Power-law running of the effective gluon mass

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Abstract

The dynamically generated effective gluon mass is known to depend non-trivially on the momentum, decreasing sufficiently fast in the deep ultraviolet, in order for the renormalizability of QCD to be preserved. General arguments based on the analogy with the constituent quark masses, as well as explicit calculations using the operator-product expansion, suggest that the gluon mass falls off as the inverse square of the momentum, relating it to the gauge-invariant gluon condensate of dimension four. In this article we demonstrate that the power-law running of the effective gluon mass is indeed dynamically realized at the level of the non-perturbative Schwinger-Dyson equation. We study a gauge-invariant non-linear integral equation involving the gluon self-energy, and establish the conditions necessary for the existence of infrared finite solutions, described in terms of a momentum-dependent gluon mass. Assuming a simplified form for the gluon propagator, we derive a secondary integral equation that controls the running of the mass in the deep ultraviolet. Depending on the values chosen for certain parameters entering into the Ansatz for the fully-dressed three-gluon vertex, this latter equation yields either logarithmic solutions, familiar from previous linear studies, or a new type of solutions, displaying power-law running. In addition, it furnishes a non-trivial integral constraint, which restricts significantly (but does not determine fully) the running of the mass in the intermediate and infrared regimes. The numerical analysis presented is in complete agreement with the analytic results obtained, showing clearly the appearance of the two types of momentum-dependence, well-separated in the relevant space of parameters. Several technical improvements, various open issues, and possible future directions, are briefly discussed.

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I. INTRODUCTION

The possibility that the non-perturbative dynamics of QCD generate an effective gluon mass, first elaborated in the pioneering work of Cornwall [1, 2], has received considerable theoretical [3, 4, 5, 6, 7, 8] and phenomenological [9, 10, 11] attention in recent years [12]. As advocated in [2], the origin of this effective mass is purely dynamical and preserves the local $SU(3)_c$ invariance of QCD, in close analogy to what happens in QED_2 (Schwinger model) [13], where the photon acquires a mass without violating the abelian gauge symmetry. The gluon mass is not a directly measurable quantity, and its value is determined, at least in principle, by relating it to other dimensionful non-perturbative parameters, such as the string tension, glueball masses, gluon condensates, and the vacuum energy of QCD [14].

Since gluon mass generation is a purely non-perturbative effect, the most standard way for studying it in the continuum is through the Schwinger-Dyson equations (SDE) governing the relevant Green's functions. These equations have been treated from the very beginning [2] in a manifestly gauge-invariant way, by resorting to the systematic rearrangement of graphs implemented by the pinch technique (PT) [2, 15, 16]. Subsequently, the powerful all-order connection [16] between the PT and the Feynman gauge of the Background Field Method (BFM) [17] gave rise to the "PT-BFM" truncation scheme [8, 18] that guarantees crucial properties, such as gauge-invariance, gauge-independence, and invariance under the renormalization-group (RG). What one characterizes as gluon mass generation at the level of the SDE governing the PT-BFM gluon self-energy is essentially the existence of solutions that reach a finite (non-vanishing) value in the deep infrared (IR). These solutions may be successfully fitted by a "massive" propagator of the form $\Delta^{-1}(q^2) = q^2 + m^2(q^2)$; the crucial characteristic, enforced by the SDE itself, is that $m^2(q^2)$ is not "hard", but depends non-trivially on the momentum transfer q^2. Specifically, $m^2(q^2)$ is a monotonically decreasing function, starting at a non-zero value in the IR ($m^2(0) > 0$) and dropping "sufficiently fast" in the deep ultraviolet (UV). When the RG logarithms are properly taken into account, one obtains in addition the non-perturbative generalization of $g^2(q^2)$, the QCD running coupling (effective charge). The presence of $m^2(q^2)$ in the corresponding logarithms tames the Landau singularity associated with the perturbative β function, and the resulting effective charge is asymptotically free in the UV, "freezing" at a finite value in the IR [2, 15, 19, 20, 21, 22, 23].

The running of $m^2(q^2)$ is of central importance for the self-consistency of this entire approach. Roughly speaking, the value of $\Delta^{-1}(0)$ is determined by integrals involving $\Delta(q^2)$, $m^2(q^2)$, and $g^2(q^2)$ over the entire range of (Euclidean) momenta. The UV convergence of these integrals depends crucially on how $m^2(q^2)$ behaves as $q^2 \to \infty$. If $m^2(q^2)$ drops off asymptotically faster than a logarithm, then $\Delta^{-1}(0)$ is finite. This is crucial because the finiteness of $\Delta^{-1}(0)$ guarantees essentially the renormalizability of QCD. Had the mass been constant instead, these integrals would diverge quadratically; to absorb such a divergence one would have to introduce a counterterm of the form $m_0^2(\Lambda^2_{UV})A^2$ at the level of the fundamental QCD Lagrangian, which is clearly forbidden by the local gauge invariance.

The UV behavior of $m^2(q^2)$ is not imposed by hand, but is instead determined dynamically from the corresponding SDE. The situation is conceptually very similar to the case of the dynamically generated (constituent) quark masses, whose momentum dependence is controlled by the gap equation. Notice, however, that there is an important technical difference. Due to its Dirac structure the gap equation may be separated into two independent components, one determining the wave function and one the mass of the quark self-energy. In the case of the SDE for the gluon self-energy there is no such direct separation: instead an appropriate matching of the contributions on both sides of the equations must be carried out.

In previous studies of linear SDE equations [2, 8] the UV running of $m^2(q^2)$ has been found to be logarithmic, of the general form $m^2(q^2) \sim (\ln q^2)^{-1-\gamma}$, with $\gamma > 0$ in order for the aforementioned integrals determining $\Delta^{-1}(0)$ to converge. Of course, it is natural to ask whether the QCD dynamics allows also for a $m^2(q^2)$ displaying power-law running, i.e. $m^2(q^2) \sim q^{-2}(\ln q^2)^{\gamma-1}$, as happens in the case of the dynamically generated quark masses [24]. This possibility was first envisaged by Cornwall [2, 25], based on the aforementioned studies of chiral symmetry breaking. A decade later, Lavelle [26] wrote down the operator-product expansion (OPE) for the (partially dressed) one-loop PT gluon self-energy, expressing it solely in terms of the gauge-invariant gluon condensate $\langle G^2\rangle = \langle 0|:G^a_{\mu\nu}G^{\mu\nu}_a:|0\rangle$ of dimension four (no quarks were considered) [27]. The resulting self-energy was then identified with an effective gluon mass, $m^2(q^2) \sim \langle G^2\rangle/q^2$, i.e. a mass displaying power-law running. However, to date, the power-law running of the effective gluon mass has never been demonstrated as an explicit dynamical possibility at the level of a (non-linear) SDE.

In this article we show that the non-linear SDE for the gluon self-energy in the PT-BFM formalism has two distinct types of solutions: (i) solutions of the type already encountered in the linear studies, with $m^2(q^2)$ running as an inverse power of a logarithm, and (ii) solutions found for the first time, where the effective gluon mass drops asymptotically as an *inverse power of the momentum* (multiplied by powers of logarithms). Which of the two types of solutions will be actually realized is a complicated dynamical problem, depending mainly on the details of the (fully dressed) three-gluon vertex, $\widetilde{\mathrm{I}\Gamma}_{\mu\alpha\beta}$, entering into the SDE for the gluon self-energy (see Fig. 1).

The article is organized as follows: In Sec. II, after setting up the notation, we present the derivation of the SDE for the PT-BFM gluon propagator. We restrict our analysis to the subset of "one-loop dressed" gluonic contributions, namely the two fully-dressed gluonic diagrams shown in Fig. 1. In addition, we explain briefly how the naive all-order Ward identity (WI) satisfied by the full three-gluon vertex in this formalism enforces the transversality of the gluon self-energy even in the absence of ghost loops. In Sec. III we manipulate the SDE derived in the previous section further, with the motivation to search for infrared-finite solutions. An important ingredient at this stage is the Ansatz introduced for the full three gluon vertex, based on the gauge-technique; this vertex satisfies, by construction, the simple all-order WI characteristic of the PT-BFM, and contains sufficient structure to give rise to a non-vanishing $\Delta^{-1}(0)$. In addition, we discuss in detail the technical adjustments implemented to the SDE, in order to endow it with the correct RG behavior. The next two sections contain the main results of this article. Specifically, in Sec. IV we extract from the SDE an integral equation that determines the running of the effective gluon mass in the UV, and study its solutions. We derive an important constraint, in the form of an integral boundary condition, relating the value of $\Delta^{-1}(0)$ with $m^2(q^2)$ in the entire range of momenta; this condition must be necessarily satisfied in order for the mass equation to have solutions that vanish asymptotically in the deep UV. Then we demonstrate that, depending on the values of two basic parameters, one finds solutions for the masses that drop as inverse powers of a logarithm of q^2, or much faster, as an inverse power of q^2. In Sec. V we carry out a numerical analysis of the SDE, which fully corroborates the existence of the two aforementioned types of solutions. In Sec. VI we present our conclusions and a discussion of various open issues. Finally, in an Appendix we present all technical points related to the modifications one must introduce to the standard angular approximation in order to capture correctly the leading $m^2(q^2)$ behavior, encoded in the exact SDE.

II. SDE FOR THE PT-BFM GLUON PROPAGATOR

In this section we derive in detail a non-linear SDE equation for the gluon propagator in the PT-BFM formalism. As has been explained in the literature, this scheme is essentially founded on the all-order correspondence between PT and BFM [16]: the (gauge-independent) PT effective n-point functions coincide with the (gauge-dependent) BFM n-point functions provided that the latter are computed in the Feynman gauge. One of the most powerful features of this formalism is the special way in which the transversality of the all-order PT-BFM self-energy is realized. Specifically, by virtue of the Abelian-like WIs satisfied by the vertices involved, gluonic and ghost contributions are *separately* transverse, within *each* order in the "dressed-loop" expansion of the corresponding SDE [8]. This property, in turn, allows for a systematic truncation of the full SDE [28] that preserves the crucial property of gauge invariance. In particular, instead of a system of two coupled equations involving gluon and ghost propagators, one may consider only the subset containing gluons, without compromising the transversality of the gluon self-energy. Therefore, in what follows, we will consider only the gauge-invariant subset of *"one-loop dressed" gluonic* diagrams, given by the two graphs of Fig. 1.

Within this formalism there are two distinct gluon propagators, the background gluon propagator $\widehat{\Delta}_{\mu\nu}(q)$ (which, in the Feynman gauge, coincides with the PT gluon propagator) and the quantum gluon propagator $\Delta_{\mu\nu}(q)$, appearing inside the loops. By virtue of a powerful all-order identity [29, 30], one may express $\Delta_{\mu\nu}(q)$ in terms of $\widehat{\Delta}_{\mu\nu}(q)$ and auxiliary (unphysical) Green's functions involving anti-fields and background sources. As a first approximation, in this work we will neglect the effects of the aforementioned auxiliary Green's functions, and carry out the substitution $\Delta_{\mu\nu}(q) \to \widehat{\Delta}_{\mu\nu}(q)$ throughout (and drop the "hats" to simplify the notation).

