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On the singular behaviour of scattering amplitudes in quantum field theory

Sebastian Buchta [(a)*], **Grigorios Chachamis** [(a)†], **Petros Draggiotis** [(b)‡], **Ioannis Malamos** [(a)§], **and Germán Rodrigo** [(a)¶]

[(a)] Instituto de Física Corpuscular, Universitat de València – Consejo Superior de Investigaciones Científicas, Parc Científic, E-46980 Paterna, Valencia, Spain

[(b)] Institute of Nuclear and Particle Physics, NCSR ”Demokritos”, Agia Paraskevi, 15310, Greece

Abstract

We analyse the singular behaviour of one-loop integrals and scattering amplitudes in the framework of the loop–tree duality approach. We show that there is a partial cancellation of singularities at the loop integrand level among the different components of the corresponding dual representation that can be interpreted in terms of causality. The remaining threshold and infrared singularities are restricted to a finite region of the loop momentum space, which is of the size of the external momenta and can be mapped to the phase-space of real corrections to cancel the soft and collinear divergences.

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*E-mail: sbuchta@ific.uv.es
†E-mail: grigorios.chachamis@ific.uv.es
‡E-mail: petros.draggiotis@gmail.com
§E-mail: ioannis.malamos@ific.uv.es
¶E-mail: german.rodrigo@csic.es

1 Introduction

The recent discovery of the Higgs boson at the LHC represents a great success of the Standard Model (SM) of elementary particles. While at the same time, the absence so far of a clear signal of physics beyond the SM leaves a certain degree of dissatisfaction. These two facts, together with the high quality of data that the LHC will provide in the next run, increases the relevance of high-precision theoretical predictions for the analysis of known phenomena and for finding innovative strategies to achieve new discoveries.

The domain of perturbative calculations in quantum field theories, e.g. the SM and beyond, has shown an extraordinary progress in the recent years. Today, $2 \to 4$ processes at next-to-leading order (NLO) are state of the art [1, 2, 3, 4, 5], and even higher multiplicities are affordable [6]. Several tools for the automated calculation of NLO differential cross sections are available [7, 8], including the merging with parton showers [9]. There has been also a lot of advances in next-to-next-to-leading order (NNLO) calculations [10, 11, 12, 13, 14]. Still, besides ultraviolet singularities which are easily removed by renormalization, the cancellation of infrared singularities by the coherent sum over different real and virtual soft and collinear partonic configurations in the final state is at the core and the main source of cumbersomeness of any perturbative calculation at higher orders [15, 16, 17, 18, 19].

The loop–tree duality method [20, 21, 22, 23] establishes that generic loop quantities (loop integrals and scattering amplitudes) in any relativistic, local and unitary field theory can be written as a sum of tree-level objects obtained after making all possible cuts to the internal lines of the corresponding Feynman diagrams, with one single cut per loop and integrated over a measure that closely resembles the phase-space of the corresponding real corrections. This duality relation is realized by a modification of the customary +i0 prescription of the Feynman propagators. At one-loop, the new prescription compensates for the absence of multiple-cut contributions that appear in the Feynman Tree Theorem [24, 25]. The modified phase-space raises the intriguing possibility that virtual and real corrections can be brought together under a common integral and treated with Monte Carlo techniques at the same time. In this paper we analyse the singular behaviour of one-loop integrals and scattering amplitudes in the framework of the loop–tree duality method. On the one hand, working in the loop momentum space is an attractive approach because it allows a rather direct physical interpretation of the singularities of the loop quantities [26]. On the other hand, the possibility to relate virtual and real corrections opens an interesting line to understand explicitly the cancellation of infrared singularities.

The outline of the paper is as follows. In Section 2 we discuss the singular behaviour of scalar loop integrals in the loop momentum space. In Section 3 we prove that there is a partial cancellation of singularities at the integrand level among different contributions of the dual representation of a loop integral. In Section 4, collinear factorization is used to sketch a phase-space mapping between virtual and real corrections for the local cancellation of infrared divergences. Finally, conclusions and outlook are presented in Section 5.

2 The singular behaviour of the loop integrand

We consider a general one-loop N-leg scalar integral

$$L^{(1)}(p_1, p_2, \dots, p_N) = \int_\ell \prod_{i\in\alpha_1} G_F(q_i) \,, \qquad \int_\ell \bullet = -i \int \frac{d^d\ell}{(2\pi)^d} \;\bullet \;, \tag{1}$$

where

$$G_F(q_i) = \frac{1}{q_i^2 - m_i^2 + i0} \tag{2}$$

are Feynman propagators that depend on the loop momentum ℓ, which flows anti-clockwise, and the four-momenta of the external legs p_i, $i \in \alpha_1 = \{1, 2, \dots N\}$, which are taken as outgoing and are ordered clockwise. We use dimensional regularization with d the number of space-time dimensions. The momenta of the internal lines $q_{i,\mu} = (q_{i,0}, \mathbf{q}_i)$, where $q_{i,0}$ is the energy (time component) and $\mathbf{q}_i$ are the spacial components, are defined as $q_i = \ell + k_i$ with $k_i = p_1 + \ldots + p_i$, and $k_N = 0$ by momentum conservation. We also define $k_{ji} = q_j - q_i$.

