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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM

No. 1114

THE CALCULATION OF COMPRESSIBLE FLOWS WITH LOCAL REGIONS OF SUPERSONIC VELOCITY

By B. Göthert and K. H. Kawalki

Translation

Berechnung kompressibler Strömungen mit örtlichen Überschallfeldern

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THE CALCULATION OF COMPRESSIBLE FLOWS WITH LOCAL REGIONS OF SUPERSONIC VELOCITY*

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ABSTRACT: The following report is concerned with a method for the approximate calculation of compressible flows about profiles with local regions of supersonic velocity. The flow around a slender profile is treated as an example.

I. STATEMENT OF THE PROBLEM

Several methods are known for the calculation of compressible flows at high subsonic velocities. The resulting approximate solutions are quite useful as long as sound velocity is not exceeded at any point of the flow field. However, apparently all these approximate calculations, without exception, cease to converge or to render useful flow patterns if the condition of purely subsonic flow is no longer satisfied. Moreover, numerous tests in wind tunnels confirmed the result that the reconversion

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of local supersonic flows into subsonic flows in real flows takes place not steadily but generally by means of a compression shock which completely changes the whole flow pattern.[1] Therefore, the question arises whether a continuous process from supersonic to subsonic flow is at all possible for bodies in parallel flow, even though the formation of a boundary layer on the surface of the body is at first neglected.

An approximation method was tested in the DVL in order to clarify these questions; this method makes possible the calculation of flows with local regions of supersonic velocity.

II. SURVEY OF THE METHOD USED

The development of the method started from the fact that the known approximation methods for compressible flows, which are without exception based on a step-by-step improvement of the incompressible flow, are quite appropriate for the limited domain of the pure subsonic flow extending from infinity to the sonic velocity boundary in the flow field near the body. However, for the region of the local supersonic flow a method based on the properties of supersonic flow will be subsequently used, as for instance the method of characteristics of Prandtl-Busemann. Accordingly the partial areas of supersonic and subsonic flow, respectively, are calculated separately by different methods which in each case are treated according to the peculiarities of the partial flow to be calculated. The partial flows that were thus determined must then be joined in such a manner that the flows agree on the surface of contact of the two regions, that is, on the sonic velocity boundary, with respect to magnitude as well as to direction of the velocity. Figure 1 shows a schematic representation of the boundary between the two flow areas.

1. Subsonic Region with Small Perturbation Velocities

Prandtl's rule will represent a good approximation for a great part of the outer subsonic flow area, since the perturbation velocities caused by the profile are sufficiently small up to some distance from the sonic boundary so that the assumptions of

[1]No account will be taken here of flows which slightly exceed sound velocity because for them there is no certain way of distinguishing between a steady transition and a compression shock; neither by pressure-distribution measurements nor by optical observations, for instance according to the schlieren method.

Prandtl's approximation are rather well satisfied. The subsonic flow of this region can, of course, not be obtained by simply distorting the incompressible flow around the profile according to Prandtl's rule. First, the local supersonic velocity field requires essentially more space than the subsonic flow. The outer subsonic streamlines, therefore, are widened outward not only by the body in the flow but also by the additional displacement due to the supersonic velocity field. However, this additional widening of the streamlines by the supersonic velocity field can only affect the subsonic flow like a modification of the boundary conditions at the sonic velocity boundary; such a modification may be represented in a simple way by source and sink distributions, dipoles, and so forth.

2. Subsonic Region with High Perturbation Velocities

As mentioned before, the part of the subsonic flow which has to be calculated according to Prandtl does not extend as far as the sonic boundary; it only reaches up to a boundary line near the sonic boundary determined by an agreed sufficiently small value of the perturbation velocities. (This boundary line is represented in fig. 1 by a dashed line.) The calculation for the region from this line to the sonic velocity boundary must generally be carried out by an improved subsonic method. A numerical method of calculation seems to be particularly appropriate for this intermediate region. The outer flow according to Prandtl, which is assumed as known, may by this method by continued a little further, namely up to the sonic velocity boundary.

One assumes, for instance, that the flow field is covered by a rectangular grid of selected points. The velocity components v_x and v_y at these points in the outer field are known for each case according to Prandtl's approximation. Then the velocities at the inner grid points may be calculated from the known values at the outer grid points if the differential quotients which are decisive for this continuation are approximated by the corresponding difference quotients.[2] Figure 2 shall be considered as an example. Magnitude and direction of the velocities at all grid points outside of the boundary lines are assumed as known. The air density shall be known also. Then the exact equations of the continuity and the irrotationality are, for the indicated grid point "2":

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[2] I wish to express my thanks here to Dr. H. Schubert/DVL for his suggestion to treat compressible flows according to the difference method.

$$\left.\begin{aligned} v_{x_2} \times \left(\frac{\partial \rho}{\partial x}\right)_2 + \rho_2 \times \left(\frac{\partial v_x}{\partial x}\right)_2 + v_{y_2} \times \left(\frac{\partial \rho}{\partial y}\right)_2 + \rho_2 \times \left(\frac{\partial v_y}{\partial y}\right)_2 = 0 \\ \left(\frac{\partial v_x}{\partial y}\right)_2 - \left(\frac{\partial v_y}{\partial x}\right)_2 = 0 \end{aligned}\right\} \quad A$$

or, in the notation of difference quotients:

$$\left.\begin{aligned} v_{x_2} \times \frac{\rho_A - \rho_4}{x_A - x_4} + \rho_2 \frac{v_{x_A} - v_{x_4}}{x_A - x_4} + v_{y_2} \times \frac{\rho_3 - \rho_1}{y_3 - y_1} + \rho_2 \times \frac{v_{y_3} - v_{y_1}}{y_3 - y_1} = 0 \\ \frac{v_{x_3} - v_{x_1}}{y_3 - y_1} - \frac{v_{y_A} - v_{y_4}}{x_A - x_4} = 0 \end{aligned}\right\} \quad B.$$

