The dynamical equation of the effective gluon mass

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Abstract

In this article we derive the integral equation that controls the momentum dependence of the effective gluon mass in the Landau gauge. This is accomplished by means of a well-defined separation of the corresponding "one-loop dressed" Schwinger-Dyson equation into two distinct contributions, one associated with the mass and one with the standard kinetic part of the gluon. The entire construction relies on the existence of a longitudinally coupled vertex of nonperturbative origin, which enforces gauge invariance in the presence of a dynamical mass. The specific structure of the resulting mass equation, supplemented by the additional requirement of a positive-definite gluon mass, imposes a rather stringent constraint on the derivative of the gluonic dressing function, which is comfortably satisfied by the large-volume lattice data for the gluon propagator, both for $SU(2)$ and $SU(3)$. The numerical treatment of the mass equation, under some simplifying assumptions, is presented for the aforementioned gauge groups, giving rise to a gluon mass that is a non-monotonic function of the momentum. Various theoretical improvements and possible future directions are briefly discussed.

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I. INTRODUCTION

A large body of recent high-quality lattice results indicate that the gluon propagator and the ghost dressing function of pure Yang-Mills theories, computed in the conventional Landau gauge, are infrared (IR) finite, both in $SU(2)$ [1–5] and in $SU(3)$ [6–9]. These important results have sparked a renewed interest in the important issue of dynamical mass generation in non-Abelian gauge theories, and especially in QCD [10–15]. Specifically, as has been suggested in a series of works, the finiteness of these quantities may be interpreted as a direct consequence of the generation of a non-perturbative (momentum-dependent) gluon mass, which acts as an IR cutoff of the theory [16, 17]. In the picture put forth in these articles, the fundamental Lagrangian of the Yang-Mills theory (or that of QCD) is never altered; the generation of the gluon mass takes place dynamically, without violating any of the underlying symmetries [12, 14, 16].

Given the non-perturbative nature of the mass generating mechanism, its study in the continuum proceeds through the Schwinger-Dyson equations (SDEs) that govern the dynamics of the various Green's functions of the theory [11, 13, 18–20], and especially of the gluon propagator, $\Delta(q^2)$. The main conceptual and technical challenge in this context is to obtain as a solution of these integral equations an IR-finite gluon propagator [$i.e.$, $\Delta^{-1}(0) = m^2(0)$], without interfering with the gauge invariance (or the BRST symmetry) of the theory, encoded in the Ward identities (WIs) and Slavnov-Taylor identities (STIs) satisfied by the Green's functions under study [11–13]. A self-consistent framework for enforcing the crucial property of gauge invariance at the level of the *truncated SDEs* is provided by the synthesis of the pinch technique (PT) [14, 16, 21–23] with the background field method (BFM) [24].

In the presence of a dynamically generated mass, the (inverse) Euclidean gluon propagator assumes the form $\Delta^{-1}(q^2) = q^2 J(q^2) + m^2(q^2)$, where the first term corresponds to the "kinetic term", or "wave function" contribution, whereas the second is the (positive-definite) momentum-dependent mass [15]. However, to date, practically all studies attempting to determine the IR behavior of the gluon propagator from SDEs eventually boil down to the solution of some integral equation involving the entire gluon propagator $\Delta(q^2)$ [12, 14, 16], rather than its two components, $J(q^2)$ and $m^2(q^2)$. This is to be contrasted to what happens in the analogous studies of chiral symmetry breaking, where one derives a system of two coupled equations, one determining the "wave function" ("kinetic part") of the quark self-

energy, and one determining the dynamical (constituent) quark mass [25, 26]. Of course, in the case of the quark self-energy the above separation of both sides of the corresponding SDE (quark gap equation) is realized in a direct way, due to the distinct Dirac properties of the two quantities appearing in it, while in the case of the gluon propagator no such straightforward separation is possible. However, a well-defined procedure, first outlined in [15], and explained here in more detail, allows for an analogous separation even in the case of the gluon propagator. The purpose of the present article is to identify and isolate from the SDE of the (Landau gauge) gluon propagator the dynamical equation that determines the evolution of the gluon mass, study its main properties, and find approximate solutions for $m^2(q^2)$.

As has been emphasized in some of the literature cited above, a crucial condition for the realization of the gluon mass generation scenario is the existence of a *longitudinally coupled vertex*, to be denoted by V, which must be added to the conventional (fully-dressed) three-gluon vertex, denoted by $\mathbb{\Gamma}$ [15]. Specifically, the vertex $\mathbb{\Gamma}' = \mathbb{\Gamma} + V$ satisfies the same STIs as $\mathbb{\Gamma}$, but now replacing the gluon propagators appearing on their rhs by a massive ones (schematically, $\Delta \to \Delta_m$). The dynamical reason for the emergence of this special vertex, as well as its diagrammatic realization in terms of Feynman graphs, is intimately connected to the well-known Schwinger mechanism [27, 28], which enables the non-perturbative generation of a gauge-boson mass. In particular, one assumes that the strong QCD dynamics give rise to longitudinally-coupled composite (bound-state) massless poles [29–35]. These poles act like Nambu-Goldstone excitations, in the sense that they preserve the form of the STIs of the theory in the presence of a mass, but they are not associated with the breaking of any local or global symmetry.

It turns out that the way the vertex V generates the mass at the level of the SDE is by introducing a "deviation" from the so-called "seagull identity" [given in Eq. (4.3)]. The role of this identity is to enforce the masslessness of a gauge boson (gluon or photon) when massive propagators appear inside its loops, assuming always that the WI and STI's are maintained, *i.e.*, the transversality of the (gluon or photon) self-energy is preserved. For example, as explained in [15], in scalar QED it is exactly this identity that enforces the masslessness of the photon at the level of the "one-loop dressed" SDE; in this case the massive propagator entering into the loop is that of the charged scalar field. The crucial point is that if the "massive" STI were to be enforced by only modifying $\mathbb{\Gamma}$ [*i.e.*, by carrying

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out the replacement $\Delta \to \Delta_m$ in the closed expressions obtained for $\mathbb{\Gamma}$ by solving the STIs it satisfies, see Eq. (A3)], then the "seagull identity" would force the (would-be) gluon mass to vanish, *i.e.*, would lead to the invalidation of the entire mass generation mechanism. The fact that the missing part for satisfying the "massive" STI is instead provided by the longitudinally coupled V has the far-reaching consequence of finally furnishing a non-trivial equation for the mass. Thus, the equation for the gluon mass is determined as *the amount by which the seagull cancellation is distorted due to the presence of the vertex* V.

To be sure, one could in principle determine the closed form of V by resorting to a procedure similar to that described in [36] for $\mathbb{\Gamma}$, namely write down the most general structure allowed for a longitudinal vertex with three Lorentz indices, and then determine the corresponding form factors from the WI and STI that this vertex must satisfy. It turns out, however, that V enters into the SDE for the Landau gauge gluon propagator (in the PT-BFM scheme) in a very particular way, which renders its closed form unnecessary; all one needs for the derivation of the mass equation is to postulate the existence of V (*i.e.*, assume that it is not identically zero) and that it satisfies the required WI and STIs.

In principle, the mass equation obtained in Eq. (5.23) must be accompanied by the corresponding equation determining the kinetic term $J(q^2)$; the solution of the resulting system of two coupled integral equations will then furnish the behavior of $m^2(q^2)$ and $J(q^2)$, and therefore that of $\Delta(q^2)$. The technical limitation in realizing these procedure is the dependence of the equation for $J(q^2)$ on the various form factors comprising the ghost-gluon kernel; the latter enters into play through the form of the vertex $\mathbb{\Gamma}$ [see Eqs. (A3)-(A4)] . The way to circumvent this problem is to actually solve Eq. (5.23) for $m^2(q^2)$ using as input for the Δ appearing in it the available lattice data, both for $SU(2)$ [1] and $SU(3)$ [6].

It turns out that the specific form of the mass equation in Eq. (5.23) introduces a non-trivial constraint on the precise behavior that Δ must display in the region between (1-5) GeV^2. Specifically, in order for the gluon mass to be positive definite, the first derivative of the quantity $q^2\Delta(q^2)$ (the "gluon dressing function") must furnish a sufficiently *negative* contribution in the aforementioned range of momenta. Interestingly enough, the Δ obtained from the lattice has indeed this particular property. This is to be contrasted to what happens, for example, in the case of a simple massive propagator $1/(q^2 + m^2)$ or with the Gribov-Zwanziger propagator $q^2/(q^4 + m^4)$ (with m constant) [37, 38]; the derivatives of the corresponding dressing functions, $q^2/(q^2 + m^2)$ and $q^4/(q^4 + m^4)$, respectively,

are positive in the entire range of (Euclidean) momenta, thus excluding the possibility of a positive-definite gluon mass.

The article is organized as follows. In Section II we introduce the necessary notation and review briefly the aspects of the PT-BFM formalism relevant to this work. In Section III we explain in detail the modifications that must be introduced to the three-gluon vertex of the theory in order to treat the generation of a gluon mass in a gauge invariant way (*i.e.*, preserving the STIs of the theory). In particular, the importance of the nonperturbative vertex V and its special properties are emphasized, and the changes introduced to the corresponding SDE during the transition from massless to massive solutions are discussed in detail. In Section IV we outline the precise criteria that will lead to the separation of the SDE for the gluon propagator into two equations, one for the kinetic part and one for the mass. The central role of the "seagull-identity" in carrying out this separation is stressed, and some explicit characteristic calculations are presented. Then, in Section V we combine the ingredients introduced in the previous sections and derive the final form of the dynamical equation for the gluon mass in the Landau gauge. In Section VI we first study the implications of the gluon mass equation in the limit of vanishing physical momentum. Then we solve an approximate form of this equation, using lattice data as input for the "unknown" quantity Δ. The solution for the gluon mass so obtained is then appropriately "subtracted out" from Δ, giving rise to an estimate for the quantity $J(q^2)$. These ingredients are then combined to construct the renormalization-group (RG) invariant gluon mass, appearing in the usual definition of the effective QCD charge within the PT-BFM framework. Our conclusions and discussion of the results appear in Section VII. Finally, some technical points are presented in the Appendix.

II.   GENERAL FRAMEWORK

In this section, we set up the necessary notation and review some of the most salient features of the PT-BFM framework, putting particular emphasis on the form of the SDE for the gluon propagator, and the various field-theoretic ingredients appearing in it.

The (full) gluon propagator $\Delta_{\mu\nu}^{ab}(q) = \delta^{ab}\Delta_{\mu\nu}(q)$ in the renormalizable R_ξ gauges is defined
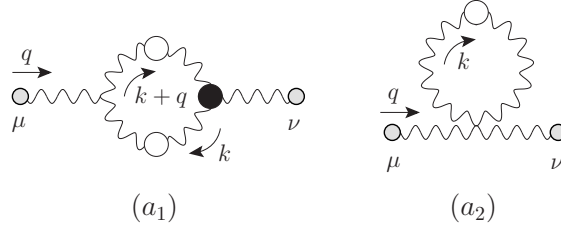
(a_1) (a_2)

FIG. 1: The "one-loop dressed" gluon contribution to the PT-BFM gluon self-energy. White (respectively, black) blobs represent connected (respectively, 1-PI) Green's functions; a gray circle on the external legs indicates background gluons. Notice that within the PT-BFM framework these two diagrams alone constitute a transverse subset of the full gluon SDE.

as

$$i\Delta_{\mu\nu}(q) = -i\left[P_{\mu\nu}(q)\Delta(q^2) + \xi\frac{q_\mu q_\nu}{q^4}\right], \qquad \Delta^{-1}_{\mu\nu}(q) = i\left[P_{\mu\nu}(q)\Delta^{-1}(q^2) + (1/\xi)q_\mu q_\nu\right], \quad (2.1)$$

with

$$P_{\mu\nu}(q) = g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}, \qquad (2.2)$$

the dimensionless transverse projector, and ξ the gauge fixing parameter. The scalar cofactor $\Delta(q^2)$ appearing above is related to the all-order gluon self-energy $\Pi_{\mu\nu}(q) = P_{\mu\nu}(q)\Pi(q^2)$ through

$$\Delta^{-1}(q^2) = q^2 + i\Pi(q^2). \qquad (2.3)$$

In addition, it is convenient to define the dimensionless function $J(q^2)$ as [39]

$$\Delta^{-1}(q^2) = q^2 J(q^2). \qquad (2.4)$$

Evidently, $J(q^2)$ coincides with the *inverse* of the gluon dressing function, frequently considered in the literature.

