Chiral condensates from tau-decay: a critical reappraisal[1]

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Abstract

The saturation of QCD chiral sum rules is reanalyzed in view of the new and complete analysis of the ALEPH experimental data on the difference between vector and axial-vector correlators (V-A). Ordinary finite energy sum rules (FESR) exhibit poor saturation up to energies below the tau-lepton mass. A remarkable improvement is achieved by introducing pinched, as well as minimizing polynomial integral kernels. Both methods are used to determine the dimension $d = 6$ and $d = 8$ vacuum condensates in the Operator Product Expansion, with the results: $\mathcal{O}_6(2.6 \text{ GeV}^2) = -(0.00226 \pm 0.00055) \text{ GeV}^6$, and $\mathcal{O}_8(2.6 \text{ GeV}^2) = -(0.0054 \pm 0.0033) \text{ GeV}^8$ from pinched FESR, and compatible values from the minimizing polynomial FESR. Some higher dimensional condensates are also determined, although we argue against extending the analysis beyond dimension $d = 8$. The value of the finite remainder of the (V-A) correlator at zero momentum is also redetermined: $\bar{\Pi}(0) = -4 \bar{L}_{10} = 0.02579 \pm 0.00023$. The stability and precision of the predictions are significantly improved compared to earlier calculations using the old ALEPH data. Finally, the role and limits of applicability of the Operator Product Expansion in this channel are clarified.

1 Introduction

More than twenty five years ago, Shifman, Vainshtein and Zakharov [1] proposed to use the Operator Product Expansion (OPE) in hadronic current-current correlators to extend asymptotic predictions of QCD to low energies. In this approach there appear universal vacuum expectation values of quark and gluon fields, the so-called vacuum condensates, which have to be extracted from experiment. This extraction is usually carried out by using methods based on dispersion relations. Ultimately, one has to relate error afflicted data in the time-like region to asymptotic QCD in the space-like region. Unfortunately, this task of analytic continuation constitutes, mathematically, a so-called ill-posed problem. In fact, extracting condensates from data is highly sensitive to data errors. Not surprisingly, results from different collaborations have not been always consistent [2]. The main reason for these inconsistencies was frequently the impossibility of estimating reliably the errors in the method. With the release of the final analysis of the precise measurements of the vector (V) and axial-vector (A) spectral functions obtained from tau-lepton decay by the ALEPH collaboration [3], an opportunity has been opened to check the validity of QCD sum rules and the extraction of condensates in the light-quark sector with unprecedented precision. It is therefore appropriate to reanalyze the data, taking into account errors and correlations with the least possible theoretical bias. In this paper we attempt such a critical and conservative appraisal for the interesting case of chiral sum rules. These sum rules involve the difference between the vector and the axial-vector correlators (V-A), which vanishes identically to all orders in perturbative QCD in the chiral limit. In fact, neglecting the light quark masses, the (V-A) two-point function vanishes like $1/q^6$ in the space-like region, where the scale $\mathcal{O}(300\ \mathrm{MeV})$ is set by the four-quark condensates. The interest in these sum rules is twofold. Apart from describing a QCD order parameter, they determine the leading contributions of the matrix elements of the electroweak penguin operators

$$Q_7 = 6(\bar{s}_L\gamma_\mu d_L) \sum_{q=u,d,s} e_q(\bar{q}_R\gamma_\mu q_R)$$

$$Q_8 = -12 \sum_{q=u,d,s} e_q(\bar{s}_L q_R)(\bar{q}_R d_L)$$

where e_q is the charge of the quark q.

In the time-like region, the chiral spectral function $\rho_{V-A}(q^2)$ should vanish for sufficiently large $Q^2 \equiv -q^2$, but judging from the ALEPH data [3] shown in Fig.1 the asymptotic regime of local duality does not seem to have been reached, i.e. the spectral function does not vanish even for the highest momenta attainable in τ-decay.

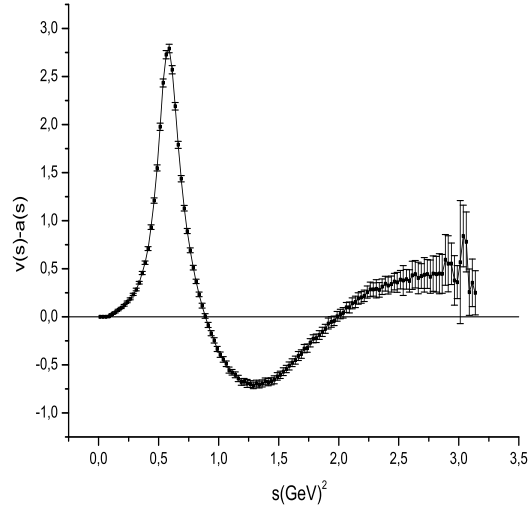Under less stringent assumptions, one would hope that at least global duality

Figure 1: The ALEPH data [3] on the vector minus axial-vector spectral function vs. perturbative QCD (solid line).

