ON FINITE $\mathcal{T}$-GROUPS

A. BALLESTER-BOLINCHES and R. ESTEBAN-ROMERO

Abstract

Characterisations of finite groups in which normality is a transitive relation are presented in the paper. We also characterise the finite groups in which every subgroup is either permutable or coincides with its permutiser as the groups in which every subgroup is permutable.

2000 *Mathematics subject classification:* 20D10 20D20 20D35.

1. Introduction

All the groups considered in the sequel are finite.

Our aim in this paper is to present characterisations of groups in which normality is transitive and to characterise the groups, all of whose subgroups are either permutable or self-permutising.

A group G is said to be a $\mathcal{T}$-*group* if every subnormal subgroup of G is normal in G. This class was investigated by several authors. The results of their investigations allow us to have a detailed picture of their structure. The account of their structure can be found in [11].

Bryce and Cossey [3] give a different account by establishing local versions of some of the results on $\mathcal{T}$-groups. For a prime p, they define classes of soluble groups $\mathcal{T}_p$, $\mathcal{D}_p$, $\mathcal{P}_p$ and $\mathcal{R}_p$ as follows:

- $\mathcal{T}_p$ is the class of all groups G for which every p'-perfect subnormal subgroup of G is normal.

- $\mathcal{D}_p$ is the class of all groups G satisfying

---

1. Sylow p-subgroups of G are Dedekind groups and

2. p-chief factors of G are cyclic and, as modules for G, form a single isomorphism class.

- $\mathcal{P}_p$ is the class of all groups G such that if S is a Sylow p-subgroup of G, then $N_G(S)$ normalises every subgroup of S.

- $\mathcal{R}_p$ is the class of all groups G for which every p-subgroup of G is pronormal in G.

They prove that the above four classes coincide and they recover the known structure results for soluble $\mathcal{T}$-groups.

On the other hand, Bianchi, Gillio Berta Mauri, Herzog and Verardi [2] present a new characterisation of soluble $\mathcal{T}$-groups using the following embedding property:

A subgroup H of G is said to be an $\mathcal{H}$-*subgroup* of G if for all $g \in G$, $N_G(H) \cap H^g \leq H$.

They prove:

THEOREM 1 ([2, Theorem 10]). *A group G is a soluble $\mathcal{T}$-group if and only if every subgroup of G is an $\mathcal{H}$-subgroup.*

The above embedding property is closely related to the *weak normality* introduced by Müller in [6]:

A subgroup H of G is called *weakly normal* in G if $H^g \leq N_G(H)$ implies that $g \in N_G(H)$.

It is clear that each $\mathcal{H}$-subgroup of G is weakly normal in G, but the converse is not true in general as the following example shows:

EXAMPLE 1. Consider $G = \Sigma_4$ and $H = \langle (1,2,3,4) \rangle$. We have that $N_G(H) = \langle (1,2,3,4), (1,3) \rangle$, and if $g = (1,2,3)$, $H^g = \langle (1,4,2,3) \rangle$, whence $H^g \cap N_G(H) = \langle (1,2)(3,4) \rangle \not\leq H$. Therefore, H is not an $\mathcal{H}$-subgroup of G. But $N_G(H)$ has a unique cyclic subgroup of order 4. Consequently if $H^g \leq N_G(H)$, then $H^g = H$, and H is a weakly normal subgroup of G.

Weak normality is an interesting embedding property. Every pronormal subgroup is weakly normal (see Section 2, Proposition 1) and, by a result of Sementovskiĭ [13], the join of two pronormal subgroups is weakly normal. We include a proof of this result in Section 2 for the sake of completeness. However, not every weakly normal subgroup is pronormal as we show in the same section.

If H is weakly normal in G and H is normal in a subgroup K of G, then $N_G(K)$ is contained in $N_G(H)$. This fact is crucial in the proof of [2, Theorem 10] and is a subgroup embedding property introduced by Mysovskikh in [7]:

2

A subgroup H of G is said to satisfy the *subnormaliser condition* in G if for every subgroup K of G such that $H \trianglelefteq K$, it follows that $N_G(K) \leq N_G(H)$.

