Developing the Framed Standard Model

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Abstract

The framed standard model (FSM) suggested earlier, which incorporates the Higgs field and 3 fermion generations as part of the framed gauge theory structure, is here developed further to show that it gives both quarks and leptons hierarchical masses and mixing matrices akin to what is experimentally observed. Among its many distinguishing features which lead to the above results are (i) the vacuum is degenerate under a global $su(3)$ symmetry which plays the role of fermion generations, (ii) the fermion mass matrix is "universal", rank-one and rotates (changes its orientation in generation space) with changing scale μ, (iii) the metric in generation space is scale-dependent too, and in general non-flat, (iv) the theta-angle term in the QCD action of topological origin gets transformed into the CP-violating phase of the CKM matrix for quarks, thus offering at the same time a solution to the strong CP problem.

[1]Work supported by Spanish MICINN under contracts Proyecto Prometeo 2008-004 CI11-086 and FPA2008-02878.

1 Introduction

Despite its great success in explaining existing data, the standard model as usually formulated is based on a number of intricate assumptions some of which are themselves in need of explanation. These include in particular the assumption of the scalar Higgs field needed for symmetry breaking in the electroweak sector and the introduction of 3 generations of fermion fields, neither of which has a theoretical foundation in a theory otherwise quite geometrically grounded. Even more mysterious is the injection from experiment of the hierarchical fermion mass spectrum and the peculiar mixing pattern between up and down fermion states, which together account for some two-thirds of the model's twenty-odd empirical parameters. For these reasons, among others, it is generally expected that a more fundamental theory exists from which the present standard model can be derived with all the mysterious patterns that it contains.

What we now call the framed standard model (FSM) is an attempt initiated in [1] at constructing just such a theory. It is based on what one can call the framed gauge theory (FGT) framework [2] in which, in addition to the gauge and matter fields in standard formulations of gauge theories, one also includes the frame vectors in internal symmetry space as dynamical variables. That frame vectors can appear as dynamical variables is familiar in gravity, where vierbeins are often taken as such in place of the metric. So it seems reasonable to consider this possibility also in the particle physics context. The immediate attraction of this is three-fold. First, the frame vectors in internal symmetry space transform as fundamental representations of the gauge symmetry but are Lorentz scalars, and so can function as the Higgs fields needed for breaking the flavour symmetry. Secondly, they carry by definition, in addition to indices referring to the local gauge frame, "dual" indices referring to a global reference frame which, in the case of colour $su(3)$ symmetry, can play the role of fermion generations. Thirdly, since physics should be independent of the choice of the reference frame, the action containing these frame vectors (or "framons") as dynamical variables should be invariant under both the global "dual" symmetry and the original local gauge symmetry, thus greatly reducing the freedom in the form that the action can take. Indeed, it was shown in [1], and more succinctly and with greater transparency in [2], that applying this idea to a theory with the gauge symmetry $su(3) \times su(2) \times u(1)$ yields a structure, namely the framed standard model (FSM), which is standard model-like, but now with both the Higgs field and

3 fermion generations already built in, as desired.

The purpose of the present paper is to develop further this FSM to see whether it could yield for the 3 generations of fermions a hierarchical mass spectrum and mixing matrices both with the distinctive features experimentally observed. Our strategy for doing so is based on the observation that if the fermion mass matrix is of rank one, "universal", and rotates (i.e., changes its orientation in generation space) with scale, then both mass hierarchy and mixing will automatically result [3]. It has already been shown in [1, 2] that in the FSM, the fermion mass matrix is indeed of rank one, being expressible in a factorized form:

$$m = m_T \boldsymbol{\alpha} \boldsymbol{\alpha}^\dagger, \tag{1}$$

in terms of a unit vector $\boldsymbol{\alpha}$ in generation space, and is "universal" in the sense that the vector $\boldsymbol{\alpha}$ is the same for all fermion types T, i.e., whether up or down, and whether leptons or quarks. Rotation of the mass matrix then just means that this $\boldsymbol{\alpha}$ rotates. In the next section §2, we shall outline for convenient reference how rotation works, and summarize some of the formulae needed for the discussion later. The main thrust of the paper, however, is directed towards the question whether the FSM will generate the rotation we seek, and if it does how it manages to do so.

Notice that neither the ideas of a rank-one mass matrix nor of its rotation with scale are new. The experimentally observed facts that fermion masses are hierarchical and that they mix with a mixing matrix close to unity (at least for quarks) has long suggested to some authors [4, 5] that a rank-one mass matrix would be phenomenologically profitable. It has also long been known [6] that even in the standard model as usually formulated the fermion mass matrix rotates with changing scales as a consequence of up-down mixing. Hence, the only really new concept here is that instead of mixing giving rise to rotation as in [6], it is rotation that gives rise to mixing, requiring thus a faster rotation than is given in the standard model by mixing. That one has to go beyond the standard model for the rotation we seek for explaining mass hierarchy and mixing is obvious since, in the standard model itself, masses and mixing angles appear as empirical parameters, meaning that the standard model is internally consistent for any choice of their values, and therefore inherently incapable of explaining them.

The manner that this faster rotation comes about in the FSM is in fact quite intriguing and deserves a brief outline before we plunge into details. As shown in [1, 2], to which the reader is referred for details, the FSM can be thought of as consisting of two sectors, the electroweak and the strong, each with its own set of gauge vector bosons and frame-vector scalar bosons (framons). What is new compared to the standard model are of course the framons and in particular their self-interaction term $V[\Phi]$, which will determine, by its minimization, the vacuum. The form that $V[\Phi]$ can take is severely constrained by invariance principles, as we already stated; its explicit form derived in [1, 2] will be given below. For the moment, we need only note that $V[\Phi]$ consists of 3 terms, a term V_W depending only on the weak framon field which is essentially the same as for the standard electroweak theory on which little need at present to be said, then a term V_S depending only on the strong framon field, and lastly a linkage term V_{WS} depending on both the weak and strong framon fields. From these, one can deduce the following. First, V_S alone on minimization gives a vacuum where the 3 strong framons form an orthonormal triad, as frame vectors are expected to do, but they are distorted from orthonormality by the linkage term V_{WS}, with the distortion depending on the direction of a vector $\boldsymbol{\alpha}$ coming from the weak sector, and this vector is exactly that which appears in the rank-one fermion mass matrix expressible as in (1). Secondly, when loop corrections are turned on, the vacuum will get renormalized and will change with scale carrying the vector $\boldsymbol{\alpha}$ along with it. And it is this change in direction of $\boldsymbol{\alpha}$ (rotation) which gives in the end the mass hierarchy and mixing patterns one wants. One sees that the rotation here actually originates in the vacuum and gets transmitted to the fermion mass matrix only through the vector $\boldsymbol{\alpha}$. It is therefore independent of the fermion type T to which $\boldsymbol{\alpha}$ is coupled, meaning that the mass matrix will remain of the factorized form (1) above and universal under rotation, both conditions needed for the rotation scheme [3] to work. Moreover, one sees that it is the strong interactions which is driving the rotation, which can therefore be fast enough to give the mixing effects one seeks.

To examine in detail the process outlined above, one will need to go successively through the following steps. First, of course, one will need

- §3 to elucidate the vacuum as derived from minimizing $V[\Phi]$.

Next, one will need to examine how this vacuum gets renormalized, and hence becomes scale-dependent. In principle, one can obtain information on this via the renormalization on any quantity which depends on the vacuum. We have chosen in this paper, mainly for historical reasons, to study the mass renormalization of certain fermion states to be specified later which are, in the present FSM framework, the analogues in the strong sector of leptons and quarks in the electroweak sector. They are hadron states. And as a sample of the renormalization effects on these, we have chosen to study those due to the insertion of a loop of what we shall call strong Higgs states which were found in a parallel earlier study [7] to give most of the rotation. These strong Higgs states are fluctuations of the strong framons about the strong vacuum, and are thus analogues in the strong sector of the standard Higgs state in the electroweak sector. To achieve this aim, one will need

- §4 to identify the strong Higgs states from framon fluctuations about the vacuum;
- §5 to derive their couplings to the chosen fermion states and evaluate their loop correction to the fermion self-energy, and hence to derive its implication on the scale-dependence of the vacuum;
- §6 to derive the rotation equation resulting from this for the vector $\boldsymbol{\alpha}$ appearing in (1) above.

Some of these steps were started in [1] but not completed or were done only to a rough approximation. Now, with much better techniques and improved understanding, the programme set out above can be carried out exactly and in full.

The resulting rotation equation for $\boldsymbol{\alpha}$, though likely to be rather limited by its mode of derivation in accuracy and range of applicability, serves nevertheless as a useful concrete example for how rotation in the FSM is generated. As already noted, the rotation still leaves the fermion mass matrix both factorized and universal as required. Besides, it is seen

- §7 to have fixed points at $\mu = 0, \infty$;
- §9 to generate automatically a CP-violating phase in the CKM matrix and to offer, at the same time, a solution to the strong CP problem;

properties which are believed to be generic, i.e., independent of much of the restrictive assumptions under which the rotation is derived here. In other words, the FSM seems to possess already those features which have been identified in the phenomenological study [3] as needed of a rotational model for a successful description of the mass and mixing data.

One other novel feature of the FSM revealed in the analysis of the rotation equation is the appearance of

- §8 a running metric for generation space,

which is of much theoretical, and perhaps even phenomenological, interest for the future, but is shown not to alter the other effects of rotation already listed.

A brief summary of the results and comparisons to other models beyond the standard model is given in the last section §10.

2 Mass Hierarchy and Mixing from Rotation

We begin, for easy reference and to introduce some notations, with a brief outline of how a rank-one rotating mass matrix (R2M2) automatically leads to mass hierarchy and to mixing between up and down fermion states, while displaying several formulae useful for later discussions. For details, the reader is referred to [3], a recent review.

We note first that any fermion mass matrix can, by a judicious relabelling of the $su(2)$ singlet right-handed fields, be cast into a form with no dependence on γ_5 [8] so that any rank-one mass matrix can be written without loss of generality in the form (1). Then the assertion that m rotates simplifies to the assertion that the vector $\boldsymbol{\alpha}$ rotates.

That an $\boldsymbol{\alpha}$ rotating with scale will automatically lead to mixing and mass hierarchy is most easily explained in the simplified situation when account is taken only of the two heaviest generations. By (1) then, taking for the moment $\boldsymbol{\alpha}$ to be real and m_T to be μ-independent for simplicity, we would have $m_t = m_U$ as the mass of t and the eigenvector $\boldsymbol{\alpha}(\mu = m_t)$ as its state vector $\mathbf{t}$. Similarly, we have $m_b = m_D$ as the mass and $\boldsymbol{\alpha}(\mu = m_b)$ as the state vector $\mathbf{b}$ of b. The vectors $\mathbf{t}$ and $\mathbf{b}$ are not aligned, being the vector $\boldsymbol{\alpha}(\mu)$ taken at two different values of its argument μ, and $\boldsymbol{\alpha}$ by assumption rotates. Let then θ_{tb} be the non-zero angle between them. Next, the state vector $\mathbf{c}$ of c must be orthogonal to $\mathbf{t}$, c being by definition an independent quantum state to t. Similarly, the state vector $\mathbf{s}$ of s is orthogonal to $\mathbf{b}$. The up dyad $\{\mathbf{t}, \mathbf{c}\}$ differs thus from the down dyad $\{\mathbf{b}, \mathbf{s}\}$ by a rotation by the angle θ_{tb} above. This gives then the following CKM mixing (sub)matrix in the situation with only the two heaviest states being considered

$$\begin{pmatrix} V_{cs} & V_{cb} \\ V_{ts} & V_{tb} \end{pmatrix} = \begin{pmatrix} \mathbf{c}\cdot\mathbf{s} & \mathbf{c}\cdot\mathbf{b} \\ \mathbf{t}\cdot\mathbf{s} & \mathbf{t}\cdot\mathbf{b} \end{pmatrix} = \begin{pmatrix} \cos\theta_{tb} & -\sin\theta_{tb} \\ \sin\theta_{tb} & \cos\theta_{tb} \end{pmatrix}, \tag{2}$$

which is no longer the identity, hence mixing.

Next, what about hierarchical masses? From (1), it follows that $\mathbf{c}$ must have zero eigenvalue at $\mu = m_t$. But this value is not to be taken as the mass of c which has to be measured at $\mu = m_c$. In other words, m_c is to be taken as the solution to the equation

$$\mu = \langle \mathbf{c}|m(\mu)|\mathbf{c}\rangle = m_U |\langle \mathbf{c}|\boldsymbol{\alpha}(\mu)\rangle|^2. \tag{3}$$

A non-zero solution exists since the scale on the LHS decreases from $\mu = m_t$ while the RHS increases from zero at that scale. Another way to see this is that since $\boldsymbol{\alpha}$ by assumption rotates so that at $\mu < m_t$, it would have rotated to some direction different from $\mathbf{t}$, and acquired a component, say $\sin\theta_{tc}$, in the direction of $\mathbf{c}$ giving thus

$$m_c = m_t \sin^2\theta_{tc}, \tag{4}$$

which is non-zero but will be small if the rotation is not too fast, hence mass hierarchy.

That the mass spectra and mixing matrices so obtained from a rank-one rotating mass matrix (R2M2) actually do resemble those observed in experiment is also readily checked in the present simplification when only the two heaviest states of each quark type are considered. By inverting the above simple formulae (4) and (2) for the masses and mixing angles, one easily derives the corresponding values of the angle θ at various scales. If R2M2 is indeed valid, then these values should all fall on a smooth curve as a function of μ representing the rotation trajectory for the vector $\boldsymbol{\alpha}$. This exercise performed in [9] gave results very well fitted by an exponential which showed that the then available data were fully consistent with the hypothesis.

One sees then that with 2 generations, rotation will give automatically both mixing and mass hierarchy, as claimed. Basically the same argument is applicable to the realistic 3-generation case, though the analysis becomes a little more intricate. We give here only the result for future reference, the detailed derivation of which can be found in, for example, [3]. For U-type quarks, the state vectors are defined in terms of the rotating vector $\boldsymbol{\alpha}$ via

$$\begin{aligned} \mathbf{t} &= \boldsymbol{\alpha}(m_t), \\ \mathbf{c} &= \mathbf{u} \times \mathbf{t}, \\ \mathbf{u} &= \frac{\boldsymbol{\alpha}(m_t) \times \boldsymbol{\alpha}(m_c)}{|\boldsymbol{\alpha}(m_t) \times \boldsymbol{\alpha}(m_c)|}. \end{aligned} \tag{5}$$

And their masses are given by

$$\begin{aligned} m_t &= m_U, \\ m_c &= m_U |\boldsymbol{\alpha}(m_c) \cdot \mathbf{c}|^2, \\ m_u &= m_U |\boldsymbol{\alpha}(m_u) \cdot \mathbf{u}|^2. \end{aligned} \tag{6}$$

Together, these 2 sets of coupled equations allow us to evaluate both the state vectors and the masses. Similar equations and remarks apply also to D-type quarks. With the state vectors so determined, the mixing matrices could then be directly evaluated, e.g., for quarks [10, 11]

$$V_{\rm CKM} \sim \begin{pmatrix} \mathbf{u} \cdot \mathbf{d} & \mathbf{u} \cdot \mathbf{s} & \mathbf{u} \cdot \mathbf{b} \\ \mathbf{c} \cdot \mathbf{d} & \mathbf{c} \cdot \mathbf{s} & \mathbf{c} \cdot \mathbf{b} \\ \mathbf{t} \cdot \mathbf{d} & \mathbf{t} \cdot \mathbf{s} & \mathbf{t} \cdot \mathbf{b} \end{pmatrix}. \tag{7}$$

The expression for the lepton mixing matrix $U_{\rm PMNS}$ [12, 13] would be similar.