should hold in the time-like region. In particular, this should be the case for the Weinberg-type sum rules [4]-[5] which involve the first and second moment of the spectral function. Surprisingly, these sum rules also appear to be poorly convergent. A likely source of duality violation could be some non-perturbative contribution to the correlator (e.g. due to instantons) which falls off exponentially in the space-like region but oscillates in the time-like region. From Fig.1 it is obvious that convergence could be improved by incorporating a weight factor which would reduce the non-asymptotic contribution to the spectral integral. This can be achieved e.g. by considering so called "pinched sum rules" [6] or "minimizing polynomial sum rules" [7]. In view of its importance we have chosen to reanalyze the issue of duality in chiral sum rules on the basis of the new ALEPH measurements. Our analysis leads to results showing a significantly improved accuracy.

2 Finite energy sum rules

We begin by defining the vector and axial-vector current correlators

$$\Pi_{\mu\nu}^{VV}(q^2) = i \int d^4x \; e^{iqx} < 0|T(V_\mu(x)V_\nu^\dagger(0))|0 > \tag{1}$$
$$= (-g_{\mu\nu} \; q^2 + q_\mu q_\nu) \; \Pi_V(q^2) \;,$$

$$\Pi_{\mu\nu}^{AA}(q^2) = i \int d^4x \; e^{iqx} < 0|T(A_\mu(x)A_\nu^\dagger(0))|0 > \tag{2}$$
$$= (-g_{\mu\nu}q^2 + q_\mu q_\nu) \; \Pi_A(q^2) - q_\mu q_\nu \; \Pi_0(q^2) \;,$$

where $V_\mu(x) =: \bar{q}(x)\gamma_\mu q(x) :$, $A_\mu(x) =: \bar{q}(x)\gamma_\mu\gamma_5 q(x) :$, and $q = (u, d)$. Here we shall concentrate on the chiral correlator $\Pi_{V-A} \equiv \Pi_V - \Pi_A$. This correlator vanishes identically in the chiral limit ($m_q = 0$), to all orders in QCD perturbation theory. Renormalon ambiguities are thus avoided. To define our normalization we note that in perturbative QCD

$$\frac{1}{\pi} \text{Im} \Pi_V^{QCD}(s) = \frac{1}{\pi} \text{Im} \Pi_A^{QCD}(s) = \frac{1}{8\pi^2} \left(1 + \frac{\alpha_s}{\pi} + ...\right) \tag{3}$$

Non-perturbative contributions due to vacuum condensates contribute to this two-point function starting with dimension $d = 6$, and involving the four-quark condensate. The Operator Product Expansion (OPE) of the chiral correlator can be written as

$$\Pi(Q^2)|_{V-A} = \sum_{N \geq 3}^{\infty} \frac{1}{Q^{2N}} \; C_{2N}(Q^2, \mu^2) \; < O_{2N}(\mu^2) > \;, \tag{4}$$

with $Q^2 \equiv -q^2$. The scale parameter μ separates the long distance non-perturbative effects associated with the condensates $< O_{2N}(\mu^2) >$ from the short distance effects which are included in the Wilson coefficients $C_{2N}(Q^2, \mu^2)$. The OPE is valid for complex q^2 and moderately large $|q^2|$ sufficiently far away from the positive real axis. Radiative corrections to the dimension $d = 6$ contribution are known [8]-[11]. They depend on the regularization scheme, implying that the value of the condensate itself is a scheme dependent quantity. Explicitly,

$$\Pi(Q^2)|_{V-A} = -\frac{32\pi}{9} \frac{\alpha_s < \bar{q}q >^2}{Q^6} \left\{1 + \frac{\alpha_s(\mu^2)}{4\pi} \left[\frac{244}{12} + \ln\left(\frac{\mu^2}{Q^2}\right)\right]\right\} + O(1/Q^8) \;, \tag{5}$$

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in the $\overline{MS}$ scheme, and assuming vacuum saturation of the four-quark conden-sate. Radiative corrections for $d \geq 8$ are not known. To facilitate comparison with current conventions in the literature it will be convenient to absorb the Wil-son coefficients, including radiative corrections, into the operators, and rewrite Eq.(4) compactly as

$$\Pi(Q^2) = \sum_{N \geq 3}^{\infty} \frac{1}{Q^{2N}} \, \mathcal{O}_{2N}(Q^2) \, , \tag{6}$$

where we are dropping the subscript (V-A) for simplicity. We will be concerned with Finite Energy Sum Rules (FESR), which are nothing but the Cauchy integral

$$\frac{1}{4\pi^2} \int_0^{s_0} ds \, P_N(s) \, [v(s) - a(s)] - f_\pi^2 P_N(m_\pi^2) = -\frac{1}{2\pi i} \oint_{|s|=s_0} ds \, P_N(s) \, \Pi^{QCD}(s) \, , \tag{7}$$

where $P_N(s)$ is an arbitrary polynomial, i.e.