There exist subgroups satisfying the subnormaliser condition which are not weakly normal, as the following example, due to V. I. Mysovskikh [8], shows:

EXAMPLE 2. Consider the natural wreath product of C_3 by A_4. Let B be the base group, $B = \langle w_1, w_2, w_3, w_4 \rangle$, where A_4 acts naturally on the indices. Let $W = \langle w_4 w_1^{-1}, w_4 w_2^{-1}, w_4 w_3^{-1} \rangle$, then A_4 acts faithfully on W. Let $G = [W]A_4$ be the corresponding semidirect product. The subgroup W, as a module over V_4, can be decomposed as a direct sum of the V_4-submodules $W_1 = \langle d \rangle$, with $d = w_1^{-1} w_2^{-1} w_3 w_4$, $W_2 = \langle e \rangle$, with $e = w_1 w_2^{-1} w_3^{-1} w_4$ and $W_3 = \langle f \rangle$, with $f = w_1^{-1} w_2 w_3^{-1} w_4$. Denote $a = (1,2,3)$, $b = (1,2)(3,4)$, $c = (1,3)(2,4)$. We have that $d^b = d$, $e^b = e^2$, $f^b = f^2$, $d^c = d^2$, $e^c = e^2$, $f^c = f$, $d^a = e$, $e^a = f$, $f^a = d$, $b^a = bc$ and $c^a = b$. Consider $D = [W_3]\langle b \rangle$, then $D \cong \Sigma_3$, and since $b^d = b$, $b^e = be^2$ and $b^f = bf^2$, it follows that $N = N_G(D) = \langle d, f, b, c \rangle = [W_1 W_3]\langle b, c \rangle$, a self-normalising subgroup. Since $D^a = \langle bc, d \rangle \leq N$ and $D^a \neq D$, we have that D is not weakly normal in G. But D satisfies the subnormaliser condition, because the intermediate subgroups between D and N are $\langle b, c, f \rangle$, $\langle b, cd, f \rangle$, $\langle b, cd^2, f \rangle$, self-normalising subgroups of order 12, and $\langle b, d, f \rangle$, a subgroup of order 18 whose normaliser is N. Consequently D satisfies the subnormaliser condition, but is not a weakly normal subgroup.

One of our aims here is to show interesting connections between the above subgroup embedding properties and to use them for characterising soluble $\mathcal{T}$-groups. We prove:

THEOREM A. *Let G be a group. The following statements are pairwise equivalent:*

1. *G is a soluble $\mathcal{T}$-group.*

2. *Every subgroup of G is weakly normal in G.*

3. *Every subgroup of G satisfies the subnormaliser condition.*

We develop local versions of the above theorem in the line of [3]. They turn out to be of interest and can be used for proving the theorem. As an application, we give an alternative proof of [2, Theorem 10].

In Section 3 we study the groups in which every subgroup H is either permutable in G or H is permutable with no cyclic subgroups not contained in H, which we call $\mathcal{PSP}$-groups. We prove that, in fact, those groups are exactly the groups in which every subgroup is permutable.

2. Finite $\mathcal{T}$-groups

We begin by showing that pronormal subgroups are weakly normal subgroups.

PROPOSITION 1. *If H is a pronormal subgroup of a group G, then H is a weakly normal subgroup of G.*

Proof. Suppose that $H^g \leq N_G(H)$. Since $H \leq N_G(H)$, we have that $\langle H, H^g \rangle \leq N_G(H)$ and there exists $x \in \langle H, H^g \rangle$ such that $H^g = H^x$. But $x \in N_G(H)$, therefore $H^x = H$ and $H^g = H$. □

REMARK 1. The converse of Proposition 1 is not true: Consider an irreducible and faithful Σ_3-module V_7 over the field of 7 elements such that the restriction of V_7 to A_3, the alternating group of degree 3, is a direct sum of two irreducible and faithful A_3-submodules W_1 and W_2 of dimension 1. Let $H = [W_1]A_3$ be the corresponding semidirect product. Since no elements of order 2 normalise H, it follows that $H \leq N_G(H) \leq [V_7]A_3$. Moreover, W_2 does not centralise H. Consequently, $H = N_G(H)$ and so H is weakly normal in G.