That the mass spectra and mixing matrices so obtained from R2M2 are still consistent with experiment when all 3 generations of fermions are taken into account is shown in [14] and is reviewed in [3] to which the reader is referred.

In the above summary, it has been assumed that all the vectors in generation space are real and that their norms and products are calculated with a flat metric. It will be seen later that if account is taken of the theta-angle term in the QCD action, famous in the old strong CP problem [15], then the vectors can become complex, giving $V_{\rm CKM}$ a CP-violating phase [11]. Also, as the present model (FSM) develops, it will be seen that the metric in generation space can become distorted from flatness. However, it will be shown that even in these circumstances, all the formulae listed above, only with certain provisos, will still remain valid.

3 The Vacuum

We turn now to our main task of seeing how rotation develops in the FSM, beginning with an elucidation of the vacuum.

The formulation in [1, 2] of the FSM as the "minimally framed" gauge theory, i.e., the framed gauge theory with the smallest number of scalar framon fields, for the gauge symmetry $su(3) \times su(2) \times u(1)$, gives two types of framons: a "weak framon" field of the form

$$\phi_r^{\tilde{r}\tilde{a}} = \alpha^{\tilde{a}}\phi_r^{\tilde{r}}, \quad r, \tilde{r} = 1,2, \quad \tilde{a} = 1,2,3, \qquad y = \pm\tfrac{1}{2}, \quad \tilde{y} = \mp\tfrac{1}{2}, \tag{8}$$

and a "strong framon" field of the form

$$\phi_a^{\tilde{r}\tilde{a}} = \beta^{\tilde{r}}\phi_a^{\tilde{a}}, \quad \tilde{r} = 1,2, \quad a, \tilde{a} = 1,2,3, \qquad y = -\tfrac{1}{3}, \quad \tilde{y} = \tfrac{1}{3}. \tag{9}$$

where $\phi_r^{\tilde{r}}$ and $\phi_a^{\tilde{a}}$ are scalar space-time x-dependent fields, while the factors $\alpha^{\tilde{a}}$ and $\beta^{\tilde{r}}$are global x-independent quantities and y and $\tilde{y}$ denote the $u(1)$ and $\tilde{u}(1)$ charges respectively. The components $\phi_r^{\tilde{r}}$ and $\phi_a^{\tilde{a}}$ are not all independent; the weak framons satisfy

$$\phi_r^{\tilde{2}} = -\epsilon_{rs}(\phi_s^{\tilde{1}})^*, \tag{10}$$

while the strong framons satisfy

$$\det(\Phi) = (\det(\Phi))^*, \tag{11}$$

where we have arranged the strong framon fields $\phi_a^{\tilde{a}}$ as a matrix, $\Phi = (\phi_a^{\tilde{a}})$. Furthermore, we introduce the 2-vector $\boldsymbol{\phi}$ as a shorthand notation for the single weak framon $\phi_r^{\tilde{1}}$ (see equation (10)).

Since physics should be independent of the choice either of the local or the global reference frame, the action constructed with these framon fields carrying both local and global indices has to be invariant under both the original local gauge symmetry $su(3) \times su(2) \times u(1)$ and its "dual", the global symmetry $\widetilde{su}(3) \times \widetilde{su}(2) \times \tilde{u}(1)$, which places severe restrictions on the form it can take. In particular, the self-interaction term of the framons, which we call the framon potential $V[\Phi]$, is restricted by invariance plus renormalizability to the form [1, 2]

$$V[\Phi] = V_W[\Phi] + V_S[\Phi] + V_{WS}[\Phi]\,, \tag{12}$$

where V_W involves only the weak framons, V_S only the strong framons, and V_{WS} both, with

$$V_W[\Phi] = -\mu_W|\boldsymbol{\phi}|^2 + \lambda_W(|\boldsymbol{\phi}|^2)^2, \tag{13}$$

$$V_S[\Phi] = -\mu_S \sum_{a,\tilde{a}} (\phi_a^{\tilde{a}*}\phi_a^{\tilde{a}}) + \lambda_S \left[\sum_{a,\tilde{a}} (\phi_a^{\tilde{a}*}\phi_a^{\tilde{a}})\right]^2 + \kappa_S \sum_{a,b,\tilde{a},\tilde{b}} (\phi_a^{\tilde{a}*}\phi_a^{\tilde{b}})(\phi_b^{\tilde{b}*}\phi_b^{\tilde{a}}), \tag{14}$$

and

$$V_{WS}[\Phi] = \nu_1|\boldsymbol{\phi}|^2 \sum_{a,\tilde{a}} \phi_a^{\tilde{a}*}\phi_a^{\tilde{a}} - \nu_2|\boldsymbol{\phi}|^2 \sum_a \left|\sum_{\tilde{a}} (\alpha^{\tilde{a}*}\phi_a^{\tilde{a}})\right|^2. \tag{15}$$

The potential $V[\Phi]$ depends on 7 real coupling parameters in all, namely $\mu_W, \lambda_W, \mu_S, \lambda_S, \kappa_S, \nu_1, \nu_2$. Although these parameters can in principle have either sign, we take $\mu_W, \lambda_W, \mu_S, \lambda_S$ all to be positive so that both the weak and strong vacua are degenerate, and also κ_S to be positive for reasons which will soon be apparent. The other 2 parameters ν_1, ν_2, however, can have either sign in the following discussion.

The object of this section is to identify the framon vacuum by minimizing this framon potential. This can be done, of course, by fixing first a gauge each for both the local and the global symmetry, then differentiating the potential with respect to the remaining 12 variables and putting the derivatives to zero. It will be straightforward, but rather complicated. The reason is that although the strong potential $V_S[\Phi]$ by itself gives a minimum for which the strong framons remain orthonormal, the term $V_{WS}[\Phi]$ which links the strong and weak sectors distorts the vacuum values of the strong framons from orthonormality, and it is the necessity of referring to these non-orthonormal frames which makes the analysis complicated. This will especially be the case when we are interested in tracing the scale-dependence of the vacuum, i.e., how the vacuum moves from one to another among the degenerate set, when all these vacua are distorted from orthonormality each in a different way.

For this reason, we adopt here a different tack. A point to note first is that the degeneracy of the strong vacuum originates from the invariance of the potential V_S under the global symmetry $\widetilde{su}(3)$. When we arrange the strong framon fields $\phi_a^{\tilde{a}}$ above as a matrix Φ, with a labelling the rows and $\tilde{a}$ the columns, then the $\widetilde{su}(3)$ transformations are represented by unitary matrices, say A^{-1}, operating from the right. Now if Φ at vacuum is orthonormal, then so is $\Phi_{\text{VAC}}A^{-1}$, but if Φ_{VAC} is not orthonormal, $\Phi_{\text{VAC}}A^{-1}$ can take many different shapes. Nevertheless, any two of all these differently shaped vacua are still related just by some A^{-1} from $\widetilde{su}(3)$.[2]

With this realization one sees that one need not actually perform the minimization analysis around a general vacuum, but only around a vacuum for which the analysis is particularly simple, since any other vacuum can be obtained from it by applying the appropriate $\widetilde{su}(3)$ transformation. Equivalently, we can think of choosing for any given vacuum an appropriate gauge so as to make it appear particularly simple, and all other vacua in the same gauge can be obtained from it by $\widetilde{su}(3)$ transformations. Indeed, one will

[2]A point already noted in [1], though not then fully utilized.

find that the analysis becomes then so simple that any vacuum can be found in this way even without doing any actual minimization.

Suppose then we choose or are given some particular vacuum in the degenerate set as our reference vacuum, quantities defined with respect to which we shall indicate by an index (either as subscript or superscript) 0. It will correspond to some value of the vector $\boldsymbol{\alpha}$, say $\boldsymbol{\alpha}_0$. Let us choose to work in the $\widetilde{su}(3)$ gauge where the vector $\boldsymbol{\alpha}_0$ is real and points in the first direction. This does not fix the $\widetilde{su}(3)$ gauge completely, but we can leave the rest unspecified for the moment. We ask now how the chosen reference vacuum will appear in this new $\widetilde{su}(3)$ gauge. To be specific, we shall need also to fix a gauge for the local $su(3)$ symmetry. We can choose [1], e.g., either the triangular gauge (where the framon matrix elements are real along the diagonal and vanishing below it) or the hermitian gauge (where the framon matrix Φ is hermitian), but with $\boldsymbol{\alpha}_0 = (1, 0, 0)$ the two gauges coincide.

To find the values of the framons Φ at the reference vacuum in the chosen gauge, let us rewrite the potential $V[\Phi]$ in (12) in terms of the vectors $\boldsymbol{\phi}^{\tilde{a}} = (\phi_a^{\tilde{a}})$ in $su(3)$ space. For $\boldsymbol{\alpha} = (1, 0, 0)$, the potential then takes the form

$$
\begin{aligned}
V[\Phi] \;\; = \;\; & -\mu_W|\boldsymbol{\phi}|^2 + \lambda_W(|\boldsymbol{\phi}|^2)^2 - \mu_S \sum_{\tilde{a}} |\boldsymbol{\phi}^{\tilde{a}}|^2 + \lambda_S \left(\sum_{\tilde{a}} |\boldsymbol{\phi}^{\tilde{a}}|^2 \right)^2 \\
& + \kappa_S \sum_{\tilde{a}} \left(|\boldsymbol{\phi}^{\tilde{a}}|^2 \right)^2 + \kappa_S \sum_{\tilde{a} \neq \tilde{b}} |\boldsymbol{\phi}^{\tilde{a}*} \cdot \boldsymbol{\phi}^{\tilde{b}}|^2 \\
& + \nu_1 |\boldsymbol{\phi}|^2 \sum_{\tilde{a}} |\boldsymbol{\phi}^{\tilde{a}}|^2 - \nu_2 |\boldsymbol{\phi}|^2 |\boldsymbol{\phi}^{\tilde{1}}|^2. \qquad (16)
\end{aligned}
$$

We note in this that only the second κ_S term depends on the orientation of the vectors $\boldsymbol{\phi}^{\tilde{a}}$, the other terms depending only on their lengths. Hence, minimizing $V[\Phi]$ with respect to their orientations, we obtain (for $\kappa_S > 0$) that these vectors will be mutually orthogonal at minimum, so that this second κ term will be zero and can be dropped from consideration.

Next, let us rewrite the ν_2 term as

$$
\nu_2 |\boldsymbol{\phi}|^2 \left[\left(-\frac{2}{3} |\boldsymbol{\phi}^{\tilde{1}}|^2 + \frac{1}{3} |\boldsymbol{\phi}^{\tilde{2}}|^2 + \frac{1}{3} |\boldsymbol{\phi}^{\tilde{3}}|^2 \right) - \frac{1}{3} \sum_{\tilde{a}} |\boldsymbol{\phi}^{\tilde{a}}|^2 \right], \qquad (17)
$$

where the second term has the same form as, and can be absorbed into, the ν_1 term in $V[\Phi]$. Combining now the remaining terms in (17) with the remaining κ_S term in $V[\Phi]$ by completing squares, we can rewrite the sum as

$$\kappa_S \left[\left(\sum_{\tilde{a}} |\boldsymbol{\phi}'^{\tilde{a}}|^2 \right)^2 - \frac{1}{6} \frac{\nu_2^2}{\kappa_S^2} (|\boldsymbol{\phi}|^2)^2 \right], \tag{18}$$

with

$$|\boldsymbol{\phi}'^{\tilde{1}}|^2 = |\boldsymbol{\phi}^{\tilde{1}}|^2 - \frac{1}{3}\frac{\nu_2}{\kappa_S}|\boldsymbol{\phi}|^2, \quad |\boldsymbol{\phi}'^{\tilde{2}}|^2 = |\boldsymbol{\phi}^{\tilde{2}}|^2 + \frac{1}{6}\frac{\nu_2}{\kappa_S}|\boldsymbol{\phi}|^2, \quad |\boldsymbol{\phi}'^{\tilde{3}}|^2 = |\boldsymbol{\phi}^{\tilde{3}}|^2 + \frac{1}{6}\frac{\nu_2}{\kappa_S}|\boldsymbol{\phi}|^2. \tag{19}$$

Again the last term in (18) has the same form as, and can be absorbed into, the λ_W term in $V[\Phi]$.

Noting that

$$\zeta_S^2 = \sum_{\tilde{a}} |\boldsymbol{\phi}'^{\tilde{a}}|^2 = \sum_{\tilde{a}} |\boldsymbol{\phi}^{\tilde{a}}|^2 \tag{20}$$

we see that the potential as a whole now resembles the old potential without the ν_2 term, only with $|\boldsymbol{\phi}^{\tilde{a}}|^2$ replaced by $|\boldsymbol{\phi}'^{\tilde{a}}|^2$ and some changes in the definition of ν_1 and λ_W. In particular, we note that the potential is symmetric in $|\boldsymbol{\phi}'^{\tilde{a}}|^2$ so that even without differentiation we can conclude that the minimum is at

$$|\boldsymbol{\phi}'^{\tilde{1}}|^2 = |\boldsymbol{\phi}'^{\tilde{2}}|^2 = |\boldsymbol{\phi}'^{\tilde{3}}|^2 = \frac{\zeta_S^2}{3}, \tag{21}$$

or at

$$|\boldsymbol{\phi}^{\tilde{1}}|^2 = \zeta_S^2(\tfrac{1+2R}{3}), \quad |\boldsymbol{\phi}^{\tilde{2}}|^2 = \zeta_S^2(\tfrac{1-R}{3}), \quad |\boldsymbol{\phi}^{\tilde{3}}|^2 = \zeta_S^2(\tfrac{1-R}{3}), \tag{22}$$

with

$$R = \frac{\nu_2 \zeta_W^2}{2\kappa_S \zeta_S^2}. \tag{23}$$

It follows then immediately that at $\boldsymbol{\alpha} = (1,0,0)$, Φ^0_{VAC} is necessarily diagonal because of the mutual orthogonality of the vectors $\boldsymbol{\phi}^{\tilde{a}}$, and that it will take the simple form

$$\Phi^0_{\text{VAC}} = V_0^0 = \zeta_S \begin{pmatrix} \sqrt{\frac{1+2R}{3}} & 0 & 0 \\ 0 & \sqrt{\frac{1-R}{3}} & 0 \\ 0 & 0 & \sqrt{\frac{1-R}{3}} \end{pmatrix}. \tag{24}$$

That the vacuum at $\boldsymbol{\alpha} = (1,0,0)$ should take this form is actually, *a posteriori*, fairly obvious. We recall that the strong potential V_S would by itself imply vacuum values for $\boldsymbol{\phi}^{\tilde{a}}$ which are mutually orthogonal and of equal lengths, and it is the term V_{WS} which distorts them from orthonormality. Choosing then $\boldsymbol{\alpha}$ to point in the $(1,0,0)$ direction means that only the lengths of the vectors will be affected, and this effect will depend on the relative strengths of the κ_S and ν_2 terms, namely on the parameter R in (23) above.