$$P_N(s) = a_0 + a_1 s + a_2 s^2 + \ldots + a_N s^N, \tag{8}$$

$f_\pi = 92.4 \pm 0.26$ MeV [12], and $v(s)$ $(a(s))$ is the vector (axial-vector) spectral function measured by ALEPH in tau-decay [3], normalized to the asymptotic value

$$v(s)_{QCD} = a(s)_{QCD} = \frac{1}{2} \left(1 + \frac{\alpha_s}{\pi} + ... \right) \, . \tag{9}$$

The axial-vector spectral function $a(s)$ does not include the pion pole contribu-tion which is added separately. For most purposes one can work in the chiral limit $m_\pi = 0$, i.e. $P_N(m_\pi^2)$ in Eq.(7) may be replaced by $a_0 f_\pi^2$. The standard FESR follow from the theorem of residues and assuming the Wilson coefficients are just numbers,

$$(-)^{(N+1)} \, \mathcal{O}_{2N}(s_0) = \frac{1}{4\pi^2} \int_0^{s_0} ds \, s^{N-1} \, [v(s) - a(s)] - f_\pi^2 \, \delta_{N1} \quad (N = 1, 2, 3...) \, , \tag{10}$$

where the index N has been rearranged for convenience. Strictly speaking, Eq.(10) only holds for the constant terms of the Wilson coefficients. Otherwise condensates of lower or higher dimension get mixed when taking into account radiative corrections due to the logarithmic terms $(\ln(\mu^2/Q^2)$ or higher). How-ever this mixing of operators of different dimensions occurs only at order α_s^2 in a given FESR [13]. For dimensional reasons the contribution of the operators of higher dimension in Eq.(10) vanishes for large s_0, while that of operators of lower dimension increases with s_0. The latter contribution is particularly disturbing for operators of high dimension. As the logarithmic terms of the relevant Wilson coefficients are not known (except the one for $\mathcal{O}_6$) we will ne-glect the contribution of operators of dimension unequal to $2N$ in Eq.(10). This approximation is inherent to every sum rule analysis of the τ-data and can only

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be justified *a posteriori* by demonstrating that the right hand side of Eq.(10) is (almost) constant. We will, however, examine the order of magnitude of the mixing to be expected by using Eq.(5) to estimate the effect of the radiative corrections of $\mathcal{O}_6$.

For $N = 1, 2$ Eq.(10) leads to the first two (Finite Energy) Weinberg sum rules, while for $N = 3, 4$ the sum rules project out the $d = 6$ and $d = 8$ vacuum condensates, respectively (notice that in the chiral limit $\mathcal{O}_2 = \mathcal{O}_4 = 0$). In order to check the convergence of the sum rules we consider the first Weinberg sum rule

$$W_1(s_0) \equiv \frac{1}{4\pi^2} \int_0^{s_0} ds \, [v(s) - a(s)] = f_\pi^2 \qquad (11)$$

Strictly speaking $s_0 \to \infty$, but precocious scaling would imply that the sum rule should be saturated at moderate values of s_0. From Fig.2, which shows $W_1(s_0)$, one can see that this is clearly not the case, even at the highest energies accessible in τ-decay. This lack of precocious scaling in the sum rule can be
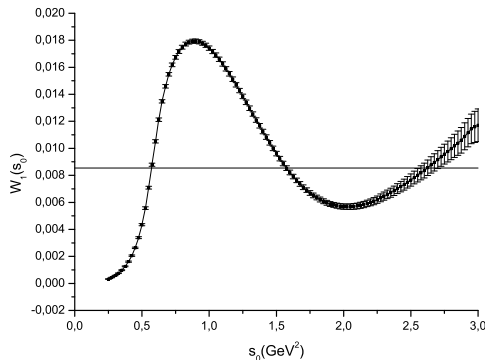

Figure 2: The first Weinberg sum rule, Eq.(11) as a function of s_0 together with $f_\pi^2 = (0.00854 \pm 0.00005) \, \text{GeV}^2$ (solid line).

simply explained by looking at the measured spectral function in Fig.1. If the spectral function had reached its approximate asymptotic value (i. e. zero) starting, let us say, from $s = 2 \, \text{GeV}^2$ then the spectral integral of Eq.(11) would have yielded f_π^2 for all $s_0 \geq 2 \, \text{GeV}^2$. This observation shows us a way out of the dilemma by turning to the more general sum rules of Eq.(7). One can choose the polynomial $P_N(s)$ in the sum rule Eq.(7) in such a way that the problematic contribution of the integration region near the endpoint of the physical cut is minimized. This method addresses two problems at the same time, the first being that experimental errors of the spectral functions grow considerably near the limit of phase space, and the second, that the asymptotic QCD formula is unreliable on the contour region near the physical cut. In following this

method we will employ two types of sum rules, pinched FESR and minimizing polynomial sum rules.

