ has order 2. Now we prove that $A$ and $B$ are $\mathfrak{N}$-connected. Since $B$ has order 4, it is enough to prove that $\langle w, y \rangle$, $\langle w, z \rangle$ and $\langle w, yz \rangle$ are nilpotent groups. First of all, we note that $v_1^{x^{-1}} = v_3v_4$, $v_2^{x^{-1}} = v_4^{-1}$, $v_3^{x^{-1}} = v_1^{-1}$, $v_4^{x^{-1}} = v_1^{-1}v_2$. We can check that the element $wy = v_1^{-1}v_3x$ has order 8 and $(wy)^y(wy) = 1$. Hence $\langle w, y \rangle = \langle wy, y \rangle$ is a dihedral group of order 16. On the other hand, $wyz = v_1^{-1}v_3xz$ has order 8 and $(wyz)^{yz}(wyz) = 1$, whence $\langle w, yz \rangle = \langle wyz, yz \rangle$ is a dihedral group of order 16. To conclude, we have that $wz = v_1^{-1}v_3xyz$ has order 4 and $(wz)^z(wz) = 1$, therefore $\langle w, z \rangle = \langle wz, z \rangle$ is a dihedral group of order 8. This shows that $A$ and $B$ are $\mathfrak{N}$-connected. But $A$ and $B$ are not cosubnormal. In order to show this, we prove that $\langle A, B \rangle$ is not a 2-group. We have that $(wy)^3(wy)^z = v_1v_3v_4$ is an element of order 3 contained in $\langle A, B \rangle$. Hence $A$ and $B$ are not cosubnormal.

A minimal counterexample must have the structure of this example. We are grateful to Stewart Stonehewer for suggesting that we try groups like this one and to Mike Newman for performing the calculations for us.

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