Having now (24) for the reference vacuum, one can obtain $\Phi_{\rm VAC}$ for any other vacuum in the degenerate set, say one corresponding to a different vector $\boldsymbol{\alpha}$, by an $\widetilde{su}(3)$ transformation A^{-1}, applied from the right, thus

$$\Phi_{\rm VAC} = V_0 = \Phi^0_{\rm VAC}\, A^{-1}, \tag{25}$$

where

$$\boldsymbol{\alpha} = A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{26}$$

though still in the same gauges as before. We notice that (25) is in general not diagonal, meaning that the framon vectors $\boldsymbol{\phi}^{\tilde{a}}$ at a general vacuum are now neither mutually orthogonal nor similarly normalized in the chosen gauges. Indeed, $\Phi_{\rm VAC}$ will in general not even be triangular nor hermitian, but can be made to be either by an appropriate change in the local gauge, i.e., by operating with an $su(3)$ transformation from the left [1]. Further, of course, $\Phi_{\rm VAC}$ can be transformed back into the canonical form (24) by appropriate gauge changes in both local $su(3)$ and global $\widetilde{su}(3)$. But in what follows, unless otherwise stated, we shall keep working in the chosen gauges as before where $\Phi^0_{\rm VAC}$ is of the form (24).

From (25), it follows that

$$\Phi^\dagger_{\rm VAC}\Phi_{\rm VAC} = \zeta_S^2(\tfrac{1+2R}{3})P_{\boldsymbol{\alpha}} + \zeta_S^2(\tfrac{1-R}{3})P^\perp_{\boldsymbol{\alpha}}, \tag{27}$$

a convenient expression to note, where $P_{\boldsymbol{\alpha}}$ and $P^\perp_{\boldsymbol{\alpha}}$ are the projection operators on to the directions parallel and orthogonal, respectively, to the vector $\boldsymbol{\alpha}$ for that vacuum.

The matrix $\Phi_{\rm VAC}$ (25) at any one of the vacua, being the vacuum (classical) value of the framon field there, would be the equivalent of the vierbeins e^a_μ in gravity for that vacuum, transforming between the local frame (here $su(3)$ labelled by the index a) and the global frame (here $\widetilde{su}(3)$ labelled by the index $\tilde{a}$). The fact that this matrix is non-unitary for any A means that at any of the degenerate vacua, the local and global frames, as also in gravity, cannot both be orthonormal. But, whereas in gravity it is the global frame indexed by a which is taken orthonormal while the local frame indexed by μ is not, here on the other hand, it would be the other way round. Since $su(3)$ colour is supposed to be confining and exact, we would want the local frame to remain orthonormal, and it would be the global $\widetilde{su}(3)$ frame which is distorted. And just as in gravity, any vector or tensor quantity can be given in either frame, i.e., labelled either by the global or the local indices, or even partly by one and partly by the other. The transformation between quantities carrying indices of one type and quantities carrying indices of the other type can be effected simply by multiplying where appropriate with the matrix (25) or its conjugate, similar to the raising and lowering of indices by the metric in gravity. There will be examples later where such switches between local and global indices are found to be convenient.

4 The Higgs States

Next, from the framon potential $V[\Phi]$ in (12), one can deduce the spectrum of the Higgs states. By Higgs states, we mean, as usual, the quanta of fluctuations of the scalar fields, i.e., in our case the framons, about the chosen vacuum, when these fluctuations do not correspond to the local gauge transformations under which the theory is by construction invariant. In the FSM, there are then two types of Higgs states. First there is the ordinary or "weak" Higgs state coming from the fluctuations of the weak framon about the weak vacuum; this is the same as in the standard electroweak theory. Secondly, there are the "strong" Higgs states, which come from the fluctuations of the strong framons about the strong vacuum, which we have now to identify for use in subsequent calculations.

In the confinement interpretation [16, 17] of symmetry-breaking that we find convenient to adopt, as explained in [1, 2], the Higgs states appear as bound states of framon-antiframon pairs, confined by "weak" $su(2)$ for the usual (weak) Higgs and by colour $su(3)$ for the strong Higgs states. In other words, the "weak" confinement being supposedly much deeper than colour confinement, the usual Higgs state will appear to us as fundamental while the strong Higgs states will appear to us as hadrons in what can be called the standard model scenario of confinement, where we can see only $su(2)$ singlets but where we have already probed inside $su(3)$ singlets and coloured objects are revealed.

The strong vacuum having been elucidated in the preceding section, it is in principle a straightforward matter to expand the framon fields about the vacuum and identify those fluctuations which do not correspond to gauge transformations. For example, to exclude those fluctuations corresponding to gauge transformations we can work in a fixed gauge, say the hermitian gauge where Φ is required to remain hermitian both before and after the fluctuations. Again, to avoid the complicated algebra when working directly with a general vacuum and non-orthonormal frames [1], we adopt the tactic of the last section and first identify the strong Higgs states for the reference vacuum where things are simple, and then deduce the same for the general vacuum by an $\widetilde{su}(3)$ transformation.

Let us then start with the vacuum $\Phi^0_{\rm VAC}$ in the simple form (24) and consider fluctuations of the framons Φ about it which we can choose to express as

$$\Phi^0_{\rm VAC} + \delta\Phi = \Phi^0_{\rm VAC}\,(1 + \epsilon S), \qquad (28)$$

where we recall that $\Phi^0_{\rm VAC}$ was chosen to be hermitian. If we take S to be $i\lambda_K,\ K = 1, ..., 8$, λ_K being the standard Gell-Mann matrices, then the fluctuation will not remain hermitian and would not then correspond, according to our stated criterion, to a Higgs mode. Indeed, it would instead generate an $\widetilde{su}(3)$ transformation taking the chosen vacuum to a neighbouring one. Equivalently, by writing

$$\Phi^0_{\rm VAC}\,(1 + i\epsilon\lambda_K) = (1 + i\epsilon\lambda_K)[(1 - i\epsilon\lambda_K)\,\Phi^0_{\rm VAC}\,(1 + i\epsilon\lambda_K)] \qquad (29)$$

we can see that it can be considered as a fluctuation from the neighbouring vacuum in the hermitian gauge (the factor inside square brackets), operated by a local $su(3)$ gauge transformation from the left, and will represent a component of the Φ scalar field which, in the popular language of symmetry-breaking, is to be eaten up by one of the gauge vector bosons to give it a mass; it will not correspond to a Higgs mode.

The Higgs modes are to be represented by the other fluctuations with S hermitian, and for which we can take S as $S = 1$ or $S = \lambda_K,\ K = 1, ..., 8$. However, for easier comparison with earlier results in [1], we choose instead to work with some linear combinations of the above and write our Higgs basis states as

$$V_1^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$
\begin{aligned}
V_2^0 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \\
V_3^0 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \\
V_4^0 &= \begin{pmatrix} 0 & \sqrt{\frac{1+2R}{2+R}}e^{i\phi_1} & 0 \\ \sqrt{\frac{1-R}{2+R}}e^{-i\phi_1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \\
V_5^0 &= \begin{pmatrix} 0 & 0 & \sqrt{\frac{1+2R}{2+R}}e^{i\phi_2} \\ 0 & 0 & 0 \\ \sqrt{\frac{1-R}{2+R}}e^{-i\phi_2} & 0 & 0 \end{pmatrix}; \\
V_6^0 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}}e^{i\phi_3} \\ 0 & \frac{1}{\sqrt{2}}e^{-i\phi_3} & 0 \end{pmatrix}; \\
V_7^0 &= i\begin{pmatrix} 0 & \sqrt{\frac{1+2R}{2+R}}e^{i\phi_1} & 0 \\ -\sqrt{\frac{1-R}{2+R}}e^{-i\phi_1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \\
V_8^0 &= i\begin{pmatrix} 0 & 0 & \sqrt{\frac{1+2R}{2+R}}e^{i\phi_2} \\ 0 & 0 & 0 \\ -\sqrt{\frac{1-R}{2+R}}e^{-i\phi_2} & 0 & 0 \end{pmatrix}; \\
V_9^0 &= i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}}e^{i\phi_3} \\ 0 & -\frac{1}{\sqrt{2}}e^{-i\phi_3} & 0 \end{pmatrix}.
\end{aligned}
\tag{30}
$$

They are chosen to form an orthonormal set when considered as vectors in a 9-dimensional space and represent the 9 independent Higgs states about the reference vacuum $\Phi^0_{\rm VAC}$ in the $\widetilde{su}(3)$ gauge where $\boldsymbol{\alpha}$ is $(1,0,0)$ and in the local $su(3)$ hermitian gauge.

In the calculation that follows, we shall need the Higgs state taken at other vacua of the degenerate set in the same gauges. These we can obtain again just by applying the appropriate $\widetilde{su}(3)$ transformation A^{-1} from the right as was done for the vacuum to obtain $\Phi_{\rm VAC}$. Thus, explicitly

$$V_K = V_K^0 A^{-1}. \tag{31}$$

It is easy to check that orthonormality of these V_K will be preserved, since the 9-dimensional inner product is given in terms of the matrices as $\mathrm{tr}(V_K V_L^\dagger)$.

Notice that these 9 Higgs states form a complete orthonormal set but they are not in general mass eigenstates. Together with the Higgs state from the electroweak sector, we have a 10×10 mass matrix for the Higgs states which can be computed by taking the second derivatives of the framon potential $V[\Phi]$, and this can then be diagonalized to find the eigenstates. Again, for the strong sector it is easiest to calculate the Higgs mass matrix first in the gauges where the vacuum (24) is diagonal and then to transform it sandwiching with A and A^{-1} as necessary, since the eigenvalues will be unchanged by this transformation.

The Higgs mass matrix and its diagonalization will not be needed in this paper directed towards the understanding of the mass and mixing patterns of quarks and leptons, and so will be relegated to Appendix A. However, for testing the FSM as a whole, the Higgs mass matrix may play in future a central role. The main new ingredients introduced by the FSM, we recall, are the strong framons which alone have no standard model counterparts, and the strong Higgs states under present consideration are their direct manifestations. Hence, the spectroscopy of these strong Higgs states would seem deserving of a close scrutiny when more is known about the parameters which enter into the model.

One consequence of the Higgs mass matrix (113) in Appendix A which could be of immediate phenomenological interest is the fact that two of the strong Higgs states have the same quantum numbers as h, the "weak " or standard model Higgs state, and so can mix with the latter. And these strong Higgs states being hadrons, with presumably hadronic decay widths and modes, any admixture of them into h can greatly alter the latter's decay characteristics deduced from the standard model and presently used experimentally as signatures for its detection. This possibility is under study.

5 The Scale-Dependence of the Vacuum

Next, we are to study the change in the (strong) vacuum with changing scales under renormalization by loops of the (strong) Higgs states, so as to derive its effects, if any, on the rotation of the quark and lepton mass matrix. We recall that since the (strong) vacuum is coupled to the quark and lepton mass matrix (1) via the vector $\boldsymbol{\alpha}$ which appears in the ν_2 term of the framon potential $V[\Phi]$, any change in it will get reflected in the quark and lepton mass matrix (1). To get $\boldsymbol{\alpha}$, a vector in $\widetilde{su}(3)$, to rotate with scale, we would want renormalization effects which are not $\widetilde{su}(3)$ invariant. And since the (strong) vacuum breaks this symmetry, as shown above, so also will the (strong) Higgs states derived as fluctuations about this vacuum. The renormalization effects from these will thus satisfy the above criterion, hence our interest.

Information on the effects on the vacuum under renormalization can in principle be derived through any quantity which depends on the vacuum and gets renormalized by the strong Higgs loops. We choose, mainly for historical reasons, to focus on the self-energy of certain fermion states to be specified on which we had some experience earlier in a similar context [7]. As will be seen, the information from this study is enough already to show that rotation will result. A parallel study on other quantities which we have not performed can in principle give further constraints on the scale-dependence of the vacuum, and hence on the rotation of $\boldsymbol{\alpha}$, but these should not be in contradiction with what we have derived if the present theory is self-consistent.

To specify these fermion states, let us first remind ourselves of the Yukawa term in the FSM for the weak framon written down in [2, 1] from which the mass matrix (1) is derived:

$$\mathcal{A}_{\mathrm{YK}}^{\mathrm{lepton}} = \sum_{[\tilde{a}][b]} Y_{[b]} \bar{\psi}^r_{[\tilde{a}]} \phi_r^{(-)\tilde{a}} \tfrac{1}{2}(1+\gamma_5)\psi^{[b]} + \sum_{[\tilde{a}][b]} Y'_{[b]} \bar{\psi}^r_{[\tilde{a}]} \phi_r^{(+)\tilde{a}} \tfrac{1}{2}(1+\gamma_5)\psi'^{[b]} + \mathrm{h.c.} \tag{32}$$

for leptons (similarly for quarks). The mass matrix is obtained by substituting, for the (weak) framon field, its vacuum value; then by a suitable relabelling of the right-handed fields, as indicated in the introduction, the mass matrix can be recast in the factorizable form (1). Furthermore, by expanding the framon field about its vacuum value to first order, one obtains the Yukawa coupling of the (weak) Higgs state h to leptons. In the confinement picture both the leptons and the Higgs state are bound states via $su(2)$ confinement, the former of a (weak) framon with a fundamental fermion ψ and the latter of a (weak) framon-antiframon pair. Similar assertions, of course, can be made about quarks.

Our object now is to write down a similar Yukawa term for the strong framon, with colour $su(3)$ now taking the place of the electroweak $su(2)$ in (32). For this, the following expression was suggested [1]

$$\mathcal{A}_{\rm YK}^{\rm strong} = \sum_{[b]} Z_{[b]} \bar{\psi}^a \boldsymbol{\phi}_a \cdot \boldsymbol{\alpha}_0 \frac{1}{2}(1+\gamma_5)\psi^{[b]} + {\rm h.c.} \tag{33}$$

We note here that in (33) the fermion fields do not carry any $\tilde{a}$ index for $\widetilde{su}(3)$ but the framon fields $\boldsymbol{\phi}_a$ do since they are vectors in generation space. Hence, to maintain $\widetilde{su}(3)$ invariance, we need a vector in $\widetilde{su}(3)$ space to saturate this $\tilde{a}$ index. There is no such vector available to play this role within the purely strong sector, but in the present FSM set-up, there is the vector $\boldsymbol{\alpha}$ coming from the weak sector which can be so employed. In introducing here a vector originating from the weak sector to construct the Yukawa term (33) for the strong framon in the strong sector, one is imitating, in spirit though not in detail, the construction of the standard Yukawa term (32) in the weak sector. In fact, as it stands, the weak Yukawa term in (32) is not explicitly invariant under $\widetilde{su}(2)$ as it ought to be, but it can be put in an explicitly invariant form [2] by writing $\phi_r^{(\pm)\tilde{a}}$ as $\boldsymbol{\gamma}^{(\pm)} \cdot \boldsymbol{\phi}_r^{\tilde{a}}$ in terms of the $\widetilde{su}(2)$ vectors $\boldsymbol{\gamma}^{(\pm)}$ originating from the electromagnetic $u(1)$ sector. This is in close parallel to the introduction above of the vector $\boldsymbol{\alpha}$ originating from the weak sector to keep (33) $\widetilde{su}(3)$ invariant. The vector $\boldsymbol{\alpha}$ here, however, does not have a definite value, but can point in any direction in $\widetilde{su}(3)$ space since the vacuum is degenerate. Nevertheless, these directions being all gauge equivalent, it should not matter which value we choose. By writing in (33) $\boldsymbol{\alpha}_0$ for the vector $\boldsymbol{\alpha}$, we have implicitly chosen that value for $\boldsymbol{\alpha}$ which corresponds to the reference vacuum in §2 above, or conversely that we have chosen the reference vacuum in §2 to be the vacuum corresponding to that vector $\boldsymbol{\alpha}$ appearing in the Yukawa coupling (33). The physical meaning for $\boldsymbol{\alpha}_0$ will be apparent later.

