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A Comment on Heavy Quark Effective Fields *

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Abstract

Effective fields defined in the heavy-quark effective theory to describe heavy quarks in heavy-light hadrons are examined in some detail in the standard formulation of a quantum field theory.

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Over the past decade, the approach to the weak decays of heavy (D and B) mesons can be characterized by the birth of more or less sophisticated phenomenological models [1] dealing with the complexity of the strong interaction dynamics. On the other hand, the current decade has witnessed the development of effective theories for the strong interaction (like the heavy-quark effective theory or HQET [2] [3]) coming from first principles, allowing very important phenomenological applications [4]. However, attention should still be paid to some formal aspects of the HQET as a proper quantum field theory as literature is sometimes confusing in this regard.

In this Letter, I shall examine some definitions for the effective fields describing heavy quarks according to the HQET, working out Feynman propagators from them. In particular I shall analyze their Fourier content in momentum space in the context of a quantum field theory.

Let us suppose a generic hadron moving with four-velocity v ($v^2 = 1$), made of a heavy quark Q along with a light component. According to Georgi's original remarks [5] the heavy quark becomes cannonball conserving its velocity, almost the same as the hadron's, unaltered as long as it interacts softly with the light degrees of freedom - without undergoing any weak or strong decay. Thereby, the heavy quark velocity can be viewed as a new *label* for heavy quarks, like flavour.

It is customary to begin any pedagogical introduction to HQET by expressing the heavy quark four-momentum p_Q as the sum $p_Q = m_Q v + k_Q$ where $m_Q v$ represents the (large) mechanical momentum whereas k_Q stands for the residual momentum of the almost on-shell heavy quark inside the hadron [1].

Let us start by writing the plane wave Fourier expansion for the fermionic field of an almost on-shell heavy quark $Q_v^{(b,\alpha)}(x)$ of flavour b and colour α, moving inside a hadron with four-velocity v

$$\begin{aligned} Q_v^{(b,\alpha)}(x) &= Q_v^{(b,\alpha)+}(x) \ + \ Q_v^{(b,\alpha)-}(x) \\ &= \int \frac{d^3\vec{p}}{J} \sum_r \left[\ b_r^\alpha(\vec{p}) \ u_r(\vec{p}) \ e^{-ip\cdot x} + \ \tilde{b}_r^{\alpha\dagger}(\vec{p}) \ v_r(\vec{p}) \ e^{ip\cdot x} \ \right] \end{aligned} \tag{1}$$

where r refers to the spin and J stands for the chosen normalization; $b_r^\alpha(\vec{p})/\tilde{b}_r^{\alpha\dagger}(\vec{p})$ is the annihilation/creation operator for a heavy quark/antiquark with three-momentum $\vec{p}$ ($p^0 \simeq +\sqrt{m_b^2+\vec{p}^2}$). Let us firstly consider the particle sector of the theory.

Particle Sector

As the heavy quark is almost on-shell we shall require for *each Fourier component* that

$$p^2 \ = \ m_b^2 \ + \ \Delta^2 \tag{2}$$

where Δ^2 is independent of $\vec{p}$ and satisfies $\lim_{m_b\to\infty}\Delta^2/m_b^2 \to 0$.

[1] Subindex Q has been added to those quantities like p_Q or k_Q to be interpreted as operator expectation values for a particular hadron state, satisfying the relation $v \simeq p_Q/m_Q$. All the components of k_Q are of the order of Λ_{QCD} and we shall express the off-shellness of the heavy quark as $p_Q^2 = m_Q^2 + 2m_Q v\cdot k_Q + k_Q^2 = m_Q^2 + \Delta^2$. In the text we substitute the index Q by b, which may be identified for most realistic applications as the bottom flavour in particular

On the other hand, spinors are normalized such that $u_r^\dagger(\vec{p})u_s(\vec{p}) = 2p^0 N\ \delta_{rs}$ and the creation/annihilation operators satisfy:

$$[b_r^\alpha(\vec{p'}), b_s^{\beta\dagger}(\vec{p})]_+ = K\ \delta^{\alpha\beta}\ \delta_{rs}\ \delta^3(\vec{p'} - \vec{p})$$

where $K,\ N$ are the corresponding normalization factors.