In the Feynman gauge (both of the usual linear renormalizable gauges as well as the BFM) the full gluon propagator $\Delta_{\mu\nu}(q)$ has the general form (we suppress color indices)

$$\Delta_{\mu\nu}(q) = -i\left[\mathrm{P}_{\mu\nu}(q)\Delta(q^2) + \frac{q_\mu q_\nu}{q^4}\right] , \tag{2.1}$$

where

$$\mathrm{P}_{\mu\nu}(q) = \; g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} , \tag{2.2}$$

is the usual transverse projector. The scalar function $\Delta(q^2)$ is related to the all-order gluon self-energy $\Pi_{\mu\nu}(q)$,

$$\Pi_{\mu\nu}(q) = \mathrm{P}_{\mu\nu}(q)\Pi(q^2)\,, \tag{2.3}$$

through

$$\Delta^{-1}(q^2) = q^2 + i\Pi(q^2)\,. \tag{2.4}$$

Notice that, since $\Pi(q^2)$ has been defined in (2.4) with the imaginary factor i pulled out in front, it is given simply by the corresponding Feynman diagrams in Minkowski space. Finally, the inverse of the full gluon propagator is given by

$$\begin{aligned}\Delta^{-1}_{\mu\nu}(q) &= ig_{\mu\nu}q^2 - \Pi_{\mu\nu}(q) \\ &= i\mathrm{P}_{\mu\nu}(q)\Delta^{-1}(q^2) + iq_\mu q_\nu\,. \end{aligned}\tag{2.5}$$

At tree-level,

$$\Delta_0^{\mu\nu}(q) = \frac{g^{\mu\nu}}{q^2}\,, \quad \Delta_0(q^2) = \frac{1}{q^2}\,. \tag{2.6}$$

The SDE in Fig. 1 reads,

$$(\Delta^{-1})^{ab}_{\mu\nu}(q) = iq^2 g_{\mu\nu}\delta^{ab} - \left[\Pi^{ab}_{\mu\nu}(q)\Big|_{a_1} + \Pi^{ab}_{\mu\nu}\Big|_{a_2}\right]\,, \tag{2.7}$$

with the closed expressions corresponding to the diagrams (a_1) and (a_2) given by

$$\begin{aligned}\Pi^{ab}_{\mu\nu}(q)\Big|_{a_1} &= \frac{1}{2}\int[dk]\,\widetilde{\Gamma}^{acx}_{\mu\alpha\beta}(q,k,-k-q)\Delta^{\alpha\rho}_{cd}(k)\widetilde{\mathrm{I}\Gamma}^{bde}_{\nu\rho\sigma}(-q,-k,k+q)\Delta^{\beta\sigma}_{xe}(k+q)\,, \\ \Pi^{ab}_{\mu\nu}\Big|_{a_2} &= \frac{1}{2}\int[dk]\,\widetilde{\Gamma}^{abcd}_{\mu\nu\rho\sigma}\Delta^{\rho\sigma}_{cd}(k)\,. \end{aligned}\tag{2.8}$$

The flow of momenta, together with the color and Lorentz indices, are shown in Fig. 1. We have assumed dimensional regularization, employing the short-hand notation $[dk] = d^dk/(2\pi)^d$, where $d = 4-\epsilon$ is the dimension of space-time.

We will use greek letters for the Lorentz indices and latin for the color indices. The tree-level vertex $\widetilde{\Gamma}^{acx}_{\mu\alpha\beta}$ appearing in (a_1) is given by ($p_1 = k$, $p_2 = -k-q$)

$$\begin{aligned}\widetilde{\Gamma}^{acx}_{\mu\alpha\beta}(q,p_1,p_2) &= gf^{acx}\widetilde{\Gamma}_{\mu\alpha\beta}(q,p_1,p_2), \\ \widetilde{\Gamma}_{\mu\alpha\beta}(q,p_1,p_2) &= (p_1-p_2)_\mu g_{\alpha\beta} + 2q_\beta g_{\mu\alpha} - 2q_\alpha g_{\mu\beta}, \end{aligned}\tag{2.9}$$

and satisfies the elementary WI

$$q^\mu\widetilde{\Gamma}_{\mu\alpha\beta}(q,p_1,p_2) = (p_2^2-p_1^2)g_{\alpha\beta} = i\left[\Delta^{-1}_{0\,\alpha\beta}(p_1) - \Delta^{-1}_{0\,\alpha\beta}(p_2)\right] = \left[\Delta^{-1}_0(p_2) - \Delta^{-1}_0(p_1)\right]g_{\alpha\beta}\,. \tag{2.10}$$

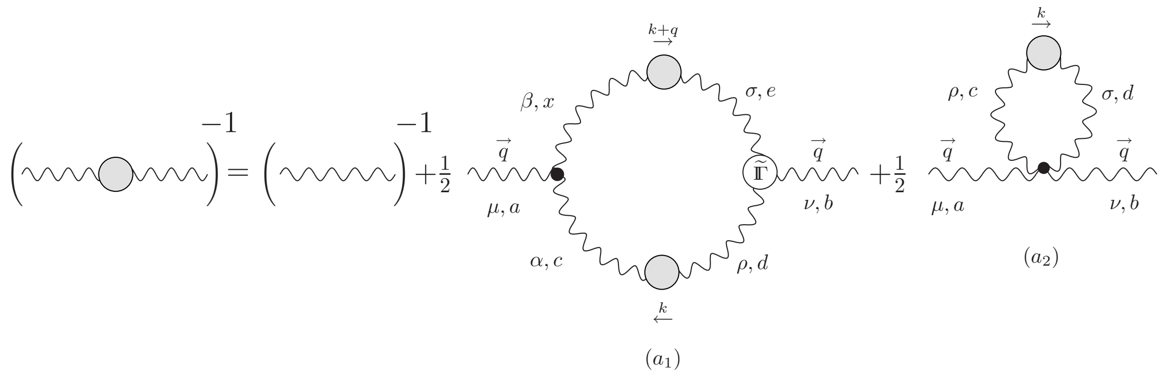

FIG. 1: The gluonic "one-loop dressed" contributions to the SDE.

The bare four-gluon vertex $\widetilde{\Gamma}^{abcd}_{\mu\nu\rho\sigma}$ appearing in (a_2) is given by

$$\begin{aligned}\widetilde{\Gamma}^{abcd}_{\mu\nu\rho\sigma} \;=\; -ig^2\Big[&f^{acx}f^{xbd}\,(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\sigma}g_{\rho\nu} + g_{\mu\rho}g_{\nu\sigma}) + f^{adx}f^{xcb}\,(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\nu}g_{\rho\sigma} - g_{\mu\sigma}g_{\rho\nu}) \\ &+ f^{abx}f^{xcd}\,(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\rho\nu})\Big]\,. \end{aligned} \tag{2.11}$$

In addition, for the fully dressed quantities $\Delta^{ab}_{\mu\nu}$ and $\widetilde{\mathbb{I}\Gamma}^{abc}_{\mu\alpha\beta}$ we will set $\Delta^{ab}_{\mu\nu} = \delta^{ab}\Delta_{\mu\nu}$ and $\widetilde{\mathbb{I}\Gamma}^{abc}_{\mu\alpha\beta} = gf^{abc}\widetilde{\mathbb{I}\Gamma}_{\mu\alpha\beta}$. In the PT-BFM formalism the all-order WI satisfied by $\widetilde{\mathbb{I}\Gamma}_{\mu\alpha\beta}$ is the naive generalization of the tree-level WI (2.10) [8, 17], i.e.

$$q^{\mu}\widetilde{\mathbb{I}\Gamma}_{\mu\alpha\beta}(q,p_1,p_2) = i[\Delta^{-1}_{\alpha\beta}(p_1) - \Delta^{-1}_{\alpha\beta}(p_2)]\,. \tag{2.12}$$

It is then elementary to verify that

$$q^{\nu}\left[\Pi_{\mu\nu}(q)\big|_{a_1} + \Pi_{\mu\nu}\big|_{a_2}\right] = 0\,. \tag{2.13}$$

Thus, the subset of graphs considered is transverse by itself, i.e. without the inclusion of ghost loops, as announced.

In order to reduce the algebraic complexity of the problem, we drop the longitudinal terms from the gluon propagators inside the integrals on the r.h.s. of (2.8), i.e. we set $\Delta_{\alpha\beta} \to -ig_{\alpha\beta}\Delta$. This may be done without compromising the transversality of the answer, provided that one drops, at the same time, the longitudinal pieces on the r.h.s. of the WI of Eq.(2.12) [31].

It is then straightforward to arrive at the following form for the SDE

$$i\mathrm{P}_{\mu\nu}(q)\Delta^{-1}(q^2) = i\mathrm{P}_{\mu\nu}(q)\,q^2 - \frac{C_{\mathrm{A}}g^2}{2}\left(\int[dk]\,\widetilde{\Gamma}^{\alpha\beta}_{\mu}\Delta(k)\widetilde{\mathbb{I}\Gamma}_{\nu\alpha\beta}\Delta(k+q) - 8\,g_{\mu\nu}\int[dk]\,\Delta(k)\right), \tag{2.14}$$

where we have used that $f^{ace}f^{bce} = \delta^{ab}C_{\rm A}$, with $C_{\rm A}$ the Casimir eigenvalue in the adjoint representation [$C_{\rm A} = N$ for $SU(N)$]. After the omission of the longitudinal parts the WI of (2.12) becomes

$$q^{\nu}\widetilde{\rm I\Gamma}_{\nu\alpha\beta}(q,p_1,p_2) = \left[\Delta^{-1}(p_2) - \Delta^{-1}(p_1)\right] g_{\alpha\beta}\,; \tag{2.15}$$

contracting by q^{ν} and using (2.15) one may easily verify that the sum of the two integrals on the r.h.s. of (2.14) is indeed transverse.

III. SDE WITH IR-FINITE SOLUTIONS

In order to proceed further with Eq.(2.14), one needs to supply some information about the form of the full vertex $\widetilde{\rm I\Gamma}$. To accomplish this, ideally one should set up the corresponding SDE governing the vertex $\widetilde{\rm I\Gamma}$, and solve a system of coupled integral equations. In practice this is very difficult and one almost always resorts to the "gauge technique" [32], expressing $\widetilde{\rm I\Gamma}^{\mu\alpha\beta}$ as a functional of Δ, in such a way as to satisfy (by construction) the appropriate WI. It is clear that this procedure leaves the transverse (i.e. identically conserved) part of the vertex undetermined, a fact that leads to the mishandling of overlapping divergences, and forces one to renormalize the resulting SD equation subtractively instead of multiplicatively.

The Ansatz we will use for the vertex is

$$\widetilde{\rm I\Gamma}^{\mu\alpha\beta} = L^{\mu\alpha\beta} + T_1^{\mu\alpha\beta} + T_2^{\mu\alpha\beta}\,, \tag{3.1}$$

with

$$\begin{aligned}
L^{\mu\alpha\beta}(q,p_1,p_2) &= \widetilde{\Gamma}^{\mu\alpha\beta}(q,p_1,p_2) + ig^{\alpha\beta}\,\frac{q^{\mu}}{q^2}\,\left[\Pi(p_2) - \Pi(p_1)\right]\,, \\
T_1^{\mu\alpha\beta}(q,p_1,p_2) &= -i\frac{c_1}{q^2}\left(q^{\beta}g^{\mu\alpha} - q^{\alpha}g^{\mu\beta}\right)\left[\Pi(p_1) + \Pi(p_2)\right]\,, \\
T_2^{\mu\alpha\beta}(q,p_1,p_2) &= -ic_2\left(q^{\beta}g^{\mu\alpha} - q^{\alpha}g^{\mu\beta}\right)\left[\frac{\Pi(p_1)}{p_1^2} + \frac{\Pi(p_2)}{p_2^2}\right]\,.
\end{aligned} \tag{3.2}$$

Several comments on the properties and role of the vertex $\widetilde{\rm I\Gamma}^{\mu\alpha\beta}$ and its individual components are now in order:

(i) When $\Pi(p_i) = 0$, $\widetilde{\rm I\Gamma}^{\mu\alpha\beta}$ goes over to the tree-level (bare) result, namely the $\widetilde{\Gamma}^{\mu\alpha\beta}$ of (2.9).

(ii) $\widetilde{\mathrm{I\Gamma}}^{\mu\alpha\beta}$ is Bose symmetric only with respect to its two legs appearing inside the loop, carrying momentum $p_1 = k$ and $p_2 = -(k+q)$; so, $\widetilde{\mathrm{I\Gamma}}^{\mu\alpha\beta}(q, p_1, p_2)$ is invariant under the simultaneous exchange $p_1 \longleftrightarrow p_2$, $\alpha \longleftrightarrow \beta$ and $b \longleftrightarrow c$.

(iii) $\widetilde{\mathrm{I\Gamma}}^{\mu\alpha\beta}$ satisfies by construction the WI of Eq.(2.12). Specifically, when the "longitudinal" part $L^{\mu\alpha\beta}$ is contracted with q_μ furnishes the r.h.s. of (2.12), whereas the "transverse" parts $T_1^{\mu\alpha\beta}$ and $T_2^{\mu\alpha\beta}$ are identically conserved.

(iv) $\widetilde{\mathrm{I\Gamma}}^{\mu\alpha\beta}$ contains massless poles, i.e. terms going as $1/q^2$; as is well-known [33], the presence of such poles allows for the possibility $\Delta^{-1}(0) \neq 0$. Note that not only the longitudinal but also one of the transverse parts contains such poles; this particular feature is motivated by the the one-loop analysis of the conventional three-gluon vertex presented in [34].

(v) Clearly, the Lorentz structures appearing in (3.1) do not exhaust all tensorial possibilities. Indeed, the most general parametrization of a vertex with three Lorentz indices and two independent momenta contains 14 linearly independent tensors [34, 35]; their number may be reduced to some extent by imposing Bose symmetry (only partial in our case) and the constraint of the corresponding WI. The simplified vertex proposed in (3.1) is only meant to capture some of the salient features expected from the full answer, most importantly the correct WI, the Bose symmetry, and the presence of poles terms.