Assume that H is pronormal in G. Let a be an element of order 2 of Σ_3. Since $[V_7]A_3$ is normal in G, we have that $\langle H, H^a \rangle \leq [V_7]A_3$. There exists an element $x \in \langle H, H^a \rangle$ such that $H^a = H^x$. Therefore $a \in [V_7]A_3$ because $ax^{-1} \in N_G(H) = H$, a contradiction.

Consequently H is a weakly normal subgroup of G which is not pronormal in G.

The join of two pronormal subgroups is not pronormal in general (see, for example, [4, Section I.6, Exercise 2]). However we have, by the following result of Sementovskiĭ, that the join of two pronormal subgroups is weakly normal.

THEOREM 2. *If A and B are pronormal subgroups of G, then $J = \langle A, B \rangle$ is a weakly normal subgroup of G.*

Proof. Suppose that A and B are pronormal subgroups of G, $J = \langle A, B \rangle$ and $J^g \leq N_G(J)$. Since $A \leq J \leq N_G(J)$ and $A^g \leq J^g \leq N_G(J)$, it follows that $\langle A, A^g \rangle \leq N_G(J)$. A similar argument shows that $\langle B, B^g \rangle \leq N_G(J)$. From the pronormality of A, there exists $x \in \langle A, A^g \rangle \leq N_G(J)$ such that $A^x = A^g$, and so $A^g \leq J^x = J$. From the pronormality of B, there exists $y \in \langle B, B^g \rangle \leq N_G(J)$ such that $B^g = B^y$, and so $B^g \leq J^y = J$. Hence

$$J^g = \langle A^g, B^g \rangle \leq \langle J, J \rangle = J,$$

whence $g \in N_G(J)$. Therefore J is weakly normal in G. □

4

The following lemma turns out to be crucial in the proof of Theorem A (see [2, Theorem 6 (ii)]).

LEMMA 1.   1. *If G is a group and H a weakly normal subgroup of G, then H satisfies the subnormaliser condition in G.*

2. *If H is a subnormal subgroup of $K \leq G$ and H satisfies the subnormaliser condition in G, then H is a normal subgroup of K (cf. [4, Lemma 6.3 (d), p. 241]).*

Proof.   1. Suppose that H is a weakly normal subgroup of G and let K be a subgroup of G containing H and contained in $N_G(K)$. Consider an element $g \in N_G(K)$. Then $K^g = K$. Hence $H^g \leq K^g = K \leq N_G(H)$. Consequently $g \in N_G(H)$ because H is a weakly normal subgroup of G.

2. Suppose that H satisfies the subnormaliser condition in G and that H is a subnormal subgroup of $K \leq G$. By induction on a length of a series from H to K, we can suppose that $H \trianglelefteq T \trianglelefteq K$. In this case, $H \leq T \leq N_G(H)$. Therefore $K \leq N_G(T) \leq N_G(H)$ by the subnormaliser condition. Hence H is a normal subgroup of K.   □

The following Lemma (see [4, Example I.6.8, page 249]) shows that for p-subgroups, pronormality, weak normality and the subnormaliser condition are equivalent properties.

LEMMA 2. *Let H be a p-subgroup of a group G. The following properties are equivalent:*

1. *H is a pronormal subgroup of G.*

2. *H is a weakly normal subgroup of G.*

3. *H satisfies the subnormaliser condition in G.*

4. *$H \trianglelefteq N_G(X)$ for every p-subgroup X such that $H \leq X$.*

5. *$H \trianglelefteq N_G(S)$ for every Sylow p-subgroup S of G such that $H \leq S$.*

Proof. By Proposition 1, it follows that 1 implies 2, and by Lemma 1, we have that 2 implies 3.