The fermion field ψ appearing in (33) above was originally meant [1] to be only generic, since it was thought that for studying renormalization effects on the vacuum it ought not to matter with which fermion field one started. The question arises, however, in the realistic situation, whether any fermion field exists which is of the generic type ψ that one wants. Now, for reasons one does not yet understand, the standard model admits only $su(2)$ doublet left-handed fermions and only $su(2)$ singlet right-handed fermions. In that case, the ψ field in (33) if interpreted as a fundamental field would appear to go against the grain, being an $su(2)$ singlet and left-handed. We recall, however, that in the confinement picture we adopted, quarks (and leptons) appear as bound states of a fundamental fermion field with a weak framon via $su(2)$ confinement. Hence, starting with the left-handed $su(2)$ doublet fundamental fermion field $\psi_r^{[\tilde{a}]}$ appearing already in (32) and the weak framon field $\phi_r^{\tilde{r}\tilde{a}}$ in (8), one easily obtains a quark field

$$\psi^{\tilde{r}} = \sum_{r,\tilde{a}} \phi_r^{\tilde{r}\tilde{a}} \psi_r^{[\tilde{a}]}, \tag{34}$$

which is a left-handed $su(2)$ singlet as required. It does carry an $\widetilde{su}(2)$ index $\tilde{r}$ but this is a global index which can be saturated in (33) with the $\tilde{r}$ index in the factor $\beta^{\tilde{r}}$ originally carried by the strong framon (9), only suppressed in (33) for convenience. In other words, there are indeed ψ fields of the generic type required in (33), only to be interpreted as quark fields, not as fundamental fields. Hence, for studying the effects of renormalization on the strong vacuum that we are after, the Yukawa term (33) is indeed admissible, though not at the fundamental level, but in the standard model scenario of interest to us.

Starting then from the Yukawa term (33), now so interpreted, one can proceed, as one did in (32) above, by inserting for the framons their vacuum values to derive a mass matrix, thus

$$\mathbf{m} = \zeta_S |v_0\rangle\langle Z| \tfrac{1}{2}(1+\gamma_5) + \zeta_S |Z\rangle\langle v_0| \tfrac{1}{2}(1-\gamma_5), \tag{35}$$

where $\langle Z| = (Z_{[1]}, Z_{[2]}, Z_{[3]})$. We can make $\mathbf{m}$ hermitian and independent of γ_5 as we did for (1), following Weinberg [8], by relabelling the right-handed fields to obtain the form

$$\mathbf{m} = \mathbf{m}_T |v_0\rangle\langle v_0|, \tag{36}$$

with

$$|v_0\rangle = V_0 \boldsymbol{\alpha}_0 \tag{37}$$

and

$$\mathbf{m}_T = \zeta_S \rho_S / v_0, \quad \rho_S = \sqrt{Z_{[1]}^2 + Z_{[2]}^2 + Z_{[3]}^2}, \quad v_0 = \sqrt{\langle v_0|v_0\rangle}. \tag{38}$$

This $\mathbf{m}$ then is the mass matrix for the fermions the self-energy of which we propose to study under renormalization.

We note that these fermions bear the same relationship to the strong framons in (33) as did the leptons to the weak framons in (32). Hence if, in the confinement picture of 't Hooft [16] and of Banks and Rabinovici [17], one interprets the leptons as bound states of the fundamental fermion ψ with the weak framon ϕ via weak $su(2)$ confinement, then the present fermions should be thought of as bound states of the fermion field ψ (quarks) with the strong framon Φ via strong $su(3)$ (i.e., colour) confinement. In other words, they are to be interpreted as hadrons, and as such will interact strongly with the strong Higgs states listed in the preceding section which are likewise hadrons.

As given in (36), both the rows and columns of the matrix $\mathbf{m}$ are labelled by colour $su(3)$ indices, since $|v_0\rangle$ according to (37) is a vector in $su(3)$ space (V_0 being a matrix with rows labelled by $su(3)$ but columns by $\widetilde{su}(3)$ indices, and $\boldsymbol{\alpha}_0$ a vector in $\widetilde{su}(3)$ space). This may seem a little surprising when considered as the mass matrix of the bound states just mentioned, of the fields ψ and Φ by colour confinement, thus

$$\boldsymbol{\chi} = \Phi^\dagger \boldsymbol{\psi} \sim \Phi_{\text{VAC}}^\dagger \boldsymbol{\psi} \tag{39}$$

which ought to be indexed by $\widetilde{su}(3)$ indices, not by colour indices which are saturated in (39), as they should be, colour being confined. However, one notes that the conversion from $\boldsymbol{\psi}$ to $\boldsymbol{\chi}$ is only via multiplication by the matrix Φ_{VAC} which, as noted before at the end of §3, plays here the role of the vierbeins e_μ^a in gravity. So just as, say, the Ricci tensor in gravity can be represented either as R^{ab} or as $R_{\mu\nu}$, the two expressions being related by contracting with the vierbeins e_μ^a, so the mass matrix $\mathbf{m}$ here can be represented either as as a matrix labelled by the local index a or the global index $\tilde{a}$, the two expressions being related by the matrix Φ_{VAC}. It just so happens that for our purpose here, it is more convenient to work with the version (36) above labelled by the local $su(3)$ indices.

Next, to examine how this mass matrix $\mathbf{m}$ renormalizes through insertions of strong Higgs loops, we shall need the couplings of these fermions with the strong Higgs states. To derive these couplings, we follow the same procedure as in (28) above and expand now to first order the strong framon field Φ in (33) about its vacuum value, thus

$$\Phi \sim \Phi_{\text{VAC}} + \sum_K H_K V_K, \tag{40}$$

with $V_K = V_K^0 A^{-1}$, V_K^0 being any one of the 9 matrices listed in (30) above, and H_K one of the strong Higgs fields. Substituting this into (33) and recalling that we have already relabelled there the right-handed fermion fields so as to derive the mass matrix $\mathbf{m}$ in the form (36), one easily obtains the desired couplings as

$$\Gamma_K = \rho_S |v_K\rangle\langle v_0| \tfrac{1}{2}(1+\gamma_5) + \rho_S |v_0\rangle\langle v_K| \tfrac{1}{2}(1-\gamma_5), \tag{41}$$

with

$$|v_K\rangle = V_K \boldsymbol{\alpha}_0. \tag{42}$$

With these couplings Γ_K, we can now evaluate the insertion of a strong Higgs loop to the fermions self-energy as

$$\Sigma(p) = \frac{i}{(4\pi)^4} \sum_K \int d^4k \frac{1}{k^2 - M_K^2} \Gamma_K \frac{(\not{p} - \not{k}) + \mathbf{m}}{(p-k)^2 - \mathbf{m}^2} \Gamma_K, \tag{43}$$

where we may for the moment take K to label the Higgs mass eigenstates. After standard manipulations and regularizing the divergence by dimensional regularization, one obtains:

$$\Sigma(p) = -\frac{1}{16\pi^2} \sum_K \int_0^1 dx \Gamma_K \{\bar{C} + \ln(\mu^2/Q^2)\} [\not{p}(1-x) + \mathbf{m}]\, \Gamma_K, \tag{44}$$

where

$$Q^2 = \mathbf{m}^2 x + M_K^2(1-x) - p^2 x(1-x), \tag{45}$$

with $\bar{C}$ being the divergent constant to be subtracted in the standard $\overline{\mathrm{MS}}$ scheme. The change to the mass matrix under renormalization, $\delta\mathbf{m}$, is obtained by first commuting the $\not{p}$ in the numerator half to the extreme left and half to the extreme right, then putting $\not{p} = \mathbf{m}$ and $p^2 = \mathbf{m}^2$. The full explicit expression for $\delta\mathbf{m}$ so obtained together with more details of the calculation can be found in [18]. Here, we shall interest ourselves only in the terms proportional to $\ln \mu^2$ and hence dependent on the scale μ. These are of two types. First, there are terms of the form

$$\Gamma_K \mathbf{m} \Gamma_K = \rho_S^2 \langle v_0|v_K\rangle |v_K\rangle\langle v_0| \tfrac{1}{2}(1+\gamma_5) + \text{c.c.} \tag{46}$$

Then there are terms of the form

$$\Gamma_K \not{p} \Gamma_K \rightarrow \tfrac{1}{2}\, \rho_S^2 \{\langle v_K|v_0\rangle |v_K\rangle\langle v_0| + \langle v_K|v_K\rangle |v_0\rangle\langle v_0|\} \tfrac{1}{2}(1+\gamma_5) + \text{ c.c.} \tag{47}$$

In (46) and (47), we have already commuted $\not{p}$ to the left and right as stipulated and used the known forms for the tree-level mass matrix (36) and Higgs couplings (41).

This then gives the renormalized mass matrix $\mathbf{m}'$ as

$$\mathbf{m}' = \zeta_S|\tilde{v}_0\rangle\langle v_0|\tfrac{1}{2}(1+\gamma_5) + \zeta_S|v_0\rangle\langle\tilde{v}_0|\tfrac{1}{2}(1-\gamma_5), \tag{48}$$

with

$$|\tilde{v}_0\rangle = |v_0\rangle + (\ln\mu^2)\, a|v_0\rangle + (\ln\mu^2)\,|u\rangle, \tag{49}$$

where

$$a = -\frac{1}{16\pi^2}\rho_S^2\left(\frac{5}{4} + \frac{1}{4}\sum_{K=1,\ldots,9}\langle v_K|v_K\rangle\right), \tag{50}$$

and

$$|u\rangle = -\frac{1}{16\pi^2}\rho_S^2\left(\frac{3}{4}\right)\sum_{K=7,8,9}\langle v_0|v_K\rangle|v_K\rangle, \tag{51}$$

where we notice that, both $\mathbf{m}$ and Γ_K being factorizable, the renormalized $\mathbf{m}'$ remains also factorizable. Again, $\mathbf{m}'$ can made hermitian and without γ_5, i.e., into the same form as $\mathbf{m}$ in (36) by relabelling the right-handed fields; so what has changed by renormalization is really just the left-hand factor $\zeta_S|v_0\rangle$, for which we shall denote temporarily as $|w\rangle$, so that we have from (49)

$$|w'\rangle = |w\rangle + (\ln\mu^2)\, a|w\rangle + (\ln\mu^2)\,\zeta_S|u\rangle, \tag{52}$$

The quantity $|w'\rangle$ is the value of $|w\rangle$ at $\mu+\delta\mu$, so that in the limit as $\delta\mu \to 0$, we get

$$\frac{d}{d\ln\mu^2}|w\rangle = \frac{d}{d\ln\mu^2}\left(\zeta_S|v_0\rangle\right) = a\zeta_S|v_0\rangle + \zeta_S|u\rangle, \tag{53}$$

where now both ζ_S and $|v_0\rangle$ are considered as varying with respect to the scale μ.

Recalling from (37) above that $|v_0\rangle = V_0\boldsymbol{\alpha}_0$ where $V_0 = \Phi_{\rm VAC}$ represents the vacuum value of the strong framon field Φ, we deduce the fact that $|v_0\rangle$ varies with μ means that the vacuum will vary with μ also, as anticipated. And since, according to (25), $\Phi_{\rm VAC}$ is given by an $\widetilde{su}(3)$ transformation on the reference vacuum $\Phi^0_{\rm VAC}$ in the chosen gauges where the latter is diagonal, the change of $\Phi_{\rm VAC}$ with respect to scale can be transferred to the change of the matrix A representing that transformation.

To exhibit explicitly how A will depend on scale according to the equation (53) derived above, let us write now

$$A = R_1 R_2 R_3 P, \tag{54}$$

with

$$\begin{aligned}
R_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & -s_1 e^{-i\sigma_1} \\ 0 & s_1 e^{i\sigma_1} & c_1 \end{pmatrix} \\
R_2 &= \begin{pmatrix} c_2 & 0 & -s_2 e^{-i\sigma_2} \\ 0 & 1 & 0 \\ s_2 e^{i\sigma_2} & 0 & c_2 \end{pmatrix} \\
R_3 &= \begin{pmatrix} c_3 & -s_3 e^{-i\sigma_3} & 0 \\ s_3 e^{i\sigma_3} & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
P &= \begin{pmatrix} e^{i\alpha_1} & 0 & \\ 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & e^{-i\alpha_1 - i\alpha_2} \end{pmatrix},
\end{aligned} \tag{55}$$

where $c_i = \cos\theta_i, s_i = \sin\theta_i$. The expression (54) is the standard parametrization in terms of Euler angles but deliberately taken in reverse order for a reason which will soon be apparent.

For the same reason, we choose to rewrite A^{-1} as

$$A^{-1} = P_1^{-1} R_3'^{-1} R_2'^{-1} R_1'^{-1}, \tag{56}$$

where

$$P_1^{-1} = P^{-1} P_2, R_3'^{-1} = P_2^{-1} R_3^{-1} P_2, R_2'^{-1} = P_2^{-1} R_2^{-1} P_2, R_1'^{-1} = P_2^{-1} R_1^{-1}, \tag{57}$$

with

$$P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & e^{-i\alpha_2} \end{pmatrix}. \tag{58}$$

Explicitly

$$P_1^{-1} = \begin{pmatrix} e^{-i\alpha_1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\alpha_1} \end{pmatrix}$$

$$
\begin{aligned}
R_3'^{-1} &= \begin{pmatrix} c_3 & s_3 e^{-i\sigma_3'} & 0 \\ -s_3 e^{i\sigma_3'} & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
R_2'^{-1} &= \begin{pmatrix} c_2 & 0 & s_2 e^{-i\sigma_2'} \\ 0 & 1 & 0 \\ -s_2 e^{i\sigma_2'} & 0 & c_2 \end{pmatrix} \\
R_1'^{-1} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 e^{-i\alpha_2} & s_1 e^{-i\sigma_1'} \\ 0 & -s_1 e^{i\sigma_1'} & c_1 e^{i\alpha_2} \end{pmatrix},
\end{aligned} \tag{59}
$$

where

$$
\sigma_3' = \sigma_3 - \alpha_2, \quad \sigma_2' = \sigma_2 + \alpha_2, \quad \sigma_1' = \sigma_1 + \alpha_2, \tag{60}
$$

and R_1' is a general element of the $\widetilde{su}(2)$ little group which leaves $\boldsymbol{\alpha}_0$ invariant.

The reason we choose to write A^{-1} in this way is that V_0, in which A appears, enters into the rotation equations only as $|v_0\rangle = V_0\boldsymbol{\alpha}_0$, so that, given the invariance of $\boldsymbol{\alpha}_0$ under R_1', we have

$$
|v_0\rangle = V_0\boldsymbol{\alpha}_0 = \Phi^0_{\rm VAC} P_1^{-1} R_3'^{-1} R_2'^{-1} \boldsymbol{\alpha}_0. \tag{61}
$$

This means that $|v_0\rangle$ and hence also the rotation equations are actually independent of the variables θ_1, σ_1' and α_2, the parameters of R_1'. Thus, of the 8 original parameters in A as befits an $\widetilde{su}(3)$ transformation, there remain only the following 5 which figure in (53), namely $\theta_2, \theta_3, \alpha_1, \sigma_2', \sigma_3'$. Writing A^{-1} in the form (56) thus allows one to remove the unnecessary variables from the equation and gives an expression of $|v_0\rangle$ in only the remaining 5 on which it really depends.