Let us redefine the momentum of each Fourier component in Eq. (1) according to HQET as the sum of a mechanical part and a *Fourier residual four-momentum* k,

$$p\ =\ m_b v\ +\ k \tag{3}$$

Hence one may write

$$Q_v^{(b,\alpha)+}(x)\ =\ e^{-im_b v\cdot x}\int \frac{d^3\vec{k}}{J}\sum_r b_r^\alpha(\vec{k})\ u_r(\vec{k})\ e^{-ik\cdot x} \tag{4}$$

The main point to be stressed is that we shall require that each Fourier component should satisfy the almost on-shell condition (2). Therefore, in the hadron rest frame $k{\cdot}v = k^0$ is related to $\vec{k}$ through the constraint

$$(k^0)^2 + 2m_b k^0 - \vec{k}^2\ =\ \Delta^2 \tag{5}$$

which yields the expected relation

$$k^0\ =\ \pm\sqrt{m_b^2 + \vec{k}^2 + \Delta^2}\ - m_b \tag{6}$$

where the positive solution can be identified as approximately the kinetic energy of the heavy quark inside the hadron; the negative solution can be rejected as yields a negative p^0. It is also interesting to note the space-like character of k.

Notice that the annihilation (and creation) operators and spinors have been simply *relabelled* in Eq. (4), satisfying the same normalization as above, though expressed in terms of the Fourier residual momentum $\vec{k}$. In particular, $b_r^\alpha(\vec{k})/b_r^{\alpha\dagger}(\vec{k})$ corresponds to annihilation/creation operators for a heavy quark with residual momentum $\vec{k}$ in a hadron moving with four-velocity v, satisfying accordingly

$$[b_r^\alpha(\vec{k}'), b_s^{\beta\dagger}(\vec{k})]_+ = K\ \delta^{\alpha\beta}\ \delta_{rs}\ \delta^3(\vec{k}' - \vec{k}) \tag{7}$$

and

$$\textstyle\sum_r u_r(\vec{k})\bar{u}_r(\vec{k})\ =\ N\ [\ m_b(1+\not{v}) + \not{k}\ ] \tag{8}$$

where the normalization factors must obey the combined relation: [6] [2]

$$I\ =\ \frac{K\ N}{J^2}\ =\ \frac{1}{(2\pi)^3\ 2p^0}\ =\ \frac{1}{(2\pi)^3\ 2(m_b v^0 + k^0)} \tag{9}$$

On the other hand, it is quite usual in literature to identify effective heavy quark subfields with those "leading" components of the Fourier expansion corresponding to momenta $p \simeq p_b$ (or equivalently with k components close to zero), so they rather look like single spinors or anti-spinors at leading order in $1/m_b$.

[2] Note that $I{\neq}1/(2\pi)^3 2k^0$ since k and p are not related through a Lorentz transformation

Instead, in this work I shall deal at first with the full $\vec{k}$ spectrum, whose components, nevertheless, are supposed to have a small off-shellness as mentioned before. Therefore, let us introduce the effective subfields in the following manner [3] [7] [3]

$$h_v^{(b,\alpha)+}(x) = e^{im_b v\cdot x}\ \frac{1+\not{v}}{2}\ Q_v^{(b,\alpha)+}(x) \tag{10}$$

$$H_v^{(b,\alpha)+}(x) = e^{im_b v\cdot x}\ \frac{1-\not{v}}{2}\ Q_v^{(b,\alpha)+}(x) \tag{11}$$

where $H_v^{(b,\alpha)+}(x)$ plays the role of the "small" (lower in the hadron reference frame) component of the corresponding spinor subfield. Thus one may identify from the expansion (4)

$$h_v^{(b,\alpha)+}(x) = \frac{1+\not{v}}{2} \int \frac{d^3\vec{k}}{J} \sum_r b_r^\alpha(\vec{k})\ u_r(\vec{k})\ e^{-ik\cdot x} \tag{12}$$

$$H_v^{(b,\alpha)+}(x) = \frac{1-\not{v}}{2} \int \frac{d^3\vec{k}}{J} \sum_r b_r^\alpha(\vec{k})\ u_r(\vec{k})\ e^{-ik\cdot x} \tag{13}$$

The relation [8]

$$H_v^{(b,\alpha)+}(x) = \frac{1-\not{v}}{2}\ \frac{\not{k}}{2m_b}\ h_v^{(b,\alpha)+}(x)$$

is verified at leading order, following from the above definitions.