Suppose now that H satisfies the subnormaliser condition in G and let X be a p-subgroup of G containing H. Then H satisfies the subnormaliser condition in $N_G(X)$ and H is subnormal in $N_G(X)$. By Lemma 2, H is a normal subgroup of $N_G(X)$. Hence 3 implies 4.

It is clear that 4 implies 5.

Finally we see that 5 implies 1. Let $g \in G$ and $J = \langle H, H^g \rangle$. Since H is a p-group, there exists a Sylow p-subgroup P of J such that $H \leq P$ and there exists a Sylow p-subgroup S of G such that $H \leq P \leq S$. It is clear that P^g is also a Sylow p-subgroup of J. Thus there exists $x \in J$ such that $P^x = P^g$. In particular, H^g is contained in P^x. This implies that $H^{gx^{-1}} \leq P \leq S$. We write $t = gx^{-1}$. It is clear that $t^{-1} \in \langle N_G(S^{t^{-1}}), N_G(S) \rangle$ because $N_G(S)$ is abnormal in G. Since $H \leq S^{t^{-1}}$, it follows that $\langle N_G(S), N_G(S^{t^{-1}}) \rangle \leq N_G(H)$ by 5. Hence $t^{-1} \in N_G(H)$ and $H^x = H^g$. Consequently, H is pronormal in G. $\qquad\square$

DEFINITION 1. Let p be a prime number.

- $\mathcal{T}_p$ is the class of all soluble groups G for which every p'-perfect subnormal subgroup of G is normal.

- $\mathcal{P}_p$ is the class of all groups G such that if S is a Sylow p-subgroup of G, then $N_G(S)$ normalises every subgroup of S.

- $\mathcal{R}_p$ is the class of all groups G for which every p-subgroup of G is pronormal in G.

We introduce the following classes of groups:

DEFINITION 2. • $\mathcal{K}_p$ is the class of all groups G such that every p-subgroup of G is weakly normal in G.

- $\mathcal{K}^p$ is the class of all groups G for which every p'-perfect subgroup of G is weakly normal in G.

- $\mathcal{S}_p$ is the class of all groups G such that every p-subgroup of G satisfies the subnormaliser condition in G.

- $\mathcal{S}^p$ is the class of all G for which every p'-perfect subgroup of G satisfies the subnormaliser condition in G.

REMARK 2. Note that the results of Bryce and Cossey cover only soluble groups. However, we do not assume that $\mathcal{P}_p$ and $\mathcal{R}_p$ are composed of soluble groups in Definition 2.

Our purpose is to prove that, in the soluble universe, $\mathcal{K}_p = \mathcal{K}^p = \mathcal{S}_p = \mathcal{S}^p = \mathcal{T}_p = \mathcal{P}_p = \mathcal{R}_p$.

In the general finite universe, Lemma 2 gives a proof of the equality $\mathcal{P}_p = \mathcal{R}_p$ (Rose [12]). Moreover, applying the same Lemma, we have that $\mathcal{K}_p = \mathcal{S}_p = \mathcal{P}_p = \mathcal{R}_p$ (see for example Peng [9] and Robinson [10]). One can wonder whether the class $\mathcal{R}_p$ is equal to the class of all groups G such that every p'-perfect subnormal subgroup is normal in G. The answer is 'no' as the alternating group of degree 5 and the prime $p = 2$ show. However $\mathcal{T}_p$ coincides with the class of all soluble $\mathcal{R}_p$-groups by [3, Theorem 2.3].

THEOREM 3. $\mathcal{K}^p = \mathcal{R}_p$.