6 The Rotation Equation: RGE for $\boldsymbol{\alpha}$

Substitution of (56) for A^{-1} into (53) then gives equations for the 5 variables on which A depends, which is all the information one needs. However, what interests us in the end is actually the vector

$$
\boldsymbol{\alpha} = A\boldsymbol{\alpha}_0, \tag{62}
$$

appearing in the mass matrix m in (1) for leptons and quarks, which we may parametrize in general as

$$\boldsymbol{\alpha} = \begin{pmatrix} \cos\theta\, e^{-i\beta_1} \\ \sin\theta\sin\phi\, e^{-i\beta_2} \\ \sin\theta\cos\phi\, e^{-i\beta_3} \end{pmatrix}. \tag{63}$$

It pays therefore to express $|v_0\rangle$, and hence subsequently also the rotation equations directly, not in terms of the variables $\theta_2, \theta_3, \alpha_1, \sigma_2', \sigma_3'$ above but in terms of the parameters of $\boldsymbol{\alpha}$. This we can do by working out (62) in terms of these parameters appearing in A above and comparing with (63), then solving for one set of variables in terms of the other. This then allows us to write:

$$|v_0\rangle = \begin{pmatrix} \sqrt{\frac{1+2R}{3}}\cos\theta\, e^{i\beta_1} \\ -\sqrt{\frac{1-R}{3}}\frac{\sin\theta\cos\theta\sin\phi}{\sqrt{\cos^2\theta+\sin^2\theta\cos^2\phi}}\, e^{-i\beta_2+i\beta_1} \\ -\sqrt{\frac{1-R}{3}}\frac{\sin\theta\cos\phi}{\sqrt{\cos^2\theta+\sin^2\theta\cos^2\phi}}\, e^{-i\beta_3} \end{pmatrix}, \tag{64}$$

with norm given by

$$v_0^2 = \langle v_0|v_0\rangle = \frac{1}{3}(1 + 2R\cos^2\theta - R\sin^2\theta), \tag{65}$$

an expression we shall need later.

Substituting (64) into (53) now gives an equation in the desired variables, namely R, θ, ϕ plus the 3 phases $\beta_1, \beta_2, \beta_3$, where the left-hand side can be written as

$$\frac{d}{d\ln\mu^2}|w\rangle = \dot{\zeta}_S|w_\zeta\rangle + \dot{\theta}|w_\theta\rangle + \dot{\phi}|w_\phi\rangle + \sum_{i=1,2,3}\dot{\beta}_i|w_{\beta_i}\rangle, \tag{66}$$

where a dot denotes differentiation with respect to $\ln\mu^2$ and where

$$|w_\zeta\rangle = \frac{\partial}{\partial\zeta_S}|w\rangle, \quad \text{etc.} \tag{67}$$

Differentiating then first with respect to the phases, we get

$$|w_{\beta_1}\rangle \;=\; i\zeta_S \begin{pmatrix} \sqrt{\frac{1+2R}{3}}\cos\theta\, e^{i\beta_1} \\ -\sqrt{\frac{1-R}{3}}\frac{\sin\theta\cos\theta\sin\phi}{\sqrt{\cos^2\theta+\sin^2\theta\cos^2\phi}}\, e^{-i\beta_2+i\beta_1} \\ 0 \end{pmatrix} \tag{68}$$

$$
|w_{\beta_2}\rangle = i\zeta_S \begin{pmatrix} 0 \\ \sqrt{\frac{1-R}{3}} \frac{\sin\theta\cos\theta\sin\phi}{\sqrt{\cos^2\theta+\sin^2\theta\cos^2\phi}}\, e^{-i\beta_2+i\beta_1} \\ 0 \end{pmatrix} \tag{69}
$$

$$
|w_{\beta_3}\rangle = i\zeta_S \begin{pmatrix} 0 \\ 0 \\ \sqrt{\frac{1-R}{3}} \frac{\sin\theta\cos\phi}{\sqrt{\cos^2\theta+\sin^2\theta\cos^2\phi}}\, e^{-i\beta_3} \end{pmatrix}, \tag{70}
$$

where we notice that apart from the equal phases on each of the 3 components on both sides, the right-hand sides of all these equations are all imaginary, whereas the left-hand side of (66) is real. Hence we conclude that

$$
\dot{\beta}_1 = \dot{\beta}_2 = \dot{\beta}_3 = 0. \tag{71}
$$

The remaining three partial derivatives give

$$
\begin{aligned}
|w_\zeta\rangle &= \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{1}{\sqrt{1+2R}} \cos\theta\, e^{i\beta_1} \\ -\frac{1}{\sqrt{1-R}} \frac{\sin\theta\cos\theta\sin\phi}{\sqrt{\cos^2\theta+\sin^2\theta\cos^2\phi}}\, e^{-i\beta_2+i\beta_1} \\ -\frac{1}{\sqrt{1-R}} \frac{\sin\theta\cos\phi}{\sqrt{\cos^2\theta+\sin^2\theta\cos^2\phi}}\, e^{-i\beta_3} \end{pmatrix} \\
|w_\theta\rangle &= \frac{\zeta_S}{\sqrt{3}} \begin{pmatrix} -\sqrt{1+2R}\sin\theta\, e^{i\beta_1} \\ -\sqrt{1-R}\frac{\sin\phi(\cos^4\theta-\sin^4\theta\cos^2\phi)}{(\cos^2\theta+\sin^2\theta\cos^2\phi)^{3/2}}\, e^{-i\beta_2+i\beta_1} \\ -\sqrt{1-R}\frac{\cos\theta\cos\phi}{(\cos^2\theta+\sin^2\theta\cos^2\phi)^{3/2}}\, e^{-i\beta_3} \end{pmatrix} \\
|w_\phi\rangle &= \frac{\zeta_S}{\sqrt{3}} \begin{pmatrix} 0 \\ -\sqrt{1-R}\frac{\sin\theta\cos\theta\cos\phi}{(\cos^2\theta+\sin^2\theta\cos^2\phi)^{3/2}}\, e^{-i\beta_2+i\beta_1} \\ \sqrt{1-R}\frac{\sin\theta\cos^2\theta\sin\phi}{(\cos^2\theta+\sin^2\theta\cos^2\phi)^{3/2}}\, e^{-i\beta_3} \end{pmatrix}.
\end{aligned} \tag{72}
$$

In order to make comparison with (53), we have to compute the following quantities:

$$
\sum_K \langle v_K | v_K \rangle = \frac{3}{2+R}(1+E) \tag{73}
$$

with

$$
E = 1 - R\cos^2\theta + 2R\sin^2\theta, \tag{74}
$$

and

$$\sum_{K=7,8,9} \langle v_0|v_K\rangle|v_K\rangle = -\frac{3R\sin\theta\cos\theta}{2+R} \left(\begin{array}{c} \sqrt{\frac{1+2R}{3}}\sin\theta\, e^{i\beta_1} \\ \sqrt{\frac{1-R}{3}}\frac{\cos^2\theta\sin\phi}{\sqrt{\cos^2\theta+\sin^2\theta\cos^2\phi}}\, e^{i\beta_1-i\beta_2} \\ \sqrt{\frac{1-R}{3}}\frac{\cos\theta\cos\phi}{\sqrt{\cos^2\theta+\sin^2\theta\cos^2\phi}}\, e^{-i\beta_3} \end{array} \right). \tag{75}$$

Notice that in computing these quantities, the state vectors $|v_K\rangle$ are summed over in a manner invariant under an orthogonal transformation among the Higgs states V_K, so that these Higgs states need not be taken as the actual mass eigenstates they started out to be in (43) but can be taken as any convenient orthonormal set, in particular just those $V_K = V_K^0 A^{-1}$ for V_K^0 listed in (30). Notice also that the phases in (72), (75) are the same as in $|v_0\rangle$. With these, equation (53) is now made explicit.

We can then extract from (53) the desired equations for $\dot{\zeta}_S, \dot{\theta}, \dot{\phi}$ by taking the inner products with the vectors $|w_\zeta\rangle, |w_\theta\rangle, |w_\phi\rangle$, obtaining

$$\dot{R} = \frac{\rho_S^2}{16\pi^2}\frac{R(1+2R)(1-R)}{E}\left[\frac{5}{2}+\frac{1}{2}\frac{3}{2+R}(1+E)\right], \tag{76}$$

$$\dot{\theta} = \frac{\rho_S^2}{16\pi^2}\frac{R\sin 2\theta}{E}\left(\frac{3}{2}\right)\left[\frac{5}{4}+\frac{3}{4}\frac{1}{2+R}\right], \tag{77}$$

$$\dot{\phi} = \frac{\rho_S^2}{16\pi^2}\frac{R\sin^2\theta\sin 2\phi}{E}\left(\frac{3}{2}\right)\left[\frac{5}{4}+\frac{3}{4}\frac{1}{2+R}\right], \tag{78}$$

where the last 2 imply that

$$\dot{\phi}/\dot{\theta} = \tfrac{1}{2}\tan\theta\sin 2\phi, \tag{79}$$

which integrates to

$$\cos\theta\tan\phi = \text{constant}, \tag{80}$$

a condition one can use in place of (78) above.

These equations (76)—(78), although derived only from a single (strong) Higgs loop insertion and therefore likely to be rather limited both in accuracy and in range of applicability, show nevertheless the crucial fact that the vector $\boldsymbol{\alpha}$, appearing in the mass matrix (1) for quarks and leptons, does indeed rotate as anticipated. The rotation is driven by the strong interactions and can thus be fast enough for the effects we want. Specifically, in the equations (76)—(78), the speed of rotation is governed by the strong coupling ρ_S between the hadron fermion state whose mass is being renormalized to the strong Higgs states, and this coupling can be adjusted to fit data if so desired. Besides, the rotation originates from the scale-dependence of the vacuum and only gets transmitted to the fermion mass matrix (1) via this $\boldsymbol{\alpha}$ by virtue of the appearance in the Yukawa coupling (32) of the weak framon field (8) of which $\boldsymbol{\alpha}$ is a factor. The value of $\boldsymbol{\alpha}$, and also the manner it rotates, is thus independent of the whether it appears in the mass matrix of the leptons or the quarks, or whether these are in the up or down flavour states. If one likes, this is because in the confinement picture of 't Hooft [16] and others [17], quarks and leptons are bound states via $su(2)$ confinement of the fundamental fermion fields ψ (which are what carry the up-down flavour and distinguish between leptons and quarks) with the weak framon (from which they acquire their dependence on the vector $\boldsymbol{\alpha}$). This means therefore that the fermion matrix (1) will remain factorized and universal even as it rotates with scale, a property that is required in [3] to give the mass hierarchy and mixing results we seek.

In addition, these equations are seen to possess a number of intriguing features, significant both for theory and for future phenomenology, which are believed to be generic and to remain valid in a more general treatment, and which will now be examined in the sections which immediately follow.

7 The Fixed Points

The first item of interest is that the equation has a number of fixed points which are likely to figure conspicuously in the mass spectrum and mixing patterns of quarks and leptons.

There will be a fixed point of the trajectory where $\dot{R}, \dot{\theta}$ and $\dot{\phi}$ all vanish. We note then the following:

- from (76), $\dot{R}$ vanishes when $R = 0, -1/2$ or $R = 1$ except for $R = 1$ and $\theta = 0$ when E in the denominator also vanishes (74);
- from (77), $\dot{\theta}$ vanishes when $R = 0$ or when $\theta = 0, \pi/2$;
- from (78), $\dot{\phi}$ vanishes when $\theta = 0$ or when $\phi = 0, \pi/2$;

where we have restricted our interest to the first octant of the unit sphere, the other octants being mere repetitions. Hence we conclude that there are fixed points of the trajectory:

- at $R = 0$ for any values of θ and ϕ ($F0$);
- at $R = -1/2$ and $\theta = 0$ for any value of ϕ ($F1$);
- at $R = 1$, $\theta = \pi/2$, and $\phi = \pi/2$ ($F2$), or $\phi = 0$ ($F3$).

If we linearize around the fixed points, taking deviations $\delta_1, \delta_2, \delta_3$ in the R, θ, ϕ directions respectively (at $F3$, e.g., $R = 1 - \delta_1, \theta = \frac{\pi}{2} - \delta_2, \phi = \delta_3$), we obtain

$$
\begin{aligned}
F0: \qquad & \dot{\delta}_1 = \frac{\rho_S^2}{4\pi^2}\delta_1,\ \dot{\theta} = 0,\ \dot{\phi} = 0 && (81)\\
F1: \qquad & \dot{\delta}_1 = -\frac{5\rho_S^2}{16\pi^2}\delta_1,\ \dot{\delta}_2 = -\frac{7\rho_S^2}{64\pi^2}\delta_2,\ \dot{\phi} = 0 && (82)\\
F2: \qquad & \dot{\delta}_1 = -\frac{9\rho_S^2}{32\pi^2}\delta_1,\ \dot{\delta}_2 = -\frac{3\rho_S^2}{32\pi^2}\delta_2,\quad \dot{\delta}_3 = -\frac{3\rho_S^2}{32\pi^2}\delta_3 && (83)\\
F3: \qquad & \dot{\delta}_1 = -\frac{9\rho_S^2}{32\pi^2}\delta_1,\ \dot{\delta}_2 = -\frac{3\rho_S^2}{32\pi^2}\delta_2,\quad \dot{\delta}_3 = \frac{3\rho_S^2}{32\pi^2}\delta_3 && (84)
\end{aligned}
$$

We can also treat this as an autonomous system of ODEs. If we evaluate the matrix

$$
\begin{pmatrix} \partial_R \dot{R} & \partial_\theta \dot{R} & \partial_\phi \dot{R} \\ \partial_R \dot{\theta} & \partial_\theta \dot{\theta} & \partial_\phi \dot{\theta} \\ \partial_R \dot{\phi} & \partial_\theta \dot{\phi} & \partial_\phi \dot{\phi} \end{pmatrix} \tag{85}
$$

at the fixed point, e.g., $F2$ we get

$$
\frac{\rho_S^2}{32\pi^2} \begin{pmatrix} -9 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix} \tag{86}
$$

and, since the eigenvalues are all negative, it is a stable node.

From these considerations, one easily concludes that $F0$ is an unstable fixed point, $F1$ and $F2$ are stable, and $F3$ is marginal. This means that our trajectory will start off at high scale $\mu = \infty$ near either $F1$ or $F2$, glance off $F3$ if it ever gets near that point, and finish eventually at $F0$ at $\mu = 0$. For this reason we shall refer to $F1$ and $F2$ as high energy fixed points and to $F0$ as the low energy fixed point. Notice that the two high energy fixed points

$F1$ and $F2$ correspond to R of different signs. Recalling the definition of R in (23) above, we see that different signs for R means also different signs for ν_2 (κ_S being by choice positive), where ν_2 is the coefficient of that term in the framon potential $V[\Phi]$ of (12) linking the strong and weak sectors, which is responsible for distorting the framons at vacuum from orthonormality. A positive sign for ν_2 means attraction, pulling the framon axes closer together, and a negative sign means repulsion pushing the framon axes further apart. The two signs, and hence the two fixed points, correspond to two different cases, only one of which need be considered depending on the choice to fit the physical conditions, which we shall have to leave later for phenomenology to decide.