3   Pinched sum rules

We begin by considering a linear combination of the first two Weinberg sum rules

$$\bar{W}_1(s_0) \equiv \frac{1}{4\pi^2} \int_0^{s_0} ds \ (1 - \frac{s}{s_0}) \ [v(s) - a(s)] = f_\pi^2. \tag{12}$$

The left hand side of Eq.(12) as a function of s_0 is shown in Fig. 3, together with the right hand side. It is very reassuring that the sum rule appears to be
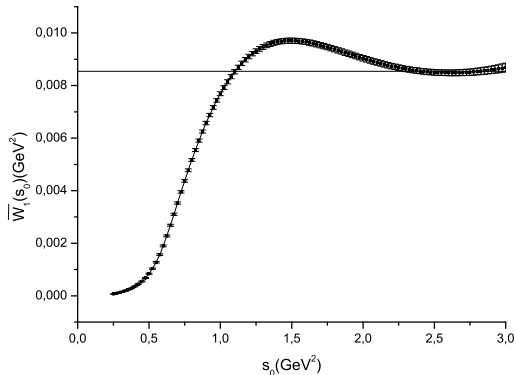

Figure 3: Pinched Weinberg sum rule, Eq.(12), as a function of s_0. The solid line is $f_\pi^2 = (0.00854 \pm 0.00005) \ \mathrm{GeV}^2$.

saturated for $s_0 > 2.3 \ \mathrm{GeV}^2$. We note that the error band is about a factor three smaller than that found in a similar analysis [14] using the old ALEPH or OPAL data [15]-[16]. The influence of the logarithmic dependence of $\mathcal{O}_6$ in this sum rule is about $4 \times 10^{-6} \ \mathrm{GeV}^2$ in the region of the saturation with s_0.

Motivated by this success, we impose the Weinberg sum rules as constraints in other pinched finite energy sum rules involving different moments. To be precise, we assume that there are no operators of dimension $d = 2$ nor $d = 4$, which is true in the chiral limit, together with the condition that the FESR involves factors of $(1 - \frac{s}{s_0})$ so as to minimize the contribution near the cut. In

this way, we write the Das-Mathur-Okubo [5] sum rule in the form

$$\bar{\Pi}(0) = \frac{1}{4\pi^2} \int_0^{s_0} \frac{ds}{s} \, (1 - \frac{s}{s_0})^2 \, [v(s) - a(s)] + \frac{2f_\pi^2}{s_0} \, , \tag{13}$$

where $\bar{\Pi}(0)$ is the finite remainder of the chiral correlator at zero momentum. It is related to $\bar{L}_{10}$, the counter term of the $O(p^4)$ Lagrangian of chiral perturbation theory [17], which has been calculated independently,

$$\bar{\Pi}(0) = -4 \, \bar{L}_{10} = \left[\frac{1}{3} f_\pi^2 < r_\pi^2 > -F_A \right] = 0.026 \pm 0.001 \, , \tag{14}$$

where $< r_\pi^2 >$ is the electromagnetic mean squared radius of the pion, $< r_\pi^2 >= 0.439 \pm 0.008 \, \text{fm}^2$ [18], and F_A is the axial-vector coupling measured in radiative pion decay, $F_A = 0.0058 \pm 0.0008$ [12]. From Fig. 4 we see that this sum rule is even more remarkably satisfied. This can be understood by noting that the sum rule emphasizes less the high-s region where duality violation competes against stability. Numerically, we find from the DMO sum rule
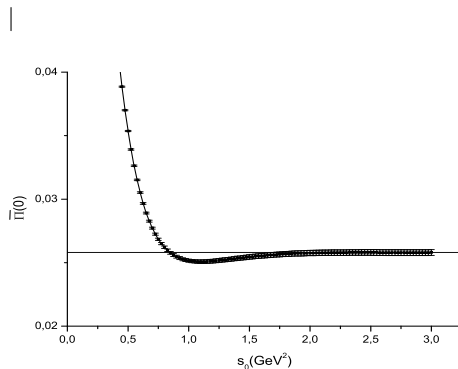

Figure 4: The finite remainder of the chiral correlator at zero momentum, $\bar{\Pi}(0)$, from Eq.(13) as a function of s_0. The solid line is the central value $\bar{\Pi}(0) = 0.02579$.

$$\bar{\Pi}(0) = 0.02579 \pm 0.00023 \, , \tag{15}$$

a result showing a remarkable accuracy for a strong interaction parameter. In this particular sum rule the contribution from the logarithmic term of $\mathcal{O}_6$ vanishes.

Next, we turn our attention to the extraction of the condensates with the help of the pinched sum rules. The philosophy of our calculation is threefold, viz.