Proof. Since p-subgroups are p'-perfect subgroups, it is clear that $\mathcal{K}^p \subseteq \mathcal{K}_p = \mathcal{R}_p$. Assume that $\mathcal{K}^p \neq \mathcal{R}_p$ and let $G \in \mathcal{R}_p \setminus \mathcal{K}^p$ be a group of minimal order. Then there exists a p'-perfect subgroup H of G such that $H^g \leq N_G(H)$ for some $g \in G$ but $g \notin N_G(H)$. Since $\mathcal{R}_p$ and $\mathcal{K}^p$ are subgroup-closed, we have that $G = \langle H, g \rangle$. Let H_p be a Sylow p-subgroup of H. Then H_p is pronormal in G and so there exists $x \in \langle H_p, H_p^g \rangle$ such that $H_p^x = H_p^g$. Since $\langle H_p, H_p^g \rangle$ is contained in $N_G(H)$, it follows that $x \in N_G(H)$. Hence H_p^x is contained in H. This means that H_p^g is a Sylow p-subgroup of H for every Sylow p-subgroup H_p of H. Since H is p'-perfect, we have that $H = \langle H_p^H \rangle = \langle H_p^{\langle H,g \rangle} \rangle = \langle H_p^G \rangle$. Consequently H is a normal subgroup of G, a contradiction. $\square$

COROLLARY 1. $\mathcal{S}^p = \mathcal{K}^p$.

Proof. If $G \in \mathcal{K}^p$, then $G \in \mathcal{S}^p$ by Lemma 1. Now if $G \in \mathcal{S}^p$, then every p-subgroup of G is pronormal by Lemma 2. Consequently $G \in \mathcal{R}_p$, which is equal to $\mathcal{K}^p$ by Theorem 3. $\square$

COROLLARY 2. $\mathcal{K}_p = \mathcal{K}^p = \mathcal{S}_p = \mathcal{S}^p = \mathcal{P}_p = \mathcal{R}_p$.

By the result of Bryce and Cossey [3, Theorem 2.3], we have that in the soluble universe the above classes are all equal to $\mathcal{T}_p$.

COROLLARY 3.

$$\bigcap_{p \in \mathbb{P}} \mathcal{S}_p = \bigcap_{p \in \mathbb{P}} \mathcal{S}^p = \bigcap_{p \in \mathbb{P}} \mathcal{K}^p = \bigcap_{p \in \mathbb{P}} \mathcal{K}_p = \mathcal{T} \cap \mathfrak{S},$$

where $\mathfrak{S}$ denotes the class of all soluble groups.

Proof. Notice that $\mathcal{K}^p = \mathcal{R}_p = \mathcal{S}^p$ from Theorem 3 and Corollary 1. Now $\bigcap_{p \in \mathbb{P}} \mathcal{R}_p = \mathcal{T} \cap \mathfrak{S}$ by [10]. $\square$

Our proof of Theorem A requires the following lemma.

LEMMA 3. *If N is a normal subgroup of G, $N \leq H \leq G$ and H/N is a weakly normal subgroup of G/N, then H is a weakly normal subgroup of G.*

Proof. Suppose that $H^g \leq N_G(H)$ with $g \in G$. Then

$$(H/N)^{gN} \leq N_{G/N}(H/N) = N_G(H)/N,$$

and from the weak normality of H/N, it follows that $gN \in N_G(H)/N$, that is, $g \in N_G(H)$. Consequently, H is a weakly normal subgroup of G. $\square$

7

Proof of Theorem A. 1 implies 2. Suppose that there exists a soluble $\mathcal{T}$-group with a non weakly normal subgroup H. We choose for G a counter-example of least order. Then there exists an element $g \in G$ such that $H^g \leq N_G(H)$ but $g \notin N_G(H)$. Let $S = \langle H, g \rangle$. Then S is a $\mathcal{T}$-group by [11, 13.4.7]. If $G \neq S$, then, by the minimal choice of G, it follows that H is a weakly normal subgroup of S. Hence $g \in N_S(H) \leq N_G(H)$, a contradiction. Therefore $G = \langle H, g \rangle$.