The starting point of our dynamical analysis is the SDE governing the gluon propagator. Within the PT-BFM framework that we employ [10–17, 21–24], one may safely truncate the SDE series down to its "one-loop dressed version" containing gluonic contributions only, given by the diagrams (a_1) and (a_2) shown in Fig. 1. Specifically, due to the special Feynman rules of the PT-BFM, and in particular the QED-like Ward identities satisfied by the fully-dressed vertices, gauge invariance remains exact, in the sense that the resulting (approximate) gluon self-energy $\Pi_{\mu\nu}(q)$ is still transverse, *i.e.*,
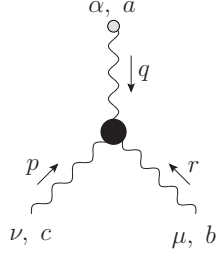
$$q^\nu \Pi_{\mu\nu}(q) = 0. \qquad (2.5)$$

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FIG. 2: The BQQ three-gluon vertex.

The PT-BFM equation for the conventional propagator reads, in this case,

$$\Delta^{-1}(q^2)P_{\mu\nu}(q) = \frac{q^2 P_{\mu\nu}(q) + i\,[(a_1) + (a_2)]_{\mu\nu}}{[1 + G(q^2)]^2},\qquad(2.6)$$

where

$$(a_1)_{\mu\nu} = \frac{1}{2}g^2 C_A \int_k \widetilde{\Gamma}^{(0)}_{\mu\alpha\beta}(q,k,-k-q)\Delta^{\alpha\rho}(k)\Delta^{\beta\sigma}(k+q)\widetilde{\Gamma}_{\nu\rho\sigma}(q,k,-k-q)$$

$$(a_2)_{\mu\nu} = g^2 C_A \left[g_{\mu\nu}\int_k \Delta^\rho_\rho(k) + (1/\xi - 1)\int_k \Delta_{\mu\nu}(k)\right],\qquad(2.7)$$

with C_A being the Casimir eigenvalue of the adjoint representation [$C_A = N$ for $SU(N)$], and the d-dimensional integral measure (in dimensional regularization) is defined according to

$$\int_k \equiv \frac{\mu^\epsilon}{(2\pi)^d}\int \mathrm{d}^d k.\qquad(2.8)$$

The vertex $\widetilde{\Gamma}$ is the fully dressed version of the trilinear vertex involving one background and two quantum gluons (BQQ vertex for short, see Fig. 2); at tree-level (all momenta entering)

$$\Gamma^{(0)}_{\alpha\mu\nu}(q,r,p) = \widetilde{\Gamma}^{(0)}_{\alpha\mu\nu}(q,r,p) + (1/\xi)\Gamma^{\mathrm{P}}_{\alpha\mu\nu}(q,r,p),\qquad(2.9)$$

with

$$\begin{aligned}
\Gamma^{(0)}_{\alpha\mu\nu}(q,r,p) &= g_{\mu\nu}(r-p)_\alpha + g_{\alpha\nu}(p-q)_\mu + g_{\alpha\mu}(q-r)_\nu,\\
\widetilde{\Gamma}^{(0)}_{\alpha\mu\nu}(q,r,p) &= g_{\mu\nu}(r-p)_\alpha + g_{\alpha\nu}(p-q+r/\xi)_\mu + g_{\alpha\mu}(q-r-p/\xi)_\nu,\\
\Gamma^{\mathrm{P}}_{\alpha\mu\nu}(q,r,p) &= g_{\alpha\mu}p_\nu - g_{\alpha\nu}r_\mu.
\end{aligned}\qquad(2.10)$$

Finally, the function $G(q^2)$ appearing in (2.6) is of central importance in this entire formalism. It is defined as the scalar co-factor of the $g_{\mu\nu}$ component of the special two-point

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FIG. 3: Diagrammatic representation of the functions Λ, H and, for later convenience, $\widetilde{H}$. Gray blobs represent 1-PI kernels with respect to vertical cuts.

function $\Lambda_{\mu\nu}(q)$, defined as (see also Fig. 3)

$$\Lambda_{\mu\nu}(q) = -ig^2 C_A \int_k \Delta_\mu^\sigma(k) D(q-k) H_{\nu\sigma}(-q, q-k, k)$$
$$= g_{\mu\nu} G(q^2) + \frac{q_\mu q_\nu}{q^2} L(q^2), \tag{2.11}$$

where we have introduced the ghost propagator $D^{ab}(q^2) = \delta^{ab} D(q^2)$, which is related to the ghost dressing function $F(q^2)$ through

$$D(q^2) = \frac{F(q^2)}{q^2}. \tag{2.12}$$

Notice that in the Landau gauge, an important exact (all-order) relation exists, linking $G(q^2)$ and $L(q^2)$ to the ghost dressing function $F(q^2)$, namely [40–43]

$$F^{-1}(q^2) = 1 + G(q^2) + L(q^2). \tag{2.13}$$

In addition, the function $G(q^2)$ participates in a set of BRST-driven identities, known as Background-Quantum identities (BQIs) [44, 45], obtained within the Batalin-Vilkovisky formalism [46, 47]. These powerful identities relate among each other the three types of gluon propagators that appear naturally in the BFM formalism, namely: (i) the conventional gluon propagator (two quantum gluons entering, QQ), denoted by $\Delta(q^2)$; (ii) the background gluon propagator (two background gluons entering, BB), denoted by $\widehat{\Delta}(q^2)$; and (iii) the mixed background-quantum gluon propagator (one background and one quantum gluons entering, BQ), denoted by $\widetilde{\Delta}(q^2)$. The corresponding BQIs are

$$\Delta(q^2) = [1 + G(q^2)]^2 \widehat{\Delta}(q^2),$$

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$$\Delta(q^2) = [1 + G(q^2)]\widetilde{\Delta}(q^2),$$

$$\widetilde{\Delta}(q^2) = [1 + G(q^2)]\widehat{\Delta}(q^2). \tag{2.14}$$

Notice that it is the first of these identities that allows the rewriting of the conventional SDE into the PT-BFM form (2.6) [11–13].

For the rest of the article we will study the gluon SDE in the Landau gauge, $\xi = 0$. The limit of Eq. (2.6) as $\xi \to 0$ is rather subtle, and has been presented in [12]. The final answer is

$$\widehat{\Pi}^{\mu\nu}(q) = [(a_1) + (a_2)]^{\mu\nu}_{\xi=0}$$

$$= g^2 C_A \sum_{i=1}^{5} A_i^{\mu\nu}(q), \tag{2.15}$$

with

$$A_1^{\mu\nu}(q) = \frac{1}{2} \int_k \Gamma^{(0)\mu}_{\alpha\beta} P^{\alpha\rho}(k) P^{\beta\sigma}(k+q) \widetilde{\mathbb{\Gamma}}^{\nu}_{\rho\sigma} \Delta(k)\Delta(k+q),$$

$$A_2^{\mu\nu}(q) = \int_k P^{\alpha\mu}(k) \frac{(k+q)^\beta \Gamma^{(0)\nu}_{\alpha\beta}}{(k+q)^2} \Delta(k),$$

$$A_3^{\mu\nu}(q) = \int_k P^{\alpha\mu}(k) \frac{(k+q)^\beta \widetilde{\mathbb{\Gamma}}^{\nu}_{\alpha\beta}}{(k+q)^2} \Delta(k),$$

$$A_4^{\mu\nu}(q) = -\frac{(d-1)^2}{d} g^{\mu\nu} \int_k \Delta(k),$$

$$A_5^{\mu\nu}(q) = \int_k \frac{k^\mu (k+q)^\nu}{k^2 (k+q)^2}. \tag{2.16}$$

The vertex $\mathbb{\Gamma}$ appearing above is the fully-dressed PT-BFM vertex studied in detail in [36], and which is related to the full BQQ vertex $\widetilde{\Gamma}$ appearing in the BFM through

$$\widetilde{\mathbb{\Gamma}}_{\alpha\mu\nu}(q, r, p) = \widetilde{\Gamma}_{\alpha\mu\nu}(q, r, p) + (1/\xi)\Gamma^{\mathrm{P}}_{\alpha\mu\nu}(q, r, p). \tag{2.17}$$

Evidently, $\widetilde{\mathbb{\Gamma}}_{\alpha\mu\nu}(q, r, p)$ and $\widetilde{\Gamma}_{\alpha\mu\nu}(q, r, p)$ differ only at tree level; specifically, one sees immediately that

$$\widetilde{\mathbb{\Gamma}}^{(0)}_{\alpha\mu\nu}(q, r, p) = \Gamma^{(0)}_{\alpha\mu\nu}(q, r, p). \tag{2.18}$$

In the rest of this paper, we will refer indifferently to both $\widetilde{\Gamma}$ and $\widetilde{\mathbb{\Gamma}}$ as the BQQ vertex; in addition, in order to simplify the notation, we will drop the "tilde" superscript.

The vertex $\mathbb{\Gamma}$ satisfies a (ghost-free) WI when contracted with the momentum q_α of the background gluon, whereas it satisfies a STI when contracted with the momentum of the

quantum gluons (r_μ or p_ν). In particular,

$$
\begin{aligned}
q^\alpha \mathbb{T}_{\alpha\mu\nu}(q,r,p) &= p^2 J(p^2) P_{\mu\nu}(p) - r^2 J(r^2) P_{\mu\nu}(r), \\
r^\mu \mathbb{T}_{\alpha\mu\nu}(q,r,p) &= F(r^2) \left[q^2 \widetilde{J}(q^2) P^\mu_\alpha(q) H_{\mu\nu}(q,r,p) - p^2 J(p^2) P^\mu_\nu(p) \widetilde{H}_{\mu\alpha}(p,r,q) \right], \\
p^\nu \mathbb{T}_{\alpha\mu\nu}(q,r,p) &= F(p^2) \left[r^2 J(r^2) P^\nu_\mu(r) \widetilde{H}_{\nu\alpha}(r,p,q) - q^2 \widetilde{J}(q^2) P^\nu_\alpha(q) H_{\nu\mu}(q,p,r) \right], \quad (2.19)
\end{aligned}
$$

and the function $\widetilde{J}$ is related to the conventional one defined in (2.4) precisely through the second equation in (2.14), namely

$$
\widetilde{J}(q^2) = \left[1 + G(q^2) \right] J(q^2). \tag{2.20}
$$

In addition, as shown in Fig. 3, the auxiliary ghost function $\widetilde{H}$ is the same as H after converting the external gluon leg into a background leg. An explicit form in terms of J, $\widetilde{J}$, H and $\widetilde{H}$ of the (longitudinal) form factors characterizing this vertex has been obtained in [36] and reported in Appendix A.

One may finally use Eq. (2.4) to re-express the relations (2.19) in terms of the (inverse) scalar functions Δ, *i.e.*,

$$
q^\alpha \mathbb{T}_{\alpha\mu\nu}(q,r,p) = \Delta^{-1}(p^2) P_{\mu\nu}(p) - \Delta^{-1}(r^2) P_{\mu\nu}(r), \tag{2.21}
$$

with analogous expressions holding for the remaining two STIs of (2.19). At this level this appears as a simple rewriting, but this form of writing will facilitate the clarification of certain conceptual issues that become relevant when dynamical mass generation is turned on (see next section).

III.   VERTICES IN THE PRESENCE OF A DYNAMICAL MASS

In order to generate a dynamical mass without interfering with gauge invariance and the BRST symmetry, one must resort to the Schwinger mechanism [27, 28]. The general idea is to assume that a longitudinally coupled bound-state pole has been formed dynamically, which will modify the structure of the full vertices of the theory [29–35]. This modification, in turn, will be responsible for obtaining massive type of solutions from the SDE of the gluon where this new vertices will be inserted [12]. It is important to be very precise regarding the nature and role of the various ingredients that enter in the ensuing analysis. We will
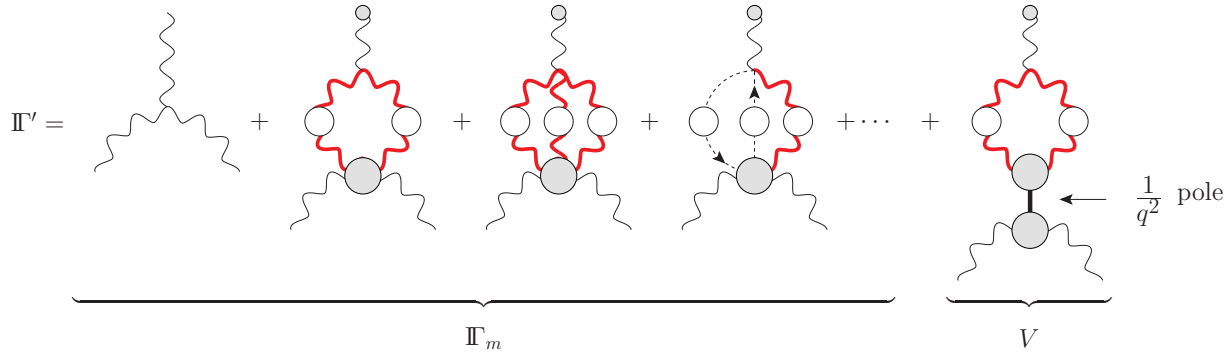
FIG. 4: The $\mathbb{\Gamma}'$ three-gluon vertex. Thick (red online) internal gluon lines indicates massive propagators Δ_m, as explained in the text.

therefore devote this section to the development and elaboration of the various key concepts needed.