If one assumes the grid to be quadratic with meshes of equal width then $\Delta s = \Delta y$ and the equations given above are simplified as follows:

$$v_{x_2} \times \Delta\rho_{A,4} + \rho_2 \times \Delta v_{x_{A,4}} = -v_{y_2} \times \Delta\rho_{3,1} - \rho_2 \times \Delta v_{y_{3,1}} \qquad \text{(1a)}$$

$$\Delta v_{y_{A,4}} = \Delta v_{x_{3,1}} \qquad \text{(1b)}$$

From the second equation one immediately obtained the velocity component v_{y_A} at the point A, while the first equation gives a relation between the velocity component v_{x_A} and the air density at the point A. Velocity and air density are connected in a rather complicated way by the adiabatic relation

$$\frac{\rho}{\rho_c} = \left[1 - \frac{k-1}{k+1} \times \left(\frac{v}{a^x}\right)^2\right]^{\frac{1}{k-1}} \qquad C$$

(ρ_o here represents the air density in the gas at rest a^* the critical sound velocity). Therefore, it will be practical to use the linearized relation $\Delta\rho = \frac{d\rho}{dv} \Delta v$ instead of the exact function, the error being negligible, because of the small width of the meshes:

$$\frac{d(\rho/\rho_o)}{d(v/a^x)} = \frac{-2}{k+1} \times \frac{v}{a^x} \times \left(\frac{\rho}{\rho_o}\right)^{2-k}$$

$$= \frac{-2}{k+1} \times \frac{v}{a^x} \times \left[1 - \frac{k-1}{k+1}\left(\frac{v}{a^x}\right)^2\right]^{\frac{2-k}{k-1}} = f\left(\frac{v}{a^x}\right) \qquad D$$

Compare figure 3.

With $dv = \frac{v_x}{v} \times dv_x + \frac{v_y}{v} \times dv_y$ there results

$$\Delta\rho_{A,4} = \frac{\rho_o}{a^x} \times \left[\frac{d(\rho/\rho_o)}{d(v/a^x)}\right]_2 \times \left(\frac{v_{x2}}{v_2} \times \Delta v_{x_{A,4}} + \frac{v_{y2}}{v_2} \times \Delta v_{y_{A,4}}\right) \qquad (2)$$

and by substitution

$$\Delta v_{x_{A,4}} = - \frac{v_{y_2}\, \Delta\rho_{3,1} + \rho_2\, \Delta v_{y_{3,1}} + \rho_o \frac{v_{x2}}{a^x} \frac{v_{y2}}{v_2} \left[\frac{d(\rho/\rho_o)}{d(v/a^x)}\right]_2 \Delta v_{x_{3,1}}}{\rho_2 + \rho_o \frac{v_{x2}}{a^x} \frac{v_{x2}}{v_2} \left[\frac{d(\rho/\rho_o)}{d(v/a^x)}\right]_2} \qquad (3)$$

The velocity components v_{x_A} and v_{y_A} for the new grid point A are known, according to those calculations, from equations (1 b) and (3). Therefore the air density also is known according to equation (2). Thus the intermediate region between the subsonic flow with small perturbation velocities (calculation according to Prandtl) and the boundary may be determined and the position of the sonic velocity boundary and the velocity direction along that boundary will be obtained as final result. It will have to be further investigated whether a calculation of the whole supersonic velocity field beyond the sonic boundary up to the surface of the body by this method would be practical. In the main, the time spent on the calculations and the accuracy of the method which has been developed here as contrasted with the method of characteristics of Prandtl-Busemann will have to be considered.

The difference method which has been developed here does not, however, permit a start of the numerical calculation at an arbitrary distance from the body in the flow; for at a very great distance from the body the calculation would not be sufficiently accurate since the meshes of the grid would be too small. This conclusion can be drawn from the fact that at a great distance the flow about any body without external forces can be replaced, except for small deviations, by the flow field around a single dipole. It is precisely the small deviations from the dipole flow, which exist even at the greatest distance, that permit a continuation of the flow, for instance according to the difference method, not about a cylinder but around the special profile.

3. Supersonic Region

From the calculations described above, which were carried out in the subsonic area according to Prandtl's rule, or by the difference method, the boundary line of the supersonic velocity field was obtained. The velocity along this line equals the sound velocity and the direction of the flow at every point of this line is known. The next problem consists in joining the corresponding supersonic velocity field to this boundary line; for instance, by the method of characteristics of Prandtl-Busemann. According to the method of characteristics it is known that the changes of state occurring in a supersonic flow are manifested in expansion or compression waves. Therefore, when the flow passes through these lines of disturbance, the flow velocity and the flow direction are modified by definite values which can easily be determined from the graphical representation of the characteristics.[3] It can be inferred from the condition of constant velocity at the sonic boundary line that two waves must originate at every point of the sonic boundary, namely an expansion and a compression wave. (See fig. 4.) The velocity direction jumps from one starting point of these waves at the sonic boundary to the next by a fixed amount for each case. Since these starting points of the waves can be shifted in any way along the sonic velocity boundary, the beginning of a graphical representation of characteristics may be drawn for any given distribution of direction on the sonic boundary. Likewise the starting points of the waves may be shifted not only on the sonic boundary but also towards points outside of the sonic boundary, so that the graphical representation of characteristics may be adjusted to any shape of the sonic boundary line. Therewith it has been demonstrated that at least in certain cases there exists a possibility of continuing the subsonic flow into a corresponding supersonic flow; finally there results the new profile contour of the body in the flow.[4]

[3]Compare L. Prandtl: "Führer durch die Strömungslehre." Verlag Vieweg and Sohn, Braunschweig (1942) S. 257 und folgende.

[4]In order to render possible sufficiently accurate drawings of grids of characteristics for local supersonic velocity fields, a graphical representation of characteristics with an interval of 1/5° has been completed at the DVL; copies may be obtained.

III. CALCULATION OF AN EXAMPLE FOR A MIXED SUPERSONIC-SUBSONIC FLOW

A simple example will demonstrate the combined effect of the calculation methods for various flow regions. The velocity field for a two-dimensional slender body was determined in the proximity of the body by means of conformal transformation and the associated distortion according to Prandtl for the Mach number $M = 0.86$. Thus there resulted a sonic velocity boundary as indicated in figure 5(a). This sonic boundary, which was obtained according to Prandtl's method, could be further improved by continuing the flow step by step to the improved sonic velocity boundary; one would have to start from a boundary line with sufficiently small perturbation velocities using the difference method described above. This correction of the sonic velocity boundary was at first disregarded for the sake of simplicity, since in general, it can be neglected for the basic calculations planned for this report. By means of the directions of velocity along the sonic boundary, the net of characteristics for the supersonic flow can be drawn so that the contour of the body in the flow is a streamline of this supersonic field. Only expansion waves start from the surface of the body, while compression waves, which are necessary for a reconversion of the supersonic flow into a subsonic flow start without exception from the sonic boundary. This behaviour is an essential criterion of the grid of characteristics. The manner of reconversion of the supersonic into a subsonic flow (that is, for instance, whether steadily or by means of a compression shock) is decisively influenced by the sonic velocity boundary.