(i) to assume that dimension $d = 2$ and $d = 4$ operators are absent in the OPE of the chiral current , (ii) to require that the polynomial projects out only one operator of the OPE at a time, and (iii) to require that the polynomial and its first derivative vanish on the integration contour of radius $|s| = s_0$. For the caveats on point (ii) of this approach see the text above. In this way one obtains for $N \geq 3$ the sum rules (ignoring the energy dependence in the Wilson coefficients)

$$
\mathcal{O}_{2N}(s_0) = (-1)^{N-1} \left\{ \frac{1}{4\pi^2} \int_0^{s_0} ds [(N-2)s_0^{N-1} - (N-1)s_0^{N-2}s + s^{N-1}] \right.
$$

$$
\left. \times [v(s) - a(s)] - (N-2)s_0^{N-1}f_\pi^2 \right\} \qquad (N \geq 3) . \tag{16}
$$

Note that there is always a pinch factor $(s - s_0)^2$ in the polynomial. We use once again the new ALEPH spectral function and error correlations in this sum rule. The crucial point of the extraction of the condensates is a careful inspection of the stability of the result with respect to the variation of all parameters in the analysis. In our case there is only one parameter, namely the radius s_0. This fact contrasts positively with other approaches based on Laplace sum rules which involve at least two parameters, and in addition do not project out just one single operator, even for $s_0 \to \infty$. As for stability, if a meaningful value of $\mathcal{O}_{2N}$ is to be extracted we would expect the r.h.s. of Eq.(16) to be constant for all s_0 larger than some some critical value . We can call this requirement **strong stability**. It is best discussed on the basis of the figures below which show the predictions for various condensates. Figure 5 shows the result for the dimension $d = 6$ condensate. There is an obvious stability region: $2.3 \leq s_0(\text{GeV}^2) \leq 3$
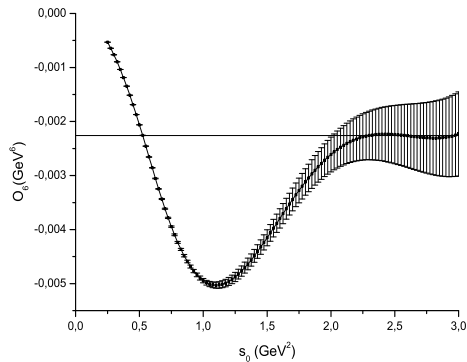

Figure 5: The dimension $d = 6$ condensate, $\mathcal{O}_6$, from Eq.(16) as a function of s_0. The solid line is the central value $\mathcal{O}_6 = -0.00226 \, \text{GeV}^6$.
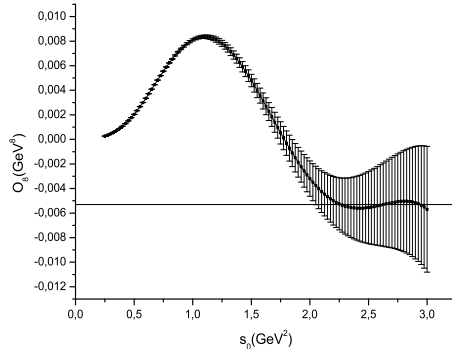
Figure 6: The dimension $d = 8$ condensate, $\mathcal{O}_8$, from Eq.(18) as a function of s_0. The solid line is the central value $\mathcal{O}_6 = -0.0053\,\mathrm{GeV}^8$.

from where we find

$$\mathcal{O}_6(2.7\,\mathrm{GeV}^2) = -(0.00226 \pm 0.00055)\,\mathrm{GeV}^6 \ . \tag{17}$$

This value is consistent with the one found from the vacuum saturation approximation $\mathcal{O}_6^{\mathrm{VS}} = -0.0020\,\mathrm{GeV}^6$ from Eq.(5) with $< \bar{q}q > (s_0) = -0.019\,\mathrm{GeV}^3$, and $\alpha(s_0)/\pi = 0.1$, but it is significantly lower than the one found in some earlier analyses based on the old, incomplete ALEPH data; e.g. $\mathcal{O}_6 = -(0.004 \pm 0.001)\,\mathrm{GeV}^6$, obtained in [14] using a similar stability criterion as here. The contribution of the logarithmic term from the $\mathcal{O}_6$ coefficient in (17), in the region of s_0 considered here, is about $8 \times 10^{-6}\,\mathrm{GeV}^6$ and hence negligible within the errors.

To facilitate a comparison with an alternative type of sum rules discussed in the next section, we give the sum rule for $\mathcal{O}_8$ explicitly

$$\mathcal{O}_8(s_0) = -\frac{1}{4\pi^2} \int_0^{s_0} ds\, [2s_0^3 - 3s_0^2 s + s^3][v(s) - a(s)] + 2s_0^3 f_\pi^2 \ . \tag{18}$$

(Notice that $2s_0^3 - 3s_0^2 s + s^3 = (s_0 - s)^2\,(s + 2s_0)$). The result for $\mathcal{O}_8$ is shown in Fig.6. In spite of the larger errors there is still a distinct region of duality in the interval: $2.3 \leq s_0(\mathrm{GeV}^2) \leq 3$, which yields

$$\mathcal{O}_8(2.6\,\mathrm{GeV}^2) = -(0.0054 \pm 0.0033)\,\mathrm{GeV}^8 \tag{19}$$