From the kinematic point of view we will describe the transition from a massless to a massive gluon propagator by carrying out the replacement (in Minkowski space)

$$\Delta^{-1}(q^2) = q^2 J(q^2) \quad \longrightarrow \quad \Delta_m^{-1}(q^2) = q^2 J_m(q^2) - m^2(q^2). \tag{3.1}$$

The symbol J_m indicates that effectively one has now a mass inside the corresponding expressions: for example, whereas perturbatively $J(q^2) \sim \ln q^2$, after dynamical gluon mass generation has taken place, one has $J_m(q^2) \sim \ln(q^2 + m^2)$. As a consequence, since J_m will be the main component in the definition of the QCD effective charge [15], the presence of the mass term in the argument of its logarithm will tame the perturbative Landau pole [16, 41, 43]. Of course, as $q^2 \to 0$, $q^2 J_m(q^2) \to 0$; therefore, if we are to ensure that this procedure will give rise to a non vanishing IR value for the gluon propagator, *i.e.*, $\Delta_m^{-1}(0) \neq 0$, we must have that $m^2(0) \neq 0$.

From the dynamical point of view, it is clear that the full three-gluon vertex must be also appropriately modified [29–31]. Specifically, we will consider a new vertex, to be denoted by $\mathbb{\Gamma}'$, and carry out the replacement

$$\mathbb{\Gamma} \quad \longrightarrow \quad \mathbb{\Gamma}' = \mathbb{\Gamma}_m + V. \tag{3.2}$$

The new vertex $\mathbb{\Gamma}_m$ is given by the same (fully dressed) graphs that make up the SDE of the BQQ vertex $\mathbb{\Gamma}$ (Fig. 4); however, now all internal (virtual) fully-dressed gluon propagators are massive *i.e.*, in the non-pole part of the vertex SDE we have $\Delta \to \Delta_m$. In addition (or

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as a result thereof), $\mathbb{\Gamma}_m$ satisfies exactly the set of WI and STIs given in (2.19), but with the replacement $J \to J_m$ throughout. So, the WI becomes

$$q_\alpha \mathbb{\Gamma}_m^{\alpha\mu\nu}(q,r,p) = p^2 J_m(p^2) P^{\mu\nu}(p) - r^2 J_m(r^2) P^{\mu\nu}(r), \tag{3.3}$$

and exactly analogous expressions for the remaining STIs satisfied when $\mathbb{\Gamma}_m$ is contracted by either r or p. Note that all other Green's functions, such as H and $\widetilde{H}$, must be replaced by the corresponding H_m and $\widetilde{H}_m$, in the same sense as before (but we will suppress their 'm' subindex throughout); thus, the diagrams defining these two ghost functions, shown in Fig. 3, will now contain massive internal gluon propagators.

On the other hand, the vertex V represents the pole part of $\mathbb{\Gamma}'$; it is *totally longitudinally coupled*, i.e., it vanishes identically when contracted by the three transverse projectors

$$P^{\alpha'\alpha}(q)P^{\mu'\mu}(r)P^{\nu'\nu}(p)V_{\alpha\mu\nu}(q,r,p) = 0, \tag{3.4}$$

and must satisfy the WI and STI of (2.19), with the replacement $k^2 J(k) \to -m^2(k)$, e.g.,

$$q^\alpha V_{\alpha\mu\nu}(q,r,p) = -m^2(p^2)P_{\mu\nu}(p) + m^2(r^2)P_{\mu\nu}(r). \tag{3.5}$$

Exactly analogous expressions will hold for the STIs satisfied when contracting with the momenta r or p.

An explicit example of such a vertex (which, however, we will not use here), has been given in [48], namely

$$
\begin{aligned}
V^{\alpha\mu\nu}(q,r,p) &= \frac{q^\alpha r^\mu (q-r)_\rho}{2q^2 r^2} P^{\rho\nu}(p) m^2(p^2) - \frac{p^\nu}{p^2} \left[m^2(r^2) - m^2(q^2) \right] P^\alpha_\rho(q) P^{\rho\mu}(r) \\
&+ \frac{r^\mu p^\nu (r-p)_\rho}{2r^2 p^2} P^{\rho\alpha}(q) m^2(q^2) - \frac{q^\alpha}{q^2} \left[m^2(p^2) - m^2(r^2) \right] P^\mu_\rho(r) P^{\rho\nu}(p) \\
&+ \frac{p^\nu q^\alpha (p-q)_\rho}{2q^2 p^2} P^{\rho\mu}(r) m^2(r^2) - \frac{r^\mu}{r^2} \left[m^2(q^2) - m^2(p^2) \right] P^\nu_\rho(p) P^{\rho\alpha}(q). \tag{3.6}
\end{aligned}
$$

The totally longitudinal nature of this vertex is manifest[1].

At this point it is clear that the full vertex $\mathbb{\Gamma}'$ will satisfy the same WI and STIs (2.19) satisfied by the $\mathbb{\Gamma}$ vertex before the introduction of any masses, but now with the replacement

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[1] Note that this vertex is totally Bose symmetric, satisfying (3.5) with respect to all its legs; instead, the vertex considered here satisfies an STI with respect to the quantum legs (r and p).

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$\Delta \to \Delta_m$. Therefore, using Eqs. (3.2), (3.3), and (3.5), one gets for $\mathbb{\Gamma}'$ the WI

$$\begin{aligned} q_\alpha \mathbb{\Gamma}'^{\alpha\mu\nu}(q,r,p) &= q_\alpha \left[\mathbb{\Gamma}_m^{\alpha\mu\nu}(q,r,p) + V^{\alpha\mu\nu}(q,r,p) \right] \\ &= [p^2 J_m(p^2) - m^2(p^2)]P^{\mu\nu}(p) - [r^2 J_m(r^2) - m^2(r^2)]P^{\mu\nu}(r) \\ &= \Delta_m^{-1}(p^2)P^{\mu\nu}(p) - \Delta_m^{-1}(r^2)P^{\mu\nu}(r). \end{aligned} \quad (3.7)$$

Similarly

$$r_\mu \mathbb{\Gamma}'^{\alpha\mu\nu}(q,r,p) = F(r^2) \left[\Delta_m^{-1}(q^2)P_\alpha^\mu(q)H_{\mu\nu}(q,r,p) - \Delta_m^{-1}(p^2)P_\nu^\mu(p)\widetilde{H}_{\mu\alpha}(p,r,q) \right], \quad (3.8)$$

$$p_\nu \mathbb{\Gamma}'^{\alpha\mu\nu}(q,r,p) = F(p^2) \left[\Delta_m^{-1}(r^2)P_\mu^\nu(r)\widetilde{H}_{\nu\alpha}(r,p,q) - \Delta_m^{-1}(q^2)P_\alpha^\nu(q)H_{\nu\mu}(q,p,r) \right]. \quad (3.9)$$

It is very important to emphasize that, even though the new (massive) WI is obtained from the old (massless) one through the replacement $\Delta \to \Delta_m$, the new vertex $\mathbb{\Gamma}'$ is *not* obtained from the old one, $\mathbb{\Gamma}$, by means of the same replacement *only*. Indeed, turning to the explicit expression for $\mathbb{\Gamma}$ given in the Appendix A, it would certainly be wrong to use there the replacement $\Delta \to \Delta_m$ (or $J(q^2) \to \Delta_m(q^2)/q^2$). Instead, the correct procedure is that outlined above: the vertex $\mathbb{\Gamma}_m$ is indeed obtained from the expressions in the Appendix, by replacing $J \to J_m$ (but with no explicit mass terms); all explicit mass terms are next added through the totally longitudinally coupled non-perturbative vertex V.

Actually, it is interesting to ponder about what would happen if one were to introduce the gluon mass through the (wrong) procedure of identifying the vertex $\mathbb{\Gamma}'$ by the simple replacement $\Delta \to \Delta_m$ carried out inside $\mathbb{\Gamma}$. In such a case one would conclude (after some steps) that the self-consistency of the theory would force $m^2(q^2)$ to vanish identically. The precise way how this "self-correction" takes place is intimately related to the so-called "seagull identity" [15], and will be discussed at the end of the Section V.

IV. GENERAL FEATURES OF THE GLUON MASS EQUATION

Let us now consider the gluon SDE of Eq. (2.6) under the light of the analysis presented in the previous section. After dynamical gluon mass generation has taken place, one needs to consider the modified SDE, which is obtained from (2.6) after (i) replacing the Δ^{-1} appearing on the lhs with the Δ_m^{-1} of Eq. (3.1), and (ii) replacing $\Delta \to \Delta_m$ and $\mathbb{\Gamma} \to \mathbb{\Gamma}'$ inside the integrals of the rhs (see also Fig. 5).
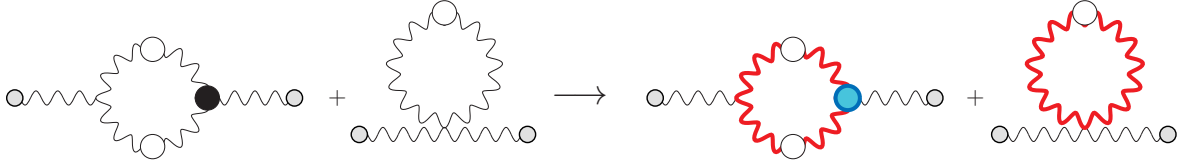
FIG. 5: Diagrammatic representation of the gluon one-loop dressed diagrams before and after dynamical gluon mass generation has taken place: the propagators and vertices on the rhs have now become massive.

From this new SDE one can obtain two separate equations, the first one governing the behavior of $J_m(q^2)$ [to be later involved in the definition of the effective charge, see Eq. (6.17)] and the second one describing the dynamical mass $m^2(q^2)$. The general idea is the following: the terms appearing on the rhs of the SDE may be separated systematically into two contributions, one that vanishes as $q \to 0$ and one that does not; the latter contribution must be set equal to the corresponding non-vanishing term on the lhs, namely $-m^2(q)$, while the former will be set equal to the vanishing term of the lhs, namely $q^2 J_m(q^2)$, the so-called "kinetic term".

Specifically (taking the trace of both sides of (2.6) to eliminate the Lorentz indices), the rhs may be schematically cast in the form

$$q^2 J_m(q^2) - m^2(q^2) = q^2 \left[1 + \mathcal{K}_1(q^2, m^2, \Delta_m) \right] + \mathcal{K}_2(q^2, m^2, \Delta_m), \tag{4.1}$$

such that $q^2 \mathcal{K}_1(q^2, m^2, \Delta_m) \to 0$, as $q^2 \to 0$, whereas $\mathcal{K}_2(q^2, m^2, \Delta_m) \neq 0$ in the same limit. Thus, for example, a term of the form $q^2 \int_k \Delta_m(k) \Delta_m(k + q)$ contributes to $\mathcal{K}_1$, whereas a term of the form $m^2(q^2) \int_k \Delta_m(k) \Delta_m(k + q)$ should be assigned to $\mathcal{K}_2$. Then, the two equations determining $J_m(q^2)$ and $m^2(q^2)$ will read (still Minkowski space)

$$\begin{aligned} J_m(q^2) &= 1 + \mathcal{K}_1(q^2, m^2, \Delta_m), \\ m^2(q^2) &= -\mathcal{K}_2(q^2, m^2, \Delta_m). \end{aligned} \tag{4.2}$$

Of course, there is an obvious subtlety that must be addressed at this point. Specifically, one may easily envisage the possibility of a term that approaches zero as $(q^2)^a$, with $0 < a < 1$. In this case, given that we must factor out a q^2 in order to obtain the equation for $J_m(q^2)$, such a term would furnish an IR divergent contribution to $J_m(q^2)$. This would be an undesirable feature, given that the $J_m(q^2)$ is intimately related to the effective charge

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of QCD, which is believed to be finite. The way to treat such a possibility is to state that, should such a term appear, it ought to be directly allotted (in its entirety, without factoring out a q^2) to the equation for $m^2(q^2)$. The presence of such a term in the mass equation will not affect the value of the mass at $q^2 = 0$, but will in general affect the shape of the resulting curve. Keeping this mathematical possibility in mind, let us point out that the terms emerging in the analysis of Section V have a very characteristic structure [see Eq. (4.15) and the related discussion], and, at least for them, the scenario contemplated above [$(q^2)^a$, with $0 < a < 1$] is not realized.

There is an additional point related to the mass equation, which is instrumental for the self-consistency of the entire approach. Specifically, a crucial condition for the mechanism of dynamical gluon mass generation, developed in a series of articles [10–13, 16], is the cancellation of all seagull-type of divergences, *i.e.*, divergences produced by integrals of the type $\int_k \Delta(k^2)$, or variations thereof [15]. The precise cancellation of such terms proceeds by means of the identity [15]

$$\int_k k^2 \Delta'_m(k) + \frac{d}{2} \int_k \Delta_m(k) = 0, \qquad (4.3)$$

where the "prime" denotes differentiation with respect to k^2, *i.e.*, $\Delta'_m(k) \equiv \partial \Delta_m(k^2)/\partial k^2$. Thus, all the ingredients entering into the SDE (most importantly, the vertex) must be such that, after taking the limit of the SDE as $q^2 \to 0$, all seagull-type contributions must conspire to appear exactly in the combination given on the lhs of Eq. (4.3).