On the other hand, H is supersoluble because it is a $\mathcal{T}$-group ([5]). Let p be the largest prime number dividing $|H|$. Then, if H_p is a Sylow p-subgroup of H, we have that $H_p \trianglelefteq H$. Now $G \in \mathcal{K}_p$. This means that H_p is weakly normal in G. Since $H_p^g \leq H^g \leq N_G(H) \leq N_G(H_p)$, it follows that $g \in N_G(H_p)$ and so H_p is a normal subgroup of G. The minimal choice of G implies that H/H_p is weakly normal in G/H_p. From Lemma 3, it follows that H is weakly normal in G, a contradiction.

By Lemma 1, we have that 2 implies 3.

3 implies 1. Assume that every subgroup of G satisfies the subnormaliser condition. Then $G \in \bigcap_{p \in \mathbb{P}} \mathcal{S}^p$ and so G is a soluble $\mathcal{T}$-group by Corollary 3. □

As a corollary, we can give an alternative proof of [2, Theorem 10] which does not use Gaschütz's theorem about the structure of soluble $\mathcal{T}$-groups. We need the following theorem.

THEOREM 4. *If G is a supersoluble group and H is a weakly normal p-subgroup of G, then H is an $\mathcal{H}$-subgroup of G.*

Proof. Suppose that the theorem is false. We can consider a group G of least order with a weakly normal subgroup H which is not an $\mathcal{H}$-subgroup of G.

Suppose that $O_{p'}(G) \neq 1$. Let N be a minimal normal subgroup of G contained in $O_{p'}(G)$. Denote $P_0 = H^g \cap N_G(H)$. From the minimality of G it follows that $P_0N/N \leq HN/N$, whence $P_0 \leq HN$. Since H is a p-subgroup and N is a p'-subgroup, we have that $P_0 \leq H^n \cap N_G(H)$ for some $n \in N$. Let $S = HN$. If $S < G$, from the minimality of G it follows that $H^n \cap N_G(H) = H^n \cap N_S(H) \leq H$, a contradiction. If $S = G$, then H is a maximal subgroup of G, and in this case H is an $\mathcal{H}$-subgroup of G, another contradiction.

Therefore we can suppose that $O_{p'}(G) = 1$. If q is the largest prime dividing $|G|$, then G has a normal Sylow q-subgroup Q. Assume that $p \neq q$, then $Q \leq O_{p'}(G) = 1$, a contradiction. Therefore $q = p$. Since $H \leq Q \trianglelefteq G$, it follows that H is subnormal in G. Hence H is normal in G by Lemma 1, a contradiction. □

Example 1 shows that we cannot extend Lemma 2 to $\mathcal{H}$-subgroups. However we have the following result.

THEOREM 5. *If G is a supersoluble group and H is a weakly normal subgroup of G such that every subgroup of H is weakly normal in G, then H is an $\mathcal{H}$-subgroup of G.*

Proof. Suppose that the theorem is false. We can consider a group G of least order with a weakly normal p-subgroup H whose subgroups are weakly normal in G but which is not an $\mathcal{H}$-subgroup of G.

Let p be the largest prime dividing $|G|$. Suppose that p does not divide $|H|$. Since $O_p(G) \neq 1$, we can consider a minimal normal p-subgroup N of G. Denote $Q_0 = H^g \cap N_G(H)$. From the minimality of $|G|$, we have that $Q_0 N/N \leq HN/N$. Hence $Q_0 \leq HN$. Since Q_0 and H are p'-subgroups, N is a p-group and G is soluble, there exists $n \in N$ such that $Q_0 \leq H^n \cap N_G(H)$. Denote $S = HN$. If $S < G$, from the minimality of G it follows that $H^n \cap N_G(H) = H^n \cap N_S(H) \leq H$, a contradiction. If $S = G$, then H is a maximal subgroup of G, and hence an $\mathcal{H}$-subgroup, another contradiction.