The loop integrand becomes singular in regions of the loop momentum space in which subsets of internal lines go on-shell, although the existence of singular points of the integrand is not enough to ensure the emergence in the loop integral of divergences in the dimensional regularization parameter. Nevertheless, numerical integration over integrable singularities still requires a contour deformation [27, 28, 29, 30, 31, 32, 33, 34], namely, to promote the loop momentum to the complex plane in order to smoothen the loop matrix elements in the singular regions of the loop integrand. Hence, the relevance to identify accurately all the integrand singularities.

In Cartesian coordinates, the Feynman propagator in Eq. (2) becomes singular at hyperboloids with origin in $-k_i$, where the minimal distance between each hyperboloid and its origin is determined by the internal mass m_i. This is illustrated in Fig. 1, where for simplicity we work in $d = 2$ space-time dimensions. Figure 1 (left) shows a typical kinematical situation where two momenta, k_1 and k_2, are separated by a time-like distance, $k_{21}^2 > 0$, and a third momentum, k_3, is space-like separated with respect to the other two, $k_{31}^2 < 0$ and $k_{32}^2 < 0$. The on-shell forward hyperboloids ($q_{i,0} > 0$) are represented in Fig. 1 by solid lines, and the backward hyperboloids ($q_{i,0} < 0$) by dashed lines. For the discussion that will follow it is important to stress that Feynman propagators become positive inside the respective hyperboloid and negative outside. Two or more Feynman propagators become simultaneously singular where their respective hyperboloids intersect. In most cases, these singularities, due to normal or anomalous thresholds [35, 36] of intermediate states, are integrable. However, if two massless propagators are separated by a light-like distance, $k_{ji}^2 = 0$, then the overlap of the respective light-cones is tangential, as illustrated in Fig. 1 (right), and leads to non-integrable collinear singularities. In addition, massless propagators can generate soft singularities at $q_i = 0$.

The dual representation of the scalar one-loop integral in Eq. (1) is the sum of N dual integrals [20, 21]:

$$L^{(1)}(p_1, p_2, \dots, p_N) \;=\; -\sum_{i\in\alpha_1} \int_\ell \tilde{\delta}(q_i) \prod_{\substack{j\in\alpha_1\\ j\neq i}} G_D(q_i; q_j) \,, \tag{3}$$

where

$$G_D(q_i; q_j) = \frac{1}{q_j^2 - m_j^2 - i0\,\eta\, k_{ji}} \tag{4}$$

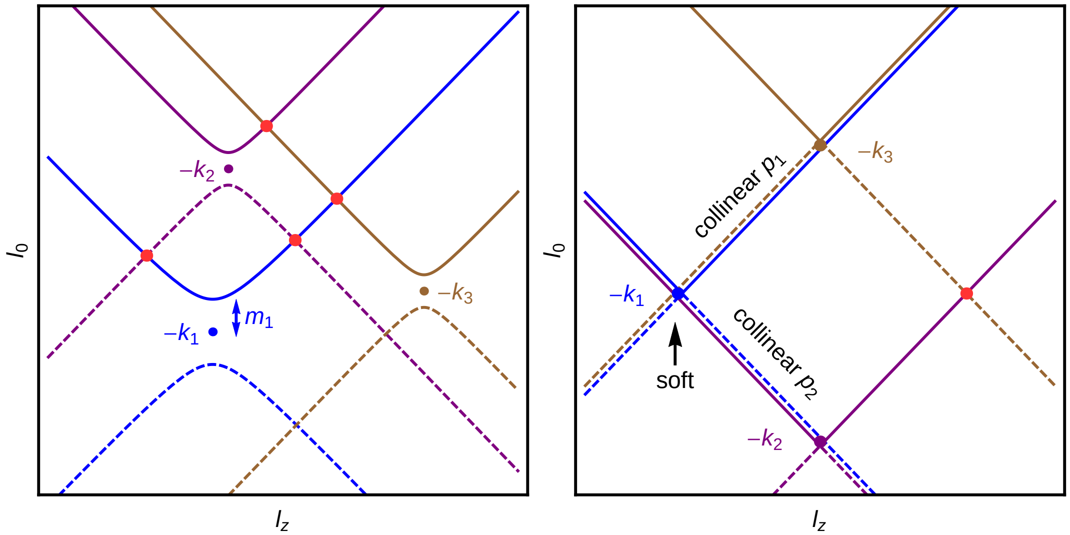

Figure 1: On-shell hyperboloids for three arbitrary propagators in Cartesian coordinates in the (ℓ_0,ℓ_z) space (left). Kinematical configuration with infrared singularities (right). In the latter case, the on-shell hyperboloids degenerate to light-cones.