The relevance and function of the identity (4.3) becomes evident when we consider the term $I(q)$, given by

$$-iI(q) = \int_k k^2 \Delta_m(k) \Delta_m(k+q) \frac{(k+q)^2 J_m(k+q) - k^2 J_m(k)}{(k+q)^2 - k^2} + c \int_k \Delta_m(k), \qquad (4.4)$$

with c (for the moment) an arbitrary real number. This term appears naturally in the PT-BFM framework, and in fact we will find it in the case of the Landau gauge studied in the next section.

Using Eq. (3.1) one may then rewrite $I(q)$ as

$$I(q) = I_1(q) + I_2(q), \qquad (4.5)$$

with

$$-iI_1(q) = -\int_k k^2 \frac{\Delta_m(k+q) - \Delta_m(k)}{(k+q)^2 - k^2} + c \int_k \Delta_m(k)$$

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$$= - \left[\int_k k^2 \frac{\Delta_m(k+q) - \Delta_m(k)}{(k+q)^2 - k^2} + \frac{d}{2} \int_k \Delta_m(k) \right] + \left(c + \frac{d}{2} \right) \int_k \Delta_m(k), \quad (4.6)$$

and

$$- i I_2(q) = \int_k k^2 \Delta_m(k) \Delta_m(k+q) \frac{m^2(k+q) - m^2(k)}{(k+q)^2 - k^2}. \quad (4.7)$$

In order to establish how the above terms must be assigned among the $\mathcal{K}_1$ and $\mathcal{K}_2$ introduced above, let us now take their limit as $q^2 \to 0$. Carrying out the appropriate Taylor expansions [see Eq. (4.14)], one finds

$$
\begin{aligned}
- i I_1(0) &= - \left[\int_k k^2 \Delta'_m(k) + \frac{d}{2} \int_k \Delta_m(k) \right] + \left(c + \frac{d}{2} \right) \int_k \Delta_m(k) \\
&= \left(c + \frac{d}{2} \right) \int_k \Delta_m(k),
\end{aligned}
\quad (4.8)
$$

where in the second step we have employed Eq. (4.3), and

$$- i I_2(0) = \int_k k^2 \Delta_m^2(k) [m^2(k)]'. \quad (4.9)$$

Thus, according to the rules introduced above, the contribution of $I(q)$ to the kinetic term is

$$i I_{\text{kt}}(q) = \int_k k^2 \frac{\Delta_m(k+q) - \Delta_m(k)}{(k+q)^2 - k^2} + \frac{d}{2} \int_k \Delta_m(k), \quad (4.10)$$

given that $I_{\text{kt}}(0) = 0$, while the contribution of $I(q)$ to the mass equation is

$$- i I_{m^2}(q) = \int_k k^2 \Delta_m(k) \Delta_m(k+q) \frac{m^2(k+q) - m^2(k)}{(k+q)^2 - k^2} + \left(c + \frac{d}{2} \right) \int_k \Delta_m(k). \quad (4.11)$$

It is clear now that the second term on the rhs of (4.11) is quadratically divergent (and of the seagull type). The only way to avoid this divergence is if the coefficient multiplying $\int_k \Delta_m(k)$ vanishes, *i.e.*, if $c = -d/2$. It turns out that, by virtue of the PT-BFM Feynman rules, and the fact that gauge invariance is preserved at every level of this approximation, the coefficient c comes out precisely equal to $-d/2$; we emphasize that this result can be realized *only* within the PT-BFM framework. Thus, after the seagull cancellation, one is left with the first term only, which is perfectly convergent, provided that the mass decreases in the deep ultraviolet. As we will see in the next section, in the Landau gauge this term accounts for the bulk of the gluon mass equation.

Even though the term $I_{\text{kt}}(q)$ of Eq. (4.10) does not appear in the rest of our analysis, it is important to gain some further intuition on its structure and its behavior for small values of q^2, especially in the light of the discussion following Eq. (4.2).

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To this end, let us introduce spherical coordinates through the definitions $q^2 = x$, $k^2 = y$, $(k+q)^2 = z$; we then have that $z = y + x + 2\sqrt{xy}\cos\theta$, and we define $w \equiv (k+q)^2 - k^2 = z - y = x + 2\sqrt{xy}\cos\theta$. The d-dimensional integral measure will read in this case

$$\int_k = \frac{1}{(2\pi)^d}\frac{\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}\int_0^\pi d\theta \sin^{d-2}\theta \int_0^\infty dy\, y^{\frac{d}{2}-1}, \tag{4.12}$$

and we finally recall the elementary integral

$$\int_0^\pi d\theta\, \sin^m\theta\cos^n\theta = \begin{cases} \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+n+2}{2}\right)}, & n \text{ even} \\ \\ 0, & n \text{ odd} \end{cases} \tag{4.13}$$

It turns out that the $I_{\mathrm{kt}}(q)$ of Eq. (4.10) may be expanded systematically as a power series in q^2. To see this in detail, we consider the Taylor expansion of an arbitrary finite function $f(z)$ around $w = 0$, given by

$$\frac{f(z) - f(y)}{w} = f'(y) + \frac{w}{2!}f''(y) + \frac{w^2}{3!}f'''(y) + ... \tag{4.14}$$

where the primes denote differentiations with respect to y (evidently we are assuming finite derivatives in the origin). Then, under the integral sign on the rhs of Eq. (4.10) one must collect pieces of a given order in q^2 from the various powers of w, using (4.13).

It is clear that when the term $f'(y)$ on the rhs of (4.14) is inserted into the integral, it generates the seagull identity (4.3); all the remaining terms will be proportional to positive powers of w, and thus, $I_{\mathrm{kt}}(0) = 0$. For example, the q^2 term in this expansion is obtained by appropriately combining contributions proportional to $f''(y)$ and $f'''(y)$. Using again (4.13), after a sequence of partial integrations, we find

$$i I_{\mathrm{kt}}(q) = \frac{q^2}{6}\left(\frac{d-4}{d}\right)\int_k k^2\Delta_m''(k) + \mathcal{O}(q^4). \tag{4.15}$$

In order to check the validity of Eq. (4.15) let us compute $I_{\mathrm{kt}}(q)$ for the simple case of a massive propagator with a "hard" (momentum-independent) mass

$$\Delta_m(q) = \frac{1}{q^2 - m^2}. \tag{4.16}$$

The integrand in the first integral on the rhs of Eq. (4.11) simplifies to

$$k^2\frac{\Delta_m(k+q) - \Delta_m(k)}{(k+q)^2 - k^2} = -\frac{k^2}{(k^2 - m^2)[(k+q)^2 - m^2]}. \tag{4.17}$$

Then, using the dimensional regularization identity

$$2m^2 \int_k \frac{1}{(k^2 - m^2)^2} = (d-2) \int_k \frac{1}{k^2 - m^2}, \tag{4.18}$$

it is relatively straightforward to demonstrate that

$$I_{\mathrm{kt}}(q) = \frac{m^2}{16\pi^2} \int_0^1 \mathrm{d}x \ln\left(1 + \frac{q^2 x(x-1)}{m^2}\right). \tag{4.19}$$

Evidently, $I_{\mathrm{kt}}(0) = 0$, as expected, while the expansion of the logarithm furnishes immediately the result

$$I_{\mathrm{kt}}(q) = -\frac{1}{16\pi^2}\frac{q^2}{6} + \mathcal{O}(q^4). \tag{4.20}$$

On the other hand, substitution into the general formula (4.15) of the propagator in (4.16) yields

$$I_{\mathrm{kt}}(q) = -i\frac{q^2}{6}\left(\frac{d-4}{d}\right) 2 \int_k \frac{k^2}{(k^2 - m^2)^3} + \mathcal{O}(q^4). \tag{4.21}$$

In dimensional regularization, around $d = 4$, we have that $d = 4 - \epsilon$, and therefore, only the divergent part of the integral contributes, $i.e.$,

$$\begin{aligned}
I_{\mathrm{kt}}(q) &= -i\frac{q^2}{6}\left(\frac{-\epsilon}{d}\right) 2 \left[\frac{1}{16\pi^2}\left(\frac{id}{4}\right)\left(\frac{2}{\epsilon}\right)\right] + \mathcal{O}(q^4) \\
&= -\frac{1}{16\pi^2}\frac{q^2}{6} + \mathcal{O}(q^4),
\end{aligned} \tag{4.22}$$

which indeed coincides with (4.20).

Notice finally that the main contribution to the kinetic term does not originate from $I_{\mathrm{kt}}(q)$, but rather from a term of the form

$$q^2 \int_k \frac{\Delta_m(k+q) - \Delta_m(k)}{(k+q)^2 - k^2}, \tag{4.23}$$

which, for the simple massive propagators of (4.16) may be easily calculated, giving rise to the standard logarithmic correction associated with the RG, with the additional feature of being IR safe due to the presence of the mass in the argument of the logarithm.

V. THE GLUON MASS EQUATION IN THE LANDAU GAUGE

We now proceed to the actual derivation of the explicit form of the mass equation in the Landau gauge. Specifically, in this gauge the rhs of Eq. (2.6) will be given by the terms A_i listed in Eq. (2.16), where now we must carry out the replacements $\Delta \to \Delta_m$ and $\mathbb{\Gamma} \to \mathbb{\Gamma}'$.

Following the rules explained in the previous section, and defining

$$A_i(q) = \text{Tr}\,[A_i^{\mu\nu}(q)], \tag{5.1}$$

the mass equation is given by

$$m^2(q^2) = -i\frac{g^2 C_A}{d-1}\frac{\left[\sum_{i=1}^{5} A_i(q)\right]_{m^2}}{[1+G(q^2)]^2}, \tag{5.2}$$

and therefore, one should determine the closed form of the quantities $[A_i(q)]_{m^2}$.

There is a simple observation, particular to this gauge, which simplifies the entire procedure considerably. Specifically, in the Landau gauge, the derivation of the gluon mass equation does not require the knowledge of the closed form of the vertex V, which captures the effects of the massless bound-state poles.

To see why this is so, let us first note that the vertex V appears only in the terms $A_1^{\mu\nu}(q)$ and $A_3^{\mu\nu}(q)$, the only place where the replacement $\mathbb{\Gamma} \to \mathbb{\Gamma}'$ may be carried out. Given that the vertex $\mathbb{\Gamma}'_{\nu\alpha\beta}$ appearing in the term $A_3^{\mu\nu}(q)$ is contracted by $(k+q)^\beta$, the result of this operation is the STI satisfied by $\mathbb{\Gamma}'$, namely

$$p^\nu \mathbb{\Gamma}'_{\alpha\mu\nu} = F(p^2)\left[\Delta_m^{-1}(r^2)P_\mu^\nu(r)\widetilde{H}_{\nu\alpha}(r,p,q) - \widetilde{\Delta}_m^{-1}(q^2)P_\alpha^\nu(q)H_{\nu\mu}(q,p,r)\right]. \tag{5.3}$$

whose validity assumes the existence of V but does not depend on the details of its closed form.

As for the term $A_1^{\mu\nu}(q)$ one starts by noticing that (i) the V is already contracted by two projection operators $P^{\alpha\rho}(k)P^{\beta\sigma}(k+q)$ and (ii) since in the PT-BFM formulation the truncated $\widehat{\Pi}^{\mu\nu}(q)$ (defined in terms of $A_1 - A_5$) is transverse, one may contract both sides of Eq. (2.15) by the projection operator $P_{\nu'}^\nu(q)$ *for free*, *i.e.*, write

$$\begin{aligned}
\widehat{\Pi}^{\mu\nu}(q) &= \widehat{\Pi}^{\mu\nu'}(q)P_{\nu'}^\nu(q) \\
&= g^2 C_A \sum_{i=1}^{5} A_i^{\mu\nu'}(q)P_{\nu'}^\nu(q).
\end{aligned} \tag{5.4}$$

The main effect of this operation, as far as the term $A_1^{\mu\nu}(q)$ is concerned, is to trigger Eq. (3.4), and so, all explicit reference to V vanishes.

In order to forestall any possible confusion, we hasten to emphasize that one should not conclude from the above argument that the existence of the vertex V is irrelevant for the entire construction. On the contrary, the vertex V is crucial for the implementation of this

particular approach. In particular, if the V did not exist (*i.e.*, if it were vanishing identically) the WI of (3.5) would be invalidated, and, as a result, $\widehat{\Pi}^{\mu\nu}(q)$ would fail to be transverse, in which case, evidently, one could no longer contract both sides of Eq. (2.15) by the projection operator $P^{\nu}_{\nu'}(q)$ for free.