Suppose that p divides $|H|$. Consider a Sylow p-subgroup H_p of H and P a Sylow p-subgroup of G. Since H_p is a weakly normal subgroup of G and H_p is a subnormal subgroup of P and P is a normal subgroup of G, we have that $H_p \trianglelefteq G$ by Lemma 1. From the minimality of G, it follows that H/H_p is an $\mathcal{H}$-subgroup of G/H_p, and from [2, Lemma 2] we have that H is an $\mathcal{H}$-subgroup of G. $\qquad\square$

Now Theorem 1 follows as a corollary of Theorems A and 5: If G is a supersoluble $\mathcal{T}$-group, given a subgroup H of G, then H and all its subgroups are weakly normal in G by Theorem A. Hence H is an $\mathcal{H}$-subgroup of G, by Theorem 5. Suppose that every subgroup of G is an $\mathcal{H}$-subgroup. Then all the subgroups of G are weakly normal. Therefore G is a $\mathcal{T}$-group by Theorem A.

3. Groups in which every subgroup is permutable or coincides with its permutiser

Let H be a subgroup of a group G. The *permutiser* $P_G(H)$ is defined to be the subgroup generated by all cyclic subgroups of G that permute with H. According to [1], a group G is said to be a *$\mathcal{P}$-group* if $H \neq P_G(H)$ for every proper subgroup H of G. The main result on the structure of $\mathcal{P}$-groups [1, Theorem A] shows that they are soluble, their chief factors have order 4 or a prime and G induces the full automorphism group on their chief factors of order 4.

This section is devoted to study the groups in which every subgroup is either permutable or coincides with its permutiser and it was motivated by the results contained in Section IV of [2], where a characterisation of groups in which every subgroup is either normal or self-normalising is presented.

9

DEFINITION 3.   1. A group G is said to be a $\mathcal{PSP}_p$-group (p a prime) if every p-subgroup H of G is either permutable or $P_G(H) = H$.

2. A group G is a $\mathcal{PSP}$-group if the above condition holds for every subgroup of G.

It is clear that groups in which every subgroup is permutable are $\mathcal{PSP}$-groups, and if G is a $\mathcal{PSP}$-group, then G is a $\mathcal{PSP}_p$-group for all primes p.

THEOREM 6. *Let G be a $\mathcal{PSP}_p$-group. Then every chief factor of G whose order is divisible by p is cyclic, that is, G is p-supersoluble.*

Proof. Suppose that G is a $\mathcal{PSP}_p$-group of order $p^k m$, where m is coprime to p. If $H < G$ and $|H| = p^l$, where $l < k$, then H is properly contained in $N_G(H)$, and hence properly contained in $P_G(H)$. Since G is a $\mathcal{PSP}_p$-group, it follows that H is permutable. Now let L be the product of all subgroups of G of order p^{k-1}. Then L is a normal p-subgroup of G, and is either a Sylow subgroup or the unique subgroup of order p^{k-1}. Suppose first that G has a normal Sylow p-subgroup P, so that all p-subgroups of G are permutable. Let H be a normal subgroup of P. If X is any p'-subgroup of G, then XH is a group, and $H = XH \cap P$ is normal in XH. Hence all p'-elements of G normalise H, and since also P normalises H, it follows that H is normal in G. Consequently the chief factors of P are chief factors of G. These factors are all cyclic. Since P is a Sylow subgroup, G has no other chief factors of order divisible by p. Suppose, on the other hand, that G has a unique subgroup L of order p^{k-1} and that each Sylow p-subgroup P of G is its own permutiser. Note that P must by cyclic, because otherwise there would be more than one subgroup of order p^{k-1}. Now P, $N_G(P)$ and the centre of $N_G(P)$ all coincide, and so G has a normal p-complement (by Burnside's Theorem). The result follows.   □