(vi) The transverse parts of $\widetilde{\mathrm{I\Gamma}}^{\mu\alpha\beta}$ are multiplied by the constants c_1 and c_2, which, at this level of approximation, will be treated as arbitrary adjustable parameters. These constants affect not only the integral equation but also the two crucial boundary conditions [(4.7) and (4.28)]. As we will see in detail in the next sections, the existence or not of self-consistent solutions for the SDE (i.e. solutions satisfying all necessary constraints) depends crucially on the values chosen for these constants. At first sight this seems to run contrary to the standard lore of the gauge technique, according to which the transverse parts in the Ansätze for the vertices do not affect appreciably the solutions in the IR. Notice, however, that, in the case we consider here, the distinction between what is of UV and what of IR origin is not so sharp. Indeed, as we will see shortly, the value of $\Delta^{-1}(0)$ – clearly an IR quantity – is determined by integrals of the full propagator together with the running mass and coupling over the entire range of momenta; there the UV behavior of these quantities is essential. For example, the running of the masses in the UV is controlled by the anomalous dimensions, which depend themselves on c_1 and/or c_2. To be sure, in a more complete treatment, the values of c_1 and c_2 should be uniquely determined by the QCD dynamics, in particular by the SDE governing the vertex $\widetilde{\mathrm{I}\Gamma}^{\mu\alpha\beta}$; unfortunately, such an analysis is beyond our powers at the moment.

We next write (2.14) in the Euclidean space; to that end, we set $-q^2 = q^2_{\rm E}$, define $\Delta_{\rm E}(q^2_{\rm E}) = -\Delta(-q^2_{\rm E})$, and for the integration measure we have $[dk] = i[dk]_E = id^4k_E/(2\pi)^4$. Then, using for $\widetilde{\mathrm{I}\Gamma}_{\nu\alpha\beta}$ the Ansatz of (3.1), and suppressing the subscript "E", (2.14) becomes

$$
\begin{aligned}
\Delta^{-1}(q^2) \;&=\; q^2 - \frac{6\tilde{b}g^2}{5\pi^2}\left[\int d^4k\left(q^2+\frac{2}{3}\left[k^2-\frac{(k\cdot q)^2}{q^2}\right]\right)\Delta(k)\Delta(k+q) - \int d^4k\Delta(k)\right] \\
&- \frac{6\tilde{b}g^2}{5\pi^2}c_1\left[\int d^4k\,k^2\,\Delta(k)\Delta(k+q) - \int d^4k\Delta(k)\right] \\
&- \frac{6\tilde{b}g^2}{5\pi^2}c_2q^2\int d^4k\,[\Delta(k)-\Delta_0(k)]\,\Delta(k+q)\,, \qquad (3.3)
\end{aligned}
$$

where $\tilde{b} \equiv 10C_A/48\pi^2$ is the contribution of the gluons to the one-loop β function in the PT-BFM scheme; the discrepancy from the correct value $b = 11C_A/48\pi^2$ is due to the omission of the ghosts [8, 17].

The measure in spherical coordinates is given by

$$
\int d^4k = 2\pi\int_0^\pi d\chi\sin^2\chi\int_0^\infty dyy\,; \qquad (3.4)
$$

introducing $q^2 \equiv x$, $k^2 \equiv y$ and $(k+q)^2 \equiv z$, we have that $k\cdot q = \sqrt{xy}\cos\chi$, and so $(k\cdot q)^2/q^2 = y\cos^2\chi$, and $z = x+y+2\sqrt{xy}\cos\chi$. Due to the dependence of the unknown function on $\Delta(z)$, to solve (3.3) one should carry out (numerically) the two integrals defined in (3.4). Alternatively, one resorts to standard approximations (see (A-4)) for the integral over $d\chi$, thus reducing the numerics to a single one-dimensional integral. In our analysis we will use a modified version of the usual angular approximation (see Appendix).

The first important issue is the value that (3.3) furnishes for $\Delta^{-1}(0)$; in particular, in order to obtain IR-finite solutions, $\Delta^{-1}(0)$ should be non-vanishing. It turns out that $\Delta^{-1}(0)$ may be determined from (3.3) exactly, *before* doing any approximation. Specifically,

$$
\begin{aligned}
\int d^4k\frac{(k\cdot q)^2}{q^2}\Delta(k)\Delta(k+q)\bigg|_{q^2\to 0} \;&=\; 2\pi\int_0^\pi d\chi\sin^2\chi\cos^2\chi\int_0^\infty dyy^2\Delta^2(y) \\
&=\; \frac{1}{4}\int d^4k\,k^2\,\Delta^2(k)\,. \qquad (3.5)
\end{aligned}
$$

Thus, one obtains from (3.3)

$$
\Delta^{-1}(0) = \frac{3\tilde{b}g^2}{5\pi^2}\left[2(1+c_1)\int d^4k\,\Delta(k) - (1+2c_1)\int d^4k\,k^2\,\Delta^2(k)\right]. \qquad (3.6)
$$

Since the r.h.s. of (3.6) is divergent, in the next section this expression will be appropriately regularized [8], following the rules of dimensional regularization; there it will become clear why the UV running of the effective mass is of central importance.

Having determined the exact expression $\Delta^{-1}(0)$, one might be tempted to carry out the angular integration by employing the usual approximation of (A-4); however, particular care is needed, since the straightforward application of (A-4) would misrepresent quantitatively some essential features of the original Eq.(3.3), and most importantly the terms determining the running of the mass. To remedy this, in the Appendix we implement judicious modifications to some of the relevant results of the angular approximation.

Let us denote the r.h.s. of (3.3) by $I(q^2)$. Then we first write (3.3) schematically in the form

$$\begin{aligned} \Delta^{-1}(x) &= [I(x) - I(0)] + \Delta^{-1}(0) \\ &\approx [I_{MA}(x) - I_{MA}(0)] + \Delta^{-1}(0)\,, \end{aligned} \tag{3.7}$$

where the exact $\Delta^{-1}(0)$ is given by (3.6), and the subscript "MA" denotes that the (modified) angular approximation has been employed. Then, using for all terms appearing in (3.3) the expressions given in (A-3) and (A-21), after some algebra we arrive at the SDE

$$\Delta^{-1}(x) = Kx + \tilde{b}g^2 \sum_{i=1}^{8} A_i(x) + \Delta^{-1}(0)\,, \tag{3.8}$$

with

$$\begin{aligned} A_1(x) &= -\left(1 + \frac{6c_2}{5}\right) x \int_x^\infty dy\, y\Delta^2(y)\,, \\ A_2(x) &= \frac{6c_2}{5}\, x \int_x^\infty dy \Delta(y)\,, \\ A_3(x) &= -\left(1 + \frac{6c_2}{5}\right) x\Delta(x) \int_0^x dy\, y\Delta(y)\,, \\ A_4(x) &= \left(-\frac{1}{10} - \frac{3c_2}{5} + \frac{3c_1}{5}\right) \int_0^x dy\, y^2\Delta^2(y)\,, \\ A_5(x) &= -\frac{6}{5}\Big(1 + c_1\Big)\Delta(x) \int_0^x dy\, y^2\Delta(y)\,, \\ A_6(x) &= \frac{6c_2}{5} \int_0^x dy\, y\Delta(y)\,, \\ A_7(x) &= \frac{2}{5}\,\frac{\Delta(x)}{x} \int_0^x dy\, y^3\Delta(y)\,, \\ A_8(x) &= \frac{1}{5x} \int_0^x dy\, y^3\Delta^2(y)\,, \end{aligned} \tag{3.9}$$

and K is the wave-function renormalization constant, whose value is fixed by the renormalization condition $\Delta^{-1}(\mu^2) = \mu^2$, with $\mu^2 \gg \Lambda^2$, where Λ is the mass-scale of QCD.

As has been discussed extensively in the literature, due to the Abelian WIs satisfied by the PT effective Green's functions, $\Delta^{-1}(q^2)$ absorbs all the RG-logs [2, 15, 16, 36], exactly as happens in QED with the photon self-energy. Equivalently, since Z_g and $Z_{\widehat{A}}$, the renormalization constants of the gauge-coupling and the gluon self-energy, respectively, satisfy the QED relation $Z_g = Z_{\widehat{A}}^{-1/2}$ [17], the product $d(q^2) = g^2(\mu^2)\Delta(q^2, \mu^2)$ forms a RG-invariant (μ-independent) quantity; for large momenta q^2,

$$d(q^2) = \frac{g^2_{pert}(q^2)}{q^2}\,, \tag{3.10}$$

where $g^2_{pert}(q^2)$ is the perturbative limit of the RG-invariant effective charge of QCD, i.e.

$$g^2_{pert}(q^2) = \frac{g^2(\mu^2)}{1 + bg^2(\mu^2)\ln(q^2/\mu^2)} = \frac{1}{b\ln(q^2/\Lambda^2)}\,. \tag{3.11}$$

Notice however that Eq.(3.8) does not encode the correct RG behavior, in the sense that it cannot be cast in a form containing only RG invariant quantities, such as $d(q^2)$ and $g^2_{pert}(q^2)$. This can be ultimately traced back to the fact that the various gauge technique inspired Ansätze for the full vertices (in our case the three-gluon vertex) tend to mishandle their transverse (identically conserved parts); this, in turn, forces one to renormalize subtractively instead of multiplicatively. We emphasize that there exist systematic methods for the construction of transverse parts with the correct UV properties [37], but have not been extended to the case of the three-gluon vertex. Therefore, to restore the correct RGI properties, we follow the heuristic procedure first proposed in [2, 25], used recently also in [8]. Specifically, every $\Delta(t)$ $(t = y, x)$ appearing inside the integrals on the r.h.s. of (3.8) is to be multiplied (by hand) by a factor $1 + \tilde{b}g^2\ln(t/\mu^2) = g^2/g^2_{pert}(t)$, i.e. we carry out the replacement $\Delta(t) \to \left[g^2/g^2_{pert}(t)\right]\Delta(t)$ (with $b \to \tilde{b}$). Then, one may rewrite (3.8) in terms of two RG invariant quantities, $d(q^2)$, and

$$\mathcal{L}(q^2) \equiv g^{-2}_{pert}(q^2) = \tilde{b}\ln\left(q^2/\Lambda^2\right)\,, \tag{3.12}$$

as follows:

$$d^{-1}(x) = K'x + \tilde{b}\sum_{i=1}^{8}\widehat{A}_i(x) \,+ d^{-1}(0)\,, \tag{3.13}$$

with

$$
\begin{aligned}
\widehat{A}_1(x) &= -\left(1+\frac{6c_2}{5}\right) x \int_x^\infty dy\, y\, \mathcal{L}^2(y) d^2(y)\,, \\
\widehat{A}_2(x) &= \frac{6c_2}{5} x \int_x^\infty dy\, \mathcal{L}(y) d(y)\,, \\
\widehat{A}_3(x) &= -\left(1+\frac{6c_2}{5}\right) x\, \mathcal{L}(x) d(x) \int_0^x dy\, y\, \mathcal{L}(y) d(y)\,, \\
\widehat{A}_4(x) &= \left(-\frac{1}{10} - \frac{3c_2}{5} + \frac{3c_1}{5}\right) \int_0^x dy\, y^2\, \mathcal{L}^2(y) d^2(y)\,, \\
\widehat{A}_5(x) &= -\frac{6}{5}\left(1+c_1\right) \mathcal{L}(x) d(x) \int_0^x dy\, y^2\, \mathcal{L}(y) d(y)\,, \\
\widehat{A}_6(x) &= \frac{6c_2}{5} \int_0^x dy\, y\, \mathcal{L}(y) d(y)\,, \\
\widehat{A}_7(x) &= \frac{2}{5}\, \mathcal{L}(x)\, \frac{d(x)}{x} \int_0^x dy\, y^3\, \mathcal{L}(y) d(y)\,, \\
\widehat{A}_8(x) &= \frac{1}{5x} \int_0^x dy\, y^3\, \mathcal{L}^2(y) d^2(y)\,,
\end{aligned}
\tag{3.14}
$$

where $K' = K/g^2$, given in closed form by

$$
K' = 1 - \frac{\tilde{b}}{\mu^2} \sum_{i=1}^{8} \widehat{A}_i(\mu^2)\,, \tag{3.15}
$$

and

$$
d^{-1}(0) = \frac{3\tilde{b}}{5\pi^2} \left[2(1+c_1)\overline{\mathcal{T}}_0 - (1+2c_1)\overline{\mathcal{T}}_1 \right], \tag{3.16}
$$

with

$$
\begin{aligned}
\overline{\mathcal{T}}_0 &= \int d^4k\, \mathcal{L}(k^2)\, d(k^2)\,, \\
\overline{\mathcal{T}}_1 &= \int d^4k\, k^2\, \mathcal{L}^2(k^2)\, d^2(k^2)\,.
\end{aligned}
\tag{3.17}
$$

IV. THE UV BEHAVIOR OF THE EFFECTIVE GLUON MASS

In this section we will discuss in detail the UV behavior of the effective gluon mass obtained from the SDE of (3.13). As we will see, depending on the values of the parameters c_1 and c_2, originally appearing in the vertex Ansatz of (3.2), one obtains logarithmic or power-law running as two distinct dynamical possibilities.