We can now proceed with the actual calculation. It is clear that the term $A^{\mu\nu}_5(q)$ cannot possibly contribute to the mass equation, since $A^{\mu\nu}_5(0) = 0$. Furthermore, with the exception of $A^{\mu\nu}_3(q)$, which will yield a direct contribution, the remaining three terms $A^{\mu\nu}_1(q)$, $A^{\mu\nu}_2(q)$, and $A^{\mu\nu}_4(q)$ contribute to the mass equations an amount that arises as the deviation from the seagull cancellation, *i.e.*, they furnish a term analogous to the I_2 of Eq. (4.7).

To see this, let us first retain the contributions of these three terms that survive *individually* the $q^2 \to 0$ limit. The term A_1 reads

$$A_1(q) = \frac{1}{2}\int_k \Gamma^{(0)}_{\mu\alpha\beta}P^{\alpha}_{\rho}(k)P^{\beta}_{\sigma}(k+q)\mathbb{I}^{\nu'\rho\sigma}_m P^{\mu}_{\nu'}(q)\Delta_m(k)\Delta_m(k+q). \tag{5.5}$$

Now, using for the vertex $\mathbb{I}_m$ the tensor decomposition (A2) with $(\alpha, \mu, \nu) \to (\nu', \rho, \sigma)$ and $r \to k$, $p \to -k - q$, it is straightforward to establish that the tensorial structures ℓ_2, ℓ_5 and ℓ_8 will be annihilated by the transverse projectors appearing in (5.5), while, ignoring terms that will again vanish due to the transverse projectors, ℓ_1, ℓ_3, ℓ_7, and ℓ_9 are at least of order q. Finally, since $\ell_{10} = 0$, we find the result

$$\begin{aligned}\mathbb{I}^{\nu'\rho\sigma}_m(q, k, -k-q) &= 2k^{\nu'}g^{\rho\sigma}\left[X_4(q, k, -k-q) + k^2 X_6(q, k, -k-q)\right] + \mathcal{O}(q) \\ &= 2k^{\nu'}g^{\rho\sigma}\frac{(k+q)^2 J_m(k+q) - k^2 J_m(k)}{(k+q)^2 - k^2} + \mathcal{O}(q).\end{aligned} \tag{5.6}$$

In addition, since

$$\Gamma^{(0)}_{\mu\alpha\beta}(q, k, -k-q)k^{\nu'}P^{\mu}_{\nu'}(q) = 2g_{\alpha\beta}\left[k^2 - \frac{(k \cdot q)^2}{q^2}\right] + \mathcal{O}(q), \tag{5.7}$$

we finally obtain

$$A_1(q) = 2(d-1)\int_k \left[k^2 - \frac{(k \cdot q)^2}{q^2}\right]\frac{(k+q)^2 J_m(k+q) - k^2 J_m(k)}{(k+q)^2 - k^2}\Delta_m(k)\Delta_m(k+q) + \mathcal{O}(q). \tag{5.8}$$

Similarly, from $A^{\mu\nu}_2(q)$ we obtain

$$A_2(q) = -\int_k \left[d - 2 + \frac{(k \cdot q)^2}{k^2 q^2}\right]\frac{k^2 \Delta_m(k)}{(k+q)^2}, \tag{5.9}$$

while $A^{\mu\nu}_4(q)$ contributes simply

$$A_4(q) = -\frac{(d-1)^3}{d}\int_k \Delta_m(k). \tag{5.10}$$

The terms in Eqs. (5.8), (5.9) and (5.10) are individually non-vanishing as $q^2 \to 0$, but their final contribution to the mass equation is controlled by the seagull identity, which forces a large part of their sum to vanish, thus reassigning them to the kinetic term. Specifically, if we use Eq. (3.1) to substitute the terms containing J_m in the numerator of the integral on the rhs of Eq. (5.8), [i.e., $k^2 J(k) = \Delta_m^{-1}(k) + m^2(k)$] the sum of these three terms gives

$$[A_1 + A_2 + A_4](q) = [A_1 + A_2 + A_4]_{\text{kt}}(q) + [A_1 + A_2 + A_4]_{m^2}(q), \qquad (5.11)$$

where

$$
\begin{aligned}
[A_1 + A_2 + A_4]_{\text{kt}}(q) =\ & -2(d-1) \int_k \left[k^2 - \frac{(k \cdot q)^2}{q^2} \right] \frac{\Delta_m(k+q) - \Delta_m(k)}{(k+q)^2 - k^2} \\
& - \int_k \left[d - 2 + \frac{(k \cdot q)^2}{k^2 q^2} \right] \frac{k^2 \Delta_m(k)}{(k+q)^2} - \frac{(d-1)^3}{d} \int_k \Delta_m(k), \quad (5.12)
\end{aligned}
$$

and leaves as residual contribution

$$[A_1 + A_2 + A_4)]_{m^2}(q) = 2(d-1) \int_k \left[k^2 - \frac{(k \cdot q)^2}{q^2} \right] \frac{m^2(k+q) - m^2(k)}{(k+q)^2 - k^2} \Delta_m(k) \Delta_m(k+q). \quad (5.13)$$

It is now easy to verify that, by virtue of the seagull identity, the rhs of (5.12) vanishes as $q^2 \to 0$. Indeed

$$[A_1 + A_2 + A_4]_{\text{kt}}(0) = -2(d-1) \int_k \sin^2 \theta \, k^2 \Delta_m'(k) - \int_k \left[(d-1) - \sin^2 \theta + \frac{(d-1)^3}{d} \right] \Delta_m(k), \qquad (5.14)$$

and using that [see also Eqs. (4.12) and (4.13) above]

$$\int_k \sin^2 \theta \, f(k) = \frac{d-1}{d} \int_k f(k), \qquad (5.15)$$

it is elementary to demonstrate that the rhs is exactly proportional to the expression on the rhs of Eq. (4.3), and therefore vanishes.

We next consider the term A_3. After taking the trace we find

$$A_3(q) = \int_k P^{\alpha\mu}(k) \frac{(k+q)^\beta \mathbb{\Gamma}'_{\nu'\alpha\beta}}{(k+q)^2} P_\mu^{\nu'}(q) \Delta_m(k). \qquad (5.16)$$

When inserted into the expression for $A_3(q)$, the first term on the rhs of (3.9) will give the result

$$\int_k \frac{F(k+q)}{(k+q)^2} \widetilde{a}(q, -k-q, k) + \mathcal{O}(q), \qquad (5.17)$$

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which contributes to the kinetic term, since in the $q \to 0$ limit vanishes due to the second identity in (A5), which in this limit gives [36]

$$\widetilde{a}(0, -k, k) = F^{-1}(k). \tag{5.18}$$

The second term on the rhs of (3.9) yields instead a surprisingly simple contribution to the mass equation. Specifically, using the definition (2.11) we obtain

$$
\begin{aligned}
[A_3]_{m^2}(q) &= \widetilde{m}^2(q^2) \int_k \frac{F(k+q)}{(k+q)^2} \Delta_\mu^\rho(k) H_{\sigma\rho}(q, -k-q, k) P^{\sigma\mu}(q) \\
&= \widetilde{m}^2(q^2) \frac{i\Lambda_{\sigma\mu}(q)}{g^2 C_A} P^{\sigma\mu}(q) \\
&= i\frac{d-1}{g^2 C_A} \widetilde{m}^2(q^2) G(q^2).
\end{aligned} \tag{5.19}
$$

On the other hand, the second of the background quantum identities (2.14) implies (see also Appendix B for an alternative derivation of this result)

$$\widetilde{m}^2(q^2) = [1 + G(q^2)]m^2(q^2), \tag{5.20}$$

so that one finally finds the contribution

$$[A_3]_{m^2}(q) = i\frac{d-1}{g^2 C_A} G(q^2)[1 + G(q^2)]m^2(q^2). \tag{5.21}$$

The next step is to substitute the above results on the rhs of the mass equation of Eq. (5.2). In doing so, we move to the Euclidean space, by setting $\int_k = i\int_{k_{\rm E}}$ and $q_{\rm E}^2 = -q^2$, and using

$$\Delta_{\rm E}(q_{\rm E}^2) = -\Delta(-q_{\rm E}^2); \qquad m_{\rm E}^2(q_{\rm E}^2) = m^2(-q_{\rm E}^2); \qquad G_{\rm E}(q_{\rm E}^2) = G(-q_{\rm E}^2). \tag{5.22}$$

Then, from Eq. (5.13) and (5.21), we arrive at the final form of the mass equation, namely

$$m^2(q^2) = \frac{2g^2 C_A}{1 + G(q^2)} \int_k \left[k^2 - \frac{(k \cdot q)^2}{q^2} \right] \frac{m^2(k+q) - m^2(k)}{(k+q)^2 - k^2} \Delta_m(k)\Delta_m(k+q), \tag{5.23}$$

where we have suppressed the suffix "E".

Finally, we are now in position to address the question posed at the end of Section III, namely what would happen if we were to introduce the gluon mass by the simple replacement $q^2 J(q^2) \to \Delta_m^{-1}(q^2)$ carried out inside $\mathbb{\Gamma}$, *i.e.*, without resorting explicitly to the vertex V (with the crucial properties assigned to it). The basic observation is that the main bulk of the mass equation, namely the rhs of Eq. (5.13), emerges as a residual contribution that

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survives the seagull cancellation. However, within this hypothetical scenario, the term $A_1(q)$ in Eq. (5.8) would be instead given by

$$
\begin{aligned}
A_1(q) &= 2(d-1) \int_k \left[k^2 - \frac{(k \cdot q)^2}{q^2} \right] \frac{\Delta_m^{-1}(k+q) - \Delta_m^{-1}(k)}{(k+q)^2 - k^2} \Delta_m(k) \Delta_m(k+q) \\
&= -2(d-1) \int_k \left[k^2 - \frac{(k \cdot q)^2}{q^2} \right] \frac{\Delta_m(k+q) - \Delta_m(k)}{(k+q)^2 - k^2} ,
\end{aligned}
\tag{5.24}
$$

thus, participating in the cancelation of Eq. (5.14), as before, but leaving no residual contribution, i.e., $[A_1 + A_2 + A_4]_{m^2}(q) = 0$. Then, the only contribution to the rhs of the mass equation would be that of $[A_3]_{m^2}(q)$ in Eq. (5.21); this contribution would be still there, because within this alternative scenario the full vertex $\mathbb{\Gamma}'$ is still assumed to satisfy the full STIs of Eq. (3.9) [but with no reference to V]. Therefore, the resulting mass equation [the equivalent of Eq. (5.23)] would read

$$
\frac{m^2(q^2)}{[1 + G(q^2)]} = 0,
\tag{5.25}
$$

which would simply imply $m^2(q^2) = 0$, i.e., no dynamical mass generation.

VI. NUMERICAL ANALYSIS

In this section, we will first derive an approximate version of the mass equation (5.23), which will facilitate the numerical treatment while retaining the main features of the full equation. Then, using as input for the functions $\Delta(q^2)$ and $F(q^2)$ [appearing in (5.23)] the available lattice data, we solve the equation numerically for the gauge groups $SU(2)$ and $SU(3)$, thus obtaining the (approximate) form of $m^2(q^2)$. Then, using Eq. (3.1), together with the $\Delta(q^2)$ of the lattice and the $m^2(q^2)$ obtained from the mass equation, we will extract the (approximate) form of $J_m(q^2)$. As a basic application, these ingredients will be subsequently combined to form the gluon mass entering in the RG-invariant combination associated with the definition of a non-Abelian effective charge.

A. Approximate version of the mass equation

We now proceed to the analysis of the mass equation (5.23). The difficulty in dealing with this equation in its full version resides in the fact that the unknown function m^2 appearing on the rhs depends on both the angular and the radial coordinates (θ and y,

respectively). To circumvent this problem we will employ certain standard approximations, in order to eliminate the angular integration. However, before embarking into the derivation of the approximate version of (5.23), we can extract useful of information about the global behavior of m^2 from its $q^2 \to 0$ limit.

Specifically, let us employ the notation introduced in (4.12), and consider the limit of Eq. (5.23) as $q^2 \to 0$. Since it is known that $L(0) = 0$ in four dimensions [41], Eq. (2.13) implies that $1 + G(0) = F^{-1}(0)$, so that we get

$$
\begin{aligned}
m^2(0) &= \frac{3}{2} g^2 C_A F(0) \int_k k^2 [m^2(k)]' \Delta^2(k) \\
&= -3 g^2 C_A F(0) \int_k m^2(k) \Delta(k) \left[k^2 \Delta(k) \right]'.
\end{aligned}
\tag{6.1}
$$

Obviously, in the kernel of above equation there is no dependence on θ, so that the angular integral can be done exactly, and one is left with the final equation

$$
m^2(0) = -\frac{3 C_A}{8\pi} \alpha_s F(0) \int_0^\infty \mathrm{d}y \, m^2(y) [y^2 \Delta^2(y)]',
\tag{6.2}
$$

where $\alpha_s = g^2/4\pi$ and, as usual, $y = k^2$ (the prime indicates now derivatives with respect to y).