are the so-called dual propagators, as defined in Ref. [20], with η a *future-like* vector, $\eta^2 \geq 0$, with positive definite energy $\eta_0 > 0$. The delta function $\tilde{\delta}(q_i) \equiv 2\pi\, i\, \theta(q_{i,0})\, \delta(q_i^2 - m_i^2)$ sets the internal lines on-shell by selecting the pole of the propagators with positive energy $q_{i,0}$ and negative imaginary part. In the following we take $\eta_\mu = (1, \mathbf{0})$, and thus $-i0\, \eta\, k_{ji} = -i0\, k_{ji,0}$. This is equivalent to performing the loop integration along the on-shell forward hyperboloids. Let us mention that in the light-cone coordinates (ℓ_+, ℓ_-, $\mathbf{l}_\perp$), where $\ell_\pm = (\ell_0 \pm \ell_{d-1})/\sqrt{2}$, Feynman propagators vanish at hyperboloids in the plane (ℓ_+,ℓ_-) which are similar to those depicted in Fig. 1 but rotated by 45 degrees. Consequently, by selecting the forward hyperboloids the integration limits of either ℓ_+ or ℓ_- are restricted and the restrictions are different for each dual integral. For this reason, although Eq. (3) is valid for any system of coordinates, we will stick for the rest of the paper to Cartesian coordinates where all the dual integrals share the same integration limits for the loop three-momentum.

A crucial point of our discussion is the observation that dual propagators can be rewritten as

$$\tilde{\delta}(q_i)\; G_D(q_i; q_j) = i\, 2\pi\, \frac{\delta(q_{i,0} - q_{i,0}^{(+)})}{2 q_{i,0}^{(+)}}\, \frac{1}{(q_{i,0}^{(+)} + k_{ji,0})^2 - (q_{j,0}^{(+)})^2}\;, \tag{5}$$

where

$$q_{i,0}^{(+)} = \sqrt{\mathbf{q}_i^2 + m_i^2 - i0} \tag{6}$$

is the loop energy measured along the on-shell hyperboloid with origin at $-k_i$. By definition we have $\mathrm{Re}(q_{i,0}^{(+)}) \geq 0$. The factor $1/q_{i,0}^{(+)}$ can become singular for $m_i = 0$, but the integral $\int_\ell \delta(q_{i,0} - q_{i,0}^{(+)})/q_{i,0}^{(+)}$ is still convergent by two powers in the infrared. Soft singularities require two dual propagators, where each of the two dual propagators contributes with one power in the infrared. From Eq. (5) it is obvious that dual propagators become singular, $G_D^{-1}(q_i; q_j) = 0$, if one of the following conditions is fulfilled:

$$q_{i,0}^{(+)} + q_{j,0}^{(+)} + k_{ji,0} = 0\;, \tag{7}$$

$$q_{i,0}^{(+)} - q_{j,0}^{(+)} + k_{ji,0} = 0\;. \tag{8}$$

The first condition, Eq. (7), is satisfied if the forward hyperboloid of $-k_i$ intersects with the backward hyperboloid of $-k_j$. The second condition, Eq. (8), is true when the two forward hyperboloids intersect each other.

In the massless case, Eq. (7) and Eq. (8) are the equations of conic sections in the loop three-momentum space; $q_{i,0}^{(+)}$ and $q_{j,0}^{(+)}$ are the distance to the *foci* located at $-\mathbf{k}_i$ and $-\mathbf{k}_j$, respectively, and the distance between the foci is $\sqrt{\mathbf{k}_{ji}^2}$. If internal masses are non-vanishing, Eq. (6) can be reinterpreted as the distance associated to a four-dimensional space with one "massive" dimension and the foci now located at $(-\mathbf{k}_i, -m_i)$ and $(-\mathbf{k}_j, -m_j)$, respectively. Then, the singularity arises at the intersection of the conic sections given by Eq. (7) or Eq. (8) in this generalized space with the zero mass plane. This picture is useful to identify the singular regions of the loop integrand in the loop three-momentum space.

The solution to Eq. (7) is an ellipsoid and clearly requires $k_{ji,0} < 0$. Moreover, since it is the result of the intersection of a forward with a backward hyperboloid the distance between the two propagators has to be future-like, $k_{ji}^2 \geq 0$. Actually, internal masses restrict this condition. Bearing in mind the image of the conic sections in the generalized massive space so we can deduce intuitively that Eq. (7) has solution for

$$k_{ji}^2 - (m_j + m_i)^2 \geq 0 \;, \qquad k_{ji,0} < 0 \;, \qquad \text{forward with backward hyperboloids} \;. \tag{9}$$

The second equation, Eq. (8), leads to a hyperboloid in the generalized space, and there are solutions for $k_{ji,0}$ either positive or negative, namely when either of the two momenta are set on-shell. However, by interpreting the result in the generalized space it is clear that the intersection with the zero mass plane does not always exist, and if it exists, it can be either an ellipsoid or a hyperboloid in the loop three-momentum space. Here, the distance between the momenta of the propagators has to be space-like, although also time-like configurations can fulfil Eq. (8) as far as the time-like distance is small or close to light-like. The following condition is necessary:

$$k_{ji}^2 - (m_j - m_i)^2 \leq 0 \;, \qquad \text{two forward hyperboloids} \;. \tag{10}$$

In any other configuration, the singularity appears for loop three-momenta with imaginary components.

3 Cancellation of singularities among dual integrands

In this section we prove one of the main properties of the loop–tree duality method, namely the partial cancellation of singularities among different dual integrands. This represents a significant advantage with respect to the integration of regular loop integrals in the d-dimensional space, where one single integrand cannot obviously lead to such cancellation.