We begin by briefly reviewing the importance of the running of the gluon mass for obtaining a finite value for $d^{-1}(0)$; the finiteness of $d^{-1}(0)$ is intimately linked to the renormalizability of the theory; if $d^{-1}(0)$ turned out to be divergent, there would be no consistent way to eliminate this divergence by absorbing it into the renormalization of the parameters appearing in the fundamental QCD Lagrangian.

As has been explained in [8], under special assumptions on the form of $d(k^2)$, the r.h.s. of (3.16) can be made finite by simply employing standard dimensional regularization results. Specifically, let us write $d(q^2)$ as

$$d(q^2) = g^2(q^2)\tilde{\Delta}(q^2)\,, \tag{4.1}$$

where

$$\tilde{\Delta}(q^2) = \frac{1}{q^2 + m^2(q^2)}\,, \tag{4.2}$$

and

$$g^2(q^2) = \left[\tilde{b}\ln\left(\frac{q^2 + f(q^2, m^2(q^2))}{\Lambda^2}\right)\right]^{-1}. \tag{4.3}$$

For large values of q^2, $\tilde{\Delta}(q^2) \to 1/q^2$ and $g^2(q^2) \to g^2_{pert}(q^2)$, and $d(q^2)$ reduces to its perturbative expression of (3.10). Then, it is relatively straightforward to show, using the elementary result

$$\int \frac{dz}{z\,(\ln z)^{1+\gamma}} = -\frac{1}{\gamma\,(\ln z)^{\gamma}}\,, \tag{4.4}$$

that the difference $\overline{\mathcal{T}}_0 - \overline{\mathcal{T}}_1$, given by

$$\begin{aligned} \overline{\mathcal{T}}_0 - \overline{\mathcal{T}}_1 \;=\; & \int d^4k\, \mathcal{L}^2(k^2)\, m^2(k^2)\, d^2(k^2) \\ & + \tilde{b}\int d^4k\, \mathcal{L}(k^2)\, g^2(k^2)\, d(k^2) \ln\left(1 + \frac{f(k^2, m^2(k^2))}{k^2}\right)\,, \end{aligned} \tag{4.5}$$

is finite, provided that $m^2(k^2)$ drops asymptotically at least as fast as $\ln^{-a}(k^2)$, with $a > 1$, and $f(k^2, m^2(k^2))$ as $\ln^{-c}(k^2)$, with $c > 0$.

Then, $\overline{\mathcal{T}}_0$ may be regularized simply by subtracting from it $\int d^4k/k^2 = 0$, i.e.

$$\begin{aligned} \overline{\mathcal{T}}_0^{\,\text{reg}} \;&=\; \int d^4k\left(\mathcal{L}(k^2)\, d(k^2) - \frac{1}{k^2}\right) \\ &=\; -\int d^4k \frac{m^2(k^2)\,\tilde{\Delta}(k^2)}{k^2} - \tilde{b}\int d^4k\; d(k^2)\,\ln\left(1 + \frac{f(k^2, m^2(k^2))}{k^2}\right). \end{aligned} \tag{4.6}$$

Thus, the regularized expression for $d^{-1}(0)$ is given by

$$d^{-1}_{\text{reg}}(0) = \frac{3\tilde{b}}{5\pi^2}\left[\overline{\mathcal{T}}_0^{\,\text{reg}} + (1 + 2c_1)(\overline{\mathcal{T}}_0 - \overline{\mathcal{T}}_1)\right]. \tag{4.7}$$

When solving (3.13) one must impose (4.7) as an additional constraint; we will refer to (4.7) as the "seagull-condition". The way this is done is by first choosing an arbitrary finite value for the $d^{-1}(0)$ appearing in (3.13) and then solving it numerically. The solution obtained for $d(q^2)$ must be first decomposed following (4.1)–(4.3), and then be substituted into the r.h.s. of (4.7); self-consistency requires that the resulting expression for $d_{\rm reg}^{-1}(0)$ must coincide with the initial value $d^{-1}(0)$. Evidently, the way that $m^2(q^2)$ runs is essential for this procedure, and can affect considerably the quantitative predictions.

In order to obtain from (3.13) the equation that determines the behavior of $m^2(x)$ at large x, first set in the r.h.s. of (3.14) $x\mathcal{L}(x)d(x) \to 1$, $\mathcal{L}(x)d(x) \to 1/x$, and $\mathcal{L}(y)d(y) = \tilde{\Delta}(y)$. Next, use the identity $y\tilde{\Delta}(y) = 1 - m^2(y)\tilde{\Delta}(y)$ in all $\widehat{A}_i(x)$, keeping only terms linear in m^2 (terms quadratic in m^2 are subleading and may be safely neglected). Then separate all contributions that go like x from those that go like m^2 on both sides, and match them up. This gives rise to two independent equations, one for the "kinetic" term, which simply reproduces the asymptotic behavior $x\ln x$ on both sides, and one for the terms with $m^2(x)$, given by

$$
\begin{aligned}
m^2(x)\ln x \;&=\; \tilde{b}^{-1}d^{-1}(0) + a_1\int_0^x dy\, m^2(y)\tilde{\Delta}(y) + \frac{a_2}{x}\int_0^x dy\, y\, m^2(y)\tilde{\Delta}(y) \\
&+\; \frac{a_3}{x^2}\int_0^x dy\, y^2 m^2(y)\tilde{\Delta}(y) + a_4 x\int_x^\infty dy\, m^2(y)\tilde{\Delta}^2(y)\,, \qquad (4.8)
\end{aligned}
$$

with

$$
a_1 = \frac{6}{5}(1+c_2-c_1)\,, \quad a_2 = \frac{4}{5}+\frac{6c_1}{5}\,, \quad a_3 = -\frac{2}{5}\,, \quad a_4 = 1+\frac{6c_2}{5}\,. \qquad (4.9)
$$

Then, rewrite the first integral on the r.h.s of (4.10) as $\int_0^x = \int_0^\infty - \int_x^\infty$ to obtain

$$
\begin{aligned}
m^2(x)\ln x \;&=\; \mathcal{C} - a_1\int_x^\infty dy\, m^2(y)\tilde{\Delta}(y) + \frac{a_2}{x}\int_0^x dy\, y\, m^2(y)\tilde{\Delta}(y) \\
&+\frac{a_3}{x^2}\int_0^x dy\, y^2 m^2(y)\tilde{\Delta}(y) + a_4 x\int_x^\infty dy\, m^2(y)\tilde{\Delta}^2(y)\,, \qquad (4.10)
\end{aligned}
$$

with

$$
\mathcal{C} \equiv \tilde{b}^{-1}d^{-1}(0) + a_1\int_0^\infty dy\, m^2(y)\tilde{\Delta}(y)\,. \qquad (4.11)
$$

As we will see in a moment, the two possible asymptotic solutions of physical interest for $m^2(x)$ are given by

$$
\begin{aligned}
m_1^2(x) \;&=\; \lambda_1^2(\ln x)^{-(1+\gamma_1)}\,, \qquad (4.12) \\
m_2^2(x) \;&=\; \frac{\lambda_2^4}{x}(\ln x)^{\gamma_2-1}\,, \qquad (4.13)
\end{aligned}
$$

where λ_1 and λ_2 are two mass-scales, and $\gamma_i > 0$, $i = 1, 2$. The possibility of power-law running for the effective gluon mass, as expressed by (4.13), was first conjectured in [2, 25], motivated by similar results in the study of chiral symmetry breaking. Indeed, notice the similarity between the solutions given in (4.12) and (4.13) for the effective gluon mass and those appearing in the more familiar context of the SDE (gap equation) for the quark self-energy; there, one finds the following two asymptotic solutions for the dynamically generated quark mass $M(x)$:

$$\begin{aligned} M_1(x) &= \mu_1(\ln x)^{-\gamma_f} \,, \\ M_2(x) &= \frac{\mu_2^3}{x}(\ln x)^{\gamma_f-1} \,, \end{aligned} \tag{4.14}$$

where $\gamma_f = 3C_f/16\pi^2 b$, with C_f the Casimir eigenvalue of the fundamental representation $[C_f = (N^2-1)/2N$ for $SU(N)]$ [38].

Now, the important point to appreciate is that, in order for (4.10) to have solutions vanishing in the UV, it is necessary that the constant term on the r.h.s. vanishes, i.e. $\mathcal{C} = 0$. If, for some reason, this condition cannot be implemented, the solution obtained will reach a constant value in the deep UV, thus invalidating the basic characteristic of the dynamically generated mass. Given that both $d^{-1}(0)$ and the integral appearing in (4.11) are manifestly positive quantities, one obvious necessary condition for obtaining $\mathcal{C} = 0$ is that $a_1 < 0$. This requirement, in turn, restricts the possible values of the parameters c_1 and c_2, through the equation defining a_1 (first in (4.9)). Assuming the correct sign for a_1, the way to actually enforce $\mathcal{C} = 0$ will be completely dynamical: one must look for masses with the appropriate momentum dependence such that the r.h.s. of (4.11) can be made equal to zero. As we will see in the next section, the condition (4.11) [or, its improved version, (4.28)] constrains the behavior of the dynamical mass in the IR and intermediate momentum regimes.

Next, we set $\mathcal{C} = 0$ in (4.10) and verify that indeed $m_1(x)$ or $m_2(x)$ [Eqs.(4.12) and (4.13)] satisfy it. The upshot of this analysis will be that when $m_1(x)$ is substituted into the r.h.s of (4.10) the first integral provides the solution, while all others are subleading, whereas for $m_2(x)$ the leading contribution comes from the second integral, and the other three are subleading. (The third and fourth integrals are thus subleading for both types of solutions).

In our demonstration we will employ the asymptotic property of the incomplete Γ function. The latter is defined as [39]

$$\Gamma(\alpha, u) = \int_u^\infty dt\, e^{-t}\, t^{\alpha-1} \,, \tag{4.15}$$

(with no restriction on the sign of α), and its asymptotic representation for large values of

$|u|$ is given by

$$\Gamma(\alpha, u) = u^{\alpha-1}e^{-u} + \mathcal{O}(|u|^{-1})\,. \tag{4.16}$$

To see in detail what happens in the case of the logarithmic running, substitute (4.12) into both sides of (4.10) and use that asymptotically $\tilde{\Delta}(y) \to y^{-1}$. Then

$$\begin{aligned}
\int_x^\infty dy\, \frac{m_1^2(y)}{y} &= \gamma_1^{-1}\, m_1^2(x)\, \ln x\,, && (4.17)\\
\frac{1}{x}\int_0^x dy\, m_1^2(y) &= m_1^2(x) + \mathcal{O}\left(1/\ln x\right)\,, && (4.18)\\
\frac{1}{x^2}\int_0^x dy\, y m_1^2(y) &= \frac{m_1^2(x)}{2} + \mathcal{O}\left(1/\ln x\right)\,, && (4.19)\\
x\int_x^\infty dy\, \frac{m_1^2(y)}{y^2} &= m_1^2(x) + \mathcal{O}\left(1/\ln x\right)\,. && (4.20)
\end{aligned}$$

In evaluating the integral of (4.17) we have used (4.4); for the remaining three we have set $y = e^{-t}$ in (4.18), $y = e^{-t/2}$ in (4.19), and $y = e^t$ in (4.20), to cast them into the form of the incomplete Γ function given in (4.15), and have subsequently used the asymptotic expression of (4.16).

Thus, for asymptotic values of x the dominant contribution comes from (4.17) Substituting into (4.10), we see that both sides can be made equal provided that

$$\gamma_1 = -a_1\,. \tag{4.21}$$

Since γ_1 must be positive, it follows that $a_1 < 0$, consistent with the requirement imposed by Eq. (4.11), as discussed after Eq. (4.14).