	R	θ	ϕ	$\boldsymbol{\alpha}^\dagger$	μ	stability
$F0$	0	any	any	traj. dep.	0	2 flat directions
$F1$	$-1/2$	0	any	(1,0,0)	∞	1 stable + 1 flat
$F2$	$+1$	$\pi/2$	$\pi/2$	(0,1,0)	∞	stable
$F3$	$+1$	$\pi/2$	0	(0,0,1)	∞	2 stable + 1 unstable

Table 1: RGE fixed points

Our conclusions above for the fixed points are summarized in Table 1. That two particular values of ϕ, namely 0 and $\pi/2$, as listed, should be picked out as fixed points may seem surprising given the original symmetry of the problem under rotation about the $\theta = 0$ axis. This will not be the case when we recall the fact that there still remains in the problem an arbitrary rotation called R'_1 in (59) above by which the trajectory can be rigidly rotated about the $\theta = 0$ axis. By means of this rotation, the two fixed points $F2$ and $F3$ can be placed at any values of ϕ, so long as these stay $\pi/2$ apart.

Although the fixed points listed in the Table 1 were deduced above from the equations (76)—(78), which themselves were derived with only one (strong) Higgs loop, we notice that they all correspond to very special locations of the rotation trajectory, either when the framons are orthogonal ($R = 1, -1/2$) though not of equal lengths, or else when the framons are actually orthonormal ($R = 0$), as can be seen from (25) and (24) above. And this conclusion is a consequence directly of the form of the framon potential (12), not of the subsequent one-loop approximation made in deriving the rotation equations. One believes therefore that the presence of these fixed points is generic and much more general than its derivation given here.

That there are fixed points on the trajectory for $\boldsymbol{\alpha}$ is of crucial importance for the success of the FSM, and indeed of any rotation model, in explaining mass hierarchy and mixing for the following reason. The idea all along is that both mixing and the lower generation masses come from rotation. So, when $\boldsymbol{\alpha}$ approaches a fixed point, rotation will slow down progressively and give smaller and smaller effects. This is easiest to visualize at the high scale end, where the existence of a fixed point would predict that the mass leakage to lower generations will become progressively smaller the higher the mass scale, hence $m_c/m_t < m_s/m_b < m_\mu/m_\tau$, as is experimentally observed. Indeed, the proximity of the heavier quark states to the high energy fixed point is such that the rotation angles involved are small enough for some well-known differential formulae [19] of space curves to apply to the rotation trajectory, leading immediately to most of the salient features in the CKM matrix [20] which are observed in experiment [21]. These effects are so pronounced that even a glance at the data, when interpreted in terms of rotation, would already suggest the existence of such an asymptote [9, 3]. Furthermore, the existence of a rotational fixed point at $\mu = \infty$ would imply that mixing angles are in general smaller for quarks, these being heavier, and therefore nearer to the fixed point, than for leptons, which is again as observed in the experimental CKM and PMNS matrices. The other fixed points on the trajectory listed in Table 1 are of perhaps no less significance but their physical implications are not yet entirely clear to us.

8 The Scale-Dependent Metric

The equations (76), (77) and (78) governing the rotation of the vector $\boldsymbol{\alpha}$ are derived from the manner the vacuum changes under a change of scale, where the vacuum at any scale in turn specifies a set of values for the strong framons Φ. The latter, having themselves been given the geometric significance of frame vectors to begin with, will then specify a metric. We conclude therefore that the metric too will depend on scale. The purpose of this section is to clarify this dependence and some of its physical implications.

Recall first that we have started with a theory invariant under $su(3) \times \widetilde{su}(3)$, but, the vacuum being degenerate, the choice of a particular vacuum breaks this symmetry. Our contention, as argued in [2], was that colour is confining so that the local $su(3)$ symmetry should still be exact. What is broken by the choice of vacuum is thus only the global symmetry $\widetilde{su}(3)$.

The breaking of this $\widetilde{su}(3)$ is reflected in the fact that the vacuum values of the strong framon field Φ, as given in (25) above, are distorted from orthonormality, or that the metric is no longer flat. We wish now to find out explicitly what form this departure from flatness of the metric in $\widetilde{su}(3)$ will take.

Let us start with the reference vacuum in which the value of Φ takes the particularly simple form in (24). This gives the metric in $\widetilde{su}(3)$ as usual as

$$g^{\tilde{a}\tilde{b}} = \sum_a (\phi_a^{\tilde{a}})^* \phi_a^{\tilde{b}}, \tag{87}$$

or in matrix form as

$$\tilde{G}^0 = (\Phi^0_{\mathrm{VAC}})^\dagger \Phi^0_{\mathrm{VAC}} = \frac{\zeta_S^2}{3} \begin{pmatrix} 1+2R & 0 & 0 \\ 0 & 1-R & 0 \\ 0 & 0 & 1-R \end{pmatrix}, \tag{88}$$

which we see is still diagonal but no longer flat.

Next, we recall from (25) that the vacuum value Φ_{VAC} for the general vacuum Φ is obtainable from that of the reference vacuum Φ^0_{VAC} by an $\widetilde{su}(3)$ transformation A^{-1} from the right, so that for the general vacuum,

$$\tilde{G} = A\tilde{G}^0 A^{-1}, \tag{89}$$

which is now not even diagonal.

As the scale μ changes, both the matrix A and the quantity R in $\tilde{G}^0$ will change, and so will the metric. It thus follows that in evaluating any metric-dependent quantities, such as lengths and inner (dot) or outer (cross) products of vectors in generation space, a metric at the appropriate scale will have to be adopted. And as noted already in (5) and (6) in §2, such quantities are required in the rotation scenario for calculating the masses, state vectors and mixing angles of the various quark and lepton states.

To evaluate the lengths or the products of two vectors at the same scale, one takes the metric at that scale. As our notation goes, where vectors carry generation indices as superscripts, it is the inverse of the matrix (89) above that we need to use to evaluate the inner products of vectors, thus

$$G = 3\zeta_S^{-2}(\mu) A(\mu) \begin{pmatrix} (1+2R(\mu))^{-1} & 0 & 0 \\ 0 & (1-R(\mu))^{-1} & 0 \\ 0 & 0 & (1-R(\mu))^{-1} \end{pmatrix} A^{-1}(\mu). \tag{90}$$

However, to evaluate the product between two vectors defined at two different scales, we need further clarification. For example, according to §2 and [3], the CKM matrix element V_{tb} is the inner product between the state vector $\mathbf{t} = \boldsymbol{\alpha}(\mu = m_t)$ of t and the state vector $\mathbf{b} = \boldsymbol{\alpha}(\mu = m_b)$ of b. Since now the metric depends on μ, one has to specify which metric is to be used to evaluate the inner product. This situation is, however, familiar in gravity where the metric also varies from point to point in space-time and one has to specify what is meant by the same (or parallel) vector at different space-time points. It is for this that the geometrical concept of parallel transport is introduced. In gravity, it is the Christoffel symbols usually denoted Γ^c_{ab} which tell us what are parallel vectors at two neighbouring points, which, once known, can be repeated to specify what are parallel vectors along any curve even at a finite distance apart. To calculate then the inner product between two vectors defined at a finite distance from each other along a curve, one can parallelly transport both vectors along the curve to the same point and take their inner product with respect to the local metric valid there. This inner product is symmetric and invariant under parallel transport, as it should be.

In view of this, the answer to our specific question above is then clear. To calculate V_{tb}, we take $\mathbf{t}$ and parallelly transport it along the rotation trajectory from $\mu = m_t$ to $\mu = m_b$, then take its inner product with $\mathbf{b}$ using the metric at $\mu = m_b$. Indeed, since the inner product is symmetric and invariant under parallel transport, one can equally well do the reverse, namely parallelly transport $\mathbf{b}$ from $\mu = m_b$ to $\mu = m_t$ and take its inner product there with $\mathbf{t}$ using the metric at $\mu = m_t$, or else parallelly transport both vectors $\mathbf{t}$ and $\mathbf{b}$ to an arbitrary common scale and evaluate their inner product using the metric valid there. The answer will be the same.

The only question left is how exactly to effect parallel transport in our system, or in other words, what are our equivalents of the Christoffel symbols in gravity. A theorem in metric geometry says that if a metric is covariantly constant (inner products invariant under parallel transport) and torsion free (the metric g_{ab} is symmetric), then the Christoffel symbols are given in terms of the metric by the formula

$$\Gamma^c_{ab} = \tfrac{1}{2} g^{cd}(\partial_a g_{db} + \partial_b g_{ad} - \partial_d g_{ab}), \tag{91}$$

familiar in gravity. This can be applied to our system here, treating $\widetilde{su}(3)$ as "space" and the scale μ as the "time-coordinate", to deduce the corresponding Christoffel symbol and hence parallel transport, as is done in Appendix

B. However, our system being so simple, it is not hard to guess the answer directly, as we actually did first, without going through this calculation. To parallelly transport a vector from the point μ to another point μ' along a trajectory parametrized by μ, one needs just to multiply the vector by the following matrix

$$\begin{aligned}\Pi(\mu \to \mu') \quad &= \quad A(\mu') \begin{pmatrix} \zeta_S' P' & 0 & 0 \\ 0 & \zeta_S' Q' & 0 \\ 0 & 0 & \zeta_S' Q' \end{pmatrix} A^{-1}(\mu') \\ &\times A(\mu) \begin{pmatrix} (\zeta_S P)^{-1} & 0 & 0 \\ 0 & (\zeta_S Q)^{-1} & 0 \\ 0 & 0 & (\zeta_S Q)^{-1} \end{pmatrix} A^{-1}(\mu), \quad (92)\end{aligned}$$

where we have introduced the shorthand notation

$$P = \sqrt{\frac{1+2R}{3}}, \quad Q = \sqrt{\frac{1-R}{3}}, \tag{93}$$

two quantities which will occur frequently in what follows. This is easily seen to preserve inner products between vectors which is, after all, the essential element of the above cited theorem. Then with parallel transport as given by Π in (92) above, it is easy now to evaluate the inner product between two vectors even defined at two different scales.

The cross product between vectors, say **a** and **b**, defined at different scales can most easily be evaluated as follows. We first parallelly transport by Π in (92) both vectors **a** and **b** from the scales where they are defined to the scale corresponding to $R = 0$ where we see from (90) that the metric is flat. We then take the cross product of the two transported vectors there with respect to the flat metric, thus

$$c^i = \epsilon^{ijk} a^j b^k. \tag{94}$$

The cross product **c** can then be parallelly transported to any desired scale by (92) which, as already noted, will preserve both its length and orthogonality with **a** and **b** parallelly transported to the same scale with respect to the metric appropriate for that scale.

Having now worked out how the norms and the products (both dot and cross) of vectors in generation space are to be taken, the latter even between vectors defined at different scales, one can proceed now to evaluate the masses and mixing matrices of both leptons and quarks according to the rules summarized in §2 with the effects of the metric folded in. It would appear at first sight that this might affect much the previous conclusions, e.g., in [3] deduced with the flat metric, but this turns out surprisingly not to be the case.

To be explicit, let us introduce at every scale μ as our "local" reference frame the "Darboux triad" [19] consisting of, first, the vector $\boldsymbol{\alpha}(\mu)$, secondly the tangent vector to the trajectory $\boldsymbol{\tau}(\mu)$ at that scale, and thirdly the normal $\boldsymbol{\nu}(\mu)$ to both the above, all three being normalized and mutually orthogonal with respect to the original flat metric. We take also the matrix $A(\mu)$ explicitly to be that matrix which takes the reference vectors at the reference vacuum to the Darboux triad at μ, thus

$$\boldsymbol{\alpha}(\mu) = A(\mu) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{\tau}(\mu) = A(\mu) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{\nu}(\mu) = A(\mu) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (95)$$

With respect to the "local" metric at μ, namely (90), the Darboux triad is no longer normalized

$$\langle\boldsymbol{\alpha}|\boldsymbol{\alpha}\rangle = \zeta_S^{-2} P^{-2}, \quad \langle\boldsymbol{\tau}|\boldsymbol{\tau}\rangle = \zeta_S^{-2} Q^{-2}, \quad \langle\boldsymbol{\nu}|\boldsymbol{\nu}\rangle = \zeta_S^{-2} Q^{-2}, \quad (96)$$

but remains orthogonal by virtue of the special form of the metric (90).

The state vector $\mathbf{t}$ of t is defined at $\mu = m_t$ to be, as before, the vector parallel to $\boldsymbol{\alpha}$, but has now to be normalized with respect to the local metric at μ, hence

$$\mathbf{t} = \zeta_{St} P_t \boldsymbol{\alpha}(\mu = m_t) = \zeta_{St} P_t A(\mu = m_t) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad (97)$$

where a subscript, t say, on scalar quantities such as ζ_S and P denotes their values evaluated at $\mu = m_t$, but vector quantities which are subject to parallel transport have the value of μ at which they are evaluated explicitly stated. The state vectors $\mathbf{c}$ of c and $\mathbf{u}$ of u remain both orthogonal to $\mathbf{t}$ and to each other by (90), so that

$$\mathbf{c} \propto \Omega_U \boldsymbol{\tau}, \quad \mathbf{u} \propto \Omega_U \boldsymbol{\nu}, \quad (98)$$

where Ω_U is a rotation about the $\mathbf{t}$ vector:

$$\Omega_U = A(\mu = m_t) \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega_U & -\sin\omega_U \\ 0 & \sin\omega_U & \cos\omega_U \end{pmatrix} A^{-1}(\mu = m_t)$$

$$= \quad A(\mu = m_t)\, \Omega_U^0\, A^{-1}(\mu = m_t). \tag{99}$$

Normalizing then the vectors in (98) with respect to the local metric at $\mu = m_t$, we have

$$\begin{aligned} \mathbf{c} &= \zeta_{St} Q_t A(\mu = m_t) \Omega_U^0 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \\ \mathbf{u} &= \zeta_{St} Q_t A(\mu = m_t) \Omega_U^0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \end{aligned} \tag{100}$$

Applying the same arguments to the D quarks as we did to the U quarks above, and introducing the corresponding rotation matrix Ω_D^0, we obtain

$$\begin{aligned} \mathbf{b} &= \zeta_{Sb} P_b A(\mu = m_b) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \\ \mathbf{s} &= \zeta_{Sb} Q_b A(\mu = m_b) \Omega_D^0 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \\ \mathbf{d} &= \zeta_{Sb} Q_b A(\mu = m_b) \Omega_D^0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \end{aligned} \tag{101}$$

The mixing elements in the CKM matrix are given as before as the inner products between the state vectors of the U and D quarks, only now with the difference that, the U and D state vectors being defined at different scales, they have first to be parallelly transported to the same scale, and their inner products have to be evaluated with the metric (90) appropriate for that scale. For instance, as discussed above, to evaluate the CKM element V_{tb}, we can parallelly transport the vector $\mathbf{t}$ in (97) from the scale $\mu = m_t$ to $\mu = m_b$, thus

$$\mathbf{t}(\mu \to m_b) = \Pi(\mu = m_t \to \mu = m_b)\mathbf{t}, \tag{102}$$

using the parallel transport operator given in (92), and then take its inner product with the vector $\mathbf{b}$ defined at $\mu = m_b$ with respect to the metric (90) at the scale $\mu = m_b$. Hence

$$V_{tb} = \mathbf{t}^\dagger(\mu \to m_b) A(\mu = m_b) \begin{pmatrix} \zeta_{Sb}^{-2} P_b^{-2} & 0 & 0 \\ 0 & \zeta_{Sb}^{-2} Q_b^{-2} & 0 \\ 0 & 0 & \zeta_{Sb}^{-2} Q_b^{-2} \end{pmatrix} A^{-1}(\mu = m_b) \mathbf{b}. \tag{103}$$

Substituting the expressions obtained before, one then easily obtains that

$$V_{tb} = (1, 0, 0) A^{-1}(\mu = m_t) A(\mu = m_b) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \mathbf{t} \cdot \mathbf{b}, \tag{104}$$

namely, exactly the same answer as was given before in (7) without incorporating the scale-dependent non-flat metric.