Let's consider first two Feynman propagators separated by a space-like distance, $k_{ji}^2 < 0$ (or more generally fulfilling Eq. (10)). In the corresponding dual representation one of these propagators is set on-shell and the other becomes dual, and the integration occurs along the respective on-shell forward hyperboloids. See again Fig. 1 (left) for a graphical representation of this set-up. There, the two forward hyperboloids of $-k_1$ and $-k_3$ intersect at a single point. Integrating over ℓ_z along the forward hyperboloid of $-k_1$ we find that the dual propagator $G_D(q_1; q_3)$, which is negative below the intersection point where the integrand becomes singular, changes sign above this point as we move from outside to inside the on-shell hyperboloid of $-k_3$. The opposite occurs if we set q_3 on-shell; $G_D(q_3; q_1)$ is positive below the intersection point, and negative above. The change of sign leads to the cancellation of the common singularity. Notice that also the dual $i0$ prescription changes sign. In order to prove analytically this cancellation, we define $x = q_{i,0}^{(+)} - q_{j,0}^{(+)} + k_{ji,0}$. In the limit $x \to 0$:

$$\lim_{x\to 0} \left(\tilde{\delta}\,(q_i)\; G_D(q_i;q_j) + (i \leftrightarrow j) \right) = \left(\frac{1}{x} - \frac{1}{x} \right) \frac{1}{2q_{j,0}^{(+)}}\, \tilde{\delta}\,(q_i) + \mathcal{O}(x^0)\;, \tag{11}$$

and thus the leading singular behaviour cancels among the two dual contributions. The cancellation of these singularities is not altered by the presence of other non-vanishing dual propagators (neither by numerators) because

$$\lim_{x\to 0} G_D(q_j;q_k) = \lim_{x\to 0} \frac{1}{(q_{j,0}^{(+)} + k_{ki,0} - k_{ji,0})^2 - (q_{k,0}^{(+)})^2} = \lim_{x\to 0} G_D(q_i;q_k)\;, \tag{12}$$

where we have used the identity $k_{kj,0} = k_{ki,0} - k_{ji,0}$. If instead, the separation is time-like (in the sense of Eq. (9)), we define $x = q_{i,0}^{(+)} + q_{j,0}^{(+)} + k_{ji,0}$, and find

$$\lim_{x\to 0} \left(\tilde{\delta}\,(q_i)\; G_D(q_i;q_j) + (i \leftrightarrow j) \right) = -\theta(-k_{ji,0})\, \frac{1}{x}\, \frac{1}{2q_{j,0}^{(+)}}\, \tilde{\delta}\,(q_i) + (i \leftrightarrow j) + \mathcal{O}(x^0)\;. \tag{13}$$

In this case the singularity of the integrand remains because of the Heaviside step function.

We should consider also the case in which more than two propagators become simultaneously singular. To analyse the intersection of three forward hyperboloids, we define

$$\lambda\, x = q_{i,0}^{(+)} - q_{j,0}^{(+)} + k_{ji,0}\;, \qquad \lambda\, y = q_{i,0}^{(+)} - q_{k,0}^{(+)} + k_{ki,0}\;. \tag{14}$$

As before, we use the identity $k_{kj,0} = k_{ki,0} - k_{ji,0}$, and thus $q_{j,0}^{(+)} - q_{k,0}^{(+)} + k_{kj,0} = \lambda\,(y - x)$. In the limit in which the three propagators become simultaneously singular:

$$\begin{aligned} &\lim_{\lambda\to 0} \left(\tilde{\delta}\,(q_i)\; G_D(q_i;q_j)\, G_D(q_i;q_k) + \text{perm.} \right) = \\ &\frac{1}{\lambda^2} \left(\frac{1}{x\,y} + \frac{1}{x\,(x-y)} + \frac{1}{y\,(y-x)} \right) \frac{1}{2q_{j,0}^{(+)}}\, \frac{1}{2q_{k,0}^{(+)}}\, \tilde{\delta}\,(q_i) + \mathcal{O}(\lambda^{-1})\;, \end{aligned} \tag{15}$$

and again the leading singular behaviour cancels in the sum. Although not shown for simplicity in Eq. (15), also the $\mathcal{O}(\lambda^{-1})$ terms cancel in the sum, thus rendering the integrand finite in the limit $\lambda \to 0$. For three propagators there are also more possibilities: two forward hyperboloids might intersect simultaneously with a backward hyperboloid, or two backward hyperboloids might intersect with a forward hyperboloid. In the former case, we define $\lambda\, x = q_{i,0}^{(+)} + q_{k,0}^{(+)} + k_{ki,0}$, and $\lambda\, y = q_{j,0}^{(+)} + q_{k,0}^{(+)} + k_{kj,0}$, with $k_{ki,0} < 0$ and $k_{kj,0} < 0$, and hence $q_{i,0}^{(+)} - q_{j,0}^{(+)} + k_{ji,0} = \lambda(x - y)$. In the $\lambda \to 0$ limit

$$\begin{aligned} &\lim_{\lambda\to 0} \left(\tilde{\delta}\,(q_i)\; G_D(q_i;q_j)\, G_D(q_i;q_k) + \text{perm.} \right) = \\ &\theta(-k_{ki,0})\, \theta(-k_{kj,0})\, \frac{1}{\lambda^2} \left(\frac{1}{x\,(y-x)} + \frac{1}{y\,(x-y)} \right) \frac{1}{2q_{j,0}^{(+)}}\, \frac{1}{2q_{k,0}^{(+)}}\, \tilde{\delta}\,(q_i) + \mathcal{O}(\lambda^{-1})\;. \end{aligned} \tag{16}$$