Next I shall derive the explicit space-time representation of the Feynman propagator for the $h_v^{(b,\alpha)+}$ subfield. From the relations (7-9) one may easily find that [4]

$$\begin{aligned} iS_{h_v}^+(x-y) &= <0|h_v^{(b,\alpha)+}(x)\bar{h}_v^{(b,\beta)+}(y)|0> \\ &= \delta^{\alpha\beta}\ \frac{1+\not{v}}{2} \left(\int \frac{d^3\vec{k}}{I}\ [(1+\not{v})m_b + \not{k})]\ e^{-ik\cdot(x-y)} \right) \frac{1+\not{v}}{2} \end{aligned} \tag{14}$$

where the invariant quantity $I = KN/J^2$ is given by Eq. (9).

With the aid of the gamma commutation relations and the fact that $(1\pm\not{v})/2$ are projectors, one obtains easily that

$$iS_{h_v}^+(x-y) = \delta^{\alpha\beta}\ \frac{1+\not{v}}{2} \left(\int \frac{d^3\vec{k}}{I}\ [2m_b + k\cdot v]\ e^{-ik\cdot(x-y)} \right) \tag{15}$$

Using the Dirac delta properties the above result can be expressed as a four-dimensional integral

$$iS_{h_v}^+(x-y) = \delta^{\alpha\beta}\ \frac{1+\not{v}}{2} \left(\int \frac{d^4k}{(2\pi)^3}\ \theta(k\cdot v)\ \delta(k^2+2m_b v\cdot k-\Delta^2)\ [2m_b+k\cdot v]\ e^{-ik\cdot(x-y)} \right) \tag{16}$$

[3] Literature is sometimes not completely clear in their interpretation as quantum fields. I define the subfields in standard notation of quantum field theory, where the superscripts + and − label positive frequencies (associated to annihilation operators of quarks) and negative frequencies (associated to creation operators of antiquarks) respectively

[4] In order to avoid a large and misleading notation the extra minus sign as a superscript in the $\bar{h}_v^+$ subfield has been omitted which, however, should be implicitly understood since negative frequencies and creation operators are involved. The same omission will occur afterwards for the $\bar{H}_v^+$ subfield as well

In the infinite mass limit the kinetic energy of the heavy quark inside the hadron will be neglected but not its three-momentum $\vec{k}$. (This is consistent with the space-like character of the four-momentum k.) In effect, I shall assume from Eq. (6) for the positive root k^0,

$$\frac{k\cdot v}{2m_b} = \frac{k^0}{2m_b} \simeq \frac{\vec{k}^2+\Delta^2}{4m_b^2} \to 0 \tag{17}$$

In the real world the above limit makes sense for $\vec{k}^2 << m_b^2$. Neglecting those Fourier components above m_b, which acts as an ultraviolet cutoff, probably makes sense for the bottom quark but not for the charm quark.

The two roots of Eq. (5) showed in expression (6) correspond to the two poles in the real k^0-axis of the integrand of Eq. (16). In the $m_b \to \infty$ limit both poles are shifted to *zero* and to $-\infty$ respectively. The theta function ensures that only the former one contributes, thereby implying

$$iS_{h_v}^{+}(x-y) = \delta^{\alpha\beta}\,\frac{1+\not{v}}{2}\left(\int \frac{d^4k}{(2\pi)^3}\,\delta(2m_b v\cdot k)\,[2m_b+k\cdot v]\,e^{-ik\cdot(x-y)}\right) \tag{18}$$

Finally, making use of the property: $\delta(2m_b v\cdot k)=\delta(v\cdot k)/2m_b$, one gets for the Feynman "propagator" at leading order

$$iS_F^{+}(x-y) = \theta(v\cdot x - v\cdot y)\,iS_{h_v}^{+}(x-y) = \frac{1+\not{v}}{2}\,\theta(v\cdot x - v\cdot y)\,\delta^{\alpha\beta}\,\delta_v^3(\vec{x}-\vec{y}) \tag{19}$$

where

$$\delta_v^3(\vec{x}-\vec{y}) = \int \frac{d^4k}{(2\pi)^3}\,\delta(k\cdot v)\,e^{-ik\cdot(x-y)}$$