Of course, the contour of the body at the boundary of the supersonic field which has been determined in the foregoing calculation of the example can not coincide with the contour of the original body which was originally distorted according to Prandtl, since the supersonic field, because of the expansion of the air, requires more space than the flow field which was distorted according to Prandtl. Therefore the surface of the body in the supersonic region will be flatter than the surface of the original body in order to satisfy this increased need for space. The outer subsonic flow must be adjusted to this increased need of space of the supersonic field in order to obtain the original contour as the result of the calculation. This adjustment may be made by replacing the supersonic field with dipoles or source-sink distributions, which will widen the streamlines of the subsonic region originally obtained by Prandtl's method. The strength of this source-sink distribution is known from the condition that the space which is to be added has to cover the difference of the flow density in supersonic flow and flow according to Prandtl and that this space corresponds in magnitude to the flattening of the original body which resulted in figure 5(a). By a step-by-step approximation the contour which results at the end of the calculation may thus be adjusted to the desired contour.

However, the example treated here already permits the conclusion that when the local sound velocity is exceeded there exist solutions with reconversion of the local supersonic velocities into subsonic velocities without compression shock, at least in frictionless flow.

The pressure distribution which was found for the example treated is compared to the pressure distribution of another slender body (fig. 5(b)) which was determined according to the same law of conformal mapping and should agree well with the body of the example. (See fig. 5(a).) The extreme flatness of the pressure distribution in the supersonic region is a special characteristic for the body with local region of supersonic velocity. This behaviour corresponds to the strong curvature of the contour of the body immediately after the flow has entered the supersonic region. Due to this large local curvature, the body B in this region will produce even for incompressible flow higher negative pressures than the body A. For this reason the two pressure distributions which were graphically represented cannot be directly compared.

IV. APPROXIMATE TREATMENT OF LOCAL REGIONS OF SUPERSONIC VELOCITY

The representations of characteristics considered thus far were notable for the fact that expansion or compression waves running in the same direction never intersect. (Compare fig. 5(a).) An intersecting of waves running in the same direction is basically impossible for expansion waves since such waves always diverge from their starting point. But compression waves converge; therefore, they could well form an envelope and cause a compression shock.

Such overlappings of waves were found for the sample body of figure 5(a) when the free-stream velocity was increased from $M = 0.86$ to $M = 0.90$. The construction of the expansion waves was started at the sonic boundary (in opposition to their actual direction) and already for the Mach number $M = 0.86$ (according to figure 5(a)) led to a pressing together of the expansion waves near the contour; for the Mach number $M = 0.90$ they ran into each other even before reaching the contour. This condition is physically senseless, that is, a symmetrical solution for the sonic boundary which was found according to the indicated scheme is, at this Mach number $M = 0.90$, no longer possible. In this case the "expansion shock," according to a consideration of Prandtl, rather will dissolve into a group of divergent expansion waves originating at the contour, while the compression shock which limits the supersonic area persists. In this manner one easily obtains the flow picture found from many tests for which the local supersonic flow is no longer reconverted into the subsonic flow through a continuous phenomenon but by means of a compression shock.

The preceding treatment was based upon the sonic velocity boundary line which was obtained according to Prandtl's rule without additional source-sink bodies which would have corresponded to the greater need for space of the supersonic region. The resulting solution which was physically senseless might well be caused by the perhaps no longer appropriate conditions for the existence of the sonic velocity boundary line. In order to obtain a clearer view one considers the limiting case where the expansion waves converge exactly at the contour. A flow around a corner develops, as represented schematically in figure 6; in the supersonic region such a flow is possible without flow separations. This assumption and a given sonic velocity boundary offer the supersonic region the opportunity to fill the largest possible space without concave curvature of the body in the supersonic domain.

An admission of concave surfaces would presuppose a sonic velocity boundary which could no longer be produced from an incompressible flow about a profile without additional source and sink distributions in place of the supersonic region; or else one would have to drop the condition of flow symmetry. Figure 5(a) demonstrates clearly how the surface always shows a convex curvature when the incoming wave is a compression wave, the outgoing wave an expansion wave. However, for the opposite condition of concave curvature in symmetrical flow the coming in of an expansion wave and going out of a compression wave would result in the emission of expansion waves from the sonic boundary in both cases. But an expansion wave which starts at the sonic boundary cannot be obtained by simply exchanging the compression and expansion waves, for instance according to the diagram in figure 4; for there must always first appear an expansion wave and then a compression wave in the direction of the flow at the sonic velocity boundary, because the flow can pass from the sonic boundary into the supersonic region only by means of an expansion. An expansion wave starting at the sonic boundary is possible only for a certain shape of the corners in the graphical representation of characteristics: two expansion waves are sent out from the corner simultaneously. (Compare fig. 7.) There is, however, a condition: the sonic boundary in this region must take a steeper course than the expansion wave which is inclined toward the direction of the flow by the Mach angle. The streamline through such a corner point lies between the expansion waves, while for a normal corner it is tangent to the apex of the triangle formed by the expansion and compression wave.

It is an important criterion for the corner with two expansion waves that the streamlines become steeper, not flatter, with increasing distance from the body; otherwise the flow would not fit together after transition throught the two expansion waves. This result agrees with the fact that in a symmetrical supersonic region also the direction of the streamlines grows steeper with increasing distance from the profile. The cause of this phenomenon is that the streamlines (because of the maximum flow density at sonic velocity) are closest to each other at the sonic boundary while there will be the greatest distance between them for the supersonic region at the point of maximum velocity, that is, generally, at the point of greatest thickness of the body.

Such distributions of slopes occur only in supersonic regions; they are not possible in subsonic areas for a flow about bodies with convex contours. Therefore it is not surprising that the sonic velocity boundary, which was obtained by distortion of the incompressible flow about the body, will not lead to a useful supersonic flow pattern as long as there are no singularities, as for instance source-sink distributions in the incompressible flow by means of which the effect of the increased expansion of the air in the supersonic region can be calculated.