Both the sign and the numerical value of this condensate are controversial (see e.g. Table 1 in [2]). Our result agrees within errors with e.g. that of [9]-[10], and that of [19] but disagrees in sign with [20], [21], [22], and with the result of the minimal hadronic approximation of large N_c [2]. In [20] the sum rules were

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evaluated at very low values of s_0, mainly because the data at that time was considered to be too inaccurate at higher s_0. Fortunately, this state of affairs has now changed with the new ALEPH analysis. Eq.(5) can be used to estimate the effect on $\mathcal{O}_8$ due to the mixing of $\mathcal{O}_6$ arising from the logarithmic term. We find a correction of about -3×10^{-4} GeV^8 which is negligible compared to the data error in Eq.(19). The results of the sum rules for $\mathcal{O}_{10}$, and $\mathcal{O}_{12}$ as given by Eq.(16) are shown in Figs. 7 and 8, respectively. The strong stability obtained
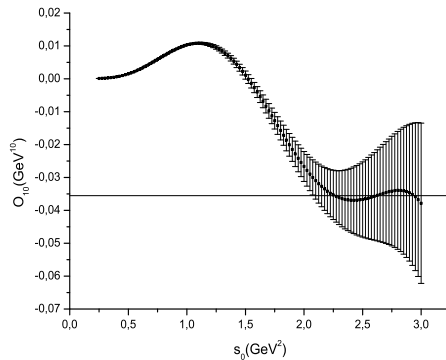

Figure 7: The dimension $d = 10$ condensate, $\mathcal{O}_{10}$, from Eq.(16) as a function of s_0. The solid line is the central value $\mathcal{O}_{10} = 0.0355$ GeV^{10}.

so far is now no longer obvious, and we find at $s_0 \approx 2.5$ GeV^2

$$\mathcal{O}_{10}(2.5 \text{ GeV}^2) = (0.036 \pm 0.014) \text{ GeV}^{10} \tag{20}$$

$$\mathcal{O}_{12}(2.5 \text{ GeV}^2) = -(0.12 \pm 0.05) \text{ GeV}^{12} . \tag{21}$$

These results, though, should be taken *cum grano salis*, to wit. Because the OPE is an asymptotic series, the upper limit of the integration range, s_0, should increase as the dimension of the operators increases. The comparison of the numerical results for $\mathcal{O}_6$, $\mathcal{O}_8$, $\mathcal{O}_{10}$, and $\mathcal{O}_{12}$ indicates that at a scale of about $|s| = 1$ GeV^2 the OPE starts to diverge at dimension $d = 10$. In addition, the problem of the mixing of operators of different dimensions becomes more severe for higher dimensional operators. For these reasons we believe that it is rather meaningless to extract quantitative results for condensates of dimension higher than $d = 8$ from τ-decay spectral functions.
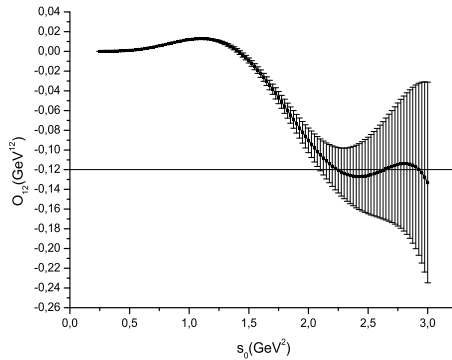
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Figure 8: The dimension $d = 12$ condensate, $\mathcal{O}_{12}$, from Eq.(16) as a function of s_0. The solid line is the central value $\mathcal{O}_{12} = -0.12 \, \mathrm{GeV}^{12}$.

4 Minimizing Polynomial Sum Rules

The starting point in this analysis is the general sum rule Eq.(7). The polynomial can be chosen in such a way that the problematic contribution of the integration region near the endpoint of the physical cut is minimized. With the normalization condition

$$P_N \left(s = 0 \right) \, = \, 1, \tag{22}$$

we require that the polynomial $P_N(s)$ should minimize the contribution of the continuum in the range $[c, s_0]$ in a least square sense, i.e.

$$\int_c^{s_0} s^k P_N(s) \, ds = 0 \qquad (k = 0, \ldots N - 1) \,, \tag{23}$$

The parameter c can be chosen freely in the interval $0 < c < s_0$. On the basis of the spectral function of Fig.1, a reasonable choice would be $2 \, \mathrm{GeV}^2 \leq c \leq s_0 \sim 3 \, \mathrm{GeV}^2$. In a sense, these polynomials are a generalization of pinched moments, and the $P_N(s)$ will approach $(s - s_0)^N$ when $c \rightarrow s_0$. While pinched moments eliminate the contribution on the physical real axis at a single point, our polynomials tend to eliminate a whole region (from c to s_0). The degree N of the polynomial can be chosen appropriately to project out certain terms in the OPE, Eq.(6). The polynomials obtained in this way are closely related to the Legendre polynomials as follows. Let us introduce the variable

$$x(s) \equiv \frac{2s - (s_0 + c)}{(s_0 - c)} = \frac{2s}{(s_0 - c)} + x(0) \,, \tag{24}$$