Let us next turn to the case of power running, substituting (4.13) into both sides of (4.10). Now the leading contribution comes from the integral proportional to a_2 in (4.10). Specifically,

$$\begin{aligned}
\int_x^\infty dy\, \frac{m_2^2(y)}{y} &= m_2^2(x) + \mathcal{O}\left(1/\ln x\right)\,, && (4.22)\\
\frac{1}{x}\int_0^x dy\, m_2^2(y) &= \gamma_2^{-1} m_2^2(x)\ln x + \frac{c'}{x}\,, && (4.23)\\
\frac{1}{x^2}\int_0^x dy\, y m_2^2(y) &= m_2^2(x) + \mathcal{O}\left(1/\ln x\right)\,, && (4.24)\\
x\int_x^\infty dy\, \frac{m_2^2(y)}{y^2} &= \frac{m_2^2(x)}{2} + \mathcal{O}\left(1/\ln x\right)\,. && (4.25)
\end{aligned}$$

In evaluating (4.23) we have used (4.4). The constant c' comes from the lower limit of the integral; it is finite, because in that limit one must use inside the integral the full

$y\tilde{\Delta}(y)$, which is infrared safe due to the presence of the mass. The term proportional to c' is suppressed by a factor $\ln^{\gamma_2} x$ (assuming $\gamma_2 > 0$) compared to the first term, and can therefore be neglected. As in the case of the logarithmic running, for the other three integrals we have used the appropriate change of variables to cast them into the incomplete Γ function, resorting again to its asymptotic expression. Thus, we conclude that $m_2^2(x)$ satisfies (4.10) provided that

$$\gamma_2 = a_2 \,. \tag{4.26}$$

This condition, in turn, constrains the possible values of c_1 appearing in the definition of a_2 (second of (4.9)).

We emphasize again that, for either of the two physically relevant possibilities given by (4.12) and (4.13) to be realized, the constant term on the l.h.s of (4.11) must be forced to vanish, by imposing the mass-condition

$$d^{-1}(0) = \gamma_1 \tilde{b} \int_0^\infty dy\, m^2(y) \tilde{\Delta}(y) \,. \tag{4.27}$$

In addition, it is important to mention that whereas the individual (4.12) and (4.13) are *separately* solutions of (4.10), a linear combination of the form $m^2(x) = C_1 m_1^2(x) + C_2 m_2^2(x)$ cannot be regarded as a solution. The reason is that for $m_1^2(x)$ to be a solution one neglects all terms in (4.18)-(4.20), which are, however, clearly larger than $m_2^2(x)$. This does not necessarily mean that the two runnings cannot coexist, it simply says that the possible coexistence cannot be self-consistently inferred from (4.10). For this reason, in the analysis of the next section the two possibilities will be treated separately. It should also be clear that the terms quadratic in $m_2^2(x)$ that have been dropped when deriving (4.8) are indeed subleading for both types of asymptotic behavior, (4.12) and (4.13).

Since in this section we have been mainly interested in the UV running of $m^2(x)$, in the analysis presented above we have used for the $g^2(q^2)$ of (4.3) its perturbative expression, given in (3.11), i.e. we have replaced $g^2(q^2) \to g^2_{pert}(q^2)$. A complete treatment, where the full expression for $g^2(q^2)$ is kept, does not affect in the least the conclusions regarding the UV behavior of $m^2(x)$, but modifies slightly the condition (4.27) in the IR. Specifically,

$$d^{-1}(0) = (1+\gamma_1)\, \tilde{b} \int_0^\infty dy\, y\, m^2(y)\, \mathcal{L}^2(y)\, d^2(y) - \tilde{b} \int_0^\infty dy\, m^2(y)\, \mathcal{L}(y)\, d(y) \,. \tag{4.28}$$

Evidently, in the limit $\mathcal{L}(y)\, d(y) \to \tilde{\Delta}(y)$, (4.28) reduces to (4.27), as it should. In the numerical analysis that follows we will always use (4.28), referring to it as the "mass-condition".

V. NUMERICAL ANALYSIS

In this section we will solve numerically the integral equation given in Eq.(3.13) supplemented by the renormalization condition (3.15), and subject to the two constrains imposed by the mass- and seagull-conditions, Eqs.(4.28) and (4.7), respectively. As mentioned already in the previous sections, and as we will see in detail in what follows, the first condition restricts the momentum-dependence of the mass in the intermediate and deep infrared regimes, while the latter furnishes essentially the value of $d^{-1}(0)$.

When dealing with this problem we have at our disposal three undetermined parameters: c_1 and c_2 appearing in the Ansatz for the three-gluon vertex [Eqs.(3.1) and (3.2)], and the value of $d^{-1}(0)$. It turns out that the simultaneous solution of the integral equation and its constrains restricts considerably the acceptable combinations of these parameters.

The strategy we will employ in our numerical analysis consists of the following main steps:

(**i**) We choose an arbitrary initial value for $d^{-1}(0)$, to be denoted by $d_{in}^{-1}(0)$, together with a set of values for c_1, c_2, and we substitute them into the integral equation (3.13), generating a solution for $d(q^2)$.

(**ii**) The solution for $d(q^2)$ obtained in (**i**) must be then decomposed as the product of $\tilde{\Delta}(q^2)$ and $g^2(q^2)$, according to Eqs.(4.1), (4.2), and (4.3). To do this, first a simple Ansatz for $m^2(q^2)$ is written down, which in the UV displays one of the two physically relevant asymptotic behaviors (logarithmic or power-law), while in the IR reaches a finite value. The generic form of these two types of Ansätze is given in (5.1) and (5.3), for logarithmic and power-law running, respectively. The anomalous dimensions γ_1 and γ_2 appearing there are linear combinations of c_1 and c_2 [given by (4.9),(4.21), and (4.26)], and control the behavior of $m^2(q^2)$ in the deep UV, whereas the parameters ρ and m_0 are free for the moment, and affect the intermediate and IR regions.

(**iii**) The integrals on the r.h.s. of the mass-condition (4.28) are evaluated numerically, using as input the Ansatz for $m^2(q^2)$ chosen in (**ii**), together with the numerical solution for $d(q^2)$; this furnishes a value for the $d^{-1}(0)$ on the l.h.s. By varying ρ and m_0 we try to make that $d^{-1}(0)$ match $d_{in}^{-1}(0)$ of step (**i**). We may or may not be able to do this, depending on the values of c_1 and c_2, and the type of Ansatz (logarithmic or power-law) chosen for the mass. In general, in the cases where (4.28) can be satisfied, one finds that this may be accomplished not just for one but for various sets of values for ρ and m_0. This, in turn, gives rise to a family of possible masses, which have a common behavior in the deep UV, but differ in the intermediate and IR regions.

(**iv**) Once the family of allowed masses has been determined from the mass-condition in (iii), one extracts from (4.1) the corresponding families of non-perturbative effective charges $g^2(q^2)$; one simply multiplies the numerical points of $d(q^2)$ by $[q^2 + m^2(q^2)]$. On physical grounds we require that the resulting effective charges should be *monotonically decreasing* functions of q^2; this requirement eliminates any member of the mass family that gives rise to effective charges with "bumps". All $g^2(q^2)$ so obtained display in the UV the logarithmic running expected from the one-loop RG (i.e. asymptotic freedom corresponding to $\beta = -\tilde{b}g^3$), and reach a finite value in the deep IR. For the implementation of the seagull-condition, (4.7), one must supply the functions $f(q^2, m^2(q^2))$, appearing in (4.3). Therefore, we must extract for each $g^2(q^2)$ the corresponding $f(q^2, m^2(q^2))$; the way this is done is by fitting the numerical points determining $g^2(q^2)$ by Eq. (4.3), assuming for $f(q^2, m^2(q^2))$ the expression of (5.2).

(**v**) We next substitute $g^2(q^2)$, $m^2(q^2)$ and $d(q^2)$ into the integrals on the r.h.s. of the seagull-condition (4.7), whose value is computed numerically; this furnishes the value for $d^{-1}(0)$ appearing on the r.h.s. If this value for $d^{-1}(0)$ does not coincide with $d_{in}^{-1}(0)$ [we require an accuracy of about 1 part in 10^3] a new set of values for c_1 and c_2 is chosen (keeping $d_{in}^{-1}(0)$ fixed), and the procedure is repeated from the beginning, until coincidence has been reached. At that point we consider to have found a solution, namely the $d(q^2)$ obtained for $d_{in}^{-1}(0)$ and the values for c_1 and c_2 used the last (and only "successful") iteration.

(**vi**) A different value for $d_{in}^{-1}(0)$ is chosen, and the procedure is repeated starting from step (**i**).

To solve the integral equation we employ a simple iterative procedure, where an initial guess is made for the solution $d(q^2)$ on a discretized momentum grid in the domain of $[0, \Lambda_{\rm UV}]$. More specifically, the grid is split in two regions $[0, \mu^2]$ and $(\mu^2, \Lambda_{\rm UV}]$ whose purpose is allow for the implementation of the renormalization condition, given by Eq. (3.15). Typically, we choose $\Lambda_{\rm UV} = 10^6\ \text{GeV}^2$, $\mu^2 = M_Z^2 = (91.18)^2\ \text{GeV}^2$ and we used as input a value of $\Lambda = 300$ MeV for the QCD mass-scale.

Our numerical analysis reveals a clear separation between the two types of asymptotic behavior for $m^2(q^2)$ depending on the values chosen for c_1 and c_2. Specifically, for $c_1 \in [0.15, 0.4]$

and $c_2 \in [-1.07, -0.92]$ the asymptotic behavior of $m^2(q^2)$ is given by Eq.(4.12), whereas for $c_1 \in [0.7, 1.3]$ and $c_2 \in [-1.35, -0.68]$ the $m^2(q^2)$ displays the power-law running of Eq.(4.13).

A. $m^2(q^2)$ with logarithmic running

When solving Eq.(3.13) choosing values for c_1 and c_2 from the intervals $c_1 \in [0.15, 0.4]$ and $c_2 \in [-1.07, -0.92]$, the constraints (4.28) and (4.7) can be simultaneously satisfied, and the $d(q^2)$ obtained may indeed be decomposed as in Eq.(4.1), with a functional Ansatz for the running mass of the form

$$m^2(q^2) = m_0^2 \left[\ln\left(\frac{q^2 + \rho\, m_0^2}{\Lambda^2}\right) \Big/ \ln\left(\frac{\rho\, m_0^2}{\Lambda^2}\right) \right]^{-1-\gamma_1}, \tag{5.1}$$

where $\gamma_1 = -a_1$ [see (4.21) and (4.9)]. Evidently, for large q^2 the above expression goes over to the logarithmic behavior described by Eq.(4.12), with $\lambda_1^2 = m_0^2 \left[\ln\left(\frac{\rho\, m_0^2}{\Lambda^2}\right)\right]^{1+\gamma_1}$. This simple Ansatz connects continuously the UV and IR regions; at $q^2 = 0$ reaches the finite value $m^2(0) = m_0^2$.

The parameters m_0 and ρ appearing in Eq.(5.1) control the way the mass runs in the intermediate and IR regions; their values are restricted by the mass-condition, Eq.(4.28). To impose the mass-condition, we first choose a random value for m_0, and then we search for values of ρ that satisfy Eq.(4.28). Even though this procedure does not single out a unique pair of values for m_0 and ρ, it restricts considerably their allowed range. In fact, the acceptable range for (m_0, ρ) gets further restricted by imposing the additional requirements that all the $m^2(q^2)$ and the effective charges generated subsequently from them (by multiplying $d(q^2)$ by $[q^2 + m^2(q^2)]$) should be monotonically decreasing functions of q^2. The combination of all these constraints leads eventually to rather stable results: if a pair (m_0, ρ) furnishes a consistent solution for a given (c_1, c_2) , then any other pair (m_0', ρ') is also a solution, provided that c_2 is only slightly adjusted (less than 5%).

The running couplings $g^2(q^2)$ obtained using Eq.(5.1) in Eq.(4.1) can be fitted very accurately by means of Eq.(4.3), with the function $f(q^2, m^2(q^2))$ given by

$$f(q^2, m^2(q^2)) = \rho_1 m^2(q^2) + \rho_2 \frac{m^4(q^2)}{q^2 + m^2(q^2)} + \rho_3 \frac{m^6(q^2)}{[q^2 + m^2(q^2)]^2}\,, \tag{5.2}$$

In Fig.(2) we present a typical solution for $d(q^2)$, $m^2(q^2)$, and the effective charge $\alpha(q^2) = g^2(q^2)/4\pi$, respectively.