Presently a method is under investigation in which the sonic velocity boundary may be adjusted to the increased need for space of the supersonic flow by arrangement of singularities in the flow; the results of these investigations will soon be published separately.

V. SYMMETRICAL AND UNSYMMETRICAL REGIONS OF SUPERSONIC FLOW

As stated in detail in the preceding chapter, concave body surfaces in the supersonic region are dependent upon the particular type of corners in the net of characteristics which is characterized by two expansion waves starting from a point of the sonic boundary. (Compare fig. 7.) The stipulation of flow symmetry then causes compression waves to start from the corresponding points of the contour which in turn cause these compression waves to proceed toward the sonic boundary in pairs. Since these compression waves which start at the contour always converge, such solutions are possible only as long as the radiated compression waves do not intersect; otherwise a compression shock will develop. Therefore the existence of symmetrical flows with local supersonic regions cannot be counted upon for strongly concave curvatures or for high Mach numbers, because the compression shock would presuppose a symmetrical expansion shock in a corresponding location; this expansion shock, however, is not physically possible.

Except for the special case described above, supersonic flows on principle tend toward flow symmetry as will be shown in the example represented in figure 8(a). There will always develop a symmetrical flow along the wall with two changes in direction for incompressible flow without separations. It is, however, known that for pure supersonic flow the flow along the wall is unsymmetrical (fig. 8(a)), if no additional guiding surfaces exist. Conditions of symmetry can be achieved through calculations for the mixed subsonic-supersonic flow around this double corner: the outer subsonic flow at the sonic boundary will supply the missing symmetrical boundary conditions (as for the example in fig. 5(a)). But it does not appear impossible that such a mixed supersonic-subsonic flow which has been made symmetrical by the outer subsonic flow may be unstable. Since the subsonic flow is produced by the form of the obstacle in the flow it will be symmetrical only for a symmetrical obstacle. On the other hand the outer flow will not be symmetrical for an unsymmetrical obstacle, as for instance a symmetrical body with a local supersonic region and a compression shock. Therefore, it seems very doubtful whether the outer flow whose form is decisively determined by the form of the supersonic region, will in turn be able to reshape decisively the supersonic flow by forcing the symmetry conditions upon it.[5]

A confirmation of the nonsymmetry of the symmetrical type of solution would explain the occurrence of unsymmetrical solutions with compression shocks which has been observed in tests.

On this occasion I should like to point out that a symmetrical supersonic flow also could be suggested for the single wall represented

[5]A solution of this stability problem shall be obtained by assuming a small unsymmetrical deformation of the sonic boundary and by then observing whether this deformation increases or diminishes.

in figure 8(a) which is physically not real but satisfies the conditions for potential flows with regard to continuity and irrotationality as well as Bernoulli's equation. This flow is represented in figure 8(b). Such a flow, as is well known, is physically real for the reason that a disturbance can not be transmitted upstream in supersonic flow. For solutions which were obtained analytically it must, therefore, always be especially verified whether this additional condition has been satisfied for the supersonic flow.

VI. SUMMARY

1. A method of calculating the approximate velocity field for compressible flows with local regions of supersonic velocity has been presented. Starting from the flow at a large distance from the profile determined according to Prandtl's rule, this outer flow was continued to the sonic velocity boundary by means of a numerical method; the method of characteristics of Prandtl-Busemann was applied for continuation beyond that boundary. These calculations result finally in the contour of the profile in the region of supersonic velocity.

2. It has been demonstrated in an example that mixed subsonic-supersonic flows about two-dimensional bodies can be calculated where not only the transition from subsonic to supersonic but also the transition from supersonic to subsonic takes place continuously, that is, without pressure jump.

3. No physically real local area of supersonic velocity could be determined for the example considered here when the sound velocity was far exceeded because then expansion or compression shocks occurred. However, there is a prospect of calculating mixed flows for such cases also: before starting the calculation one would have further to extend the outer subsonic flow by means of source and sink distributions; in this way the greater need for space of the supersonic velocity field would be satisfied.

4. The symmetry of the local area of supersonic velocity for mixed supersonic-subsonic flows may be enforced by the outer subsonic flow. Since, however, the outside subsonic flow in turn must be produced by the profile in the flow or by the local area of supersonic velocity, the assumption seems justified that

often only an unstable symmetrical flow can be produced by the outside subsonic flow; and this unstable symmetrical flow will turn even at small disturbances into the unsymmetrical case with compression shocks.