Notice that the singularity in $1/(x-y)$ cancels in Eq. (16) (also at $\mathcal{O}(\lambda^{-1})$). In the latter case, we set as before $\lambda\, x = q_{i,0}^{(+)} + q_{k,0}^{(+)} + k_{ki,0}$, and define $\lambda\, z = q_{i,0}^{(+)} + q_{j,0}^{(+)} + k_{ji,0}$, then

$$\lim_{\lambda\to 0} \left(\tilde{\delta}\,(q_i)\; G_D(q_i;q_j)\, G_D(q_i;q_k) + \text{perm.} \right) = -\theta(-k_{ki,0}) \\ \times\theta(-k_{ji,0})\, \frac{1}{\lambda^2} \, \left(\frac{1}{x\, z}\right) \, \frac{1}{2q_{j,0}^{(+)}} \, \frac{1}{2q_{k,0}^{(+)}} \, \tilde{\delta}\,(q_i) + \mathcal{O}(\lambda^{-1}) \; . \tag{17}$$

Similarly, it is straightforward to prove that four forward hyperboloids do not lead to any common singularity and more generally that the remaining multiple singularities are only driven by propagators that are time-like connected and less energetic than the propagator which is set on-shell.

Thus, we conclude that singularities of space-like separated propagators *, occurring in the intersection of on-shell forward hyperboloids, are absent in the dual representation of the loop integrand. The cancellation of these singularities at the integrand level already represents a big advantage of the loop–tree duality with respect to the direct integration in the four dimensional loop space; it makes unnecessary the use of contour deformation to deal numerically with the integrable singularities of these configurations. This conclusion is also valid for loop scattering amplitudes. Moreover, this property can be extended in a straightforward manner to prove the partial cancellation of infrared singularities.

Collinear singularities occur when two massless propagators are separated by a light-like distance, $k_{ji}^2 = 0$. In that case, the corresponding light-cones overlap tangentially along an infinite interval. Assuming $k_{i,0} > k_{j,0}$, however, the collinear singularity for $\ell_0 > -k_{j,0}$ appears at the intersection of the two forward light-cones, with the forward light-cone of $-k_j$ located inside the forward light-cone of $-k_i$, or equivalently, with the forward light-cone of $-k_i$ located outside the forward light-cone of $-k_j$, Thus, the singular behaviour of the two dual components cancel against each other, following the same qualitative arguments given before. For $-k_{i,0} < \ell_0 < -k_{j,0}$, instead, it is the forward light-cone of $-k_i$ that intersects tangentially with the backward light-cone of $-k_j$ according to Eq. (7). The collinear divergences survive in this energy strip, which indeed also limits the range of the loop three-momentum where infrared divergences can arise. If there are several reference momenta separated by light-like distances the infrared strip is limited by the minimal and maximal energies of the external momenta. The soft singularity of the integrand at $q_{i,0}^{(+)} = 0$ leads to soft divergences only if two other propagators, each one contributing with one power in the infrared, are light-like separated from $-k_i$. In Fig. 1 (right) this condition is fulfilled only at $q_{1,0}^{(+)} = 0$, but not at $q_{2,0}^{(+)} = 0$ neither at $q_{3,0}^{(+)} = 0$.

In summary, both threshold and infrared singularities are constrained in the dual representation of the loop integrand to a finite region where the loop three-momentum is of the order of the external momenta. Singularities outside this region, occurring in the intersection of on-shell forward hyperboloids or light-cones, cancel in the sum of all the dual contributions.

4 Cancellation of infrared singularities with real corrections

Having constrained the loop singularities to a finite region of the loop momentum space, we discuss now how to map this region into the finite-size phase-space of the real corrections for the cancellation of the

* Including light-like and time-like configurations such that Eq. (10) is fulfilled.

remaining infrared singularities. The use of collinear factorization and splitting matrices, encoding the collinear singular behaviour of scattering amplitudes as introduced in Ref. [37, 38], is suitable for this discussion.