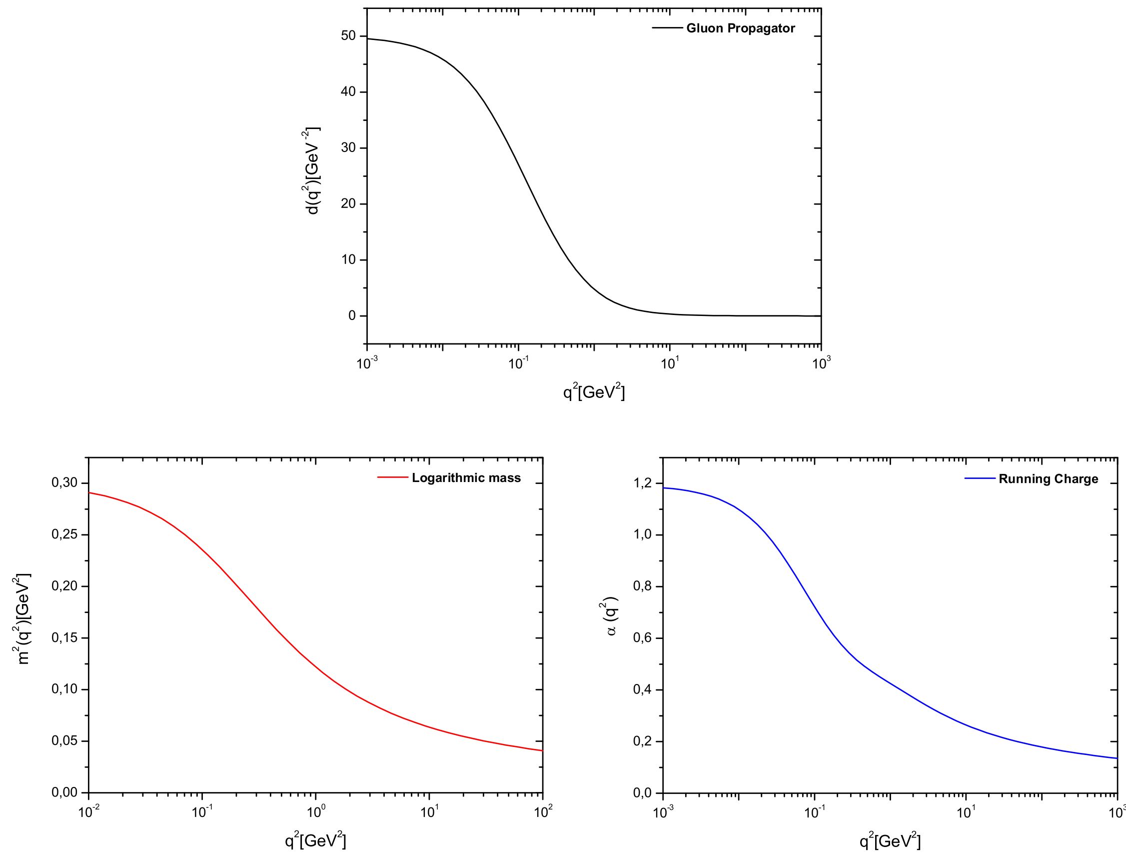

FIG. 2: Solutions obtained for the choice $d^{-1}(0) = 0.02$ GeV^2, $c_1 = 0.15$ and $c_2 = -0.9635$. Upper panel: the numerical solution for $d(q^2)$. Lower panels: the logarithmic dynamical mass, $m^2(q^2)$, for $m_0^2 = 0.3$ GeV^2 and $\rho = 1.007$ in the Eq.(5.1). On the right panel we show the running charge, $\alpha(q^2) = g^2(q^2)/4\pi$, which can be fitted using Eqs.(4.3) and (5.2) with $\rho_1 = 6.378$, $\rho_2 = -8.984$ and $\rho_3 = 3.466$.

B. $m^2(q^2)$ with power-law running

As we increase c_1, Eq.(4.28) can not be satisfied if we insist on imposing the logarithmic running for $m^2(q^2)$. In particular, for $c_1 \in [0.7, 1.3]$, $c_2 \in [-1.35, -0.68]$ and $d^{-1}(0) \in [0.01 \text{ GeV}^2, 0.04 \text{ GeV}^2]$ we have verified that the mass-condition can be satisfied only if instead of Eq.(5.1) we use the following functional form for the running mass

$$m^2(q^2) = \frac{m_0^4}{q^2 + m_0^2}\left[\ln\left(\frac{q^2 + \rho\, m_0^2}{\Lambda^2}\right) \Big/ \ln\left(\frac{\rho\, m_0^2}{\Lambda^2}\right)\right]^{\gamma_2 - 1}, \tag{5.3}$$

with $\gamma_2 = a_2$ [see (4.26) and (4.9)]. Evidently, this Ansatz corresponds to power-law running for the mass; for large q^2, the $m^2(q^2)$ goes over to the solution denoted by $m_2^2(q^2)$ in (4.13), with $\lambda_2^4 = m_0^4 \left[\ln\left(\frac{\rho\, m_0^2}{\Lambda^2}\right)\right]^{1-\gamma_2}$. Clearly, (5.3) is the simplest extension of the asymptotic expression (4.13) to the entire range of momenta, from the UV all the way down to $q^2 = 0$, where it assumes the finite value $m^2(0) = m_0^2$.

As in the previous case, only those sets of m_0 and ρ that satisfy the mass-condition (4.28) are allowed. In fact, in the case of the power-law running this condition turns out to be significantly more restrictive than in the logarithmic case. The reason is that, due to the faster decrease of the mass in the UV, the leading contribution to the mass-condition comes now from the intermediate and IR regions; therefore, the result is much more sensitivity to small variations of m_0 and ρ.

A typical solution for a choice of c_1, c_2, and $d^{-1}(0)$ within the aforementioned ranges is shown in Fig.(3). The $d(q^2)$ is decomposed according to (4.1) into an $m^2(q^2)$ of the general form given in Eq.(5.3) and an effective charge $\alpha(q^2)$; the latter is fitted using again (4.3) and (5.2).

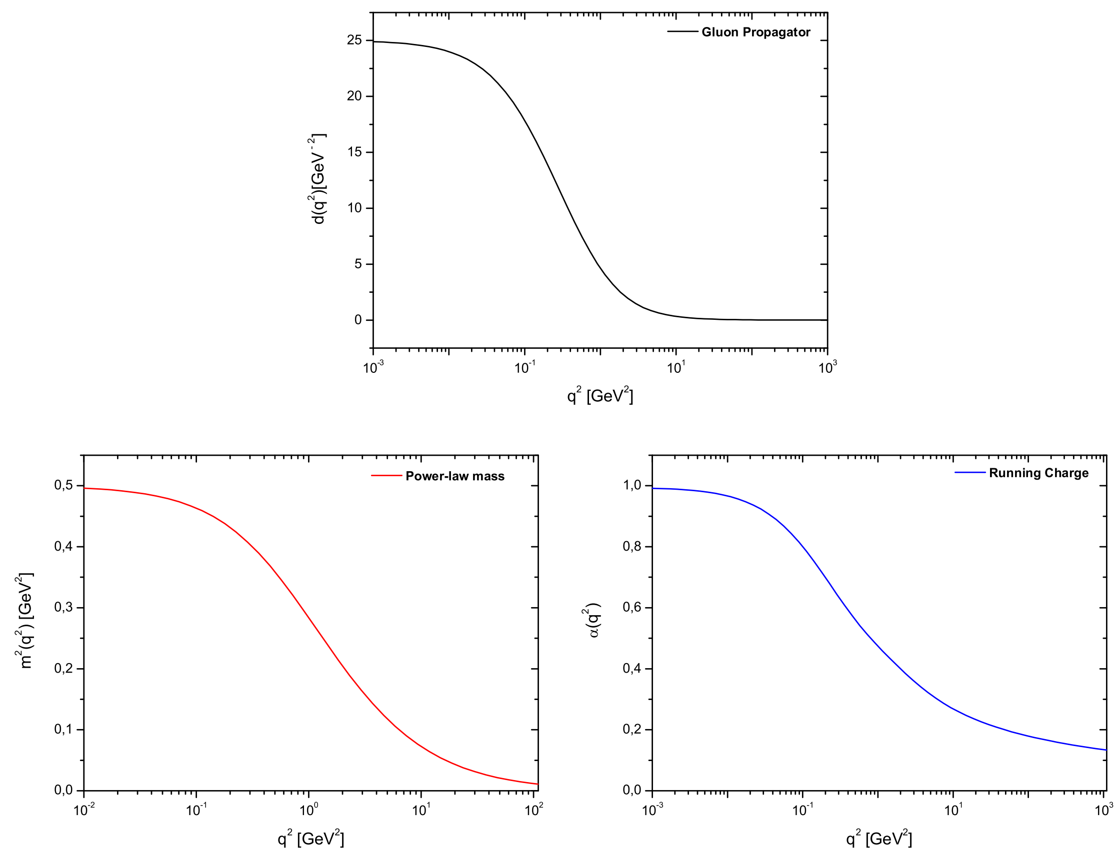

FIG. 3: Solutions obtained for the choice $d^{-1}(0) = 0.04$ GeV^2, $c_1 = 1.1$ and $c_2 = -1.121$. Upper panel: the numerical solution for $d(q^2)$. Lower panels: the power-law dynamical mass, $m^2(q^2)$ for $m_0^2 = 0.5$ GeV^2 and $\rho = 1.046$ in the Eq.(5.3). On the right panel we show the running charge, $\alpha(q^2) = g^2(q^2)/4\pi$, which can be fitted by Eqs.(4.3) and (5.2) with $\rho_1 = 1.205$, $\rho_2 = -0.690$ and $\rho_3 = 0.121$.

In the upper panels of Fig.(4) we plot a series of effective charges obtained by fixing different set of values for $d^{-1}(0)$, c_1, and c_2. All these coupling were subjected to the constraints imposed by Eqs.(4.28) and (4.7); the corresponding values for c_1 and c_2 are given in the legend. The respective logarithmic and power-law masses are shown in the lower panels. Observe that $\alpha(0)$ shows a strong dependence on the values chosen for c_1 and c_2; this is so, even if the same value for $d^{-1}(0)$ is chosen. The difference between effective charges obtained with the same $d^{-1}(0)$ is due to the fact that one is forced to use different values of m_0^2 in order to satisfy the mass-condition. Thus, by changing the value of m_0^2, one obtains different values of $\alpha(0)$ for the same $d^{-1}(0)$.

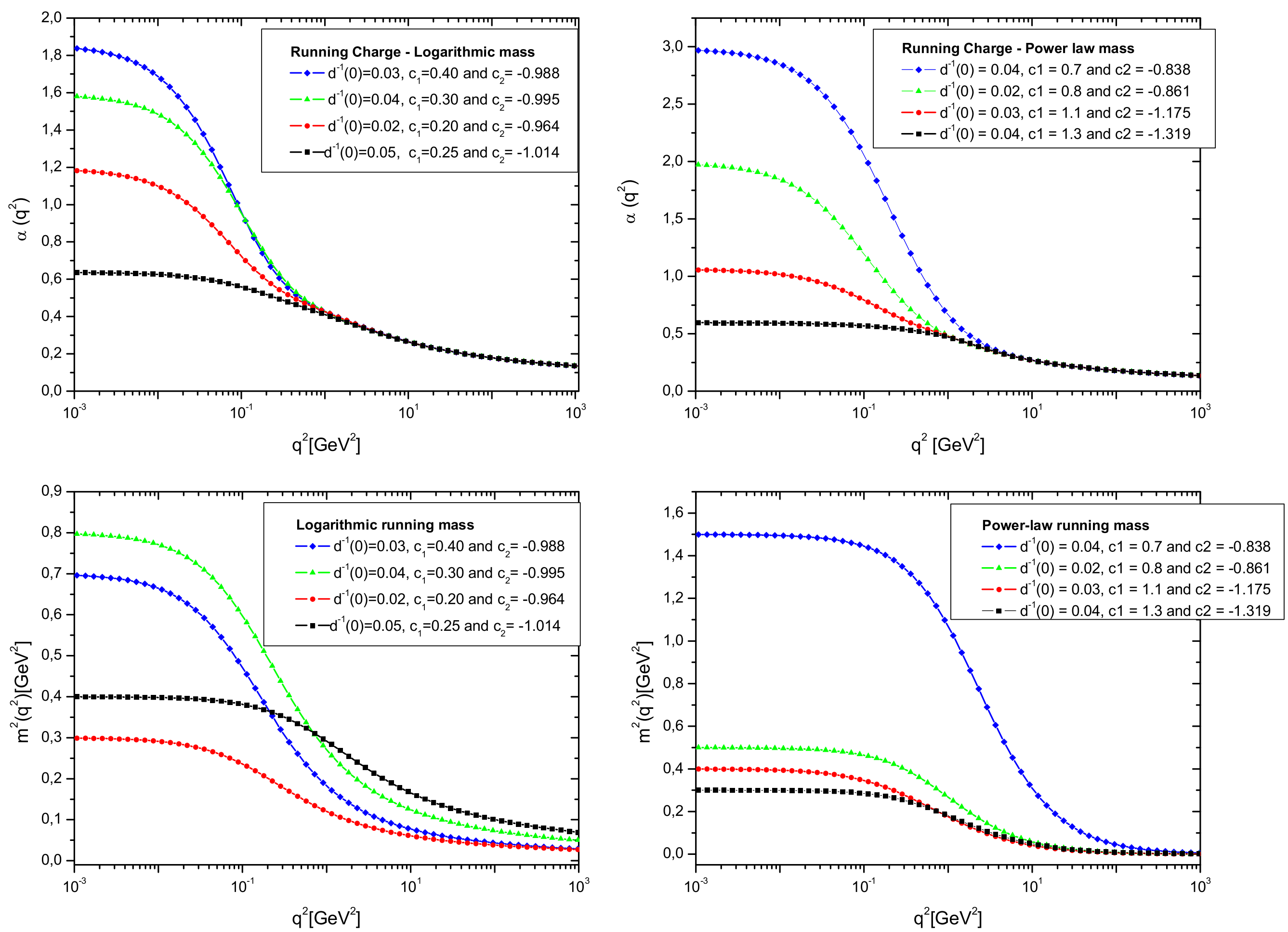

FIG. 4: Upper panels: The running charges, $\alpha(q^2) = g^2(q^2)/4\pi$, corresponding to the different choices of $d^{-1}(0)$, c_1 and c_2 for the logarithmic running mass (left panel) and for the power-law running mass (right panel). The corresponding masses are plotted in the lower panels.