Translated by Mary L. Mahler
National Advisory Committee
for Aeronautics

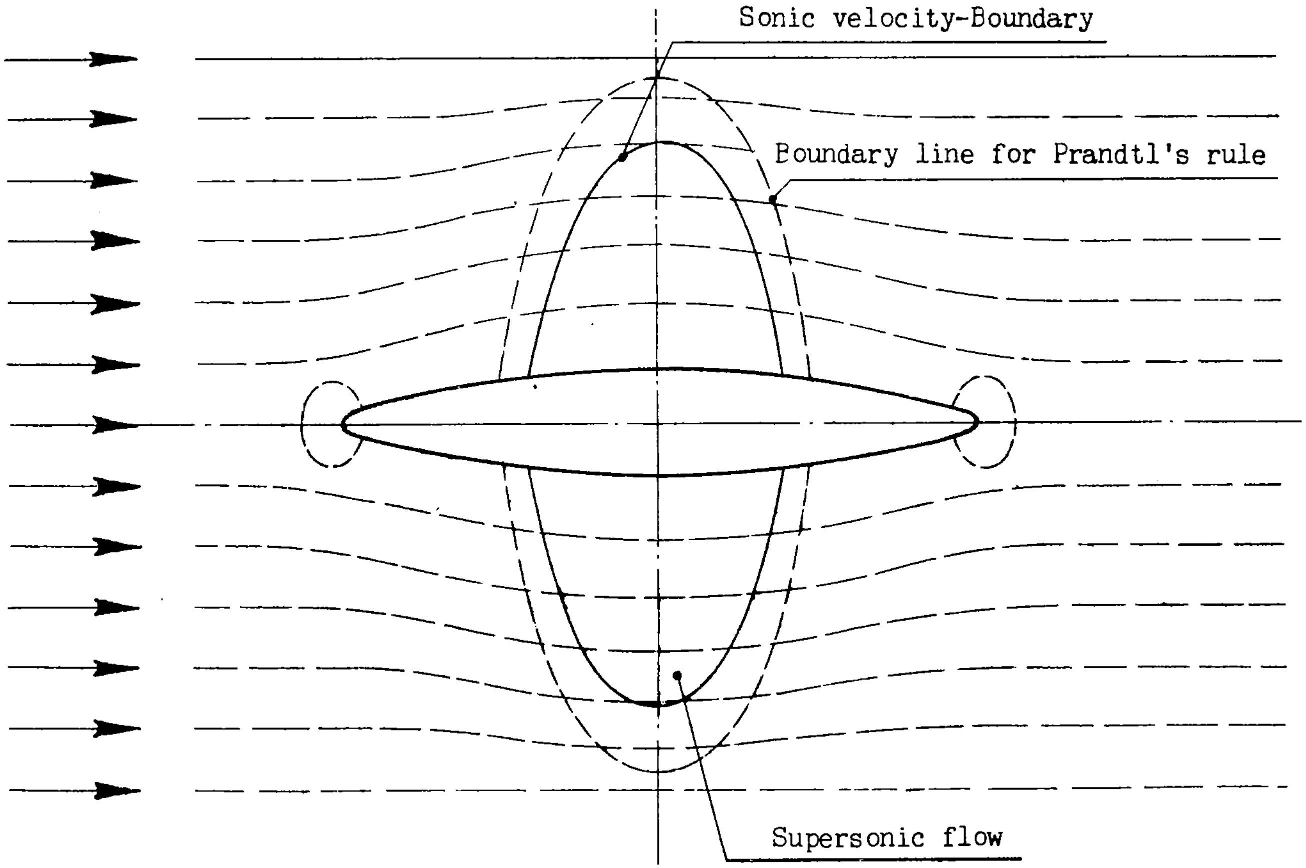

Figure 1. Schematic representation of the flow regions for mixed subsonic -supersonic flow.

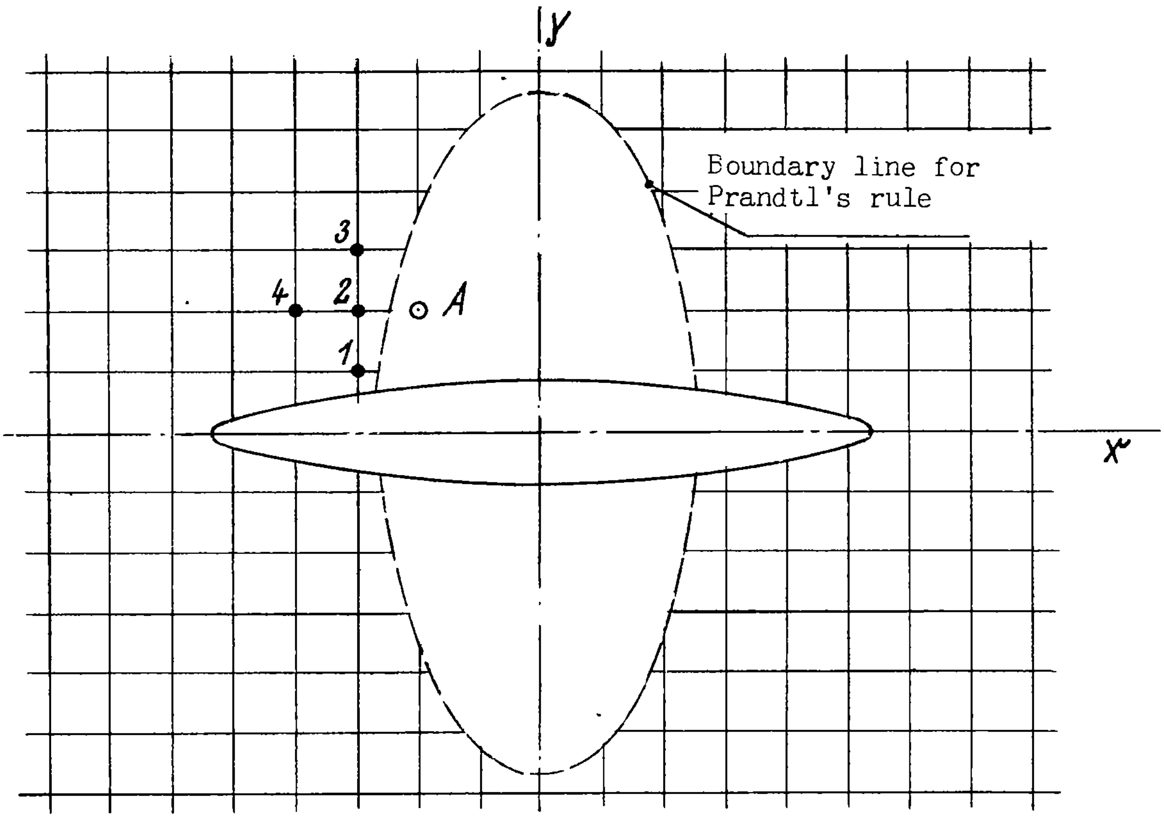

Figure 2. Partition of the flow field for the difference method.

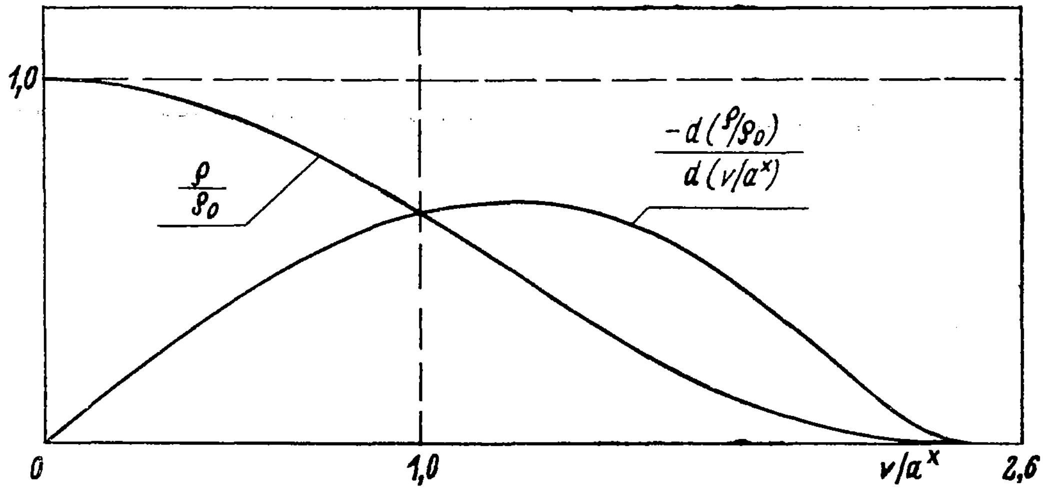

Figure 3. Relation between air density and velocity under the assumption of an adiabatic change of state.