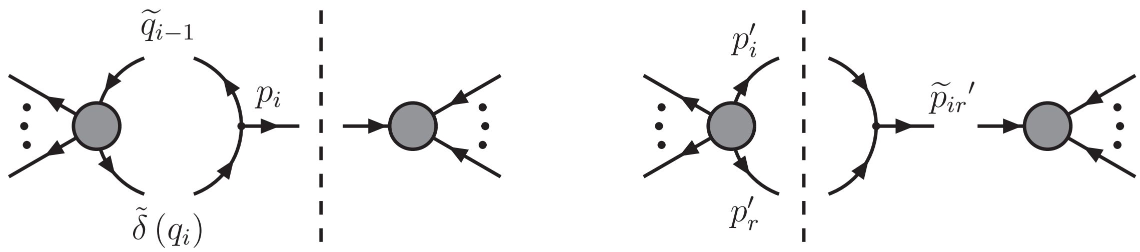

Figure 2: *Factorization of the dual one-loop and tree-level squared amplitudes in the collinear limit. The dashed line represents the momentum conservation cut.*

We consider the interference of the one-loop scattering amplitude $\mathcal{M}_N^{(1)}$ with the corresponding N-parton tree-level scattering amplitude $\mathcal{M}_N^{(0)}$, which is integrated with the appropriate phase-space factor

$$\int d\Phi_N(p_1; p_2, \ldots, p_N) = \left(\prod_{i=2}^{N} \int_{p_i} \tilde{\delta}\,(p_i) \right) \, (2\pi)^d \, \delta^{(d)}(\sum_{i=1}^{N} p_i) \, , \tag{18}$$

where we assume that only the external momentum p_1 is incoming ($p_{1,0} < 0$). Then, we select the corresponding dual contribution with the internal massless line q_i on-shell

$$\begin{aligned} I_i^{(1)} &= 2\mathrm{Re} \int d\Phi_N(p_1; p_2, \ldots, p_N) \int_\ell \tilde{\delta}\,(q_i) \; \theta(p_{i,0} - q_{i,0}^{(+)}) \\ &\times \; \langle \mathcal{M}_N^{(0)}(p_1, \ldots, p_N) | \, \mathcal{M}_{N+2}^{(0)}(\ldots, p_i, -q_i, q_i, p_{i+1}, \ldots) \rangle \, , \end{aligned} \tag{19}$$

where the loop energy in Eq. (19) is restricted by the energy of the adjacent external massless particle $p_{i,0}$ to select the infrared sector, according to the discussion of the previous sections. We also consider the $N+1$-parton tree-level scattering amplitude

$$|\mathcal{M}_{N+1}^{(0),\, ir}(p_1, p'_2, \ldots)\rangle = |\mathcal{M}_{N+1}^{(0),\, ir}(\ldots, p'_{ir} \to p'_i + p'_r, \ldots)\rangle \, , \tag{20}$$

where an extra particle is radiated from parton i, with $p'_{ir} = p'_i + p'_r$, and the complementary scattering amplitude $\mathcal{M}_{N+1}^{(0)}$ that contains all the tree-level contributions with the exception of those already included in $\mathcal{M}_{N+1}^{(0),\, ir}$. The corresponding interference, integrated over the phase-space of the final-state particles, is

$$I_{ir}^{(0)} = 2\mathrm{Re} \int d\Phi_{N+1}(p_1; p'_2, \ldots) \, \langle \mathcal{M}_{N+1}^{(0),\, ir}(p_1, p'_2, \ldots) | \, \mathcal{M}_{N+1}^{(0)}(p_1, p'_2, \ldots) \rangle \, . \tag{21}$$

For the simplicity of the presentation, we do not consider explicitly in this paper the square of $\mathcal{M}_{N+1}^{(0),\, ir}$, which is related with a self-energy insertion in an external leg and whose infrared divergences are removed by wave-function remormalization [20]. The final-state external momenta of the loop and tree amplitudes in Eq. (19) and Eq. (21), although labelled with the same indices, are constrained by different phase-space momentum conservation delta functions. A mapping between the primed (real amplitudes) and unprimed (virtual amplitudes) momenta is necessary to show the cancellation of collinear divergences.

In the limit where $\mathbf{p}_i$ and $\mathbf{q}_i$ become collinear the dual one-loop matrix element $\mathcal{M}^{(0)}_{N+2}$ in Eq. (19) factorizes as

$$|\mathcal{M}^{(0)}_{N+2}(\ldots,p_i,-q_i,q_i,\ldots)\rangle = \boldsymbol{Sp}^{(0)}(p_i,-q_i;-\widetilde{q}_{i-1})\,|\overline{\mathcal{M}}^{(0)}_{N+1}(\ldots,-\widetilde{q}_{i-1},q_i,\ldots)\rangle + \mathcal{O}(\sqrt{q^2_{i-1}})\ , \quad (22)$$

where the reduced matrix element $\overline{\mathcal{M}}^{(0)}_{N+1}$ is obtained by replacing the two collinear partons of $\mathcal{M}^{(0)}_{N+2}$ by a single parent parton with light-like momentum

$$\widetilde{q}^{\mu}_{i-1} = q^{\mu}_{i-1} - \frac{q^2_{i-1}\, n^{\mu}}{2\, nq_{i-1}}\ , \quad (23)$$

with n^μ a light-like vector, $n^2=0$. Similarly, in the limit where $\mathbf{p}'_i$ and $\mathbf{p}'_r$ become collinear the tree-level matrix element $\mathcal{M}^{(0),\,ir}_{N+1}$ factorizes as

$$\langle\mathcal{M}^{(0),\,ir}_{N+1}(p_1,p'_2,\ldots,p'_{N+1})| = \langle\overline{\mathcal{M}}^{(0)}_{N}(\ldots,p'_{i-1},\widetilde{p}'_{ir},p'_{i+1},\ldots)|\,\boldsymbol{Sp}^{(0)\dagger}(p'_i,p'_r;\widetilde{p}'_{ir}) + \mathcal{O}(\sqrt{s'_{ir}})\ , \quad (24)$$