Equations (6.1) and (6.2) furnish a rather interesting constraint on the structure of the full gluon propagator. Indeed, it is clear that due to the positive sign in front of the first line of Eq. (6.1), solutions of (5.23) leading to a positive $m^2(0)$ cannot be monotonically decreasing; or, seeing it from the point of view of Eq. (6.2), the kernel $[y^2 \Delta^2(y)]'$ must reverse sign and display a "sufficiently deep" negative region at intermediate momenta, in order to obtain $m^2(0) > 0$. This is a highly non-trivial requirement, because, to the best of our knowledge, there is no *a priori* fundamental reason why the full gluon propagator propagator should show this particular behavior.

We now proceed to the derivation of an approximate version of (5.23) that will reproduce in the $q^2 \to 0$ limit Eq. (6.1), and therefore implement the important constraint that this equation entails.

Let us then denote by $R(q)$ the integral appearing on the rhs of (5.23); using the simple identity

$$
(k \cdot q)^2 = \frac{1}{4} \left\{ [(k+q)^2 - k^2]^2 - 2q^2[(k+q)^2 - k^2] + (q^2)^2 \right\},
\tag{6.3}
$$

we see that the second term above, when inserted back into $R(q)$, vanishes upon integration,

and therefore one is left with

$$R(q) = R_1(q) + R_2(q),$$
(6.4)

where

$$R_1(q) = \int_k \left(k^2 - \frac{q^2}{4} \right) \frac{m^2(k+q) - m^2(k)}{(k+q)^2 - k^2} \Delta_m(k) \Delta_m(k+q),$$

$$R_2(q) = -\frac{1}{2q^2} \int_k m^2(k)[(k+q)^2 - k^2] \Delta_m(k) \Delta_m(k+q).$$
(6.5)

To cast $R_1(q)$ and $R_2(q)$ into a form suitable for solving the corresponding dynamical equation, we first introduce the by now familiar spherical coordinates and then split the radial integration into two intervals

$$\int_0^\infty \mathrm{d}y = \int_0^x \mathrm{d}y + \int_x^\infty \mathrm{d}y,$$
(6.6)

so that in the second integral since $y > x$ always, we can expand the integrand according to (4.14). Proceeding in this way, and observing that partial integration gives

$$\int_x^\infty \mathrm{d}y\, y^2 [m^2(y)]' \Delta^2(y) = -m^2(x)x^2 \Delta^2(x) - \int_x^\infty \mathrm{d}y\, m^2(y)[y^2 \Delta^2(y)]'$$
(6.7)

we obtain

$$16\pi^2 R_1(x) \approx \Delta(x) \int_0^x \mathrm{d}y\, y \left(y - \frac{x}{4} \right) \frac{m^2(x) - m^2(y)}{x - y} \Delta(y) - m^2(x)x^2 \Delta^2(x)$$

$$- \int_x^\infty \mathrm{d}y\, m^2(y)[y^2 \Delta^2(y)]',$$

$$16\pi^2 R_2(x) \approx \frac{1}{2} \int_0^x \mathrm{d}y\, y\, m^2(y) \left(1 - \frac{y}{x} \right) \Delta^2(y) + \frac{1}{4} \int_x^\infty \mathrm{d}y\, m^2(y)[y^2 \Delta^2(y)]'.$$
(6.8)

Finally, since as shown in [41, 42], $L(x)$ is considerably smaller than $G(x)$ in the entire range of (Euclidean) momenta, we can use the approximation $1 + G(x) \approx F^{-1}(x)$; thus, we obtain the approximate equation

$$m^2(x) = m^2(0) \frac{F(x)}{F(0)} + \frac{\alpha_s C_A}{2\pi} F(x) \overline{R}(x),$$
(6.9)

with

$$\overline{R}(x) = \frac{1}{2} \int_0^x \mathrm{d}y\, y\, m^2(y) \left(1 - \frac{y}{x} \right) \Delta^2(y) + \Delta(x) \int_0^x \mathrm{d}y\, y \left(y - \frac{x}{4} \right) \frac{m^2(x) - m^2(y)}{x - y} \Delta(y)$$

$$- m^2(x)\, x^2 \Delta^2(x) + \frac{3}{4} \int_0^x \mathrm{d}y\, m^2(y)[y^2 \Delta^2(y)]',$$
(6.10)

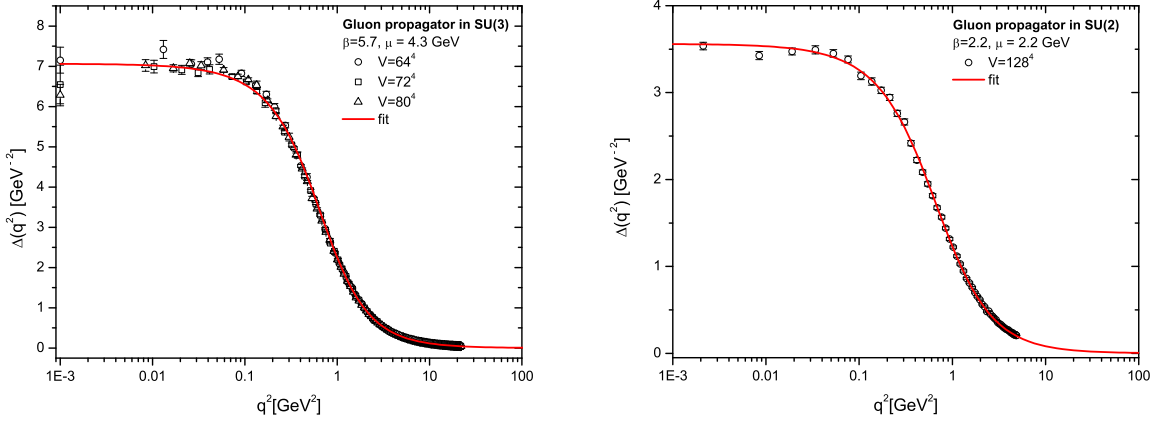and $m^2(0)$ given in Eq. (6.2). Evidently, $\overline{R}(0) = 0$.

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FIG. 6: Lattice results for the $SU(3)$ (left) and $SU(2)$ (right) gluon propagator, renormalized at $\mu = 4.3$ GeV and $\mu = 2.2$ GeV respectively. The continuous lines represents our best fits to the data obtained from Eq. (6.11).

B. Lattice ingredients: Gluon propagator and ghost dressing function

The two main ingredients of the mass equation (6.9) are the gluon propagator $\Delta(q^2)$ and the ghost dressing function $F(q^2)$. Of course, $\Delta(q^2)$ is composed by $J(q^2)$ and $m^2(q^2)$, as dictated by Eq. (3.1), but, as mentioned in the Introduction, the derivation of the corresponding equation for $J(q^2)$ is beyond our powers at this point, mainly due to lack of knowledge of certain of its ingredients. Similarly, $F(q^2)$ satisfies its own SDE (see, e.g., [41]), which would furnish yet another equation in a complicated coupled system. For the purposes of the present work, which is the preliminary scrutiny of the mass equation (6.9) appearing for the first time in the literature, we will instead resort to the high quality lattice data available, and use them as inputs inside (6.9).

In order to do that, we start by showing on the left panel of Fig. 6 the lattice data for $\Delta(q^2)$ obtained in [6], corresponding to a $SU(3)$ quenched lattice simulation, renormalized at $\mu = 4.3$ GeV; on the right panel of the same figure, we show the quenched $SU(2)$ lattice data obtained in [1], renormalized at $\mu = 2.2$ GeV.

As has been discussed in detail in the literature [10, 12, 16], both sets of lattice data can be accurately fitted in terms of a IR finite gluon propagator of the form [43]

$$\Delta^{-1}(q^2) = M^2(q^2) + q^2 \left[1 + \frac{13 C_{\mathrm{A}} g_1^2}{96\pi^2} \ln \left(\frac{q^2 + \rho_1 M^2(q^2)}{\mu^2} \right) \right], \tag{6.11}$$
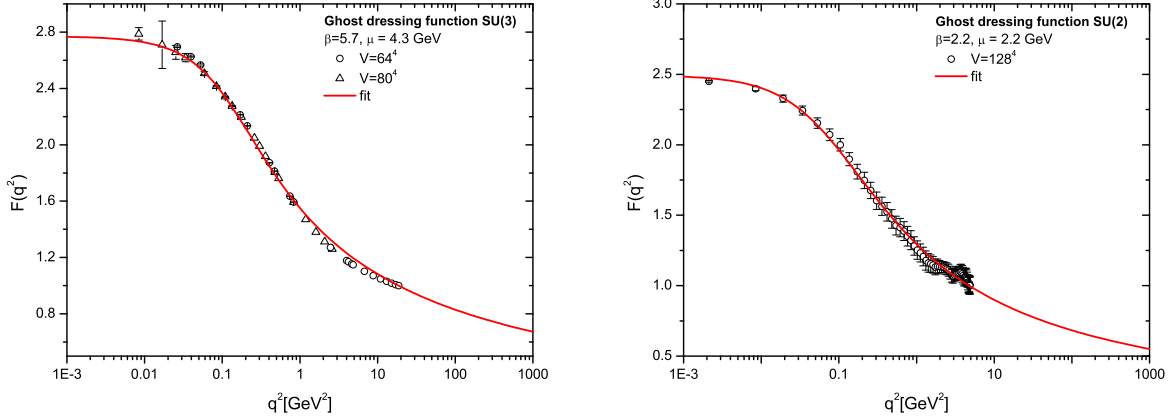
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FIG. 7: Lattice results for the $SU(3)$ (left) and $SU(2)$ (right) ghost dressing function, renormalized at $\mu = 4.3$ GeV and $\mu = 2.2$ GeV respectively. The continuous lines represent our best fits to the data obtained from Eq. (6.13).

where [49]

$$M^2(q^2) = \frac{m_0^4}{q^2 + \rho_2 m_0^2}. \tag{6.12}$$

The function $M^2(q^2)$ controls the value of $\Delta^{-1}(q^2)$ at the origin; evidently, $\Delta^{-1}(0) = M^2(0) = m_0^2/\rho_2$. The best fits (shown by the continuous lines in Fig. 6) correspond to the following values of the fitting parameters:

- $SU(3)$ case: $m_0 = 520$ MeV, $g_1^2 = 5.68$, $\rho_1 = 8.55$, $\rho_2 = 1.91$;

- $SU(2)$ case: $m_0 = 867$ MeV, $g_1^2 = 10.80$, $\rho_1 = 1.96$, $\rho_2 = 2.68$.

Turning next to the ghost dressing function, on the left panel of Fig. 7, we show the $SU(3)$ lattice results of [6], renormalized as before at $\mu = 4.3$ GeV; on the right panel we plot instead the results for the $SU(2)$ case [1], renormalized at $\mu = 2.2$ GeV. As can be clearly seen, both functions saturate in the deep IR at the constant value [12, 50, 51], and can therefore be fitted in terms of the expression

$$F^{-1}(q^2) = 1 + \frac{9}{4}\frac{C_A g_2^2}{48\pi^2} \ln\left(\frac{q^2 + \rho_3 M^2(q^2)}{\mu^2}\right), \tag{6.13}$$

with $M^2(q^2)$ given by Eq. (6.12), but changing the parameter $\rho_2 \to \rho_4$.

The best values for the fitting parameters are:

- $SU(3)$ case: $g_2^2 = 8.57$, $m = 520$ MeV, $\rho_3 = 0.25$, $\rho_4 = 0.68$;

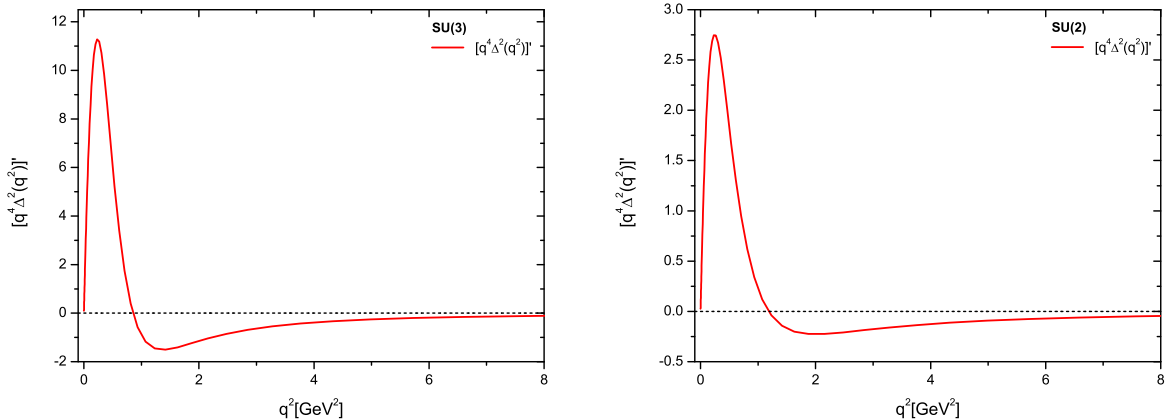- $SU(2)$ case: $g_2^2 = 15.03$, $m = 523$ MeV, $\rho_3 = 0.21$, $\rho_4 = 0.78$.