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and define the polynomials as

$$P_N(s) = \frac{L_N[x(s)]}{L_N[x(0)]} , \qquad (25)$$

where $L_N(x)$ are the Legendre polynomials

$$L_N(x) = \frac{1}{2^N N!} \frac{d^N}{dx^N} (x^2 - 1)^N . \qquad (26)$$

We give here only the first few minimizing polynomials

$$P_1(s) = 1 - \frac{2s}{(s_0 + c)} \qquad (27)$$

$$P_2(s) = \frac{3(2s - (s_0 + c))^2 - (s_0 - c)^2}{3(s_0 + c)^2 - (s_0 - c)^2} \qquad (28)$$

$$P_3(s) = \frac{5(2s - (s_0 + c))^3 - 3(s_0 - c)^2(2s - (s_0 + c))}{-5(s_0 + c)^3 + 3(s_0 - c)^2(s_0 + c)} \qquad (29)$$

If the polynomials are expressed as in Eq. (8), then from Eqs.(7) and (10) there follows the sum rule

$$a_0 d_0 + a_1 d_1 + ... + a_N d_N - f_\pi^2 = a_2 \mathcal{O}_6 - a_3 \mathcal{O}_8 + ... + (-1)^N a_N \mathcal{O}_{2N+2} , \qquad (30)$$

where the constants

$$d_N = \frac{1}{4\pi^2} \int_0^{s_0} ds\, s^N \, (v(s) - a(s)) \qquad (31)$$

are to be determined from the ALEPH data.

We begin with the $\mathcal{O}_6$ condensate and obtain from Eq.(30) the sum rule

$$d_0 + a_1 d_1 + a_2 d_2 - f_\pi^2 = a_2 \mathcal{O}_6 . \qquad (32)$$

After substituting P_2 from Eq.(28) this sum rule becomes

$$\mathcal{O}_6(s_0) = \frac{1}{6}(s_0^2 + 4s_0 c + c^2)\left(d_0 - f_\pi^2\right) - (s_0 + c)\, d_1 + d_2 . \qquad (33)$$

In the sequel we choose the initial value $s_0 = 3$ GeV^2, but will subsequently change it in the range $2.5 \leq s_0(\mathrm{GeV}^2) \leq 3$ in order to verify the criterion of **strong stability**. The condensate $\mathcal{O}_6$ from Eq.(33) is plotted in Fig. 9 as a function of c. One can appreciate a stable point for $c = 2.5$ GeV^2. Fixing c at this point of minimal sensitivity has been discussed previously in other FESR applications, e.g. in [7]. For $c = 2.5$ GeV^2, and $s_0 = 3$ GeV^2 we obtain from Eq.(33) the result:

$$\mathcal{O}_6(3\ \mathrm{GeV}^6) = -(0.0023 \pm 0.0013)\ \mathrm{GeV}^6 , \qquad (34)$$
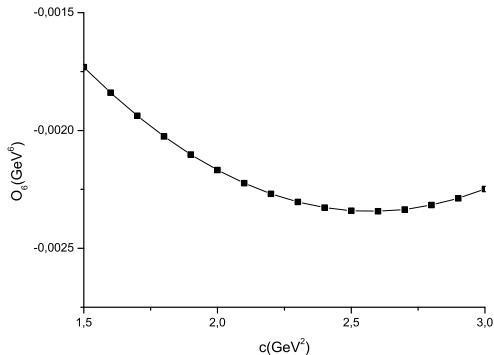
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Figure 9: The condensate $\mathcal{O}_6$ as a function of the parameter c.

which compares well within errors with the previous result from the pinched sum rule, Eq.(17). We have tested positively the stability around this point by choosing different values of s_0 in the range $2.5 \leq s_0(\mathrm{GeV}^2) \leq 3$. For instance, using $s_0 = 2.5 \mathrm{\ GeV}^2$ we obtain $O_6(2.5 \mathrm{\ GeV}^6) = -(0.00224 \pm 0.00046) \mathrm{\ GeV}^6$. Other values of s_0 in the above range lead to similar results, well in agreement within errors with Eq.(34), thus satisfying the criterion of **strong stability**.

Next, we consider the $\mathcal{O}_8$ condensate, and use Eq.(30) to obtain the sum rule

$$d_0 + a_1 d_1 + a_2 d_2 + a_3 d_3 - f_\pi^2 = a_2 \mathcal{O}_6 - a_3 \mathcal{O}_8 \ . \tag{35}$$

The presence of $\mathcal{O}_6$ in the sum rule for $\mathcal{O}_8$ can be dealt with in two ways. One could insert the numerical value of $\mathcal{O}_6$, e.g. Eq.(34), as obtained from its own sum rule, or rather substitute the analytic expression of the sum rule itself. The latter procedure yields the best possible results in terms of stability and accuracy, and leads to the sum rule

$$\mathcal{O}_8(s_0) = -\frac{1}{5} \left[s_0(s_0 + 2c)^2 + c^3 \right] (d_0 - f_\pi^2) + \frac{3}{10} \left[3(s_0^2 + c^2) + 4s_0 c \right] d_1 - d_3 \ . \tag{36}$$