In Fig.(5) we show a comparison between logarithmic and a power-law running, by choosing c_1 and c_2 from the corresponding intervals, for fixed values of $d^{-1}(0)$ and m_0^2. The corresponding $d(q^2)$ are shown in the upper panel, whereas the running masses and couplings are plotted in the lower left and right panels, respectively. The faster decrease of the power-law running mass in the UV is clearly visible. It is evident that, as already mentioned, in the power-law case the leading contribution to the mass-condition comes mainly from the IR and intermediate regions, while in the logarithmic case the UV region provides a considerable support. Since the UV behavior of the two $d(q^2)$ and $\alpha(q^2)$ is essentially fixed by asymptotic freedom, whereas their IR regimes are determined, to a large extent, by the value of m_0^2, the difference between the two cases is perceptible only in the intermediate momentum region.

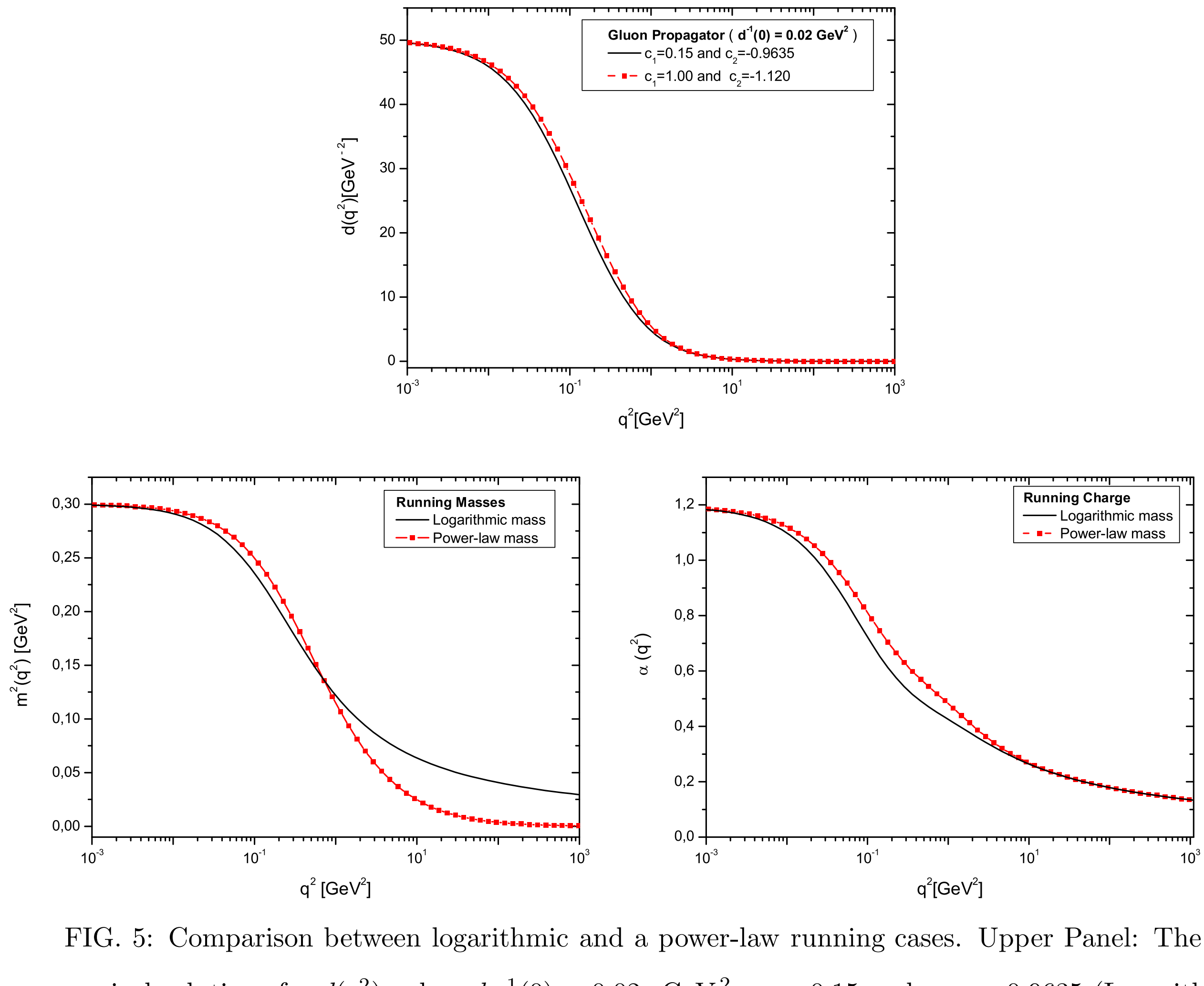

FIG. 5: Comparison between logarithmic and a power-law running cases. Upper Panel: The numerical solutions for $d(q^2)$, when $d^{-1}(0) = 0.02,\ \mathrm{GeV}^2$, $c_1 = 0.15$ and $c_2 = -0.9635$ (Logarithmic case) and for $c_1 = 1.00$ and $c_2 = -1.12$ (Power-law). Lower Panels: The corresponding logarithmic and a power-law running masses and the running charges.

VI. DISCUSSION AND CONCLUSIONS

In this article we have studied both analytically and numerically the non-linear SDE for the gluon self-energy in the PT-BFM formalism, focusing on the dynamical generation of a gauge-invariant infrared cutoff. In particular, we have established the existence of IR-finite solutions, i.e. solutions that are finite in the entire range of momenta, displaying asymptotic freedom in the UV and reaching a finite positive value at $q^2 = 0$. This non-perturbative behavior may be described in terms of an effective gluon mass, whose presence tames the perturbative Landau singularity and prevents $\Delta(q^2)$ from diverging in the IR. Just as happens with the constituent quark masses, this dynamical gluon mass depends non-trivially on the momentum. Our study of the non-linear SDE has revealed that the dynamical mass $m^2(q^2)$ may have two different types of functional dependence on q^2 in the deep UV: (i) $m^2(q^2)$ drops as an inverse power of a logarithmic; this behavior has also been found in the studies of linearized SDE, and (ii) $m^2(q^2)$ with power-law running, i.e. the mass drops off as $1/q^2$. This type of solution is found for the first time in the context of a SDE. At the level of the SDE we study either type of asymptotic behavior (logarithmic and power-law) may be obtained, depending on the details of the three-gluon vertex. The latter depends on two parameters, c_1 and c_2, which control the relative contribution of its various tensorial structures. Our numerical analysis reveals that the sets of values for c_1 and c_2 that give rise to logarithmic running belong to an interval that is disjoint and well-separated from that producing power-law running.

The possibility of gluon masses falling like the inverse square of the momentum has been anticipated [2] by analogy with the constituent quark masses generated from the standard gap equation for the quark self-energy [second equation in (4.14)]. In addition, general OPE considerations support the existence of a $m^2(q^2)$ displaying power-law running. Assuming that the OPE holds for a quantity like $m^2(q^2)$, and given that, in the absence of quarks, $\langle G^2\rangle$ is the lowest order (dimension four) local gauge-invariant condensate, then one would expect that asymptotically, and up to logarithms, $m^2(q^2) \sim \langle G^2\rangle/q^2$, exactly as was found in [26].

To be sure, the various connections between the effective gluon mass and the OPE, the gluon condensates, and the QCD sum rules, deserve a detailed, in-depth study. One issue is the type of modifications induced to the OPE predictions for observables (i.e. correlators of gauge-invariant currents) if one were to use in their calculation gluon propagators with a dynamical (or even hard) mass, as was first done in [40]. In addition, despite important contributions in this direction [26, 41, 42], a definite, first-principle relation between m^2 and $\langle G^2 \rangle$ (or other condensates [45]) still eludes us. In the context of the SDE this is mainly because the CJT formalism [43] has not been yet fully adapted to treat gluon mass generation in a consistent way. Qualitatively speaking, the CJT effective potential V is given by

$$V = -\frac{1}{2}\mathrm{Tr}\ln\left(\Delta\Delta_0^{-1}\right) + \frac{1}{2}\left(\mathrm{Tr}\Delta\Delta_0^{-1} - 1\right) + V_{2\mathrm{PI}} \tag{6.1}$$

where the trace is taken in the functional sense, and $V_{2\mathrm{PI}}$ denotes the contributions from the (appropriately dressed) two-particle irreducible graphs. In the original formulation V is a functional of the conventional gluon propagators, and higher point Green functions; its extremization with respect to any of yields the corresponding SDE's. To make reliable contact with the results of the BFM-PT, one should modify Eq.(6.1) appropriately, expressing it in terms of the gauge-invariant PT gluon propagator. Thus, the gluon mass will enter into V through the massive gluon propagators; then, the minimization of V will yield a theoretical expression for the energy density of the QCD vacuum, which must be set equal to the experimental value obtained using QCD sum rules. To date, the aforementioned modifications to V have been carried out at the two-loop level only [44]; clearly, their all-order generalization would be of great interest.

It is clear from the analysis presented that the role of the three-gluon vertex $\widetilde{\mathrm{I\!\Gamma}}_{\mu\alpha\beta}$ vertex is absolutely central. Specifically, the tensorial structure of the vertex, the presence or absence of kinematic poles, and the relative strength between the various components are determining factors for the existence of IR-finite solutions, and the type of running of the effective gluon mass. As we have mentioned in Sec. III, the Ansatz of (3.1) employed for the vertex attempts to capture some of the main features, but should be eventually obtained from an independent study of the dynamical SDE that it satisfies. The rich tensorial structure of a vertex with three Lorentz indices turns such a study into a rather complicated task, given that, in general, one has to deal with fourteen form-factors. However, as a first approximation, one can focus on those form-factors that enter into the expression determining $\Delta^{-1}(0)$, i.e. the generalized version of (3.6). As an alternative, one could try to improve the Ansätze employed for the vertex, in the spirit of [37], in an attempt to correctly incorporate the required asymptotic behavior into the SDE, i.e. without having to resort to the heuristic procedure followed here.

Another source of relative uncertainty when evaluating the SDE in question is the use of the angular approximation. As we explained in detail, the standard version of this approximation gives rise to an approximate SDE that does not capture faithfully some of the essential features of the original SDE. To ameliorate this drawback we have introduced modifications to the angular approximation, presented in the Appendix. In our opinion the qualitative conclusions of this article are robust and do not depend on its use; we expect them to persist a more complete study, where the angular integration is carried out numerically, without resorting to any approximation. Should quantitative discrepancies arise, they will mainly manifest themselves in changes of the values of c_1 and c_2, which, at this level of approximation, are free to vary. To be sure, once dynamical information for the three-gluon vertex has been furnished one would be more restricted, and the two-dimensional integration should be carried out in its entirety.

Last but not least, let us turn to the possible role of the ghost sector. As we have amply emphasized, in the PT-BFM scheme the omission of the ghost loops does not interfere with the gauge-invariance of the final answer, and in particular with the transversality of the gluon self-energy. In addition, neglecting ghost contributions only affects the RG-logarithms by about only 10%, the difference between $b = 11C_A/48\pi^2$ and $\tilde{b} = 10C_A/48\pi^2$. The possible impact of the ghosts in the IR is, however, an entirely different matter; it will greatly depend, among other things, on the structure and solutions of the corresponding SDE for the ghost propagator within the PT-BFM formalism. Actually, given that in this latter framework one is working in the Feynman gauge (of the BFM) there is no a-priori reason that would exclude the possibility of obtaining IR-finite solutions for the ghost propagator. If this turned out to be true, it would suggest that in the gluon-mass description of QCD the ghost sector may not play such a central role as in the "ghost-dominance" picture [48], obtained when working in the (conventional) Landau gauge. A preliminary study of these issues is already underway, and we hope to report its results in the near future.