This conclusion that the CKM matrix element remains formally the same in terms of the state vectors with or without incorporating the scale-dependent metric holds not just between the t and b states as above demonstrated, but between any pair of U and D states, as can readily be checked explicitly. The reason for such a simple answer is that the "local" metric at μ is diagonal in these state vectors, so that whether in parallel transport or in forming the inner product, the vectors just get simply multiplied by some factors of $\zeta_S P$ or $\zeta_S Q$, and these all eventually cancel out.

However, this result by itself does not yet mean that the actual values of the CKM matrix will remain the same with or without the scale-dependent metric, for it is still to be verified how the state vectors themselves will be affected by the introduction of the scale-dependent metric. To see this, let us work out as an example explicitly the U states. We recall from (1) that the mass matrix of the U quarks is scale-dependent, so that at $\mu = m_c$, relevant for the evaluation of the state vector of c, the mass matrix reads as

$$m(\mu = m_c) = m_U \, \boldsymbol{\alpha}(\mu = m_c) \boldsymbol{\alpha}^\dagger(\mu = m_c). \tag{105}$$

According to the analysis in §2, the physical mass m_c for the c quark is given by the diagonal element of $m(\mu = m_c)$ taken between the state vector of the c quark. But now, for the case of the scale-dependent metric, the vector $\mathbf{c}$ has first to be parallelly transported from $\mu = m_t$ where it was originally defined in (100) to $\mu = m_c$, and its matrix element of $m(\mu = m_c)$ has to be evaluated with respect to the metric (90) taken again at $\mu = m_c$. These by now familiar operations then yield

$$m_c = m_U \, \zeta_{Sc}^{-2} \, P_c^{-2} \, |\boldsymbol{\alpha}(\mu = m_c) \cdot \mathbf{c}(\mu \to m_c)|^2, \tag{106}$$

an answer differing from that obtained before in (6) in §2 without the scale-dependent metric merely by a factor $\zeta_{Sc}^{-2} P_c^{-2}$. Besides, a repetition of the argument for the masses of the t and u quarks gives the same factor, only now taken at respectively the scales m_t and m_u.

In other words, as far as the calculation of the quark masses is concerned, what the scale-dependent metric has done is to multiply the coefficient $m_T(\mu)$ by the factor $\zeta_S^{-2}(\mu) P^{-2}(\mu)$. And since in the rotation scheme, the state vectors of the lower generations, such as $\mathbf{c}$ and $\mathbf{u}$, also depend on the mass calculation, then even the CKM matrix would be affected by the introduction of the scale-dependent metric.

At least, that would be the case in theory if we know what the value of m_T is and how it varies with μ. However, at the phenomenological level at which present fits to experiment are performed, as in [3], the coefficient m_T is treated as empirical to be fitted to data. Then, it would not matter whether it was m_T or $m_T \zeta^{-2}(\mu) P^{-2}(\mu)$ that is to be fitted empirically, and the two cases, with or without the scale-dependent metric, would yield the same answer; i.e., the phenomenology reviewed in [3] would still apply now in the present case in FSM with a scale-dependent metric. Thus, for example, in [3] fairly good fits were obtained assuming m_T to be approximately independent of μ when the metric was taken to be flat and μ-independent. Then the same fit would be obtained here with the μ-dependent non-flat metric assuming instead that $m_T \zeta^{-2}(\mu) P^{-2}(\mu)$ is approximately μ-independent. Notice, however, that this statement has been shown to be valid only when we concern ourselves just with the mass hierarchy and the mixing pattern. One would hope that in probing further into physical phenomena beyond the above limited domain, then the effect of the scale-dependent metric may make itself manifest, but of this we have as yet found no clear example.

We end this section by noting that although we have performed the analysis on the scale-dependent metric by starting with the rotation equation (76)—(78), very little of the result depend in fact on them. The form of the metric (90) and of the parallel transport (92) in terms of the quantity R is a consequence merely of the framon potential (12), which is in turn the consequence of the double invariance under $su(3) \times su(2) \times u(1)$ and $\widetilde{su}(3) \times \widetilde{su}(2) \times \tilde{u}(1)$ plus renormalizability. And from these premises already all the results discussed would follow. Only the details of how R actually varies with μ would depend on the rotation equations, such as those above. We stress therefore that the result deduced above for the scale-dependent metric is generic for the FSM scheme and not subject to the limitations of the approximations used to derive the equations (76)—(78).

9 The KM CP-violating Phase

One special feature of the rotation equation derived in §6 for the vector $\boldsymbol{\alpha}$ is that the phases of the elements remain unchanged with changing μ. Since the state vectors of the various fermion states are all themselves derived eventually from $\boldsymbol{\alpha}$, though each at some specific value of μ, the above observation would mean that their elements would carry the same phases also. Hence, in taking the inner products between these state vectors to calculate the mixing matrices according to (7) for quarks and a similar expression for leptons, these phases will all cancel and one will obtain real values for all entries. In other words, these mixing matrices will have no Kobayashi-Maskawa phase [11] and be CP-conserving. If this were to be the final answer—and for some time we thought it was—then it would be disappointing, for the possibility of having such a CP-violating phase is one of the most intriguing properties of the 3-generation mixing matrix.

At first sight, it might appear that the above result is just an accident of the particular manner the equation was derived, namely from the insertion of a single strong Higgs-loop into the fermion self-energy as specified in section 5. On reflection, it is soon realized that this is not the case. We recall that the idea all along is that rotation is driven by renormalization effects in the strong sector and only gets transmitted to $\boldsymbol{\alpha}$ in the weak sector via the linkage term V_{WS} in the framon potential $V[\Phi]$ in (12). And it is this rotation which gives rise to the CKM and PMNS mixing matrices. Thus, if these matrices were to develop a Kobayashi-Maskawa phase and hence CP-violations, it would mean that strong interactions where the effect originates, though CP-conserving themselves, are capable somehow of generating CP-violating effects via rotation in the weak sector. This does not seem reasonable. It would appear that for such a mechanism to give CP-violating phases in the mixing matrices, one will have to start with a framework where the strong interactions themselves are CP-violating.

Surprisingly, this last conclusion is not as hopeless as it might seem. We have to recall first that strong interaction as embodied in QCD is *a priori* not CP-conserving since gauge and Lorentz invariance admit in QCD in principle a CP-violating term of the form

$$\mathcal{L}_\theta = -\frac{\theta}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{107}$$

of topological origin, where θ can take any arbitrary value [15]. Simply because of the absence of any strong CP-violations observed in experiment [22], it is customary to declare by fiat that θ in (107) above is zero, or to explain why it has to be less than about 3×10^{-10}. The need to do so is in fact known as the strong CP problem, a classic problem that has been with us for more than 40 years [15]. It would be much more satisfying theoretically if one could start instead with the general action including a theta-angle term, with the coefficient θ not vanishingly small, that the original invariance principles allow and find some theoretical reason why it would not necessarily lead to strong CP-violations in contradiction to experiment. This is what is meant in common usage by a solution to the strong CP-problem.

Now a very attractive feature of the rank-one rotating mass matrix (R2M2) mechanism, for which the present FSM is an example, is that, in addition to offering an explanation for the distinctive fermion mass and mixing patterns observed in experiment as outlined in the introduction, it offers as a by-product also a neat solution to the strong CP problem, transforming the unwanted theta-angle into a CP-violating phase in the CKM matrix where it is actually wanted. How this comes about is as follows.

It has long been known that if one were to make a chiral transformation on a fermionic variable, thus

$$\psi \to \exp(i\alpha\gamma_5)\psi, \tag{108}$$

then the Feynman integral will acquire from the Jacobian of the transformation a factor of the same form as the theta-angle term (107) above, only with θ there replaced by 2α. Since physics should not be changed by a change in integration variables it follows that any theta-angle term at first present in the action can thus be eliminated by a judicious chiral transformation on the quark fields. The trouble, however, is that the chiral transformation will affect also other terms in the action depending on ψ, in particular the quark mass term, which in general will go complex, thus

$$m\bar{\psi}\psi \to m\exp(2i\alpha)\bar{\psi}\tfrac{1}{2}(1+\gamma_5)\psi + m\exp(-2i\alpha)\bar{\psi}\tfrac{1}{2}(1-\gamma_5)\psi. \tag{109}$$

This would normally mean CP-violations again, unless m happens to be zero. But, as far as we understand at present, experiment does not seem to want any quarks to have zero mass.

It is at this point that R2M2 starts to make a difference. We recall that the fermion mass matrix there is of the form (1) which has 2 zero eigenvalues at every μ. A chiral transformation can thus be performed on either of these eigenstates without making the mass term complex, as per (109). In other words, at any μ, any theta-angle term in the action can be eliminated by a chiral transformation without making the mass term complex. Yet, as outlined in §2 and explained in more detail in, for example, [3], this does not require any of the quarks to have physical zero mass since they can all acquire masses by the "leakage" mechanism because of rotation, avoiding thus any conflict as yet with experiment.

But this is not all. One still has to check whether the chiral transformation performed to eliminate θ will affect other terms in the action and lead to CP-violations elsewhere. The interesting thing is that it does, but only in just the right place where it is needed. As analysed in [23], to keep the mass matrix hermitian at all μ, the chiral transformation for eliminating θ has to be performed on the state orthogonal to both the rotating vector $\boldsymbol{\alpha}$ and the tangent to the rotation trajectory at every μ. But this normal direction $\boldsymbol{\nu}(\mu)$ is itself also μ-dependent because of rotation, so that the CKM matrix (7) which involves vectors defined at different μ's, will acquire thereby new phases from the chiral transformations. In other words, the R2M2 mechanism allows the elimination of the theta-angle term, i.e. a solution of the strong CP problem, without making the mass term complex, but only at the cost of introducing a CP-violating phase in the CKM matrix, even with $\boldsymbol{\alpha}$ real to begin with. But this is, of course, a price one is most willing to pay, for this phase was exactly what one was looking for above at the beginning of the section.

What is perhaps most gratifying for the rotation scheme, with the above mechanism for generating the Kobayash-Maskawa phase in the CKM matrix, is that it even yields CP-violations of the correct order of magnitude. It was shown in [23, 3] that starting with a theta-angle of order unity in the strong sector, the rotation scheme will automatically end up with a Jarlskog invariant [24] of order 10^{-5} as is observed in experiment [21] provided that the rotation is adjusted to yield roughly the correct value for say m_c/m_t by leakage as per (4).

The analysis of [3, 23], of which the above is a brief paraphrase, applies in general terms to the situation here in FSM, but in detail has to be modified. The reason is that the analysis there was based on the assumption that in generation space the metric is flat. Here, in the FSM, as detailed in §8 the metric is not flat. Hence, any quantity, such as lengths and products of vectors used in the analysis will have to be recalculated here in terms of the FSM metric. However, rather than repeating the detailed analysis given in [3], inserting the FSM metric wherever appropriate, it would be sufficient here and quicker to take a new tack to indicate how the effect can be calculated.

We shall do so first for the situation when the metric is flat. We recall then that the Darboux triad, set up above in (95) and consisting of the 3 vectors $\boldsymbol{\alpha}$, $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$, forms an orthonormal basis in generation space at every point μ of the rotation trajectory. At every μ, according to the preceding analysis, a chiral transformation is to be performed to eliminate the theta-angle term on the state in the direction of $\boldsymbol{\nu}$ so as to keep both $m(\mu)$ and $m(\mu + \delta\mu)$ hermitian [23]. Since we are now concentrating on the CKM mixing matrix, where only left-handed fields occur, we can replace the chiral transformation (108) by just the phase factor $\exp(-i\theta/2)$.

Writing out then the state vectors, say, of the U-quarks in terms of the Darboux triad as basis at $\mu = m_t$, we have

$$\begin{aligned}
\tilde{\mathbf{t}} &= \boldsymbol{\alpha}(\mu = m_t), \\
\tilde{\mathbf{c}} &= \cos\omega_U \boldsymbol{\tau}(\mu = m_t) + \sin\omega_U \boldsymbol{\nu}(\mu = m_t) e^{-i\theta/2}, \\
\tilde{\mathbf{u}} &= -\sin\omega_U \boldsymbol{\tau}(\mu = m_t) + \cos\omega_U \boldsymbol{\nu}(\mu = m_t) e^{-i\theta/2},
\end{aligned} \tag{110}$$

using the notation introduced in (99) above. Similar expressions are obtained for the D-type quarks with ω_U changed to ω_D and the Darboux triad evaluated instead at $\mu = m_b$.

The CKM mixing matrix can now be expressed as the inner products of the chirally rotated quark states (cf. equation (7))

$$V_{CKM} = \begin{pmatrix} \tilde{\mathbf{u}} \cdot \tilde{\mathbf{d}} & \tilde{\mathbf{u}} \cdot \tilde{\mathbf{s}} & \tilde{\mathbf{u}} \cdot \tilde{\mathbf{b}} \\ \tilde{\mathbf{c}} \cdot \tilde{\mathbf{d}} & \tilde{\mathbf{c}} \cdot \tilde{\mathbf{s}} & \tilde{\mathbf{c}} \cdot \tilde{\mathbf{b}} \\ \tilde{\mathbf{t}} \cdot \tilde{\mathbf{d}} & \tilde{\mathbf{t}} \cdot \tilde{\mathbf{s}} & \tilde{\mathbf{t}} \cdot \tilde{\mathbf{b}} \end{pmatrix}. \tag{111}$$

And because the direction $\boldsymbol{\nu}$ in which the chiral phase occurs varies with scale μ, and the U and D vectors are evaluated at different scales, i.e., m_t and m_b respectively, the inner products appearing in (111) will in general be complex therefore leading to a nonvanishing Jarlskog invariant and hence CP-violation, as concluded before in [23, 14].

What happens now when the metric is not flat in the present FSM situation? With respect to the FSM metric, as already noted, none of the vectors in the Darboux triad are now of unit length although they remain mutually orthogonal. Nevertheless, one can construct a new triad orthonormal with respect to the FSM metric as follows:

$$\begin{aligned}
\boldsymbol{\rho}' &= \boldsymbol{\alpha}/|\boldsymbol{\alpha}| \\
\boldsymbol{\nu}' &= \frac{\boldsymbol{\alpha} \times \boldsymbol{\tau}}{|\boldsymbol{\alpha} \times \boldsymbol{\tau}|} \\
\boldsymbol{\tau}' &= \boldsymbol{\nu}' \times \boldsymbol{\rho}',
\end{aligned} \tag{112}$$

where all lengths and products are to be evaluated in terms of the FSM metric. Again, the state vectors of the various quark states, as constructed in the preceding section with respect to the FSM metric, can be written out in terms of the 3 vectors $\boldsymbol{\rho}', \boldsymbol{\tau}', \boldsymbol{\nu}'$ as in (110) above except that every vector will now have to be primed to indicate that it is defined with respect to the non-flat FSM metric. The same applies to both the U and the D quarks, from which one concludes that the CKM matrix too will look the same as in (111) above, with only the proviso that all inner products are to be evaluated with the FSM metric and that the two vectors involved in the product have first to be parallelly transported to a common scale before the product is taken.