Observe that there is only a single pole in the real k^0-axis at $k^0 = 0 - i\epsilon$, leading to

$$S_F^{+}(x-y) = \frac{1+\not{v}}{2}\,\delta^{\alpha\beta}\int \frac{d^4k}{(2\pi)^4}\,\frac{1}{k\cdot v + i\epsilon}\,e^{-ik\cdot(x-y)} \tag{20}$$

The Dirac delta in (19) reflects the fact that the heavy quark remains immovable inside the hadron moving with four-velocity v, as already pointed out in Ref. [9]. Finally, I remark that the propagator in momentum space indeed shows the usual form in HQET [2] [5]

$$S_F^{+}(k) = \frac{1+\not{v}}{2}\,\delta^{\alpha\beta}\,\frac{1}{k\cdot v+i\epsilon} \tag{21}$$

In a similar way one gets for the propagator corresponding to the $H_v^{(b,\alpha)+}$ subfield for example,

$$iS_{H_v}^{+}(x-y) = <0|H_v^{(b,\alpha)+}(x)\bar{H}_v^{(b,\beta)+}(y)|0> \;\equiv 0 \tag{22}$$

[5] Obviously the same expression can be obtained directly from the equation satisfied at leading order by the propagator in HQET:

$$iv\cdot\partial\; S_F^{+}(x-y) = \frac{1+\not{v}}{2}\,\delta^4(x-y)$$

and then making use of the Fourier transform technically coming from the fact that the following matrix element:

$$\frac{1-\not{v}}{2}\,\frac{[(1+\not{v})m+\not{k}]}{2m_b}\,\frac{1-\not{v}}{2}\;=\;-\frac{1-\not{v}}{2}\,\frac{k\cdot v}{2m_b} \tag{23}$$

vanishes in the $m_b\to 0$ limit. If we do not neglect $(\vec{k}^2+\Delta^2)/4m_b^2$ the expression (22) is not identically zero, showing explicitly that indeed the "small" components of the effective heavy quark field yield $1/m_b$ effects.

Vacuum expectation values of the combined products of the $h_v^{(b,\alpha)+}$ and $H_v^{(b,\beta)+}$ subfields lead to further contributions [6] to the full propagator similarly suppressed. Therefore, at leading order the only "propagator" for a heavy quark is that coming from the $h_v^{(b,\alpha)+}$ subfield as expected.

Anti-Particle Sector

Proceeding in a parallel way as in the particle sector, let us remark however that now the Fourier expansion of $Q_v^{(b,\alpha)-}(x)$ involves negative frequencies. Therefore, starting from Eq. (1) I shall introduce the Fourier residual momentum in this case as

$$p\;=\;m_b v\;-\;k \tag{24}$$

so k^0 will explicitly exhibit its negative sign. In effect, let us write

$$Q_v^{(b,\alpha)-}(x)\;=\;e^{im_b v\cdot x}\int\frac{d^3\vec{k}}{J}\sum_r\tilde{b}_r^{\alpha\dagger}(\vec{k})\;v_r(\vec{k})\;e^{-ik\cdot x} \tag{25}$$

The slightly off-shellness condition (2) now implies from Eq. (24)

$$(k^0)^2\;-\;2m_b k^0\;-\;\vec{k}^2\;=\;\Delta^2 \tag{26}$$

whose roots are

$$k^0\;=\;\pm\sqrt{m_b^2+\vec{k}^2+\Delta^2}\;+m_b \tag{27}$$

but only the negative solution will contribute and a purely advanced Green function in space-time will appear at the end of the calculation.