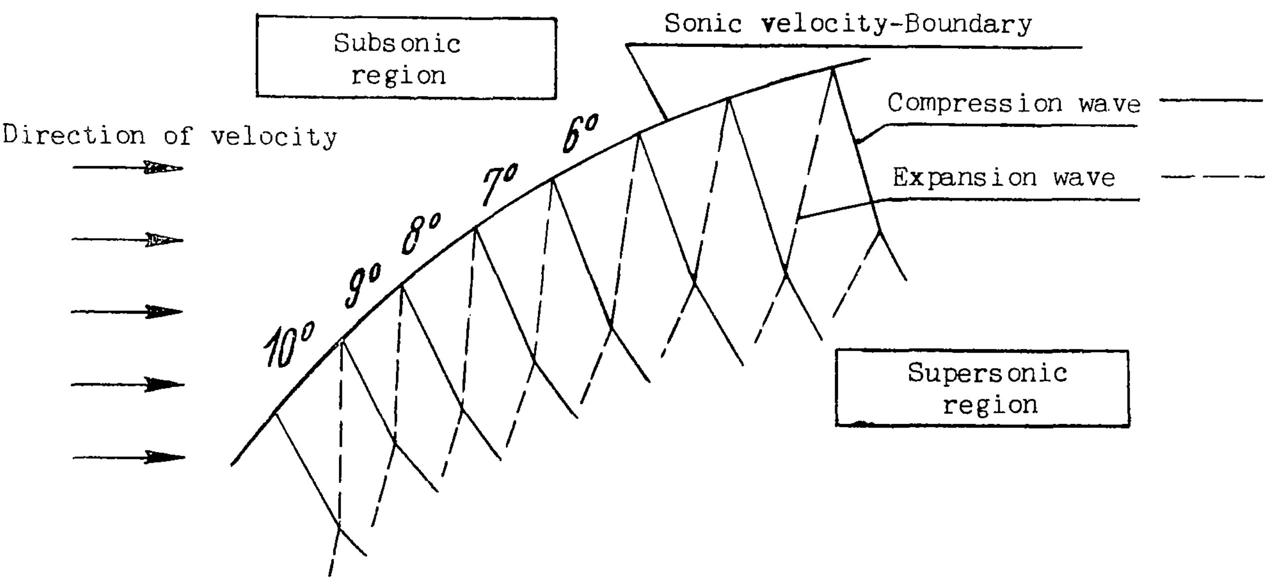

Figure 4. Sonic boundary with the field of expansion and compression waves, respectively.

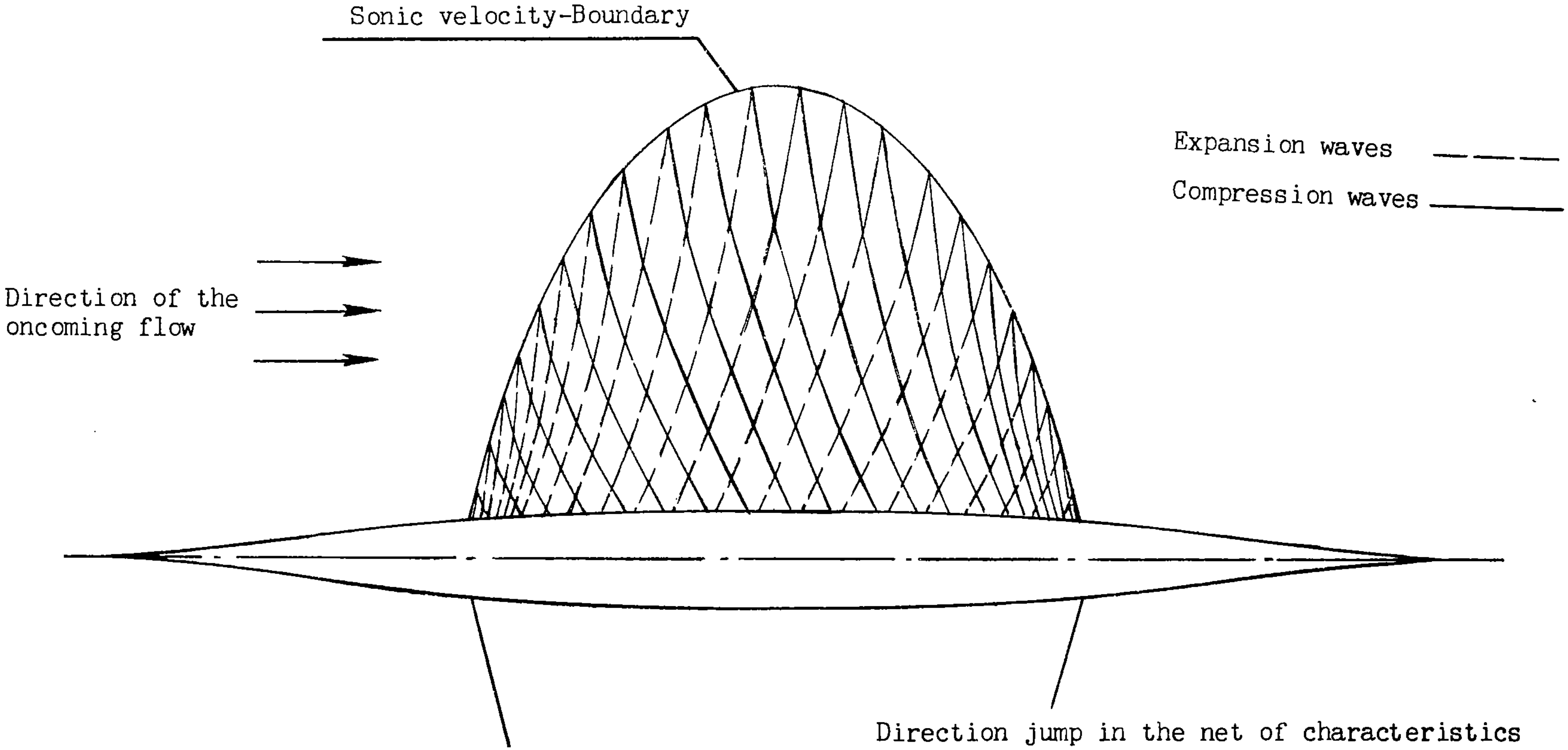

Figure 5a. Grid chacteristics in the local supersonic region for a two-dimensional body B (relation of thickness $d/l = 0.0715$ at the Mach number $M = 0.86$.