The amusing thing is that, as noted in the preceding section at the end, these inner products actually appear the same whether evaluated with or without the FSM metric. This then means that not only the conclusion that elimination of the theta-angle will lead to a Kobayashi-Maskawa CP-violating phase in the CKM matrix, but also the result that it will lead to a Jarlskog invariant of the same order of magnitude as observed in experiment for a theta-angle of order unity, obtained before in [23] with a flat metric, will both still be preserved in the present case with the FSM metric. And this result is again generic, dependent only on the properties of the vacuum, not on the approximations on which the particular rotation equations (76)—(78) were derived.

10 Summary and Remarks

Let us first briefly summarize what seems to have been achieved by formulating the standard model as a framed gauge theory as has been done in [2] and developed in this paper.

By its very nature as frame vectors, framon fields carry in addition to indices referring to the local gauge symmetries $su(3)$, $su(2)$, and $u(1)$, also indices referring to the global symmetries $\widetilde{su}(3)$, $\widetilde{su}(2)$, and $\tilde{u}(1)$. The action for the framed standard model is to be invariant under both these local and global symmetries. The occurrence of the 3 global symmetries are welcome since they can play the role of fermion generations, up-down flavour, and baryons-lepton number respectively [2], while offering a geometric interpretation for them which was previously unavailable in the usual formulation of the standard model.

The scalar framon fields introduced by (minimal) framing are of two types [2], weak and strong. The weak framon is of the form $\phi_r^{\tilde{r}\tilde{a}} = \alpha^{\tilde{a}}\phi_r^{\tilde{r}}$. It contains a global factor $\boldsymbol{\alpha}$, a vector in 3-D generation space, in addition to the scalar field $\phi_r^{\tilde{r}}$, which is basically the same as that which occurs in the standard electroweak theory. Hence, both leptons and quarks which are, in the confinement picture of 't Hooft [16] and others [17], bound states of the weak framon with fundamental fermion fields, also carry the global factor $\boldsymbol{\alpha}$ and acquire thereby the index $\tilde{a}$ to play the role of the generation index.

The Yukawa couplings constructed with the weak framon will thus automatically give rise to mass matrices of the factorizable (rank-one) form (1) for both quarks and leptons, with $\boldsymbol{\alpha}$ as a factor, which is universal, being a property of the framon, not of the fermion to which it is bound. Such a mass matrix has long been regarded by phenomenologists as a good starting point for understanding the fermion mass hierarchy and mixing [4, 5].

This same vector $\boldsymbol{\alpha}$ which appears in the mass matrices of quarks and leptons gets coupled to the strong framon Φ in the framon potential (12) simply by virtue of the double invariance required under both the local and global symmetries via the so-called ν_2 term. Minimization of this potential (§3) gives a degenerate vacuum which depends on $\boldsymbol{\alpha}$. Hence, if the vacuum changes with scale the vector $\boldsymbol{\alpha}$ will change also (i.e., it rotates).

An explicit sample calculation carried out in §4—6 shows that under renormalization in the strong sector, the vacuum changes with the renormalization scale μ. It then follows that $\boldsymbol{\alpha}$ will rotate with μ. This rotation is a matter only of the vacuum, hence universal, i.e., independent of the fermion type in the mass matrix (1) of which $\boldsymbol{\alpha}$ appears. The rotation is found (§7) further to have fixed points at $\mu = \infty$ and $\mu = 0$.

The mass matrix (1) thus possesses all the properties (i.e., rank-one, rotating, universal and endowed with fixed points) which have been identified in an earlier analysis [3] as needed to reproduce the hierarchical mass and mixing patterns observed in experiment. And, in common to all such rotation schemes, it offers also a solution to the strong CP problem, transforming the theta-angle there into a CP-violating phase in the CKM matrix, giving a Jarlskog invariant of the appropriate order of magnitude for θ of order unity (§9). These observations are not affected by the appearance (§8) of a non-flat scale-dependent metric in generation space in spite of its potential significance in theory.

It seems thus that simply by implementing the idea of framing, i.e. promoting frame vectors into dynamical variables, an idea borrowed from gravity, one seems to have already gone quite some way towards understanding the unusual features of the standard model which have so far been taken for granted. One has yet to see whether the mass and mixing parameters observed in experiment can actually be accommodated in the FSM, and whether the model might lead to some consequences at variance with experiment elsewhere, the latter in relation to the strong framons in particular, as mentioned at the end of §4. Some work has been done already towards those ends, which we hope to report later. But to answer these questions with confidence will clearly be a long process which will ultimately require the participation and scrutiny of the community.

For conclusion, a word of comparison between the present FSM with other models or theories probing what underlies the standard model may be in order. Compared with models or theories of the type known as beyond the standard model (BSM), the FSM is obviously much more modest both in scope and in aim. For example, superstring theory, the prime example of the BSM theories, starts with higher dimensions both of space-time and of the fundamental object (i.e., from point particles to strings or branes) and extensions of the gauge symmetry (e.g., to SUSY), while the FSM remains in 4-dimensional space-time with point particles and the same local gauge symmetry $su(3) \times su(2) \times u(1)$ as the standard model itself. And while superstring theory opens up a new world with almost boundless implications way beyond the confines of particle physics, in cosmology and cosmogony etc., the aims of the FSM remain within particle physics, at least for the present. The virtue of a limited scope, however, is economy, so that for example the FSM, by explaining the mass hierarchy of fermions and their mixing patterns, can look to reducing, and even calculating, some of the many empirical parameters of the standard model in the future; whereas in BSM theories the number of parameters tends further to increase (in SUSY alone, there are already more than a hundred). Nevertheless, there is no obvious contradiction of the tenets of the FSM with any BSM theories, nor is there any obvious obstacle in incorporating the FSM into those larger theories, i.e., if ever one so desires.

However, perhaps the most distinguishing (some might even say revolutionary) feature of the framed standard model is its suggestion that the origin of all those baffling intricacies in flavour physics is to be found not in the far ultraviolet region as most theories would advocate but at the energy accessible already to us today, only still unrecognized by us because it is hidden cleverly by nature from our view. Fermion generation itself is said to be the dual of colour, while the rotation of the fermion mass matrix, which is thought to lead to both mixing and the mass hierarchy, is seen in §3—§6 to be driven by hadronic interactions. And even the CP-violating phase in the CKM matrix is assigned a hadronic origin §9 in the theta-angle term of the old strong CP problem. If that is indeed the case, then it can in future lead to a phenomenological bonanza, for the tests on its tenets can no longer be deferred to infinity as they can be for some other theories, but will have to be confronted by us today.

We are greatly indebted to James Bjorken for many exchanges over the last two years on the subject of mass matrix rotation. Although his approach to deriving rotation is quite different from the FSM here, he has given us great encouragement on our approach and has sharpened considerably our own ideas by probing us with some questions that we should have asked ourselves but did not.

Appendix A

The Higgs mass spectrum can be found straightforwardly by computing the second derivatives of the framon potential $V[\Phi]$. It is not needed in this paper but will be useful in future applications. The resulting 10×10 matrix is block diagonal, with the lower 6×6 block actually diagonal. With a little more manipulation (by elementary row operations), we can further reduce the upper block, so that in the end we get

$$M_H = \begin{pmatrix} 4\lambda_W\zeta_W^2 & 2\zeta_W\zeta_S(\nu_1-\nu_2)\sqrt{\frac{1+2R}{3}} & 2\sqrt{2}\zeta_W\zeta_S\nu_1\sqrt{\frac{1-R}{3}} & 0 \\ * & 4(\kappa_S+\lambda_S)\zeta_S^2\left(\frac{1+2R}{3}\right) & 4\sqrt{2}\lambda_S\zeta_S^2\frac{\sqrt{(1+2R)(1-R)}}{3} & 0 \\ * & * & 4(\kappa_S+2\lambda_S)\zeta_S^2\left(\frac{1-R}{3}\right) & 0 \\ 0 & 0 & 0 & D \end{pmatrix} \tag{113}$$

where

$$D = \kappa_S\zeta_S^2 \begin{pmatrix} 4(\frac{1-R}{3}) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4(\frac{1-R}{3}) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4(\frac{1-R}{3}) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(\frac{2+R}{3}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(\frac{2+R}{3}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(\frac{2+R}{3}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2(\frac{2+R}{3}) \end{pmatrix}, \tag{114}$$

where an $*$ denotes the corresponding symmetric entry, and where the first row (and column) corresponds to the electroweak state h.

The fact that if we wish to find the Higgs masses we need only diagonalize a 3×3 matrix makes it theoretically possible. However, short of actually finding the eigenvalues (other than numerically), which involves solving cubic equations, we can usefully find conditions for which the eigenvalues are positive.

An elementary result from linear algebra says that a real symmetric 3×3 matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ * & a_{22} & a_{23} \\ * & * & a_{33} \end{pmatrix} \tag{115}$$

has positive eigenvalues if and only if

1. $a_{11} > 0$,

2. $\det \begin{pmatrix} a_{11} & a_{12} \\ * & a_{22} \end{pmatrix} > 0$,

3. $\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ * & a_{22} & a_{23} \\ * & * & a_{33} \end{pmatrix} > 0$.

Assuming all coupling constants (except possibly ν_1, ν_2) to be positive, and also $-1/2 < R < 1$, we find necessary and sufficient conditions on the coupling constants for positive Higgs masses:

$$\begin{aligned} 4\lambda_W(\lambda_S + \kappa_S) &> (\nu_1 - \nu_2)^2, \\ 4\lambda_W\kappa_S(\kappa_S + 3\lambda_S) &> \kappa_S(\nu_1 - \nu_2)^2 + 2\lambda_S\nu_2^2 + 2\kappa_S\nu_1^2. \end{aligned} \tag{116}$$

These conditions are satisfied when either all coupling constants are 1, or when ν_i are small compared to the other coupling constants.

We can replace the above two necessary and sufficient conditions by two neater sufficient conditions

$$\begin{aligned} 4\lambda_W(\lambda_S + \kappa_S) &> (\nu_1 - \nu_2)^2, \\ 4\lambda_W\lambda_S\kappa_S &> \kappa_S\nu_1^2 + \lambda_S\nu_2^2. \end{aligned} \tag{117}$$

Note that the above quoted result about positivity of eigenvalues is a direct generalization of the conditions for a local minimum of a surface, in dimension 2, and has a straightforward generalization to dimension $n > 3$.

Appendix B

The space of degenerate vacua is parametrized by $\widetilde{SU}(3)$, but at the moment we are interested in the classes of vacua corresponding to various $\boldsymbol{\alpha}$. Also, so far we have essentially only real $\boldsymbol{\alpha}$, so that the above metric is worked out implicitly in three real dimensions.

However, as $\boldsymbol{\alpha}$ runs, we should take into account the parameter $t = \ln \mu^2$, so that we are really not in $\mathbb{R}^3$, but in $\mathbb{R}^3 \times \mathbb{R}$. This is clear if we think of where the RGE curve lies. In fact, we should think of a metric which is flat in the t or 0 direction, and that the other components depend only on this coordinate, so that the $t =$ constant surfaces have constant metric. This is a (Riemannian) metric of Bianchi Type I, the simplest type.

Below we shall work out explicitly, by calculating the Christoffel symbols, the parallel transport matrix (92). We shall do so, for simplicity, only for the case $A = 1$, i.e., when there is no rotation with scale. In the following we use superscripts to denote vector components, $\boldsymbol{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$.

Take coordinates (t, x, y, z) indexed by $(0, 1, 2, 3)$. We start with the metric

$$\tilde{G}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{1+2R} & 0 & 0 \\ 0 & 0 & \frac{1}{1-R} & 0 \\ 0 & 0 & 0 & \frac{1}{1-R} \end{pmatrix}. \tag{118}$$

From this we find, using equation (91), the non-zero Christoffel symbols, for $i = 2, 3$,

$$\Gamma^0_{11} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{1}{1+2R}\right), \tag{119}$$

$$\Gamma^0_{ii} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{1}{1-R}\right), \tag{120}$$

$$\Gamma^1_{10} = \Gamma^1_{01} = \frac{1}{2}(1+2R)\frac{\partial}{\partial t}\left(\frac{1}{1+2R}\right), \tag{121}$$

$$\Gamma^i_{i0} = \Gamma^i_{0i} = \frac{1}{2}(1-R)\frac{\partial}{\partial t}\left(\frac{1}{1-R}\right). \tag{122}$$

We have the covariant derivative of a vector $\boldsymbol{\alpha}$ in the t direction given by

$$\nabla_t \boldsymbol{\alpha} = \left(\frac{\partial \alpha^i}{\partial t} + \alpha^k \Gamma^i_{0k}\right) e_i. \tag{123}$$

If we now consider just the x component of this we find, using $\Gamma^i_{0j} = 0$ for $i \neq j$,

$$(\nabla_t \boldsymbol{\alpha})^1 = \frac{\partial \alpha^1}{\partial t} + \alpha^1 \Gamma^1_{01}. \tag{124}$$

Now parallel transport means that $(\nabla_t \boldsymbol{\alpha})^1 = 0$ so

$$\frac{\partial \alpha^1}{\partial t} = -\frac{1}{2}(1+2R)\frac{\partial}{\partial t}\left(\frac{1}{1+2R}\right)\alpha^1. \tag{125}$$

Since $\boldsymbol{\alpha}$ and R are functions of t only the partial derivatives are in fact total derivatives and we can now easily integrate

$$\int_t^{t'} \frac{d}{dt}\left(\ln \alpha^1\right) dt = \int_t^{t'} \frac{d}{dt}(\ln(1+2R)^{\frac{1}{2}}) dt, \tag{126}$$

$$\left(\frac{\alpha^{1'}}{\alpha^1}\right) = \left(\frac{1+2R'}{1+2R}\right)^{\frac{1}{2}}. \tag{127}$$

Similarly if we consider the y and z components of the covariant derivative we find

$$\frac{\partial \alpha^2}{\partial t} = -\frac{1}{2}(1-R)\frac{\partial}{\partial t}\left(\frac{1}{1-R}\right)\alpha^2, \tag{128}$$

$$\frac{\partial \alpha^3}{\partial t} = -\frac{1}{2}(1-R)\frac{\partial}{\partial t}\left(\frac{1}{1-R}\right)\alpha^3, \tag{129}$$

which we can integrate to give

$$\left(\frac{\alpha^{2'}}{\alpha^2}\right) = \left(\frac{\alpha^{3'}}{\alpha^3}\right) = \left(\frac{1-R'}{1-R}\right)^{\frac{1}{2}}. \tag{130}$$

We can now write parallel transport (for $A = 1$) in the t direction as

$$\boldsymbol{\alpha}' = \begin{pmatrix} \frac{P'}{P} & 0 & 0 \\ 0 & \frac{Q'}{Q} & 0 \\ 0 & 0 & \frac{Q'}{Q} \end{pmatrix} \boldsymbol{\alpha}. \tag{131}$$

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