Nevertheless, we cannot ensure that if G is a $\mathcal{PSP}_p$-group, then all the p-subgroups of G are permutable. The alternating group of degree 4, A_4, is a $\mathcal{PSP}_3$-group, but its Sylow 3-subgroups are not permutable. However, the following property holds:

THEOREM 7. *The following statements are pairwise equivalent:*

1. *G is a $\mathcal{PSP}$-group.*

2. *G is a $\mathcal{PSP}_p$-group for every prime p.*

3. *Every subgroup of G is permutable.*

Proof. Suppose that G is a $\mathcal{PSP}$-group, then given $H \leq G$, H is a permutable subgroup of G or $P_G(H) = H$. In particular, given a p-subgroup H of G, H is a permutable subgroup of G or $P_G(H) = H$, and G is a $\mathcal{PSP}_p$-group. Thus 1 implies 2.

Suppose that G is a $\mathcal{PSP}_p$-group for every prime number p. From Theorem 6, we have that G is p-supersoluble for every prime number p. Hence G is supersoluble. From [1], we have that G is a $\mathcal{P}$-group. Since G is a $\mathcal{PSP}_p$-group, every p-subgroup of G is permutable for all primes p. Therefore every subgroup of G is permutable. Hence 2 implies 3.

To conclude, we observe that if every subgroup of G is permutable, then G is a $\mathcal{PSP}$-group, and hence 3 implies 1. $\qquad\square$

Acknowledgements

References

[1] J. C. Beidleman and D. J. S. Robinson, 'On finite groups satisfying the permutizer condition', *J. Algebra* **191** (1997), 686–703.

[2] M. Bianchi, A. Gillio Berta Mauri, M. Herzog and L. Verardi, 'On finite solvable groups in which normality is a transitive relations', *J. Group Theory* (2) **3** (2000), 147–156.

[3] R. A. Bryce and J. Cossey, 'The Wielandt subgroup of a finite soluble group', *J. London Math. Soc.* (2) **40** (1989), 244–256.

[4] K. Doerk and T. Hawkes, *Finite Soluble Groups*, number 4 in: De Gruyter Expositions in Mathematics (Walter de Gruyter, Berlin, New York, 1992).

[5] W. Gaschütz, 'Gruppen, in dennen das Normalteilersein transitiv ist', *J. reine angew. Math.* **198** (1957), 87–92.

[6] K. H. Müller, 'Schwachnormale Untergruppen: Eine gemeinsame Verallgemeinerung der normalen und normalisatorgleichen Untergruppen', *Rend. Sem. Mat. Univ. Padova* **36** (1966), 129–157.

[7] V. I. Mysovskikh, 'Investigation of subgroup embeddings by the computer algebra package GAP', in: *Computer algebra in scientific computing—CASC'99 (Munich)* (Springer, Berlin, 1999) pp. 309–315.

[8] ——, 'Subnormalizers and embedding properties of subgroups of finite groups', *Zapiski Nauchnikh Seminarov POMI* **265** (1999), 258–280.

[9] T. A. Peng, 'Finite groups with pronormal subgroups', *Proc. Amer. Math. Soc.* **20** (1969), 232–234.

[10] D. J. S. Robinson, 'A note on finite groups in which normality is transitive', *Proc. Amer. Math. Soc.* **19** (1968), 933–937.

[11] ——, *A course in the theory of groups* (Springer-Verlag, New-York, 1982).

[12] J. Rose, *The abnormal structure of finite groups* (Ph.D. Thesis, Cambridge, 1964).

[13] V. G. Sementovskiĭ, 'Pronormal subgroups of finite groups', *Vescī Akad. Navuk BSSR Ser. Fīz.-Mat. Navuk* (4) **135** (1966), 12–16.

Departament d'Àlgebra   Departament de Matemàtica Aplicada
Universitat de València   Universitat Politècnica de València
Dr. Moliner, 50   Camí de Vera, s/n
E-46100 Burjassot (València)   E-46022 València
Spain   Spain