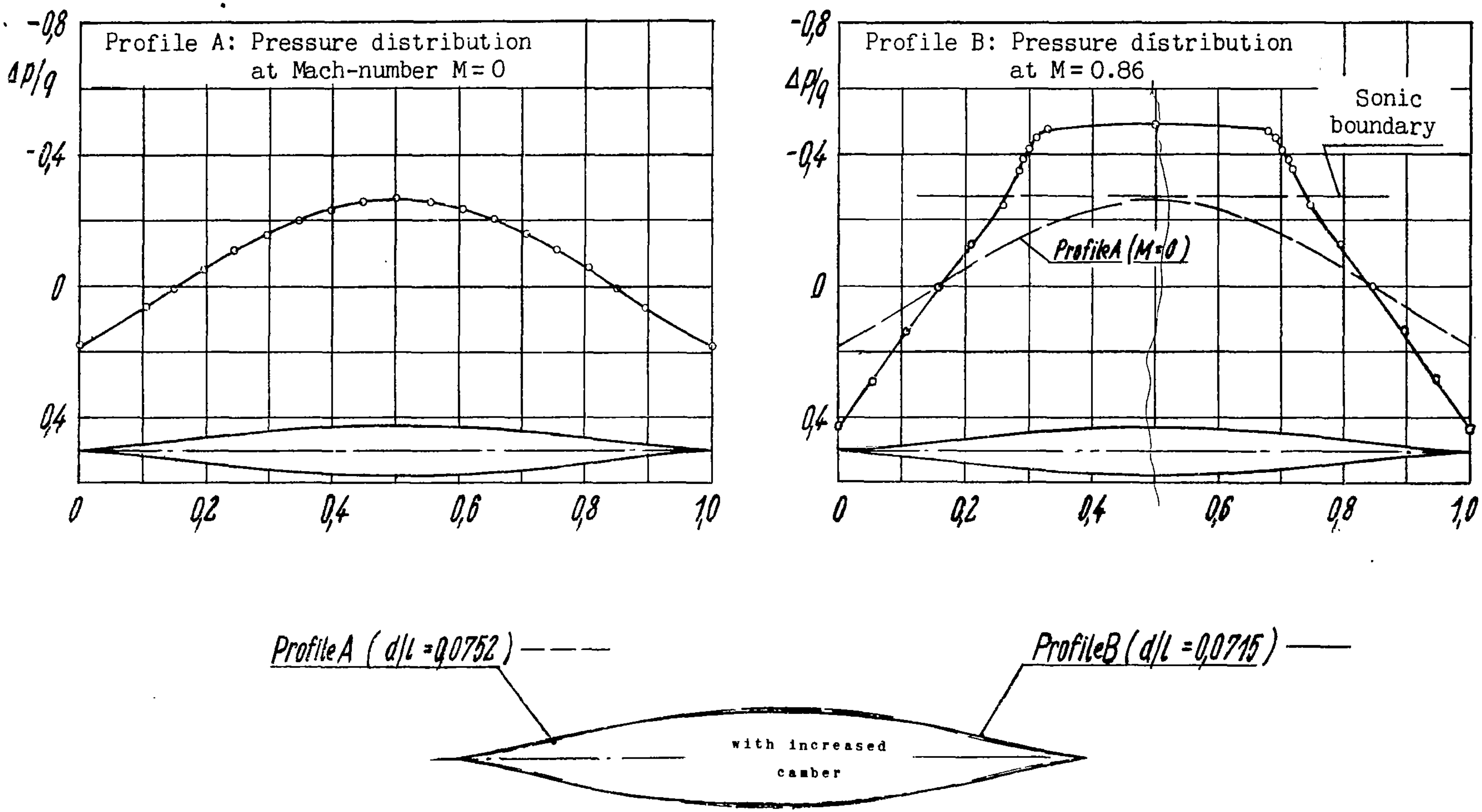

Figure 5b. Calculated pressure distribution for two two-dimensional bodies in incompressible flow and in compressible flow with local area of supersonic velocity.

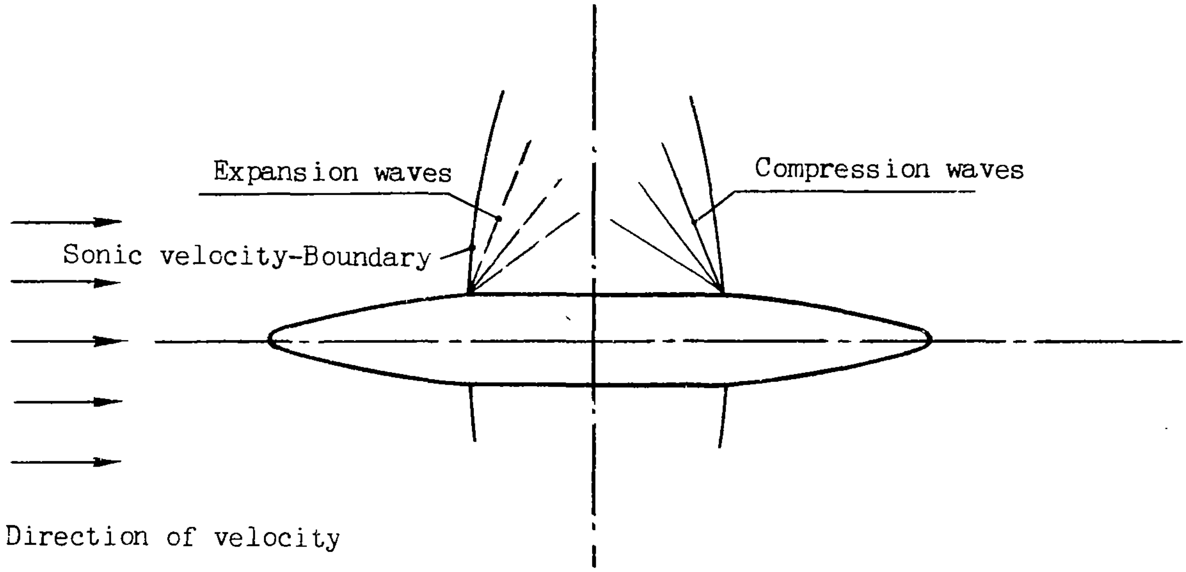

Figure 6. Mixed subsonic-supersonic flow about a body with plane surface in the supersonic region.