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FIG. 8: The kernel $[q^4\Delta^2(q^2)]'$ appearing in Eq. (6.2) obtained from the $SU(3)$ (left) and $SU(2)$ (right) lattice data. In both cases one clearly sees the behavior expected for getting a positive value for $m^2(0)$. The zero crossing happens at $q_0^2 \approx 0.85$ and $q_0^2 \approx 1.1$ respectively.

C.   Solutions of the mass equation and extraction of $J_m(q^2)$

After presenting the precise form of $\Delta(q^2)$ and $F(q^2)$, the next task is to find solutions of the approximate mass equation (6.9).

To begin with, we compute (for both gauge groups considered) the derivative of the gluon dressing squared, $[y^2\Delta^2(y)]'$, entering into the condition (6.2). As mentioned earlier, the behavior of this quantity provides a rather direct criterion for the existence or not of positive-definite mass solutions, and in particular $m^2(0) > 0$. Specifically, the absence of a negative region from this derivative immediately excludes such solutions, while a relatively shallow "well" makes their existence unlikely.

In the results shown in Fig. 8 we clearly see that both derivatives change their sign in the intermediate momenta region, which, as previously explained, constitutes precisely the required behavior. This behavior is to be contrasted with that of simple propagators, such as $1/(q^2 + m^2)$, or the Gribov-Zwanziger propagator $q^2/(q^4 + m^4)$ [37, 38], which fail to provide the necessary negative region (in fact the derivative is positive everywhere). It should be noted that, instead, the "refined" version of the Gribov-Zwanziger propagator [52] is expected to furnish a considerable negative region, given that it is known to provide a good fit to the lattice data.

Of course, the aforementioned criterion can only serve as a necessary but not sufficient
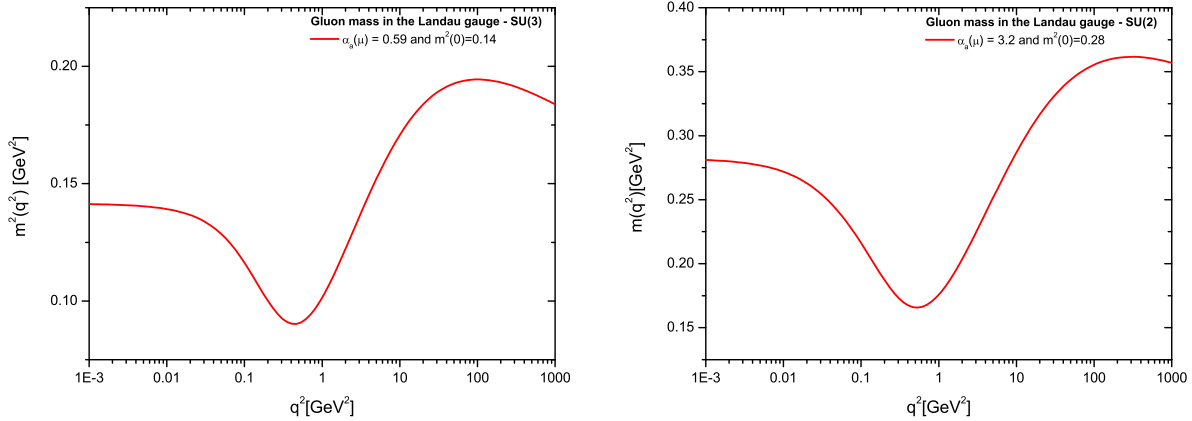
FIG. 9: The solution for $m^2(q^2)$ obtained through the approximate mass equation (6.9) for $SU(3)$ (left) and $SU(2)$ (right).

condition: to get a positive definite value for $m^2(0)$ one still needs to demonstrate that the negative region $q^2 > q_0^2$ (with q_0^2 the value where the curve is zero) furnishes more support to the integral of Eq. (6.2) than its positive region.

To proceed with the actual determination of $m^2(x)$ from Eq. (6.9), we substitute the quantities $\Delta(y)$, $F(y)$ and C_A for the $SU(3)$ and $SU(2)$ gauge groups and solve for the unknown function. In both cases the value of $m^2(0)$ is a boundary condition, fixed through the value of the corresponding lattice gluon propagator at the origin, *i.e.*, $m^2(0) = \Delta^{-1}(0)$. Specifically, for $SU(3)$ we have that $\Delta^{-1}(0) \approx 0.14$ while for $SU(2)$ $\Delta^{-1}(0) \approx 0.28$

The solutions obtained are shown in Fig. 9; the values for α_s needed to satisfy the boundary condition are $\alpha_s = 0.59$ and $\alpha_s = 3.2$ for $SU(3)$ and $SU(2)$ respectively. Notice that the masses corresponding to both gauge groups display the same qualitative behavior, and, as expected, are clearly non-monotonic functions of the momentum.

From the solutions for $m^2(q^2)$ obtained above, and the lattice results for $\Delta(q^2)$, we may now extract the approximate form of the "kinetic term", $J_m(q^2)$. Specifically, $J_m(q^2)$ can be determined (in Euclidean space) through Eq. (3.1), namely

$$J_m(q^2) = \frac{\Delta^{-1}(q^2) - m^2(q^2)}{q^2}, \tag{6.14}$$

Notice that special care must be taken in the $q^2 \to 0$ limit of Eq. (6.14). In the region of small momenta, Eq. (6.14) has a delicate cancellation between the denominator and the numerator, which also tends to zero in this limit, since $\Delta^{-1}(q^2) \to m^2(0)$. In order to avoid
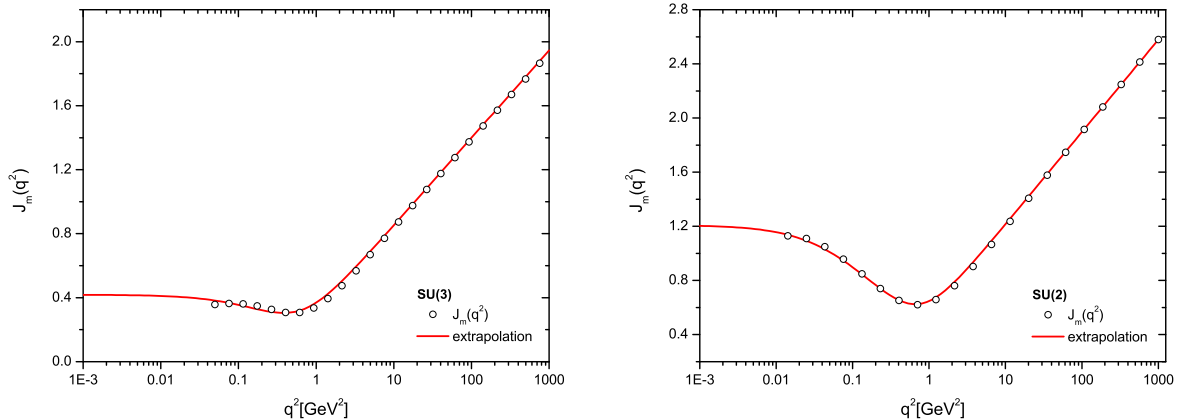
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FIG. 10: Values of $J_m(q^2)$ obtained from Eq. (6.14) (white circles) using the $SU(3)$ gluon propagator and the corresponding extrapolation towards the $q^2 \to 0$ limit (continuous line). As usual we show both the $SU(3)$ (left) and the $SU(2)$ cases.

spurious distortion in the IR behavior of $J_m(q^2)$, we will extract $J_m(q^2)$ until certain (small) value of q^2 past which we will do an extrapolation towards $q^2 \to 0$. The results of this procedure are shown in Fig. 10, where, for both the $SU(3)$ (left) and the $SU(2)$ (right) cases, we display the points obtained directly from Eq. (6.14) as well as our extrapolation curves.

Knowledge of $m^2(q^2)$ and $J_m(q^2)$ allows one to determine the approximate form of the (formally) RG-invariant gluon mass that appears naturally in the definition of the QCD effective charge [16, 21, 22, 53]. Let us recall that, due to the Abelian WIs satisfied by the PT-BFM Green's functions, the propagator $\widehat{\Delta}(q^2)$ absorbs all the RG logarithms, exactly as happens in QED with the photon self-energy. As a result, the product

$$\overline{d}_0(q^2) \equiv g_0^2 \widehat{\Delta}_0(q^2) = g^2 \widehat{\Delta}(q^2) \equiv \overline{d}(q^2), \tag{6.15}$$

forms a RG-invariant (μ-independent) quantity. As has been explained in the recent literature [15], $\overline{d}(q^2)$ may be cast in the form

$$\overline{d}(q^2) = \frac{\overline{g}^2(q^2)}{q^2 + \overline{m}^2(q^2)}, \tag{6.16}$$

with

$$\begin{aligned}
\overline{g}^2(q^2) &= g^2 \widehat{J}_m^{-1}(q^2), \\
\overline{m}^2(q^2) &= \widehat{m}^2(q^2) \widehat{J}_m^{-1}(q^2).
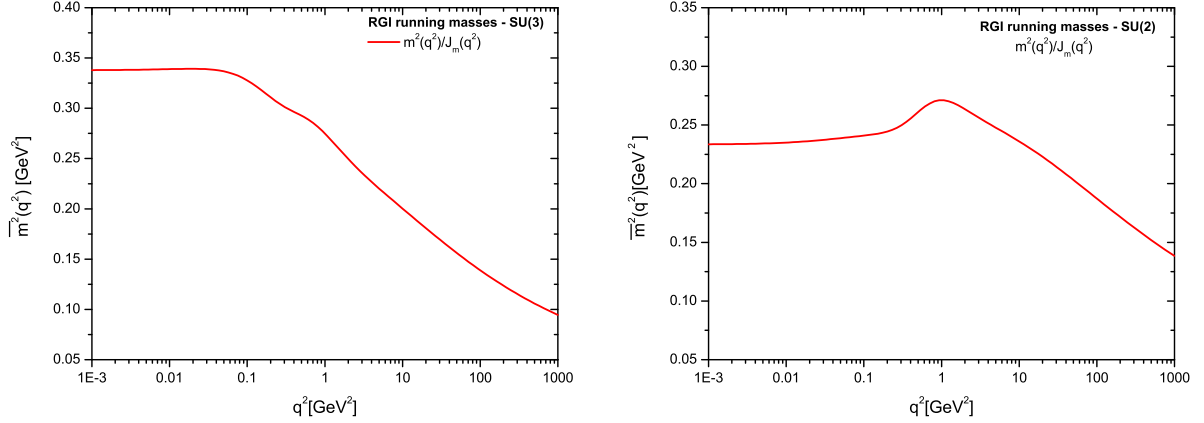\end{aligned} \tag{6.17}$$

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FIG. 11: The RG-invariant mass, $\overline{m}^2(q^2)$, defined in Eq. (6.20) for the $SU(3)$ (left) and $SU(2)$ (right) cases.

The two factors defined above are individually RG-invariant; the dimensionful quantity corresponds to a massive propagator with a momentum dependent mass, while the dimensionless factor $\overline{g}^2(q^2)/4\pi$ defines the effective charge.

Next, using the BQIs (2.14) to relate the components of $\widehat{\Delta}(q^2)$ to the corresponding ones of $\Delta(q^2)$, we get

$$\widehat{J}_m(q^2) = [1 + G(q^2)]^2 J_m(q^2),$$
$$\widehat{m}^2(q^2) = [1 + G(q^2)]^2 m^2(q^2),$$

(6.18)

and therefore

$$\widehat{m}^2(q^2)\widehat{J}_m^{-1}(q^2) = m^2(q^2)J_m^{-1}(q^2),$$

(6.19)

which finally furnishes the relation

$$\overline{m}^2(q^2) = m^2(q^2)J_m^{-1}(q^2).$$

(6.20)

We are now in the position to determine the mass $\overline{m}^2(q^2)$ by simply forming the ratio of the plots presented in Fig. 9 and 10. The result is shown in Fig. 11; as can be seen, in the $SU(3)$ case $\overline{m}^2(q^2)$ corresponds roughly to a monotonically decreasing function (see also [54]), with $\overline{m}(0) \approx 580\,\text{MeV}$. Finally, for the $SU(2)$ case we obtain $\overline{m}(0) \approx 480\,\text{MeV}$.