APPENDIX A: MODIFIED ANGULAR APPROXIMATION

In this Appendix we discuss the technical details related to the angular approximation and its modification.

It is convenient to introduce the following quantities, appearing on the r.h.s. of (3.3):

$$\begin{aligned}
I_0(q^2) &\equiv q^2 \int d^4k\, \Delta_0(k)\Delta(k+q)\,, \\
I_1(q^2) &\equiv q^2 \int d^4k\, \Delta(k)\Delta(k+q)\,, \\
I_2(q^2) &\equiv \int d^4k\, k^2\Delta(k)\Delta(k+q)\,, \\
I_3(q^2) &\equiv \frac{1}{q^2}\int d^4k\, k^2\,\left[k^2-(k+q)^2\right]\Delta(k)\Delta(k+q)\,. \qquad \text{(A-1)}
\end{aligned}$$

Note that

$$\int d^4k \frac{(k\cdot q)^2}{q^2}\Delta(k)\Delta(k+q) = \frac{1}{4}\,I_1(q^2)+\frac{1}{2}\,I_3(q^2)\,. \qquad \text{(A-2)}$$

The angular integration of $I_0(q)$ may be carried out exactly, after shifting the integration variable $k+q\to k$,

$$\begin{aligned}
I_0(q^2) &= q^2\int d^4k\,\frac{\Delta(k)}{(k+q)^2} \\
&= \pi^2\left[\int_0^x dyy\Delta(y) + x\int_x^\infty dy\Delta(y)\right]. \qquad \text{(A-3)}
\end{aligned}$$

For the remaining integrals we will use the angular approximation, appropriately modified to account correctly for their contributions to the running of the mass $m^2(x)$, i.e. terms that will enter in Eq. (4.8).

The standard angular approximation amounts to

$$\int_0^\pi d\chi\sin^2\chi\, f(z) \approx \frac{\pi}{2}\Big[\theta(x-y)f(x)+\theta(y-x)f(y)\Big], \qquad \text{(A-4)}$$

with $z=x+y+2\sqrt{xy}\cos\chi$, and $\theta(x)$ is the Heaviside step function.

Defining

$$\bar{I}_i(x)\equiv \pi^{-2}[I_i(x)-I_i(0)]\,, \qquad \text{(A-5)}$$

and using the superscript "A" to indicate that the aforementioned standard angular approximation has been employed, we obtain

$$\begin{aligned}
\bar{I}_1^{A}(x) &= x\Delta(x)\int_0^x dy\, y\Delta(y) + x\int_x^\infty dy y\Delta^2(y)\,, \\
\bar{I}_2^{A}(x) &= \Delta(x)\int_0^x dy\, y^2\Delta(y) - \int_0^x dy y^2\Delta^2(y)\,, \\
\bar{I}_3^{A}(x) &= \frac{\Delta(x)}{x}\int_0^x dy\, y^2(y-x)\Delta(y)\,.
\end{aligned} \tag{A-6}$$

Let us now check the faithfulness of these expressions for a special form of the massive propagator. Specifically, substitute in (A-6) Δ by the $\tilde{\Delta}$ of (4.2), apply the identity $y\tilde{\Delta}(y) = 1 - m^2(y)\tilde{\Delta}(y)$, keeping only the terms linear in $m^2(y)$ and set $\tilde{\Delta}(x) \to x^{-1}$, to obtain

$$\begin{aligned}
\bar{I}_1^{A}(x)|_{m^2} &= -\int_0^x dy\, m^2(y)\tilde{\Delta}(y) - x\int_x^\infty dy\, m^2(y)\tilde{\Delta}^2(y)\,, \\
\bar{I}_2^{A}(x)|_{m^2} &= 2\int_0^x dy\, m^2(y)\tilde{\Delta}(y) - \frac{1}{x}\int_0^x dy\, y\, m^2(y)\tilde{\Delta}(y)\,, \\
\bar{I}_3^{A}(x)|_{m^2} &= \frac{1}{x}\int_0^x dy\, y\, m^2(y)\tilde{\Delta}(y) - \frac{1}{x^2}\int_0^x dy\, y^2 m^2(y)\tilde{\Delta}(y)\,.
\end{aligned} \tag{A-7}$$

Now the result in (A-7) is to be compared with the direct calculation of $\bar{I}_i(x)$, *before* resorting to (A-4). We begin with $I_2(q^2)$; by substituting Δ by $\tilde{\Delta}$ into $I_2(q^2)$ of (A-1) and isolating again the mass terms, we have

$$I_2(q^2) = \int d^4k\, \tilde{\Delta}(k) - \int d^4k\, m^2(k)\tilde{\Delta}(k)\tilde{\Delta}(k+q)\,, \tag{A-8}$$

and

$$I_2(q^2) - I_2(0) = \int d^4k\, m^2(k)\tilde{\Delta}(k)\left[\tilde{\Delta}(k) - \tilde{\Delta}(k+q)\right]\,. \tag{A-9}$$

This is the *exact mass dependence*. To this last expression we now apply (A-4), to get

$$\begin{aligned}
\bar{I}_2^{m^2}(x)|_{A} &= \int_0^x dy\, y\, m^2(y)\tilde{\Delta}(y)\left[\tilde{\Delta}(y) - \tilde{\Delta}(x)\right] \\
&= \int_0^x dy\, m^2(y)\tilde{\Delta}(y) - \frac{1}{x}\int_0^x dy\, y\, m^2(y)\tilde{\Delta}(y) \; + \mathcal{O}(m^4)\,.
\end{aligned} \tag{A-10}$$

Comparing $\bar{I}_2^{m^2}(x)|_{A}$ with $\bar{I}_2^{A}(x)|_{m^2}$ [Eqs.(A-10) and (A-7)] we see a discrepancy of a factor of 2 in their first term. The simplest term, *quadratic in* Δ, that could correct this discrepancy is $\frac{1}{2}\int_0^x dy y^2\Delta^2(y)$; adding it to the result of the standard angular approximation, $\bar{I}_2^{A}(x)$ [Eq.(A-6)], will lead to a modified $\bar{I}_2^{A}(x)|_{m^2}$, that will coincide with $\bar{I}_2^{m^2}(x)|_{A}$.

We will next repeat the same exercise for $I_3(q^2)$. It is elementary to show, by substituting $\Delta \to \tilde{\Delta}$ into $I_3(q^2)$ that

$$I_3(q^2) = \frac{1}{q^2}\int d^4k \,\left[m^2(k+q) - m^2(k)\right]\, k^2\tilde{\Delta}(k)\tilde{\Delta}(k+q) \,+\, \int d^4k\,\tilde{\Delta}(k)\,, \tag{A-11}$$

and therefore, applying the angular approximation and subtracting at $x=0$,

$$\begin{aligned} \bar{I}_3^{m^2}(x)|_A &= \frac{\tilde{\Delta}(x)}{x}\int_0^x dy\, y^2\left[m^2(x) - m^2(y)\right]\,\tilde{\Delta}(y) \\ &\sim \frac{m^2(x)}{2} - \frac{1}{x^2}\int_0^x dy\, y^2 m^2(y)\,\tilde{\Delta}(y)\,. \end{aligned} \tag{A-12}$$

This term is subleading for either type of running of the mass (logarithmic or power-law). This important property is obviously *not* captured by the approximate expression of $\bar{I}_3^A(x)|_{m^2}$ in (A-7), since the first term would furnish (erroneously) a leading order contribution in the case of power-law running. This shortcoming may be remedied by simply adding to $\bar{I}_3^A(x)$, Eq.(A-6), the term $\frac{1}{2x}\int_0^x dyy^3\Delta^2(y)$; thus, the subleading nature of the term $I_3(q)$ is preserved.

We finally turn to $I_1(q^2)$. The most immediate way to see that the standard angular approximation for I_1, given by the first equation in (A-6), mistreats the running of the mass, is to set $\Delta(y) \to \tilde{\Delta}(y)$ and then $m^2(y) = m^2$ in the initial expression for $I_1(q^2)$ (second equation in (A-1)). Then $I_1(q^2)$ gets reduced to a standard one-loop integral,

$$I_1(q^2) = q^2\int\frac{d^4k}{(k^2+m^2)[(k+q)^2+m^2]}\,, \tag{A-13}$$

and thus $\bar{I}_1(q^2)$ is given by

$$\bar{I}_1(q^2) = -q^2\left[c - 2 + \ln\left(\frac{m^2}{\mu^2}\right) + D\ln\left(\frac{D+1}{D-1}\right)\right]\,, \tag{A-14}$$

where $c = -\frac{2}{\epsilon} + \gamma - \ln 4\pi$ and $D = \left(1 + \frac{4m^2}{q^2}\right)^{1/2}$. For large values of q^2 we have that the finite part of $\bar{I}_1(q^2)$ goes like

$$\bar{I}_1(x) \sim -x\ln x - 2m^2\ln x\,. \tag{A-15}$$

Notice that the last term, which determines the contribution to the running of the mass, comes with weight 2 compared to the logarithmic term determining the asymptotic running of $\Delta^{-1}(x)$ (remember that $I_1(q^2)$ is to be multiplied by $(-\tilde{b})$, so the first term of (A-14) provides the RG logarithm). Instead, setting $\Delta(y) \to \tilde{\Delta}(y)$ and $m^2(y) = m^2$ in $\bar{I}_1^A(x)$ and $\bar{I}_1^A(x)|_{m^2}$, [Eqs.(A-6) and (A-7)], we find as leading contribution

$$\begin{aligned} \bar{I}_1^A(x)|_{m^2} &\sim -m^2 \ln x \\ \bar{I}_1^A(x) &\sim -x \ln x - m^2 \ln x \,. \end{aligned} \tag{A-16}$$

Unlike the (exact) expression of (A-15), now the two logarithms appear with equal relative weight. Evidently, the angular approximation captures correctly the leading (RG) logarithm, but furnishes only half of the $m^2 \ln x$ contribution.

There is another way to verify that the correct relative weight between the two logarithms of $I_1(q^2)$ is 2. When the spectral representation for $\Delta(q)$ is used,

$$\Delta(q^2) = \int d\lambda^2 \, \frac{\rho\,(\lambda^2)}{q^2 - \lambda^2 + i\epsilon} \,, \tag{A-17}$$

the "kinetic" term of the resulting (linearized) SDE, i.e. the exact analogue to $-I_1(q^2)$, is given by

$$B(q^2) = q^2 \int d\lambda^2 \, \rho\,(\lambda^2) \int d^4k \, \Delta(k) \, \Delta(k+q) \,, \tag{A-18}$$

and may be easily cast in the form

$$B(x) = cx + x \int_0^{x/4} dy \left(1 - \frac{4y}{x}\right)^{1/2} \Delta(y) \,, \tag{A-19}$$

where c is a (divergent) constant, to be absorbed into the wave-function renormalization of the gluon field. When determining the contribution of $B(x)$ to the running of m^2 for large x, it would be wrong to simply set $\left(1 - \frac{4y}{x}\right)^{1/2} \to 1$; instead, one must expand this term to first order. Setting $\Delta(y) \to \tilde{\Delta}(y)$ we obtain

$$\begin{aligned} B(x) &= cx + x \int_0^{x/4} dy \tilde{\Delta}(y) \; - 2 \int_0^{x/4} dy \, y \tilde{\Delta}(y) \\ &\sim c'x + x \ln x + 2 \int_0^{x/4} dy \, m^2(y) \tilde{\Delta}(y) \,, \end{aligned} \tag{A-20}$$

where c' includes now some irrelevant (finite) constant term. Again, the mass term is multiplied by a factor of 2 compared to the RG logarithm.

The simplest way of remedy this discrepancy is to modify the result of the angular approximation for $I_1(q^2)$, adding the term $\frac{1}{2} \int_0^x dy y^2 \Delta^2(y)$; note that this is exactly the term that has been added to $I_2(q^2)$.

Thus, finally, we arrive at the following modified expressions, which capture correctly the leading m^2-dependence of the SDE:

$$
\begin{aligned}
\bar{I}_1^{MA}(x) &= x\Delta(x)\int_0^x dy\, y\Delta(y) + \frac{1}{2}\int_0^x dy y^2\Delta^2(y) + x\int_x^\infty dy\, y\Delta^2(y)\,,\\
\bar{I}_2^{MA}(x) &= \Delta(x)\int_0^x dy\, y^2\Delta(y) - \frac{1}{2}\int_0^x dy\, y^2\Delta^2(y)\,,\\
\bar{I}_3^{MA}(x) &= \frac{\Delta(x)}{x}\int_0^x dy\, y^2(y-x)\Delta(y) + \frac{1}{2x}\int_0^x dy\, y^3\Delta^2(y)\,.
\end{aligned}
\tag{A-21}
$$

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