where $s'_{ir}=p'^2_{ir}$, and

$$\widetilde{p}'^{\mu}_{ir} = p'^{\mu}_{ir} - \frac{s'_{ir}\, n^{\mu}}{2\, np'_{ir}} \quad (25)$$

is the light-like momentum of the parent parton. A graphical representation of the collinear limit of both virtual and real corrections is illustrated in Fig. 2. This graph suggests that in the collinear limit the mapping between the four-momenta of the virtual and real matrix elements should be such that $p_i=\widetilde{p}'_{ir}$, $p_j=p'_j(j\neq i)$, $-\widetilde{q}_{i-1}=p'_i$ and $q_i=p'_r$. Notice that p'_r is restricted by momentum conservation but q_i is not. However, the relevant infrared region is bound by $q^{(+)}_{i,0}\leq p_{i,0}$ in Eq. (19). This restriction allows to map q_i to p'_r. The mapping, nevertheless, is not as obvious as can be induced from Fig. 2 as the propagators that become singular in the collinear limit in the virtual and real matrix elements are different. Reconsidering p'_i as the parent parton momentum of the collinear splitting, we find the following relation between splitting matrices entering the real matrix elements

$$\boldsymbol{Sp}^{(0)\dagger}(p'_i,p'_r;\widetilde{p}'_{ir}) = \frac{(\widetilde{p}'_{ir}-p'_r)^2}{s'_{ir}}\,\boldsymbol{Sp}^{(0)}(\widetilde{p}'_{ir},-p'_r;p'_i)\ , \quad (26)$$

where $(\widetilde{p}'_{ir}-p'_r)^2/s'_{ir} = -np'_i/np'_{ir}$. We show now that the factor $-np'_i/np'_{ir}$ is compensated by the phase-space. By introducing the following identity in the phase-space of the real corrections

$$1 = \int d^dp'_{ir}\,\delta^{(d)}\left(p'_{ir}-p'_i-p'_r\right)\ , \quad (27)$$

and performing the integration over the three-momentum $\mathbf{p}'_i$ and the energy component of p'_{ir}, the real phase-space becomes

$$\int d\Phi_{N+1}(p_1;p'_2,\ldots) = \int d\Phi_N(p_1;\ldots,p'_{ir},\ldots)\int_{p'_r}\tilde{\delta}\left(p'_r\right)\,\frac{E'_{ir}}{E'_i}\ , \quad (28)$$

where the factor $(np'_i/np'_{ir})(E'_{ir}/E'_i)$ equals unity in the collinear limit. Inserting Eq. (22) in Eq. (19), and Eq. (24), Eq. (26) and Eq. (28) in Eq. (21) the loop and tree contributions show to have a very similar structure with opposite sign and match each other at the integrand level in the collinear limit. Correspondingly, soft singularities at $p'_r\to 0$ can be treated consistently as the endpoint limit of the collinear mapping.

5 Conclusions and outlook

The loop–tree duality method exhibits attractive theoretical aspects and nice properties which are manifested by a direct physical interpretation of the singular behaviour of the loop integrand. Integrand singularities occurring in the intersection of on-shell forward hyperboloids or light-cones cancel among dual integrals. The remaining singularities, excluding UV divergences, are found in the intersection of forward with backward on-shell hyperboloids or light-cones and are produced by dual propagators that are light-like or time-like separated and less energetic than the internal propagator that is set on-shell. Therefore, these singularities can be interpreted in terms of causality and are restricted to a finite region of the loop three-momentum space, which is of the size of the external momenta. As a result, a local mapping at the integrand level is possible between one-loop and tree-level matrix elements to cancel soft and collinear divergences. One can anticipate that a similar analysis at higher orders of the loop–tree duality relation is expected to provide equally interesting results. We leave this analysis for a future publication.

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References

[1] C. F. Berger *et al.*, Phys. Rev. Lett. **102** (2009) 222001 [arXiv:0902.2760 [hep-ph]].

[2] K. Melnikov and G. Zanderighi, Phys. Rev. D **81** (2010) 074025 [arXiv:0910.3671 [hep-ph]].

[3] G. Bevilacqua, M. Czakon, C. G. Papadopoulos and M. Worek, Phys. Rev. Lett. **104** (2010) 162002 [arXiv:1002.4009 [hep-ph]].

[4] A. Denner, S. Dittmaier, S. Kallweit and S. Pozzorini, Phys. Rev. Lett. **106** (2011) 052001 [arXiv:1012.3975 [hep-ph]].

[5] F. Campanario, M. Kerner, L. D. Ninh and D. Zeppenfeld, Phys. Rev. Lett. **111** (2013) 052003 [arXiv:1305.1623 [hep-ph]].

[6] Z. Bern, L. J. Dixon, F. Febres Cordero, S. Hoeche, H. Ita, D. A. Kosower, D. Maitre and K. J. Ozeren, Phys. Rev. D **88** (2013) 014025 [arXiv:1304.1253 [hep-ph]].