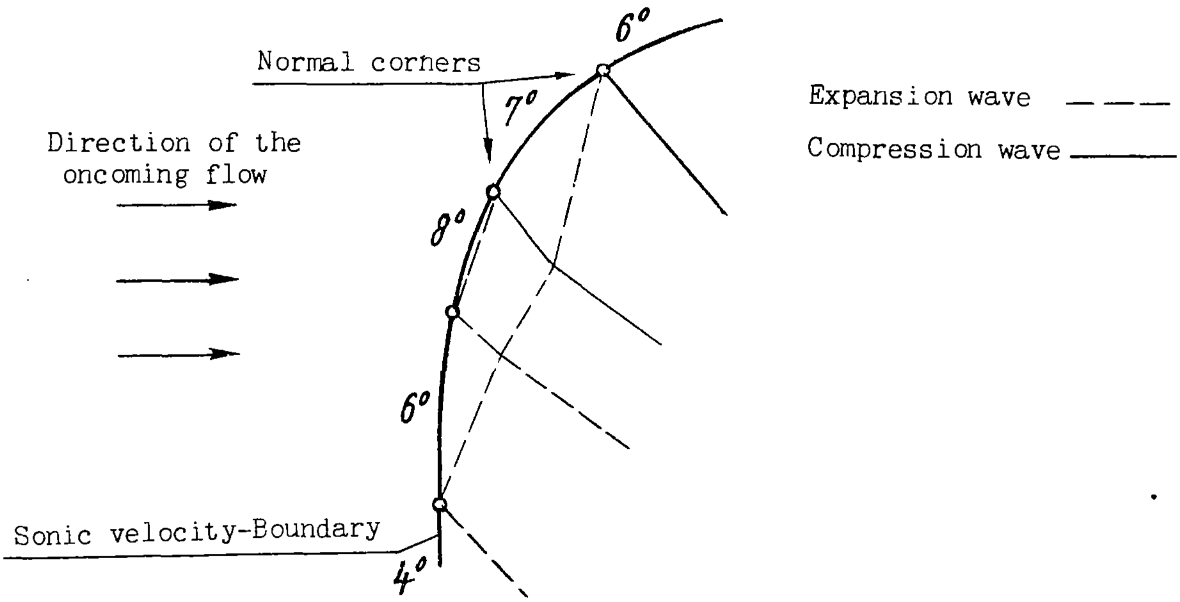

Figure 7. Schematic representation of the two different kinds of wave radiation at the sonic velocity boundary.

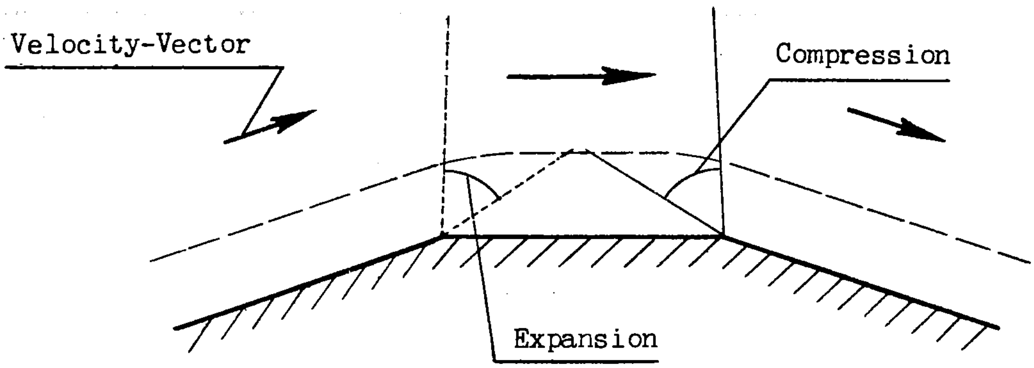

Figure 8a. Supersonic flow about a double corner.

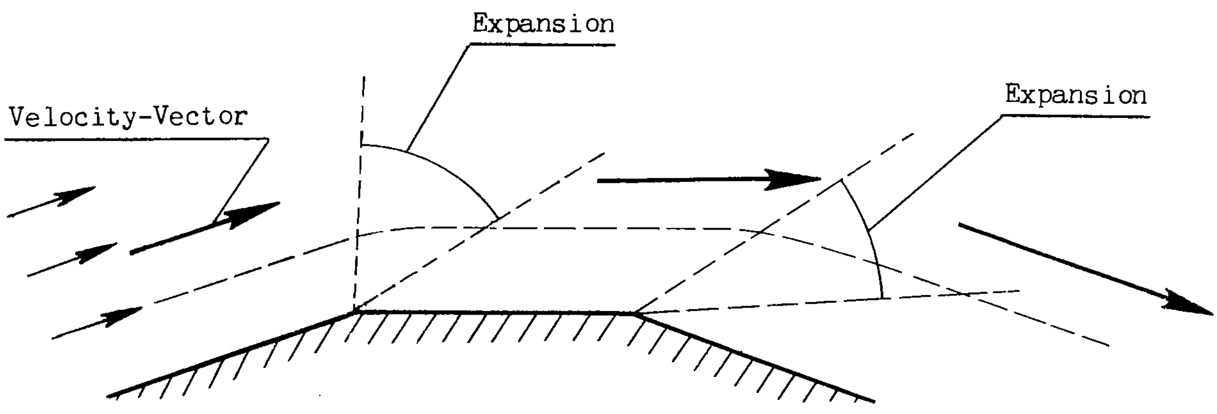

Figure 8b. Symmetrical potential flow with supersonic free-stream velocity about a physically not realizable double corner.