Notice the welcome absence of the term involving the second moment, i.e. d_2; it cancels out when substituting in Eq.(35) the sum rule for $\mathcal{O}_6$, Eq.(33). Choosing again the initial value $s_0 = 3 \mathrm{\ GeV}^2$, one obtains for $\mathcal{O}_8$ the results shown in Fig. 10. One can see again a stability region near $c = 2.5 \mathrm{\ GeV}^2$, leading to the result

$$\mathcal{O}_8(3 \mathrm{\ GeV}^2) = -(0.0048 \pm 0.0039) \mathrm{\ GeV}^8 \tag{37}$$

It is worth mentioning that the polynomial coefficients entering the sum rule Eq.(36) differ significantly from the ones in the corresponding pinched sum rule Eq.(18). It is therefore reassuring that both results for $\mathcal{O}_8$ are compatible.

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To check for **strong stability** we have, once again, varied s_0 in the range $2.5 \leq s_0(\text{GeV}^2) \leq 3$. For $s_0 = 2.5 \text{ GeV}^2$ we find $O_8(2.5 \text{ GeV}^8) = -(0.0056 \pm 0.0024) \text{ GeV}^8$, and similar results for other values of s_0 in the above range. Thus, the criterion of **strong stability** is again satisfied, albeit within large errors. Proceeding beyond dimension $d = 8$ is marred by the same problems mentioned at the end of Section 3; the minimizing polynomial FESR do not avoid the divergence of the OPE, nor the increasing importance of operator mixing.
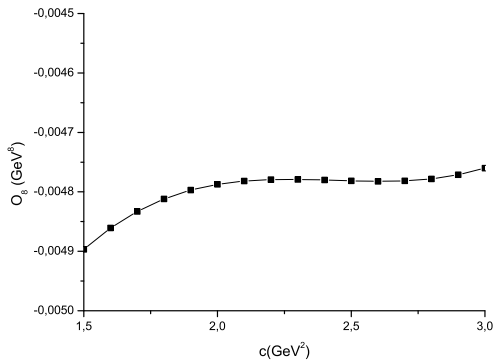

Figure 10: $\mathcal{O}_8$ as a function of the parameter c.

5 Conclusion

The final ALEPH data for the chiral spectral function $v(s) - a(s)$ shows clearly that this spectral function has not yet reached its asymptotic form dictated by perturbative QCD, i.e. it does not vanish, even at the highest energies attainable in τ-decay. If the asymptotic regime had been reached precociously, let us say at $Q^2 \simeq 2 \text{ GeV}^2$, then it would have been straightforward to calculate the non-perturbative condensates with the help of the Cauchy Integral. Since this is not the case, some method to improve convergence must be applied. We have shown that in the framework of FESR this can be done by suitably reducing the impact of the high energy region in the dispersive integral, either by using pinched sum rules or by using minimizing polynomial sum rules. We first used the data in a pinched linear combination of the first two Weinberg sum rules which follow from the fact that there are no dimension $d = 2$ and $d = 4$ operators contributing to the chiral correlator to demonstrate the precocious saturation of the sum rule and the remarkable effectiveness of the method. Motivated by

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this success, we determined a number of QCD condensates by making maximal use of the fact that there are no dimension $d = 2$ and $d = 4$ operators and requiring **strong stability** for both methods, i.e. we varied the radius s_0 in the Cauchy integral beginning at the end of τ-decay phase space and required that the condensates calculated from the data should be reasonably constant for all s_0 in some finite region including the end of phase space. We do not assume (as is done in most FESR calculations) that the dispersive integral vanishes from $s_0 \to \infty$. By showing that there is "strong stability" i.e. precocious saturation of the FESR we prove that this region contributes only negligibly. It would indeed be surprising if the observed stability would disappear for s_0 larger than the end of phase space. We do, however, have to make the assumption, inherent in all sum rule analyses of τ-decay, that unknown $O(\alpha_s^2)$ effects of mixing of operators of different dimensions are negligible for the relevant duality range $2.5 \text{ GeV}^2 \lesssim s_0 \lesssim 3 \text{ GeV}^2$. We have checked explicitly that in all the cases considered in this work this is the situation when one takes into account the logarithmic term of the dimension six Wilson coefficient. The results for $\mathcal{O}_6$ and $\mathcal{O}_8$ satisfy this strong stability criterion as is best seen from the figures. Extraction of higher condensates of dimension $d \geq 10$ leave room for interpretation, but the conclusion that they grow rapidly with dimensionality is rather obvious. Together with the increasing importance of operator mixing, it makes the extraction of these condensates a difficult exercise. Our result that $\mathcal{O}_6$ and $\mathcal{O}_8$ have the same sign is in conflict with some of the earlier determinations based on the incomplete ALEPH data but agrees with others (see e.g. [2] for a comparative summary).

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