The effective subfields are defined as

$$h_v^{(b,\alpha)-}(x)\;=\;e^{-im_b v\cdot x}\,\frac{1-\not{v}}{2}\,Q_v^{(b,\alpha)-}(x)=\;\frac{1-\not{v}}{2}\int\frac{d^3\vec{k}}{J}\sum_r\tilde{b}_r^{\alpha\dagger}(\vec{k})\;v_r(\vec{k})\;e^{-ik\cdot x} \tag{28}$$

$$H_v^{(b,\alpha)-}(x)\;=\;e^{-im_b v\cdot x}\,\frac{1+\not{v}}{2}\,Q_v^{(b,\alpha)-}(x)\;=\;\frac{1+\not{v}}{2}\int\frac{d^3\vec{k}}{J}\sum_r\tilde{b}_r^{\alpha\dagger}(\vec{k})\;v_r(\vec{k})\;e^{-ik\cdot x} \tag{29}$$

[6] The combinations $<0|h_v^+\overline{H}_v^+|0>$ and $<0|H_v^+\overline{h}_v^+|0>$ yield respectively

$$\frac{1\pm\not{v}}{2}\,\frac{\not{k}\mp k\cdot v}{k^2+2m_b k\cdot v-\Delta^2}$$

in momentum space. In the sum of all four contributions the full propagator is recovered where the anti-spinors satisfy

$$\sum_r v_r(\vec{k})\bar{v}_r(\vec{k}) \;=\; N\;[\;m_b(\not{v}-1)-\not{k}\;] \tag{30}$$

Following the same steps and limits as in the particle sector, one obtains for the negative-frequency propagator

$$iS^-_{h_v}(x-y) \;=<0|\bar{h}_v^{(b,\alpha)-}(y)h_v^{(b,\beta)-}(x)|0> = \;-\frac{1-\not{v}}{2}\;\delta^{\alpha\beta}\;\delta^3_v(\vec{x}-\vec{y}) \tag{31}$$

whereas the propagators involving the $H_v^{(b,\alpha)-}$ subfield do not contribute at leading order in analogy to Eq. (22). Therefore the Feynman propagator for a massive anti-quark is

$$iS^-_F(x-y) \;=\; -\,\theta(v{\cdot}y-v{\cdot}x)\;iS^-_{h_v}(y-x) \;=\; \frac{1-\not{v}}{2}\;\theta(v{\cdot}y-v{\cdot}x)\;\delta^{\alpha\beta}\;\delta^3_v(\vec{x}-\vec{y}) \tag{32}$$

where the single pole lies now at $k{\cdot}v=0+i\epsilon$, leading to

$$S^-_F(x-y) \;=\; \frac{\not{v}-1}{2}\;\delta^{\alpha\beta}\int\frac{d^4k}{(2\pi)^4}\;\frac{1}{k{\cdot}v-i\epsilon}\;e^{-ik\cdot(x-y)} \tag{33}$$

Let us make a final remark on the single poles of the propagators. Since the heavy quark mass was assumed infinite, the particle and anti-particle sectors of the effective theory actually decouple (for example no heavy quark pair production is allowed). This manifests in the fact that there is no single expression representing altogether the propagation forward in time of the positive energy solutions and backward in time of the negative energy solutions as otherwise happens in the Dirac theory.

In summary, in this work I have examined those effective subfields commonly introduced in HQET to deal with almost on-shell heavy quarks from the standard viewpoint of a quantum field theory (although especially simple in this case). For example, the $\overline{h}_v^+(x)$ field as defined in Eq. (12) creates a heavy quark at point x in space-time as a superposition of single-particle states with momenta ranging over the entire $\vec{k}$ domain (actually for $\vec{k}^2<m_b^2$) and not yet limited to small values of the components, of the order of Λ_{QCD}.

However, in constructing the effective theory for heavy quarks one should add a further restriction on the Fourier expansion of the fields, limiting the range of the $\vec{k}$ components to values of the order of Λ_{QCD}. Note that this condition amounts to a new constraint not completely equivalent to the small virtuality already imposed by means of Eq. (2). Indeed, the smallness of $\vec{k}$ implies the almost on-shellness condition but the converse is not necessarily true.

In fact, eliminating those components with large residual momentum in the Fourier expansion of the fields is equivalent to integrating away the high-velocity states in the functional integral formulation of Ref. [11]. It is also interesting to mention that, once restricted the $\vec{k}$ range to values sharply peaked at the origin, the propagator would spread out over coordinate space with typical width $\simeq 1/\Lambda_{QCD}$ as a consequence of the Fourier transform.

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