[7] G. Bevilacqua, M. Czakon, M. V. Garzelli, A. van Hameren, A. Kardos, C. G. Papadopoulos, R. Pittau and M. Worek, Comput. Phys. Commun. **184** (2013) 986 [arXiv:1110.1499 [hep-ph]].

[8] G. Cullen, H. van Deurzen, N. Greiner, G. Heinrich, G. Luisoni, P. Mastrolia, E. Mirabella and G. Ossola *et al.*, arXiv:1404.7096 [hep-ph].

[9] J. Alwall, R. Frederix, S. Frixione, V. Hirschi, F. Maltoni, O. Mattelaer, H.-S. Shao and T. Stelzer *et al.*, JHEP **1407** (2014) 079 [arXiv:1405.0301 [hep-ph]].

[10] P. Bolzoni, F. Maltoni, S. O. Moch and M. Zaro, Phys. Rev. Lett. **105** (2010) 011801 [arXiv:1003.4451 [hep-ph]].

[11] S. Catani, G. Ferrera and M. Grazzini, JHEP **1005** (2010) 006 [arXiv:1002.3115 [hep-ph]].

[12] S. Catani, L. Cieri, G. Ferrera, D. de Florian and M. Grazzini, Phys. Rev. Lett. **103** (2009) 082001 [arXiv:0903.2120 [hep-ph]].

[13] C. Anastasiou, G. Dissertori and F. Stockli, JHEP **0709** (2007) 018 [arXiv:0707.2373 [hep-ph]].

[14] M. Czakon, P. Fiedler and A. Mitov, Phys. Rev. Lett. **110**, 252004 (2013) [arXiv:1303.6254 [hep-ph]].

[15] S. Catani and M. H. Seymour, Nucl. Phys. B **485** (1997) 291 [Erratum-ibid. B **510** (1998) 503] [hep-ph/9605323].

[16] S. Catani and M. H. Seymour, Phys. Lett. B **378** (1996) 287 [hep-ph/9602277].

[17] S. Frixione, Z. Kunszt and A. Signer, Nucl. Phys. B **467** (1996) 399 [hep-ph/9512328].

[18] A. Gehrmann-De Ridder, T. Gehrmann and E. W. N. Glover, JHEP **0509** (2005) 056 [hep-ph/0505111].

[19] S. Catani and M. Grazzini, Phys. Rev. Lett. **98** (2007) 222002 [hep-ph/0703012].

[20] S. Catani, T. Gleisberg, F. Krauss, G. Rodrigo and J. C. Winter, JHEP **0809** (2008) 065 [arXiv:0804.3170 [hep-ph]].

[21] I. Bierenbaum, S. Catani, P. Draggiotis and G. Rodrigo, JHEP **1010** (2010) 073 [arXiv:1007.0194 [hep-ph]].

[22] I. Bierenbaum, S. Buchta, P. Draggiotis, I. Malamos and G. Rodrigo, JHEP **1303** (2013) 025 [arXiv:1211.5048 [hep-ph]].

[23] I. Bierenbaum, P. Draggiotis, S. Buchta, G. Chachamis, I. Malamos and G. Rodrigo, Acta Phys. Polon. B **44** (2013) 2207.

[24] R. P. Feynman, Acta Phys. Polon. **24** (1963) 697.

[25] R. P. Feynman, *Closed Loop And Tree Diagrams*, in *Magic Without Magic*, ed. J. R. Klauder, (Freeman, San Francisco, 1972), p. 355, in *Selected papers of Richard Feynman*, ed. L. M. Brown (World Scientific, Singapore, 2000) p. 867.

[26] G. F. Sterman, Phys. Rev. D **17** (1978) 2773.

[27] W. Gong, Z. Nagy and D. E. Soper, Phys. Rev. D **79** (2009) 033005 [arXiv:0812.3686 [hep-ph]].

[28] Z. Nagy, and D. E. Soper, Phys. Rev. D **74** (2006) 093006 [hep-ph/0610028].

[29] M. Kramer, 1 and D. E. Soper, Phys. Rev. D **66** (2002) 054017 [hep-ph/0204113].

[30] D. E. Soper, Phys. Rev. D **64** (2001) 034018 [hep-ph/0103262].

[31] D. E. Soper, Phys. Rev. D **62** (2000) 014009 [hep-ph/9910292].

[32] D. E. Soper, Phys. Rev. Lett. **81** (1998) 2638 [hep-ph/9804454].

[33] S. Becker and S. Weinzierl, Phys. Rev. D **86** (2012) 074009 [arXiv:1208.4088 [hep-ph]].

[34] S. Becker and S. Weinzierl, Eur. Phys. J. C **73** (2013) 2321 [arXiv:1211.0509 [hep-ph]].

[35] S. Mandelstam, Phys. Rev. Lett. **4** (1960) 84.

[36] H. Rechenberg and E. C. G. Sudarshan, Nuovo Cim. A **12** (1972) 541.

[37] S. Catani, D. de Florian and G. Rodrigo, Phys. Lett. B **586** (2004) 323 [hep-ph/0312067].

[38] G. F. R. Sborlini, D. de Florian and G. Rodrigo, JHEP **1401** (2014) 018 [arXiv:1310.6841 [hep-ph]].