VII. DISCUSSION AND CONCLUSIONS

In the present work we have derived the dynamical equation that determines the evolution of the gluon mass in the Landau gauge, using as our starting point the "one-loop dressed" SDE for the gluon propagator in the PT-BFM scheme. The entire construction hinges on the crucial assumption that a special vertex, denoted by V, is dynamically generated, according to the philosophy and formalism associated with the Schwinger mechanism. The role of this vertex is to maintain gauge invariance (as expressed through the STIs satisfied by the Green's functions of the theory) in the presence of a dynamical mass. Interestingly enough, the derivation of the mass equation does not depend on the specific closed form of that vertex.

The equation for the gluon mass derived here, given in (5.23), and in particular its limit in the deep IR, imposes a rather strong constraint on the form of the full gluon propagator in the region of intermediate momenta of about (1-5) GeV^2. In this specific range of momenta the shape of the gluon propagator must be such that the derivative of the square of the gluon dressing function $[q^4\Delta^2(q^2)]'$ becomes sufficiently negative, thus ensuring eventually the positivity of the gluon mass.

We emphasize that the central result of this article, Eq. (5.23), does not exhaust all possible contributions to the gluon mass equation. Specifically, Eq. (5.23) captures only the part of the equation originating from the "one-loop dressed" gluon SDE. In order to determine the corresponding contribution coming from the "two-loop dressed" gluon SDE one must identify the seagull cancellation mechanism (and the corresponding "seagull-identity") that operates at the "two-loop dressed" level. The identification of the "two-loop dressed" analogue of Eq. (4.3) requires (among other things) some very specific information on the structure of the four-gluon vertex, at least in the special kinematic limit of vanishing external momentum. Calculations in this direction are already in progress.

It is important to warn the reader about some additional limitations afflicting the present work, related to the renormalization properties of the mass equation, and the dependence of the various quantities, most importantly of the gluon mass, on the renormalization point μ. When dealing with the mass equation of Eq. (5.23) we have tacitly assumed that the multiplicative renormalization has been carried out, thus rendering all quantities finite (but μ-dependent). In carrying out the SDE renormalization one usually resorts to the momentum-

subtraction (MOM) scheme; in our case this choice is further motivated by the additional fact that this is the scheme employed for the renormalization of the lattice data that are used as input into Eq. (5.23). Of course, given the gluon mass in the Landau gauge is not a RG-invariant quantity, there is a residual dependence on μ, which, in principle, should cancel out against analogous contributions when a RG-invariant combination is formed (this type of powerful cancellation has been presented in [41] for the QCD effective charge). However, the approximations employed in the process of the renormalization may distort the exact dependence on μ. Specifically, the renormalized version of Eq. (5.23) displays a dependence on some of the renormalization constants Z involved, as happens typically in the treatment of SDEs. This fact in itself is normal, but makes the further treatment ambiguous, because the correct cancellation of the residual dependence on the UV cutoff (induced by the presence of the Z) requires the knowledge (among other things) of the transverse (automatically conserved) part of the full vertex $\mathbb{\Gamma}$ [36]. Therefore, the next step has been to set $Z = 1$, a fact which, in general, is known to alter the dependence of the solution (in this case of m^2) on μ. In fact, the situation appears to be very similar to what happens typically in the studies of chiral symmetry breaking through the standard gap equation. In this latter context, the various approximations associated with renormalization introduce characteristic artifacts; for example, the value of the anomalous dimension of the dynamical quark mass is distorted, a problem that is usually compensated by modifying accordingly (by hand) the kernel of the gap equation. Needless to say, it would be very important to improve on any of the above points, but at present this appears to be technically rather difficult.

Given that the existence of a non-trivial vertex V is of central importance, it would be absolutely essential to establish its existence. This can be done following two distinct but complementary approaches. First, one may write down the most general longitudinal structure allowed by Lorentz symmetry and then use the WI and STIs that the V is supposed to satisfy [*e.g.*, (3.5)] to actually determine the form of the various form factors, in the spirit of [36]. Second, one may address the dynamical question of whether such a nonperturbative vertex may be actually produced by the strongly coupled Yang-Mills theory. In fact, the main characteristic of the vertex V, which sharply differentiates it from ordinary vertex contribution, is that it contains massless bound-state poles. In principle, the dynamical formation of such poles must be studied by means of a homogeneous Bethe-Salpeter equation, following the methodology developed in [31–33]. We hope to be able to pursue some of these

points in the near future.

Appendix A: Explicit form of the vertex $\mathbb{\Gamma}$ and $\mathbb{\Gamma}_m$

The longitudinal part of the vertex $\mathbb{\Gamma}$ (and therefore also that of $\mathbb{\Gamma}_m$) has been constructed in [36] by simultaneously solving the Ward and Slavnov-Taylor identities presented in Eq. (2.19); for the kinematics, see Fig. 2. Specifically, the longitudinal part is written as

$$\mathbb{\Gamma}^{\alpha\mu\nu}(q,r,p) = \sum_{i=1}^{10} X_i(q,r,p)\ell_i^{\alpha\mu\nu}(q,r,p), \tag{A1}$$

in the standard basis ℓ_i of [39]

$$
\begin{aligned}
&\ell_1^{\alpha\mu\nu} = (q-r)^\nu g^{\alpha\mu} &&\ell_2^{\alpha\mu\nu} = -p^\nu g^{\alpha\mu} &&\ell_3^{\alpha\mu\nu} = (q-r)^\nu[q^\mu r^\alpha - (q\cdot r)g^{\alpha\mu}] \\
&\ell_4^{\alpha\mu\nu} = (r-p)^\alpha g^{\mu\nu} &&\ell_5^{\alpha\mu\nu} = -q^\alpha g^{\mu\nu} &&\ell_6^{\alpha\mu\nu} = (r-p)^\alpha[r^\nu p^\mu - (r\cdot p)g^{\mu\nu}] \\
&\ell_7^{\alpha\mu\nu} = (p-q)^\mu g^{\alpha\nu} &&\ell_8^{\alpha\mu\nu} = -r^\mu g^{\alpha\nu} &&\ell_9^{\alpha\mu\nu} = (p-q)^\mu[p^\alpha q^\nu - (p\cdot q)g^{\alpha\nu}] \\
&\ell_{10}^{\alpha\mu\nu} = q^\nu r^\alpha p^\mu + q^\mu r^\nu p^\alpha.
\end{aligned}
\tag{A2}
$$

and the X_i are given by

$$
\begin{aligned}
X_1(q,r,p) &= \frac{1}{4}\widetilde{J}(q^2)\left\{-p^2 b_{prq}F(r^2) + [2a_{rpq} + p^2 b_{rpq} + 2(q\cdot r)d_{rpq}]F(p^2)\right\} \\
&\quad + \frac{1}{4}J(r^2)\left[2 + (r^2 - q^2)\widetilde{b}_{qpr}F(p^2)\right] + \frac{1}{4}J(p^2)\,p^2\,\widetilde{b}_{qrp}F(r^2) \\
X_2(q,r,p) &= \frac{1}{4}\widetilde{J}(q^2)\left\{(q^2 - r^2)b_{prq}F(r^2) + [2a_{rpq} + (r^2 - q^2)b_{rpq} + 2(q\cdot r)d_{rpq}]F(p^2)\right\} \\
&\quad + \frac{1}{4}J(r^2)\left[-2 + p^2\widetilde{b}_{qpr}F(p^2)\right] + \frac{1}{4}J(p^2)\,(r^2 - q^2)\,\widetilde{b}_{qrp}F(r^2) \\
X_3(q,r,p) &= \frac{F(p^2)}{q^2 - r^2}\left\{\widetilde{J}(q^2)\left[a_{rpq} - (q\cdot p)d_{rpq}\right] - J(r^2)\left[\widetilde{a}_{qpr} - (r\cdot p)\widetilde{d}_{qpr}\right]\right\} \\
X_4(q,r,p) &= \frac{1}{4}\widetilde{J}(q^2)q^2\left[b_{prq}F(r^2) + b_{rpq}F(p^2)\right] + \frac{1}{4}J(r^2)\left[2 - q^2\widetilde{b}_{qpr}F(p^2)\right] \\
&\quad + \frac{1}{4}J(p^2)\left[2 - q^2\widetilde{b}_{qrp}F(r^2)\right]
\end{aligned}
$$

$$X_5(q,r,p) = \frac{1}{4}\widetilde{J}(q^2)(p^2 - r^2)\left[b_{prq}F(r^2) + b_{rpq}F(p^2)\right] + \frac{1}{4}J(r^2)\left[2 + (r^2 - p^2)\widetilde{b}_{qpr}F(p^2)\right]$$
$$- \frac{1}{4}J(p^2)\left[2 + (p^2 - r^2)\widetilde{b}_{qrp}F(r^2)\right]$$

$$X_6(q,r,p) = \frac{J(r^2) - J(p^2)}{r^2 - p^2}$$

$$X_7(q,r,p) = X_1(q,p,r)$$

$$X_8(q,r,p) = -X_2(q,p,r)$$

$$X_9(q,r,p) = X_3(q,p,r)$$

$$X_{10}(q,r,p) = \frac{1}{2}\left\{\widetilde{J}(q^2)\left[b_{prq}F(r^2) - b_{rpq}F(p)\right] + J(r^2)F(p^2)\widetilde{b}_{qpr} - J(p^2)F(r^2)\widetilde{b}_{qrp}\right\}. \quad (A3)$$

The functions $a_{qrp} \equiv a(q,r,p)$, etc are the form factors appearing in the tensorial decomposition of the ghost-gluon kernels $H_{\nu\mu}(p,r,q)$ and $\widetilde{H}_{\nu\mu}(p,r,q)$, namely

$$H_{\nu\mu}(p,r,q) = g_{\mu\nu}a_{qrp} - r_\mu q_\nu b_{qrp} + q_\mu p_\nu c_{qrp} + q_\nu p_\mu d_{qrp} + p_\mu p_\nu e_{qrp}, \quad (A4)$$

and similarly for $\widetilde{H}$. They satisfy the non-trivial all-order constraints

$$F(r^2)[a_{prq} - (r \cdot p)b_{prq} + (q \cdot p)d_{prq}] = F(p^2)[a_{rpq} - (r \cdot p)b_{rpq} + (q \cdot r)d_{rpq}],$$
$$F(r^2)[\widetilde{a}_{qrp} - (q \cdot r)\widetilde{b}_{qrp} + (q \cdot p)\widetilde{d}_{qrp}] = 1. \quad (A5)$$

Appendix B: On the relation between $\widetilde{m}^2(q^2)$ and $m^2(q^2)$

In Section V we have assumed that the relation (5.20) between the masses $\widetilde{m}(q)$ and $m(q)$ holds. This is tantamount to claiming that the BQIs (2.14) hold after dynamical mass generation has taken place.

To further substantiate this claim, let us consider the SDE for the QB gluon self-energy $\widetilde{\Pi}$. If we keep dressed the background side of the equation, we can still truncate meaningfully the SDE retaining only the one-loop dressed gluon contributions, which now read

$$(b_1)_{\mu\nu} = \frac{1}{2}g^2 C_A \int_k \Gamma^{(0)}_{\mu\alpha\beta}(q,k,-k-q)\Delta^{\alpha\rho}(k)\Delta^{\beta\sigma}(k+q)\widetilde{\Gamma}_{\nu\rho\sigma}(q,k,-k-q),$$
$$(b_2)_{\mu\nu} = g^2 C_A \left[g_{\mu\nu}\int_k \Delta^\rho_\rho(k) - \int_k \Delta_{\mu\nu}(k)\right]. \quad (B1)$$

The projection to the Landau gauge gives rise to three terms only, which coincide with A_1, A_2 and A_4 of Eq. (2.16). Then writing

$$\widetilde{\Delta}^{-1}(q^2) \equiv q^2\widetilde{J}(q^2) - \widetilde{m}^2(q^2) = q^2 + i\widetilde{\Pi}(q^2), \quad (B2)$$

it is relatively straightforward to establish that the rhs of the equation for $\widetilde{m}$ is determined by the the mass term of A_1 only. Specifically, using the result (5.13) one has (Euclidean space)

$$
\begin{aligned}
\widetilde{m}^2(q^2) &= 2g^2 C_A \int_k \left[k^2 - \frac{(k \cdot q)^2}{q^2} \right] \frac{m^2(k+q) - m^2(k)}{(k+q)^2 - k^2} \Delta_m(k) \Delta_m(k+q) \\
&= \frac{g^2 C_A}{d-1} [A_1 + A_2 + A_4]_{m^2} ,
\end{aligned}
\tag{B3}
$$

where in the last step we have used Eq. (5.13).

Substituting the above result, together with (5.19), into Eq. (5.2), we find

$$
\begin{aligned}
m^2(q^2) &= \frac{\widetilde{m}^2(q^2)}{[1 + G(q^2)]^2} + \frac{\widetilde{m}^2(q^2) G(q^2)}{[1 + G(q^2)]^2} \\
&= \frac{\widetilde{m}^2(q^2)}{1 + G(q^2)} ,
\end{aligned}
\tag{B4}
$$

namely the relation of Eq. (5.20).

---

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